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SPECIAL ISSUE
SCIENCE IN THE FOREST, SCIENCE IN THE PAST

Different clusters of text from ancient China, different mathematical ontologies

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Sources attesting to mathematical activities in ancient China form at least four distinct clusters of texts, bespeaking at least four different—though overlapping—ways of practicing mathematics. I will focus on two such sets of documents: the canons that in the seventh century constituted one of the two curricula taught in the Imperial "School of Mathematics," and manuscripts recently excavated from tombs sealed in the last centuries BCE. I will argue that these two sets of documents testify to two different ways of practicing mathematics, which related to different material practices. Accordingly, we can perceive that mathematical objects were shaped and explored in different ways, with significant consequences for the knowledge produced.

Keywords: arithmetical operations, division, multiplication, numbers, history of mathematics in China, practices, terminology, ontological assumptions

Dedicated to the memory of Michel Kerszberg, whose mind faded away all too early.

In the different contexts in which mathematics has been practiced, we can observe a certain diversity in the types of entities actors' work has actually brought into play. So far as numbers are concerned, we note that some groups of actors in the ancient world worked only with integers as such (this is notably the case in Euclid's *Elements* [1956]), while others, as we shall see, took into consideration quantities composed with integers and fractions, and also sometimes measurement units.

Moreover, when *we* might think that different actors are dealing with the same kind of entity, that may well not be the case for *them*. For instance, we can recognize that the circle figures in all extant mathematical corpora but only in some contexts did its center play a prominent role. Further, the geometrical constructions carried out on circles, and also the statements considered, differ depending on the contexts.

So the *nature* of the entities that actors deal with in the context of a given mathematical activity cannot be taken for granted. This statement holds true in general and for the ancient world in particular, though there the scarcity of sources makes the nature of the mathematical entities considered especially difficult to address. This will be the main focus of this study.

The issue is even trickier when we ask what the actors' own ontological ideas and assumptions were, for our texts are seldom explicit on that topic. In Euclid's *Elements*, for instance, we have definitions, postulates, and axioms, followed by theorems and problems with proofs, yet no second-order statements addressing ontological issues. Tackling Euclid's own views on that subject raises a methodological problem. This was already a point of contention in antiquity.

In the framework of this volume, we can certainly not afford to project our observers' assumptions on the texts, for that would erase the diversity within and among them. We should also refrain from seeking answers in ancient Greek writings that seem to *us* to belong to the same context as Euclid's *Elements* and that yield evidence on the issue of the ontology of mathematical entities. Such a hermeneutical practice is doomed to shape Greek antiquity as more uniform than it actually was. I shall argue that this, too, would be anachronistic since, after all, *we* are those who shape the writings as pertaining to the same context

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and then interpret some of them in the light of the others. Is it surprising that as the result of such a practice, we end up speaking of "the Greeks," or elsewhere, of "the Chinese"? Perhaps our method puts the rabbit in the hat in which, as if by magic, we find it at the end of the operation. What other method can we use? I shall suggest that to address ontological questions, we should (and actually can) rely on corpora shaped by groups of actors themselves. This element of method is correlated with an assumption that holds that the answers to our ontological questions should be sought not in general but rather only in specific contexts. This principle will turn out to be justified by the facts that it will enable us to perceive.

But when corpora shaped by actors do not explicitly discuss the ontological questions that interest us, how should we proceed? One of the goals of this essay is to suggest a possible way ahead.

I shall use a case study to show how actors' ontological positions are reflected in their technical language and their material practices insofar as they can be reconstructed. This case study will rely on the corpus of Chinese mathematical canons and commentaries that, from 656 CE on, were used as textbooks in one of the two curricula taught in the Imperial "School of Mathematics," established in the first decades of the Tang Dynasty (618–907). Some of the commentaries that had been selected and edited in this context include terse ontological statements. I will show how we can suggest an interpretation for these statements, relying on an observation of the technical language and material practices shaped to carry out mathematical activity in this framework. The interpretation that I will offer will thus reveal a correlation between ontological statements on the one hand, and features of the technical languages and material practices on the other hand. Hence, this suggests that we can rely on a close study of technical languages and practices to grasp at least some aspects of actors' ontological views.

I will then apply the same method to another cluster of Chinese mathematical texts—that is, mathematical manuscripts dating from early imperial China, some of which were recently found in tombs sealed in the last centuries BCE and others bought on the antiquities market. Again, these manuscripts do not contain any statement that makes explicit aspects of the scribes' ontological ideas. However, both the technical language these documents use and the features of the material practices to which they attest do not appear to reflect ontological assumptions similar to those to which the first corpus of writings testifies. The same conclusion holds true for another corpus of mathematical writings in Chinese that Zhu Yiwen recently uncovered (I return to a more precise description of my clusters of writings below).

This set of facts invites a first general conclusion; that is, that actors' ontological ideas in mathematics are not determined by the language they speak and write, and even do not necessarily depend on it. Indeed, in ancient China, we have different clusters of writings whose authors seem to have embraced different ontological positions even though they probably spoke the same language. More generally, I will suggest that in any given context, actors' ontological ideas, technical language, and material practices in mathematics are correlated with one another, since they were all shaped by actors while carrying out mathematical activity and they were thus produced in intimate relation to one another. My case study further invites the second (more speculative) conclusion that, in the same way as technical language and material practices change while mathematical work is carried out, ontological ideas also change accordingly.

A first cluster of mathematical texts: Canons and commentaries

Let me outline the context in which actors put together the corpus of texts on which I will rely for my main case study, since this will highlight how I suggest using it.

In 656, Li Chunfeng 李淳風 (?602–670) presented to the Tang throne an anthology of mathematical writings entitled *The ten canons of mathematics* (*suanjing shi shu*, 算經十書; hereafter, *The ten canons*).[1] This anthology was the result of a task that Li Chunfeng had fulfilled upon imperial request, together with a group of scholars who had been convened for this purpose. Li and the colleagues working under his supervision had selected ten Chinese mathematical canons, with—for some of them—ancient commentaries. They had prepared new editions for all these writings and had composed annotations on them.[2] Immediately after the anthology had

1. In the last decades, two critical editions of the anthology have been published: Qian Baocong 錢寶琮 (1963) and Guo Shuchun 郭書春 and Liu Dun 劉鈍 (1998). They organize the canons in chronological order.

2. I have examined the evidence we have about the editorial work carried out by Li Chunfeng's team in Chemla (2013a).

been completed and presented to the Throne—that is, from 656 on—its canons and commentaries, together with two additional writings, were used as textbooks in the newly established Imperial "School of Mathematics," which trained students in mathematics with a view to securing a career in the bureaucracy for those who had passed the examinations.[3] The study of eight among these canons, with their commentaries, formed the core of an elementary curriculum, while the other two canons defined a more advanced program. My argument only requires that I focus on the elementary curriculum.

This curriculum began with the study of *Mathematical canon by Master Sun* (*Sunzi suanjing* 孫子算經), a book completed in circa 400 CE and whose ancient commentaries are lost. The third book that was taught, *The nine chapters on mathematical procedures* (*Jiuzhang suanshu* 九章算術; hereafter, *The nine chapters*), was the major piece of the curriculum, in the sense that its study, together with that of the fourth book (a short tract that had been composed as a complement to the last of the nine chapters), required three years and was thus by far the longest. In fact, the title *The nine chapters* referred not only to the canon bearing that title, which had been completed in the first century CE, but also to the commentary on it that Liu Hui 劉徽 completed in 263, and finally to Li Chunfeng et al.'s subcommentary. The curriculum concluded with the study of two books with commentaries and subcommentaries: *The Gnomon of the Zhou* (*Zhoubi* 周髀), a canon, the most recent layers of which (commentaries aside) dated to the first century CE, and which was devoted to mathematics required for the calendar, and a sixth-century compilation, *Mathematical procedures for the five canons*, which gave mathematical procedures accounting for numerical values stated in historical commentaries on Confucian Canons and other related classical texts.[4]

I interpret the fact that these canons, composed during different periods, were taught in the same curriculum at the time as evidence that for seventh-century actors they could be considered as delivering a coherent body of mathematical knowledge and practices, even if one can find minor differences between them. This is a key hypothesis for my argument. It implies that before the seventh century, the specific body of mathematical knowledge and practices to which these canons testify and that will be at the center of my argument had been handed down in some milieus for centuries. Moreover, evidence shows that these ten canons were regularly reedited upon imperial order, and were used for teaching in subsequent centuries in China. These last remarks thus additionally imply that these elements of mathematical knowledge and practice continued to be handed down later.

For the sake of my argument, a second type of evidence will prove useful. The first six canons taught in the first curriculum were mainly composed of problems and mathematical procedures. In addition, their texts all refer to the use of counting rods to write down numbers on a calculating surface, which was separate from the text and on which computations were carried out. By contrast, this type of content is only part of what we find in the commentaries and subcommentaries on these canons that have survived until today—that is, only a fraction of those that Li Chunfeng et al. had selected or else further composed for the 656 edition. This holds true, in particular, for Liu Hui's commentary and Li Chunfeng et al.'s subcommentary on *The nine chapters*. These commentaries include, among other things, discussions about mathematics, and explicit references to mathematical practice. They also provide evidence on how the earliest readers we can observe read and interpreted the canons. Of particular importance for us in this study is the fact that commentaries further formulate the terse ontological statements that I mentioned above and whose interpretation we will discuss below.

For these reasons, commentaries, and the features of the canons to which they can be related, play a central role in my argument, which, accordingly, grants pride of place to *The nine chapters* and its commentaries. However, another canon will also give us essential elements

3. For details about the school, its official organization, its curricula and modes of examination, see Volkov (2014), on which I rely here.

4. About the order of study, which is an important point in my argument here, see Volkov (2012: 515–18; 2014: 61). This order differs from the one adopted in modern critical editions of *The ten canons*. The status of the critical editions is in fact not clear. Since they both include one of the additional texts studied at the School of Mathematics, it seems that they give a critical edition of the writings studied in this school. However, the order of the writings that they adopt is chronological, and hence is not related to the order of the curricula. On the specific canons just mentioned, one can consult, respectively, Lam and Ang (2004); Chemla and Guo Shuchun (2004); Cullen (1996); Zhu Yiwen 朱一文 (2016).

of information—that is, *Mathematical canon by Master Sun*, with which the elementary curriculum began.

Restoring material practices to which all canons and commentaries refer

Indeed, the fact that *Mathematical canon by Master Sun* was the first textbook taught in the elementary curriculum has important implications for us. The assumption that the corpus taught delivered a globally coherent body of mathematics entails that the elementary pieces of knowledge and practices presented in the first pages of this canon can be considered as valid for the whole corpus (but, as I will show later, not for all mathematical writings from ancient China).[5] Let us outline them.

Mathematical canon by Master Sun began with basic knowledge about measurement units and key constants, before explaining how to use counting rods to represent numbers. This material representation of numbers was formed on a surface, on which we do not have precise information, apart from the fact that canons and commentaries regularly prescribe to "put *zhi* 置" numbers on it (sometimes making explicit how to arrange them in specific positions) in order to execute computations.[6] Computations could, and did, rely on the numbers thus represented and arrayed to proceed. The way in which computations were conducted shows that they also made use of the facts that numbers written with rods could be moved on the surface, and the value of the numbers placed in a given position of the surface could likewise be changed. We will shortly see an example of this, with the first calculations presented in *Mathematical canon by Master Sun* immediately after the description of the number system.

As a consequence, in the practice of mathematics to which our corpus of texts attests, computations were carried out wholly outside the texts, and only materially (if we set aside the possibility of mental computation, which, however, our corpus never evokes). Further, when numbers were mentioned within mathematical writings, they were written using the Chinese language. It was for computations, and only for computations, that the number system using rods that *Mathematical canon by Master Sun* described was employed. Before the tenth century, we know of no graphic illustration that would have been included in a text to show how numbers were represented with rods, or how computations were actually conducted.[7]

Features of the practice of computations taught in the elementary curriculum (the practice with respect to which our corpus makes sense) are pivotal for my goal in this essay. However, the argument requires that we restore (at least part of) this practice, on the basis of the references that writings make to numbers and computations. The first pages of *Mathematical canon by Master Sun* yield precious evidence for this.

Let us examine what these pages tell us about the number system using rods. To represent numbers, *Mathematical canon by Master Sun* prescribed to "first determine the positions," which correspond to successive decimal components of the numbers, and then, to place rods for the subsequent digits, from right to left, first the units, vertically, then the tens, horizontally, and then, alternatively, vertically (for even powers of ten) and horizontally (for the odd powers) (Qian Baocong 錢寶琮 1963, 2: 282). Let us leave aside the specific way of using rods to write digits (Lam and Ang 2004: 33ff., 191ff.). In what follows, to represent computations that I restore using the descriptions given in the text, I replace rods with Hindu-Arabic numerals. What the canon describes here is indeed a place-value decimal system, in the sense that, up to the change of orientation, the same set of digits is used to write down decimal components of numbers in successive positions, and the position where a digit is placed determines the order of magnitude (the power of ten) with which it must be understood, in the same way as in the inscription 123, 1 means a hundred

5. This strategy was used in Proust (2007), in which knowledge taught to scribes in schools is used to interpret more advanced tablets.

6. Martzloff ([1997] 2006: 188) notes that we have no evidence that "counting boards" existed, by contrast with counting rods, which are mentioned explicitly in texts, and samples of which have been found in several excavations. See also Volkov (2001).

7. The earliest known documents showing illustrations of the number system to which *Mathematical canon by Master Sun* refers are Dunhuang manuscripts. Manuscript *Pelliot chinois 3349*, which bears the title *Suan jing* 筭經 (Mathematical canon), and seems to date from the second half of the tenth century, features both a description of the number system close to that in *Mathematical canon by Master Sun* and illustrations for it inserted into the writing. The same number system is recorded in the Dunhuang manuscript *Or. 8210/S.930*, with captions similar to that of *Mathematical canon by Master Sun*.

in relation to its position in the sequence of digits.[8] With our assumption of the coherence of the corpus, we can assume that this number system described in the *Mathematical canon by Master Sun* was the one to which all the other canons taught in the same curricula refer.

In the absence of any illustration of the number system in our corpus, we find evidence for our assertions about it in clues collected from computations of our corpus that rely on it. In particular, the two procedures that follow in *Mathematical canon by Master Sun* and outline the processes to multiply (*cheng*) and divide (*chu*) with this number system on the calculating surface, confirm the place-value decimal features of the number system. These procedures will play a central role in my argument (see Fig. 1a&b for how the executions of multiplication and division, respectively, are commonly restored relying on *Mathematical canon by Master Sun*). The key fact for us is that the text for the division algorithm begins *not* with a prescription but with a statement. It asserts that the algorithm for division is "exactly opposed" to that of multiplication (Qian Baocong 錢寶琮 1963, 2: 282). This assertion yields a clue though its precise meaning is not immediately clear. Interpreting this statement will highlight an important feature of the practice with counting rods on the surface.

According to the texts recorded in *Mathematical canon by Master Sun*, the algorithms for multiplication and division combine two types of "positions (*wei*)." First, numbers are written as a horizontal sequence of digits placed in successive (decimal) "positions." These positions echo a characteristic feature of the algorithms—that is, that the algorithms iterate the same subprocedure along the sequence of digits to execute the operations, exactly like in present-day practices of place-value decimal notation in multiplication and division (I use the plural, since they present variations worldwide).

Second, the execution of each operation uses three positions (upper, middle, and lower). In the lower position, the multiplier (Fig. 1a) and the divisor (*fa* 法, Fig. 1b) are respectively placed. In the context of the execution of the two operations, both the multiplier and the divisor are similarly moved leftward (multiplied by a power of ten) at the beginning of the process and, then, progressively moved rightward during the execution. The Chinese text uses a classical pair of opposed operations for this: "moving forward" 進 *jin* versus "moving backward" 退 *tui*. We return to this point below. The significant digits of the numbers placed in these lower positions thus similarly do not change, whereas their decimal position is constantly modified in their respective rows. Note that in this, the algorithms rely on the place-value number system.[9] This illustrates why, more generally, operations reveal features of the number system to which they are applied. This remark explains how operations in *Mathematical canon by Master Sun* give clues to material inscriptions with rods, which left no trace in the writings.

How mathematical practices make statements about mathematical entities

For both multiplication and division, the type of change occurring in the lower rows stands in contrast with those undergone by the numbers placed in the two rows above: the decimal position of the latter will not be shifted, whereas their numerical value will change along the process of computation. In these two rows, the starting configurations of multiplication and division both have an empty row and a full row, but which is which depends on the operation (what follows constantly refers to Fig. 1). The starting configurations for multiplication and division are thus opposed to one another, exactly like the final ones will be. Indeed, for both multiplication and division, the execution will proceed through emptying the full row while filling up the empty row. Again, here, the Chinese text evokes a classical pair of opposed processes for this: 得 *de* "yield" versus 失 *shi* "lose," when it states that the multiplication yields the result in the middle row, while the division yields it in the upper row. We also return to this pair below.

8. This is what Dunhuang manuscripts show. The fact that the orientation of the rods alternates from one position to the next has no impact on this conclusion. This point is confirmed by the nature of the arithmetic. I have dealt with this issue elsewhere, and since it is of minor importance here, I do not return to it.

9. The text of the algorithms in *Mathematical canon by Master Sun* makes clear that the positions writing down the numbers are decimal and these motions of the counting rods representing the numbers correspond to multiplications by powers of ten. They thereby yield key clues indicating the place-value decimal features of the number system underlying these procedures.

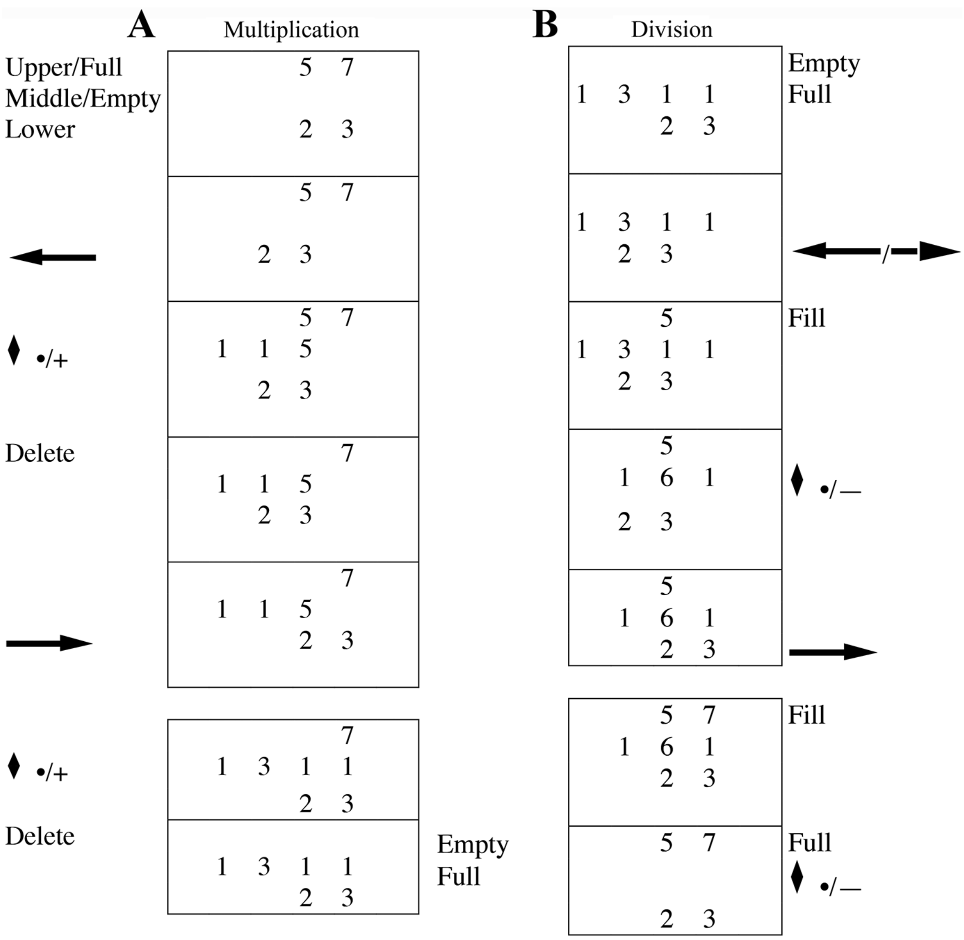

Figure 1a: Process of a multiplication with rods on the calcuting surface, according to *Mathematical canon by Master Sun*, circa 400 (the example chosen is mine)
Figure 1b: Process of a division with rods on the calcuting surface, according to *Mathematical canon by Master Sun*, circa 400 (the example chosen is mine)

In the process of multiplying, the leftmost digit of the multiplicand, in the upper row, will be multiplied by the multiplier, and the product is *added* to the middle row. Once the subprocedure is over, the leftmost digit of the multiplicand is *deleted*, the multiplier is moved one position rightward, and the subprocedure is applied again with the next leftmost digit in the upper position. By contrast, in the upper row of a division, the successive digits of the "quotient" are *inserted* at each stage, and each digit is multiplied by the divisor, in the corresponding position, the product being *subtracted* from the middle row (the "dividend" *shi* 實). We thus see that the two rows (upper and middle) in the processes of multiplication and division behave in ways exactly opposed to each other.

As a result, these executions of multiplication and division are globally devised in such a way that the processes of computation on the calculating surface, as they can be restored using the text, display a network of oppositions and similarities. In particular, the relationship of opposition between the operations translates into a row-to-row dynamic opposition between the processes of computation. Vertical positions (upper/middle/lower) and their arrangement are essential for this, since if we compare the two processes, we see rows are involved in either identical or opposed operations, and thus present identical or opposed behaviors. It is by reference to this property of the flows of computation that I suggest interpreting the statement inserted in *Mathematical canon by Master Sun* that the algorithm for division is "exactly opposed" to that of multiplication. The statement implies a more general conclusion, essential for us: processes of computations on the calculating surface are not merely means to yield a result, but they are also designed to *assert* something about the relationship between the operations thereby executed. Positions (*wei*), with the type and sequence of elementary operations applied to them, provided actors with tools that analyzed these processes. In other words, the practice with positions was used for mathematical theory.

In this case, the algorithms and the rows have one more property. Suppose we were dividing not 1,311, but 1,312, by 23. The computation would yield in the three rows, respectively, 57, 1, 23, which would be read as the exact result 57 + 1/23. The process of multiplication, applied to these three rows, would restore the original values and configuration of the division. The succession of multiplication and division on the calculating surface cyclically restores the original configuration of the previous operation. In addition, multiplication and division are operations for the execution of which algorithms are given. They are also operations that occur in other algorithms executing other operations. The property of canceling each other out holds for the operations as well as for the configurations (since results are given as exact). In conclusion, multiplication and division are both opposed to each other and cancel each other out when applied in succession.

This example illustrates an unexpected practice of computation with the calculating surface, which is specific to the mathematical culture to which our corpus of text attests. The practice is exemplified by two processes, whose relationship with each other conveys meanings that *Mathematical canon by Master Sun* makes explicit.[10] Without restoring this material practice, we would miss meanings stated in ways that are different from common modes of expression today. Moreover, we would not be able to interpret accurately the statement made about the processes of computation in *Mathematical canon by Master Sun*. What is essential for us is that the interpretation of the statement reveals how actors in this context observed processes of computation on the surface.

Remember that these algorithms and this book were learned at the beginning of the elementary curriculum. This suggests that the practice was taught at the time, including a specific way of reading material processes of computation in relation with one another and analyzing the relation of opposition between them on this basis. In fact, several other canons, taught in the same curriculum, testify to the use of a similar way of working with dynamic processes of computation on the calculating surface. I will only evoke them briefly here, to shape the background on the basis of which I will offer an interpretation of some ontological statements, and also draw a contrast between this first corpus and other corpora of mathematical texts from ancient China.

The nine chapters already contains texts for all the processes of computation for which I could identify the same practice on the calculating surface. It must be noted that this canon does not describe algorithms for multiplication and division. However, the way in which *The nine chapters* refers to multiplication *cheng* and division *chu* supports the hypothesis of the coherence of the corpus; that is, that the algorithms learned with *Mathematical canon by Master Sun* were also those on which *The nine chapters* relied, and in particular, those with respect to which processes of computation described in *The nine chapters* likewise stated relations of similarity or opposition.

For instance, in this latter canon, we find texts for algorithms executing square root and cube root extractions.[11] Like multiplication *cheng* and division *chu*, they rely on the positions of the decimal expansion of the number, whose root is extracted. Moreover, likewise, these algorithms bring into play three main positions (upper/middle/lower), called respectively "quotient," "dividend," "divisor."[12] In the same way as the names for the positions borrow the terms used for division in *Mathematical canon by Master Sun*, the elementary operations used to execute both square and cube root extractions derive for the most part from the process of division. As a result, the elementary operations applied to each of the positions "quotient," "dividend," and "divisor" present a strong similarity with those featuring in the process of a division. The use of the same name thus echoes the fact that the positions undergo correlated changes in the process of execution. As a re-

10. In this chapter, space precludes further worked examples. They can be found in Chemla (2000, 2017).

11. Chemla (1994) deals with these algorithms, and the others executing the same operations that occur in *The ten canons*. I do not repeat the details, and only state the conclusions here.

12. By contrast, fleeting positions are sometimes inserted below these three, and they receive no name.

sult, the material processes of computation on the surface, as they can be restored, appear to state a similarity between division and root extraction in exactly the same way as in *Mathematical canon by Master Sun* the related processes stated the relation of opposition between multiplication *cheng* and division *chu*. In addition, the same practice of writing processes on the calculating surface indicates that likewise, a relation of similarity between square and cube root extraction is asserted.

Other canons in our corpus contain texts for algorithms executing square and cube root extractions, which present slight variations by comparison with those in *The nine chapters*. Interestingly, these texts of algorithms appear to use exactly the same practice of stating relationships between operations using the processes of computation on the calculating surface. However, the way in which they shape similarities differs. This suggests that these processes of computation were a tool with which actors explored how one could understand the relationships between operations (Chemla 1994). Again, without reading these processes as we have seen above actors read them, we would miss part of the mathematical work carried out using these means.

To return to *The nine chapters*, we could highlight other phenomena on the basis of restoring material processes of computation and reading them in the same way as was described above. Interestingly, these phenomena correspond to phenomena affecting the terminology. We have an example of this in the two expressions "dividing this by extraction of the square root *kai fang chu zhi* 開方除之" and "dividing this by extraction of the cube root *kai lifang chu zhi* 開立方除之," which are used to prescribe square and cube root extractions, respectively. The terminology thus shows a structure in the set of three operations that is strictly parallel to what the processes of computation state.

The nine chapters introduces a fourth operation in a similar way. For us, it is a quadratic equation. In the canon, it appears as an arithmetical operation. The process of its execution is extracted from the process of computation of a square root extraction, using the latter execution from a given point onward.[13] Accordingly, on the calculating surface, the process of computation of the former is a part of the latter. The operands to which the quadratic equation as an operation is applied are precisely those featuring on the calculating surface at the point where the part of the process of square root extraction that is kept begins. In this case, positions on the calculating surface serve to introduce a new operation that builds upon one that is already known. These operands are referred to as "dividend" and "joined divisor," which evoke the names given to these positions in the context of the execution of a square root extraction. Finally, the new operation is prescribed using the expression "dividing by extraction of the square root." So, again, the relationship of similarity between the processes of computation of the quadratic equation and the square root extraction echoes the relationship expressed using the terminology.

Positions are again central in a fifth algorithm described in *The nine chapters* to solve what for us corresponds to systems of linear equations.[14] The text of the algorithm prescribes to lay out the data corresponding to each equation in a column in such a way that all the data corresponding to the same unknown in different equations are placed in the same row. Accordingly, *The nine chapters* again uses a place-value scheme. The data of each problem form a rectangle, in which the data associated to unknowns are arranged in a square, whereas the constant terms of the equation form a row under this square. The algorithm solving the problems will rely on this rectangular layout to determine the unknowns.

What is remarkable is that the algorithm combines essentially two main operations that are repeated, one between columns, and the other between positions in the upper square and the lower row. The former operation, which (relying on the physical properties of numbers represented with rods) brings into play two columns to eliminate the upper position in one of them, is prescribed as "an upright division *zhi chu* 直除," while the latter is a plain division *chu*. In this context again, the process of computation shapes a relation of similarity between this execution and that of a division. What is striking is that the data arranged in the lower row are referred to as "dividends," whereas the data arranged in the upper square are referred to using a synonym

13. On the argumentation supporting the claims about the quadratic equation that follow, and the transformations of the operation in the next centuries, see Chemla (2017), which contains a more complete bibliography on the topic.

14. What follows relies on Chemla (2000), where a more detailed argument and references are given.

of "divisor"—namely, "measure *cheng* 程."[15] This identification of the shaping of a similarity using positions and terms enables us, then, to interpret the name that *The nine chapters* gives to the operation that this algorithm executes: "divisors/measures in square *fangcheng* 方程." In other words, the operation appears to be a generalization of the division *chu*. Instead of having a single divisor and dividend, "measures in square" opposes a square of divisors to a row of dividends, and the operation is executed using the key process of division first vertically, and then horizontally.

To summarize, the same practice on the calculating surface, which relies in an essential way on how data are arranged in positions, and the same (and related) use of the terminology shape and state the operation "measures in square" as similar to that of division. In this context, we find again the same group of features that we have met in all the contexts in which these practices were in play in our corpus: positions forming place-value notations on the calculating surface; the use of the same terms or of related terms to designate the elements from two different operations that were brought into relation; the establishment of a relationship between the processes of computation.

In all the examples from *The nine chapters* that we have evoked, we have seen that one of the two fundamental operations that were opposed to each other—that is, the division *chu*—played a cardinal role, since it was used as a basis to which the processes of execution of other operations were reduced. More generally, we see that the pair of opposed operations and the practice of computation on the calculating surface that were learned at the beginning of the elementary curriculum played a key part for the knowledge that would be taught later. This knowledge included not only actual algorithms but also the understanding of a structure in a set of operations. To grasp this structure, students had to know the two operations of multiplication and division, and also to understand the way of reading their relation of opposition directly on the processes of execution. Both aspects formed the cornerstone of the knowledge they would acquire later.

So far, we have uncovered a practice of searching for relations between algorithms executing operations, whose conduct and expression used two types of tools: on the one hand, the processes of computation on the calculating surface and the relationships that could be established between them using positions in a specific way, and on the other hand, the terminology referring to positions and prescribing operations. What this search appears to have established is that a certain number of algorithms could be shown to be reducible to the division *chu*—that is, one of the two poles of a pair of opposed processes.

15. Both terms refer to the idea of "norm" with respect to which one will shape a given quantity, and thus measure it.

Ontological statements, material practices, and the differences in ontological ideas between different clusters of texts

In this section, I propose a hypothesis concerning the possible connection between the practices in *The nine chapters* and certain philosophical—indeed, ontological—statements in Liu Hui's commentary, and then point to a contrast in a second corpus of mathematical texts that have recently come to light. In both cases I must emphasize the tentative character of my proposals.

The conclusions of the previous section have an echo in a statement that the third-century commentator Liu Hui formulates in the preface to his commentary on *The nine chapters*. There he writes: "(In *The nine chapters*), I observed the dividing of *Yin* and *Yang* and I synthesized the source of mathematical procedures." Taken out of context, this terse statement is hard to interpret. But one possibility may be suggested using the background described above.

The mention of *Yin* and *Yang* evokes philosophical developments in China that took their reference point in the *Book of changes* (*Yijing* 易經) and the ancient commentaries that were handed down with this writing, notably the "Great commentary" (*Xici zhuan*), which seems to have assumed the form under which we know it in the first half of the second century BCE. Reflections putting *Yin* and *Yang* into play, such as those we can read in the "Great commentary," approached realities from the viewpoint of processes of transformation at play in them. In this context, *Yin* and *Yang* featured polarities enabling observers to account for how processes unfolded as the interplay of fundamental and general patterns of transformation opposed to one another. The scope of this type of analysis in the "Great commentary" encompasses the natural world, social interactions, and cultural artifacts. Liu Hui's mention of *Yin* and *Yang* in the context of mathematics suggests that some practitioners of this discipline also thought about mathematics from the same ontological viewpoint. This assumption is supported by the multiple quotations of the *Book of changes* and the "Great commentary" that we read in the commentaries on *The nine chapters*.[16]

In this context, one possible interpretation of Liu Hui's statement in his preface would derive from establishing a connection between the pair *Yin/Yang* and the two operations of multiplication and division, with their execution on the calculating surface. Given the analysis developed above, it seems natural to suggest that multiplication and division have embodied fundamental and general patterns of transformation, opposed to one another, by reference to which other processes of transformation could be analyzed. Another piece of information supports this interpretation: in his commentary, Liu Hui refers to the flow of transformations that algorithms carry out on numbers using one of the general terms referring to change in the "Great commentary"—that is, *bianhua* 變化.[17] According to this interpretation, Liu Hui's statement would refer (in particular, but probably not only) to how one might observe the interplay of multiplication and division in processes of computation.[18] The statement might also refer to the pair of elementary and fundamental operations that are in play in the processes executing multiplication and division as in many other natural processes. Indeed, "moving forward" 進 *jin* and "moving backward" 退 *tui* are terms that regularly occur in the "Great commentary," as are "yield" 得 *de* and "lose" 失 *shi*). Finally, when Liu Hui claims to have "synthesized the source of mathematical procedures" it may be that he has in mind his uncovering of elementary and fundamentally opposed operations to which all the other processes of computation can be reduced.

These interpretations have two main consequences of importance for my purpose here. First, whatever the precise reference of the statement might have been, the interpretations that I have sketched all suggest that mathematical realities like computations would thus systematically have been viewed as processes. This ontological assumption went hand in hand with a program of research: as was the case for other processes of change, this program aimed at identifying fundamental processes to which all other processes could be reduced, through an inquiry comparing processes with one another and searching how they related to one another. In this case, like in other contexts, this search seems to have uncovered that fundamental processes and key patterns of transformation could be arranged into pairs of opposed operations (multiplication and division, moving forward and backward, yielding and losing, etc.).

It is important to note that this search was carried out using specific practices, like the material practice of computation with rods that we have restored on the calculating surface. This brings me to the second consequence that is essential for us and that concerns the type of relationship that practices and ontological assumptions have to one another. Indeed, the ontological assumption that mathematical entities can be viewed as processes and the type of search that corresponded to this assumption are *reflected* in practices that actors shaped to work on mathematics. One such practice is the use of "positions" on the calculating surface, thanks to which flows of transformation could be shaped, analyzed, and compared. This latter practice interestingly echoes the use of trigrams and hexagrams in the context of the *Book of changes*. Another practice is the use of terminology, which shaped networks of similarities and oppositions. Perhaps, in fact, actors did *shape* these practices in *relation* to their ontological assumption and the related program that it led them to pursue. This would account for why to some extent practices bespeak the ontological assumptions actors

16. Chemla (1997) analyzes one such quotation in its context. The analysis that I develop (and will not repeat here) implies that the choice of a title in *The nine chapters*, which this quotation echoes, might entail that even in *The nine chapters*, we might perceive a reference to the "Great commentary." If this assumption holds true, the commentator would only make explicit what he reads in the canon. Moreover, the analysis developed in this other study shows that mathematical entities were also approached from the viewpoint of their transformations, and not only computations. However, here I will only focus on the aspect that will enable me to establish a contrast between the corpus under consideration and other mathematical writings.

17. In Chemla (1999), I have analyzed this reference in context. Let me insist on the fact that I do not claim to offer the only interpretation possible, nor do I mean to have exhausted the meaning of these terms and sentences in this discussion.

18. I have given an interpretation of the part played by multiplication and division in another range of phenomena in mathematics, and also on a longer time span in Chemla (2010).

held. After all, it is about the processes of multiplication and division, shaped using positions and terms in a specific way (our two practices), that *Mathematical canon by Master Sun* asserts that they are "exactly opposed" to one another—that is, that the canon inserts a statement that relates to an interest in polarities in mathematics. These practices can also be identified in *The nine chapters*, and I have emphasized that they were taught (probably with the corresponding approach to processes of computation) at the beginning of the elementary curriculum. We have seen that the commentator Liu Hui referred explicitly to the related ontological assumption. The occurrence of the same practices in *The nine chapters* invites us to assume that the same ontological assumption and the same program already existed at the time when *The nine chapters* was completed.

The hypothesis that mathematical practices reflect (at least to some extent) ontological assumptions provides to us a method to approach such assumptions in the context of writings that contain no explicit statement about them. I will now use this method to show that none of the mathematical manuscripts from early imperial China so far published (my second cluster of texts) seem to reflect ontological assumptions similar to those I have associated with the elementary curriculum.

At the present day (2017), these manuscripts include *Writings on mathematical procedures* (*Suan shu shu* 算數書), which was found in a tomb sealed circa 186 BCE, at Zhangjiashan,[19] and *Mathematics* (*Shu* 數), which was bought on the antiquities market and which its editors date from no later than circa 212 BCE.[20] Both appear to be related to the same milieus for they present tight connections with the practice of administrative regulation. They also include other manuscripts, which were not yet published. However, I will also mention the published part of the manuscripts from early imperial China (the Qin dynasty) kept at Beijing University. All these manuscripts attest to mathematical practices and knowledge presenting some similarities with those in our first corpus. They refer to the use of rods, and to the positioning of numbers on a surface to compute. The mathematical terminology they use has a lot in common with what we find in *The ten canons*.

The key point about these manuscripts concerns the operation of division. Although all other arithmetical operations (including multiplication) are usually prescribed by verbs, the division to which the manuscripts attest (by contrast with what we have described above) is only prescribed using whole sentences (this holds true of every bit of manuscript so far published).[21] In particular, even if the verb *chu* occurs in them, at the time it only had the meaning of subtraction (including repeated subtraction). This suggests that the executions of multiplication and division are not related to one another. I have offered a reconstruction of the execution of a division at the time, which confirms this assumption. But there is more.

Several manuscripts contain an algorithm to extract a square root.[22] This algorithm does not rely on a decimal expansion of the number whose root is sought, and it does not have any relationship with an algorithm of division of the type we have mentioned above for writings in our first corpus. More generally, nowhere do we have any hint that a place-value notation would be used. In particular, in contrast to the writings in the first corpus, nowhere do we find a division or a multiplication by ten carried out using a shift rightward or leftward of the rods representing a number. However, this is an aspect of a much more general phenomenon: the manuscripts do not attest to the use of positions in the execution of computations similar to what we have described for our first corpus. Finally, nowhere does an interest in the relations between algorithms come to the surface.

These elements strongly suggest that these manuscripts do not reflect any program of searching for elementary and fundamental operations within processes of computation similar to the one to which *The ten canons* attests. By contrast, these mathematical texts seem to reflect the use of operations as means to reach a result rather than as processes to be pondered. If so, this

19. Peng Hao 彭浩 (2001) published the first annotated edition of the text. English translations can be found in Cullen (2004) and Dauben (2008).

20. The first annotated edition was provided in Xiao Can 蕭燦 (2010). The slips are reproduced, transcribed, and an annotated edition is given in Zhu Hanmin 朱漢民 and Chen Songchang 陳松長 zhubian 主編 (2011).

21. I have been exploring this issue in Chemla (2013b, 2014, and forthcoming). More publications will follow.

22. See in particular Han Wei 韓巍 (2013: 38–39), which shows that the manuscripts kept at Beijing University have the same procedure as *Writings on mathematical procedures*. This suggests that this procedure enjoyed a certain stability at the time.

suggests that the ontological assumptions about processes of computation were not the same.[23]

Conclusion

Whether a similar analysis applies to other mathematical traditions must wait on further study, which I hope this study might inspire. For the time being, let me simply emphasize the general issues that the inquiry presented here invites us to ponder.

The two sets of documents that I have considered (one more extensively than the other) testify to two (partly) different ways of practicing mathematics, which related to different material and terminological practices. Accordingly, I have suggested that in the two contexts, mathematical processes of computations were shaped and explored in different ways, with significant consequences for the knowledge produced. This has led me to conclude that the related ontologies of mathematical processes were different in the social backgrounds in which the two clusters of texts were produced and used. The evidence is not enough to allow us to dig further into ontological assumptions held by the actors who used the manuscripts. However, it suffices to point out a contrast in this respect between the two clusters of texts.

The hypothesis we are led to propose on this basis is the following. Ontological assumptions are not solely determined by written or spoken language. In our case, although all the actors wrote (and most probably spoke) in Chinese, they seem to have embraced at least partly different ontological ideas about mathematical entities.

I have insisted that ontological ideas can be approached both through the technical terminologies the actors shaped and through the material practices that can be reconstructed from their writings. The reason for this is that these three facets of mathematical activity are interrelated. In this latter respect, perhaps we can go one step further.

Indeed, since the two clusters of texts that I have considered had several features in common, we know they are somehow related to each other. In case, in the future, we can prove that they are historically more closely related—that is, that the mathematical practices and knowledge to which the manuscripts attest in fact developed into mathematical knowledge and practices to which *The ten canons* testifies—this would have an interesting consequence for our topic. It would indeed point out that ontological assumptions of the type we have analyzed in *The ten canons* took shape in correlation with the shaping of mathematical practices that reflect them and enable actors to work with and explore them. In this case, we would be in a position to observe how ontological assumptions change and how this process relates to actual practices that actors design for their mathematical activity.

23. I could develop the same argument relying on the corpus of mathematical writings that Zhu Yiwen uncovered in seventh-century commentaries on Confucian canons—that is, in writings composed more or less at the same time as Li Chunfeng's annotated edition of *The ten canons* (Zhu Yiwen 朱一文 2016). In this case, actors never seem to place rods on a calculating surface for mathematical work.

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