Time-reversal symmetric Crooks and Gallavotti-Cohen fluctuation relations in driven classical Markovian systems

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Abstract. In this paper, we address an important question of the relationship between fluctuation theorems for the dissipated work $W_d = W - \Delta F$ with general finite-time (like Jarzynski equality and Crooks relation) and infinite-time (like Gallavotti-Cohen theorem) drive protocols and their time-reversal symmetric versions. The relations between these kinds of fluctuation relations are uncovered based on the examples of a classical Markovian N-level system. Further consequences of these relations are discussed with respect to the possible experimental verifications.

Keywords: Large deviations in non-equilibrium systems, Large deviation, Stochastic processes phenomena, Stationary states, Stochastic particle dynamics, Rigorous results in statistical mechanics, Dissipative systems

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1. Introduction

For the last decades, since 1970s [1] when the first fluctuation theorems generalizing the second law of thermodynamics were formulated (see review [2] and references therein), there have been discovered many variants of fluctuation relations spreading from the ones for the heat and environmental entropy production in the static conditions, either in non-equilibrium steady state (NESS) [3, 4, 5, 6] or during relaxation to equilibrium [7], to the well-known Jarzynski equality [8] and Crooks relation [9] written for the work dissipated in the system under a finite-time drive. Some work has been done on their generalizations for the periodic drive [10, 11] and for the stochastic entropy production [12, 13] which are less known (please see [14] for the extensive review).

Experimental verifications of different kinds of fluctuation relations has been initiated by measurements in biological systems [15] and then done in various different classical systems, such as mechanical [16, 17, 18], biological [19, 20], and condensed matter systems both in contact with equilibrium [10, 21, 22, 23, 24, 25] and non-equilibrium [26] environment. In most of these systems, thermodynamic variables (work, heat, or entropy) has been extracted indirectly via the measurement of the microscopic state of the system (a position of the bead in a laser tweezer, an instantaneous angular deflection of the rotation pendulum, a charge state of a Coulomb-blockaded device and so on). Direct measurements of the heat or work especially in quantum systems [27, 28] have not been done yet, but many efforts have been undertaken, especially in the most stable Coulomb-blockaded devices [29, 30, 31, 32, 33, 34].

Recently fluctuation relations have been also generalized to the case of a feedback-controlled systems [35, 36] including recent ones like [37] which has opened a path to understand the paradox of Maxwell's Demon from the Landauer's principle [38, 39] and verify these predictions experimentally in finite-time protocols [40, 41, 42, 43, 44, 45, 46, 47] and even in the steady-state conditions in the autonomous realization of a Szilard engine [31] with the direct measurements of the effect of the feedback both on the system and demon's temperatures, giving a direct access to the demon's thermodynamics (see recent reviews for the details [48, 49]).

Recent theoretical progress has already provided a more detailed information about properties of the large fluctuations both in the stochastic entropy production [50, 51, 52, 53] in NESS (with experimental verification in [54], including quantum systems [55]) and in the heat in driven systems [56] basing on a Martingale theory.

Despite the impressive progress in the understanding of physics of fluctuations until now, the relations between fluctuation theorems in the systems under finite-time drives and in NESS (or periodic NESS) has been only barely studied. For example, in the work [57] the importance of initial conditions for finite-time fluctuation theorems in NESS comparing to their asymptotic long-time counterparts has been discussed. In this paper, we address the important and demanding question of these relations between fluctuation theorems for driven systems on an example of a classical Markovian N-level system.

The paper is organized as follows. In Sec. 2, we describe the model, overview briefly main fluctuation theorems, and formulate the main question in the focus. Section 3 gives the standard method of the calculation [58] of the probability distribution of the dissipated work in a driven system and provides main equations used further. In Sec. 4, we derive the conditions when the fluctuation relations can be written in the time-reversal symmetric case and extend the class of drive protocols for which these conditions are satisfied. Section 5 is devoted to the consideration of the relations of finite-time and periodic-NESS fluctuation relations in a two-level system, where we provide an exact correspondence between finite-time and periodic-NESS fluctuation theorems. Section 6 concludes our paper.

2. Model and definitions

In this section, we consider a Markovian N-level system. The formalism of this section is standard and for more details please address, e.g., the book [58]. The system in focus is characterized by the energy levels $E_n(\lambda)$, $n = \overline{0, N-1}$, and subjected to the drive via a time-dependent control parameter $\lambda(t)$. The system is placed in contact with a bath with a certain inverse temperature, β. The Markovian dynamics of the considered system is described by the standard rate equations written in the matrix form

$$\frac{d}{dt}\left|\mathbf{p}(t)\right\rangle = \hat{\Gamma}(\lambda(t))\left|\mathbf{p}(t)\right\rangle \tag{1}$$

for the vector $|\mathbf{p}(t)\rangle = (p_0, \ldots, p_{N-1})$ of probabilities $p_n(t)$ of the system to be in the state n at a certain time instant t. In the main part of the paper, for simplicity, we consider the case when time-dependent incoming rates $\Gamma_{n,n'}(\lambda(t))$ from states n' to a certain state n satisfy the local detailed balance (LDB) condition

$$\Gamma_{n',n}(\lambda(t)) = \Gamma_{n,n'}(\lambda(t)) e^{\beta[E_n(\lambda(t)) - E_{n'}(\lambda(t))]} \ . \tag{2}$$

The normalization condition for the probability distribution $\langle \mathbf{1}|\mathbf{p}\rangle \equiv \sum_{n=0}^{N-1} p_n(t) = 1$, with $|\mathbf{1}\rangle = (1, \ldots, 1)$, is conserved by rate equations as the overall escape rate from the state n is $\Gamma_{n,n} = \sum_{n' \neq n} \Gamma_{n',n}$. Here and further, we put the Boltzmann's constant to be unity, i.e., $k_B = 1$ and measure temperature in energy units. The initial distribution $p_n(0)$ of the system is considered to be equilibrium

$$|\mathbf{p}_{eq}(\lambda(0))\rangle = e^{\beta[F(\lambda(0)) - \hat{E}(\lambda(0))]}\,|\mathbf{1}\rangle \ , \tag{3}$$

where $E_{n,n'}(\lambda) = \delta_{n,n'} E_n(\lambda)$ is a diagonal matrix of system's energy levels and $\beta F(\lambda) = -\ln \sum_n e^{-\beta E_n(\lambda)}$ is the free energy of the system at a certain value of $\lambda(t) = \lambda$.

The first law of thermodynamics $dE_{n(t)}(\lambda(t)) = \delta W + \delta Q$, written in terms of the system internal energy $E_{n(t)}(\lambda(t))$ gives the definitions of the work performed to the system

$$W = \int_0^T \left.\frac{\partial E_n}{\partial \lambda}\right|_{n(t)} \frac{d\lambda}{dt} dt = \sum_j \left[E_{n_j}(t_{j+1}^{(J)}) - E_{n_j}(t_j^{(J)})\right] \ , \tag{4}$$

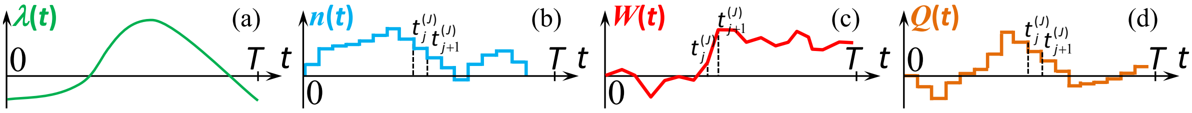

Figure 1. Sketch of (a) the general cyclic drive protocol $\lambda(t)$ (green line), (b) a single piecewise constant trajectory $n(t)$ of the system state (blue line), which jumps at time instants $t_j^{(J)}$ from state n_{j-1} to n_j, and corresponding (c) work (red line) and (d) heat (orange line) on this trajectory.

and the heat dissipated to the bath

$$Q = -\int_0^T \frac{\partial E_n}{\partial n}\bigg|_{\lambda(t)} \frac{dn}{dt} dt = \sum_j \left[E_{n_j}(t_j^{(J)}) - E_{n_{j-1}}(t_j^{(J)})\right] . \qquad (5)$$

being the changes of $E_{n(t)}(\lambda(t))$ with respect to the control parameter $\lambda(t)$ and the system state $n(t)$, respectively, see Fig. 1. Here and further, we consider the evolution of the system's state $n(t)$ as a set of jumps from n_{j-1} to n_j occurred at time instant $t_j^{(J)}$, see Fig. 1(b).

For driven systems which obey LDB (2) under a finite-time drive $\lambda(t)$, $0 \leq t \leq T$, and start from the equilibrium distribution (3), the probability distribution of work is characterized by the Jarzynski equality [8]

$$\left\langle e^{-\beta W} \right\rangle = e^{-\beta \Delta F} \qquad (6)$$

and the Crooks relation [9]

$$P(W)/\bar{P}(-W) = e^{\beta(W-\Delta F)} . \qquad (7)$$

Here, the averaging $\langle \ldots \rangle$ is performed over all microscopic realizations of the system and the bath during the protocol $\lambda(t)$, $\bar{P}(W)$ denotes the probability distribution of work in the time-reversed drive protocol $\lambda(T-t)$.

To lift the equilibrium condition on the initial distribution (3), one has to consider the large-deviation version [62] (sometimes called weak version [57]) of Crook's relation [4] for the asymptotic long-time limit

$$\lim_{t\to\infty} \frac{1}{t} \ln \frac{P(W = wt)}{\bar{P}(-W = -wt)} = \beta w , \qquad (8)$$

where $w = W/t$ is an intensive parameter of work. The free energy rate $\Delta F/t$ is negligible in infinite-time limit as the free energy difference ΔF is bounded. Note that the analogous large-deviation Crook's relation can be written for the heat rate $q = Q/t$ as the internal energy change $\Delta E_n(\lambda)$ is bounded for all finite values of λ. Further, for simplicity, we will omit the explicit dependence of E_n and $p_{eq,n}$ on $\lambda(t)$, keeping only t as an argument.

We complete the introductory part of the paper by considering briefly the stochastic entropy productions. Stochastic entropy production of the environment, Δs_m, generalizes the concept of the heat for the systems violating the LDB condition (2). Indeed, like heat Q (5), this quantity sums the jumps $\Delta s_m = \sum_i \Delta s_{m,n_{j-1}\to n_j}(t_j^{(J)})$

occurring as soon as the state $n(t)$ of the system changes (from n_{j-1} to n_j occurred at time instant $t_j^{(J)}$), however, the size of each jump

$$\Delta s_{m,n\to n'}(t) = \ln \frac{\Gamma_{n',n}(t)}{\Gamma_{n,n'}(t)} \tag{9}$$

coincides with the one $\Delta Q_{n\to n'}(t) \equiv [E_{n'}(t) - E_n(t)]$ of Q multiplied by β only when the system obeys LDB (2).

The analogue of the dissipated work, $W - \Delta F$, for this case is the total entropy production introduced in [12]. It is given by the sum

$$\Delta s_{tot} = \Delta s_m + \Delta s_{sys} \tag{10}$$

of the environmental Δs_m and system entropy change $\Delta s_{sys} = s_{sys}(T) - s_{sys}(0)$, where

$$s_{sys} = -\ln p_{n(t)}(t) \tag{11}$$

is the stochastic analogue of the Shannon's entropy given by $\langle s_{sys}\rangle = -\sum_n p_n \ln p_n$. The main property of the stochastic total entropy production Δs_{tot} is that it satisfies the generalized Jarzynski equality and the Crooks relations [12], called sometimes the integral and detailed fluctuation relations (DFR), respectively [14],

$$\langle e^{-\Delta s_{tot}}\rangle = 1 \ , \tag{12}$$

$$P(\Delta s_{tot})/\bar{P}(-\Delta s_{tot}) = e^{\Delta s_{tot}} \ . \tag{13}$$

These fluctuation relations work beyond LDB condition and for any initial distribution. However, the price paid for lifting of LDB and the equilibrium initial distribution is that Δs_{tot} depends not only on a single trajectory realized by a system, but also on its instantaneous probability distribution via $s_{sys}(t)$. However recently it have been found that certain decompositions of the stochastic entropy production provide the representation of Δs_{tot} on a single trajectory in terms of physical observables like work and particle current for some given initial conditions [63, 64].

The large-deviation variant of DFR (13)

$$\lim_{t\to\infty} \frac{1}{t} \ln \frac{P(\Delta s_{tot} = \sigma t)}{\bar{P}(-\Delta s_{tot} = -\sigma t)} = \sigma \ , \tag{14}$$

has been originally written in the paper [4] for the environmental entropy in the system in the non-equilibrium steady state (NESS), as the system entropy production is intensive quantity (as well as the internal and free energies). Note that the large-deviation Crooks relation for the work in NESS conditions is trivial as the control parameter λ is constant and the work is zero. To avoid this triviality, further, we consider the periodic-drive condition inferring periodic-NESS [65]. Thus, the free energy difference can be omitted in both Eqs. (7, 8).

Obviously in all considered variants (7, 8, 13, 14) of DFR the probability distribution $\bar{P}$ in the denominator coincides with the one in the numerator P provided the drive protocol is time-reversal symmetric (TRS), $\lambda(T-t) = \lambda(t)$ ‡

$$P(W)/P(-W) = e^{\beta(W-\Delta F)} \ , \tag{15}$$

‡ However, in the large-deviation versions it is enough that the drive would be symmetric with respect to an arbitrary finite time shift, see, e.g., Fig. 3.

$$P(\Delta s_{tot})/P(-\Delta s_{tot}) = e^{\Delta s_{tot}} , \quad (16)$$

$$\lim_{t\to\infty} t^{-1} \ln\left[P(wt)/P(-wt)\right] = \beta w , \quad (17)$$

$$\lim_{t\to\infty} t^{-1} \ln\left[P(\sigma t)/P(-\sigma t)\right] = \sigma . \quad (18)$$

This poses a certain symmetry restrictions on the distribution P and opens an intriguing possibility for the direct calculations of first-passage-time distribution for considered variables from their distributions at fixed time [59, 60]. Another issue emerging from the relations (15 – 18) is the surprising analogy of the work statistics with the multifractality of the wavefunctions close to the Anderson localization transition considered in [61].

Both for the dissipated work and for the total entropy production an important question arises: What is the relation between large deviation and finite-time versions of Crooks relations? In particular, what are the requirements on the drive beyond TRS for a system to obey Crook-like relations for the only distribution function P and what are the relations between these requirements for finite-time protocol and periodic-NESS?

To address all these questions in the next section, we describe the standard method to calculate the probability distributions by writing the rate equations for the generating functions and focus mostly on the dissipated work normalized to the temperature $w_d = \beta(W - \Delta F)$ as a variable of interest. Please see Appendix A for the general method given, e.g., in the book [58] for other thermodynamics variables mentioned above.

3. Calculation of $P(W - \Delta F)$

In order to write the rate equation of the form similar to (1) one should consider the n-resolved distribution function $|\mathbf{P}(w_d)\rangle = (P_0(w_d), \ldots, P_{N-1}(w_d))$, with the components defined as

$$P_n(\beta(W - \Delta F) = w_d) = \langle \delta(\beta W - \beta\Delta F - w_d)\delta_{n,n(t)}\rangle , \quad (19)$$

because the probability distribution itself $P(w_d) = \langle \mathbf{1}|\mathbf{P}(w_d)\rangle \equiv \sum_n P_n(w_d)$ does not determine explicitly the system state $n(t)$. To simplify the derivation even further we go to the Laplace transform of $|\mathbf{P}(w_d)\rangle$ being the n-resolved generating function §

$$|\mathbf{G}_q\rangle = \int |\mathbf{P}(w_d)\rangle\, e^{-qw_d} dw_d . \quad (20)$$

Using the standard trajectory representation of the jump Markov processes widely used in the full counting statistics (see, e.g., [66]), one can derive the rate equations of the form of (1)

$$\frac{d}{dt}|\mathbf{G}_q(t)\rangle = \hat{\Gamma}^{(q)}(t)\,|\mathbf{G}_q(t)\rangle , \quad (21)$$

with the modified rate matrix $\hat{\Gamma}^{(q)}(t)$, and the initial condition $|\mathbf{G}_q(0)\rangle = |\mathbf{p}(0)\rangle$ provided $w_d(0) = 0$. For the dissipated work which rate $\dot{w}_{d,n}(t)$ is a deterministic function of $n(t)$

§ Rate equations for the n-resolved distribution function $|\mathbf{P}(w_d)\rangle$ itself are given in Appendix A or [68].

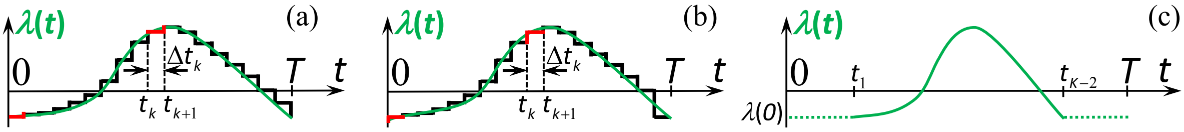

Figure 2. (a, b) Sketch of the general cyclic drive protocol $\lambda(t)$ (green lines) and its time discretized forms (black lines) (a) with plateaux followed by jumps and (b) jumps followed by plateaux (both emphasized in red); (c) Modified cyclic drive protocol with the zeroth, $0 = t_0 < t < t_1$, and last, $t_{K-2} < t < t_{K-1} = T$, intervals of a constant drive (dashed lines). The latter intervals do not contribute to the work generating function G_q as the constant drive does not change work in equilibrium ($0 = t_0 < t < t_1$) or relaxation ($t_{K-2} < t < t_{K-1} = T$) part.

only the escape rates should be modified

$$\Gamma^{(q)}_{n,n}(t) = \Gamma_{n,n}(t) + q\dot{w}_{d,n}(t) \ , \tag{22}$$

with $\dot{w}_{d,n} = \partial e_n/\partial t|_{n(t)}$ and $e_n(t) = \beta(E_n(t) - F(t))$. Note that unlike Eq. (1) the latter equation does not conserve normalization condition as $\Gamma^{(q)}_{n,n} \neq \sum_{n'\neq n} \Gamma^{(q)}_{n',n}$.

The probability distribution of w_d

$$P(w_d) = \frac{1}{2\pi i} \lim_{Q\to\infty} \int_{\chi - iQ}^{\chi + iQ} G_q(t) e^{q w_d} dq \tag{23}$$

is given by the inverse Laplace transform of the generating function

$$G_q(t) = \langle \mathbf{1}|\mathbf{G}_q(t)\rangle \equiv \sum_n G_{q,n}(t) \ . \tag{24}$$

The parameter χ is greater than real part of all singularities of $G_q(t)$ as a function of q.

The generating function (24), both for finite-time and periodic-NESS protocols with the duration or the period T can be written as follows

$$G_q(MT) = \langle \mathbf{1}| \left(\hat{U}_q(T)\right)^M |\mathbf{p}_{eq}(0)\rangle \ . \tag{25}$$

Here, $|\mathbf{p}_{eq}(0)\rangle = e^{-\hat{e}_0}|\mathbf{1}\rangle$ is the initial equilibrium probability distribution vector, with $e_{k,nn'} = \delta_{nn'} e_n(t_k) = \delta_{nn'}\beta(E_n(t_k) - F(t_k))$. The evolution operator $\hat{U}_q(t)$ satisfying the same equations (21) as $|G_q(t)\rangle$ is given by the time-ordered exponential $\hat{U}_q(t) = \mathrm{Texp}(\int_0^t \hat{\hat{\Gamma}}^{(q)}(t)dt)$ and can be written as a product

$$\hat{U}_q(T) = e^{q(\hat{e}_{K-1} - \hat{e}_0)} \hat{u}_{K-1} e^{q(\hat{e}_{K-2} - \hat{e}_{K-1})} \hat{u}_{K-2} \cdot \ldots \cdot e^{q(\hat{e}_0 - \hat{e}_1)} \hat{u}_0 \tag{26}$$

compounded of the evolution $e^{q(\hat{e}_k - \hat{e}_{k+1})} \neq \hat{I}$ of the generating function of w_d at drive jumps occurring at times t_k, $1 \leq k \leq K-1$, and of the evolution operators of the probability distribution (1) $\hat{u}_k = \exp[\hat{\Gamma}(t_k + 0^+)\Delta t_k]$. Here, we consider discrete time intervals $\Delta t_k = t_{k+1} - t_k$, $0 = t_0 < \ldots < t_K = T$, $0 \leq k \leq K-1$, chosen in such a way to neglect variations of $\hat{\Gamma}$ at each interval Δt_k, see Fig. 2(a). Further, we refer to the drive discretized in such a way as K-step drive. In Eq. (25), the number of periods M equals to unity for the finite-time protocol, $M = 1$, and goes to infinity for periodic-NESS case, $M \to \infty$.

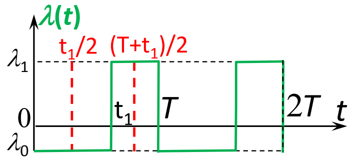

Figure 3. Sketch of the two periods of cyclic two-step drive protocol $\lambda(t)$ (green line). Red vertical lines show positions of time shifts with respect to which the drive is time-reversal symmetric.

In the periodic-NESS the quantity relevant for fluctuation relations is the cumulant generating function

$$\Delta_q = \lim_{M\to\infty} \frac{1}{M} \ln G_q(MT) = \ln \varepsilon_q \ , \tag{27}$$

which coincides with the logarithm of the largest eigenvalue ε_q of the evolution operator $\hat{U}_q$ (see, e.g., [68, 69]) and independent of the initial conditions. In terms of the above mentioned generating functions the integral fluctuation relation (12) reads as

$$\left\langle e^{-w_d} \right\rangle \equiv G_1(t) = 1 \Rightarrow \Delta_1 = 0 \ , \tag{28}$$

while the detailed ones are

$$G_q(t) = G_{1-q}(t) \text{ and } \Delta_q = \Delta_{1-q} \tag{29}$$

for the finite-time (15) and periodic-NESS (17) protocols, respectively.

4. Time-reversal symmetric drive and beyond

It is quite obvious that the time-reversal symmetry of the drive is too restrictive for satisfying the DFRs (15, 17, 29). What are more general conditions for which either or both symmetries (29) are satisfied? To answer this non-trivial question, we consider structure of the evolution operator. Due to the LDB (2), the evolution operators at each time step $t_k < t < t_{k+1}$ satisfy the symmetry

$$\hat{u}_k = e^{-\hat{e}_k} \hat{u}_k^T e^{\hat{e}_k} \tag{30}$$

and the corresponding evolution operator entering the generating function $G_{1-q}(MT) = \langle \mathbf{1} | \left(\hat{U}_{1-q}(T)\right)^M | \mathbf{p}_{eq}(0) \rangle$ takes the form after this symmetry transformation

$$\hat{U}_{1-q}(T) = e^{-\hat{e}_0} e^{q(\hat{e}_0 - \hat{e}_{K-1})} \hat{u}_{K-1}^T \cdot \ldots \cdot e^{q(\hat{e}_1 - \hat{e}_0)} \hat{u}_0^T e^{\hat{e}_0} \ . \tag{31}$$

This leads to the following expressions for the generating functions in both sides of DFR (29)

$$G_q(T) \ = \ \langle \mathbf{1} | \, e^{q(\hat{e}_{K-1} - \hat{e}_0)} \hat{u}_{K-1} \cdot \ldots \cdot \hat{u}_1 e^{q(\hat{e}_0 - \hat{e}_1)} \, | \mathbf{p}_{eq}(0) \rangle \ , \tag{32}$$

$$G_{1-q}(T) = \langle \mathbf{1} | \, e^{q(\hat{e}_1 - \hat{e}_0)} \hat{u}_1 \cdot \ldots \cdot \hat{u}_{K-1} e^{q(\hat{e}_0 - \hat{e}_{K-1})} \, | \mathbf{p}_{eq}(0) \rangle \ . \tag{33}$$

One can easily see that the only difference between two expressions is in the inverse order of indices corresponding to the time intervals.

In the particular case of $K = 2$ the only evolution operator entering the latter expressions is $\hat{u}_1$ and $K - 1 = 1$, thus the generating functions are (trivially) equal. Physically in this case of $K = 2$, the corresponding two-step drive is TRS with respect to a certain time shift. Indeed, in this case $\lambda(0 < t < t_1) = \lambda_0$ and $\lambda(t_1 < t < T) = \lambda_1$ and the time shifts $t_1/2$ and $(T + t_1)/2$ put the initial time to the middle of one of two plateaux thus making the drive TRS, see Fig. 3. As the generating function (32) does not depend on $\hat{u}_0$ and, thus, on the zeroth time interval Δt_0, the symmetry for it is valid in the same way as for the TRS drive protocol. It may be confusing why $G_q(T)$ is independent of the zeroth time interval Δt_0, but explicitly depends on the last one Δt_{K-1}. The answer to this question is hidden in the choice of the time discretization. Indeed, we have chosen the discretized protocol to start with the plateau followed by the instantaneous jump at t_{k+1} at each time interval $t_k < t \leq t_{k+1}$, Fig. 2(a). As the system is in equilibrium initially for the finite-time protocol, $M = 1$, the absence of the drive in $0 < t < t_1$ changes nothing. In an alternative discretization shown in Fig. 2(b), when the jumps in drive are followed by plateaux, $G_q(T)$ depends explicitly on Δt_0, but not on Δt_{K-1} as the relaxation at $t_{K-1} < t < T$ does not affect the dissipated work. In general continuous drive both possible plateaux in the beginning and in the end of the drive do not affect dissipated work as the control parameter is constant, Fig. 2(c). Here and further, we stick to the first variant of discretization shown in Fig. 2(a) for clarity.

For the general TRS drive all time intervals are coupled in pairs $\hat{u}_k = \hat{u}_{K-k}$, $\hat{e}_{K-k} = \hat{e}_k$ and, thus, the expressions (32, 33) are equal and finite-time DFR in (29) is obviously satisfied. The corresponding evolution operators $\hat{U}_q(T)$ and $\hat{U}_{1-q}(T)$ simply relate to each other

$$\left(\hat{U}_{1-q}^T\right)^M (T) = \hat{C}\left(\hat{U}_q(T)\right)^M \hat{C}^{-1} , \tag{34}$$

with $\hat{C} = e^{\hat{e}_0}\hat{u}_0$ for any M. Thus, asymptotic DFR in (29) is also satisfied as both the initial conditions and the evolution $\hat{C}$ give only subleading contributions to Δ_q in the limit $t \to \infty$.

This asymptotic DFR in (29) is also preserved in more general case, when the relation (34) between evolution operators $\hat{U}_{1-q}(T)$ and $\hat{U}_q(T)$ holds with an arbitrary matrix $\hat{C}$ which depends on q and on the protocol at one period, but not on the number of periods M. If on top of that we initialize the system in such a way that the vectors $|\mathbf{p}_{eq}(0)\rangle \equiv e^{-\hat{e}_0}|\mathbf{1}\rangle$ and $\langle\mathbf{1}|$ are the right and left eigenvectors of $\hat{C}$, respectively, with the same eigenvalue c

$$\hat{C}\,|\mathbf{p}_{eq}(0)\rangle = c\,|\mathbf{p}_{eq}(0)\rangle , \quad \langle\mathbf{1}|\,\hat{C} = \langle\mathbf{1}|\,c , \tag{35}$$

the expressions (32, 33) become equal.

From this perspective one might come to a quite natural conclusion that the symmetry of the cumulative function $\Delta_q = \Delta_{1-q}$ in periodic-NESS protocols is less restrictive than the one of the generating function $G_q = G_{1-q}$ in finite-time protocols as the former does not have any conditions on the initial distribution (cf. the discussion of the role of initial conditions in NESS [57]). However, in general it is not so clear.

	Crooks	TRS Crooks	G_q symmetry	G_q expression
Finite-time drive	$\frac{P(W)}{\tilde{P}(-W)} = e^{\beta(W-\Delta F)}$	$\frac{P(W)}{P(-W)} = e^{\beta(W-\Delta F)}$	$G_q = G_{1-q}$	$G_q \overset{\Delta t_0\to\infty}{\longleftarrow} \mathrm{tr}\,\hat{U}_q\|$
Periodic NESS	$\frac{1}{t}\ln\frac{P(wt)}{\tilde{P}(-wt)} \overset{t\to\infty}{\longrightarrow} \beta w$	$\frac{1}{t}\ln\frac{P(wt)}{P(-wt)} \overset{t\to\infty}{\longrightarrow} \beta w$	$\Delta_q = \Delta_{1-q}$	$\Delta_q = \max\mathrm{spec}[\hat{U}_q]$

Table 1. Summary of finite-time and periodic NESS fluctuation theorems. The notation "max spec" means the maximal eigenvalue in the spectrum of an operator.

Indeed, the condition (29) for $M > 1$ crucially depends on Δt_0 via the step evolution operator $\hat{u}_0$, while expressions (32, 33) do not. To clarify this statement, we derive a general relation between the generating function $G_q(T)$ and the trace of the evolution operator for $\Delta t_0 \to \infty$

$$G_q(T) \equiv \langle \mathbf{1}|\, \hat{U}_q(T)\, |\mathbf{p}_{eq}(0)\rangle = \lim_{\Delta t_0\to\infty} \mathrm{tr}\, \hat{U}_q(T) \ . \tag{36}$$

This is the main result of our paper, which works for any classical Markovian N-level system obeying rate equations (1).

The origin of this relation lies in the structure of rate equations with constant tunneling rates $\Gamma_{nn'}$, for example, at a certain step $t_k < t < t_{k+1}$. Indeed, the eigenvalues $\gamma_m(t_k) \leq 0$ of the rate matrix $\hat{\Gamma}(t_k + 0)$ are negative, except one single zero value $\gamma_0 = 0$ corresponding to the unit left eigenvector $\langle \mathbf{1}|$ and to the instantaneous equilibrium distribution vector $|\mathbf{p}_{eq}(t_k)\rangle$ as a right eigenvector. Thus, the evolution operator reads

$$\hat{u}_k(\Delta t_k) = |\mathbf{p}_{eq}(t_k)\rangle \langle \mathbf{1}| + e^{-|\gamma_{\min}|\Delta t_k} \hat{\delta u}_k(\Delta t_k) \ , \tag{37}$$

where $\gamma_{\min}(t_k) = \max_{m\neq 0}\gamma_m(t_k) < 0$ and $\hat{\delta u}_k(\Delta t_k)$ is the matrix with non-increasing matrix elements. The second term in (37) decays exponentially fast to zero with increasing Δt_0. Thus, considering the limit $\Delta t_0 \to \infty$ in r.h.s. of (36) and substituting expressions (32) and (26) in l.h.s. and r.h.s., one can easily prove the relation (36).

From Eq. (36) one can conclude that the finite-time fluctuation relation (15) is satisfied as soon as the trace of the evolution operator in the limit $\Delta t_0 \to \infty$ satisfies the symmetry

$$\lim_{\Delta t_0\to\infty} \mathrm{tr}\, \hat{U}_q(T) = \lim_{\Delta t_0\to\infty} \mathrm{tr}\, \hat{U}_{1-q}(T) \ . \tag{38}$$

On the other hand, the validity of the asymptotic fluctuation relation (17) depends not only on the evolution operator trace symmetry, but on the symmetry of its maximal eigenvalue (27). This shows that neither of DFRs in (29) implies the other. The general results on the detailed fluctuation theorems for the dissipated work known in the literature or derived in this section are summed up in Table 1.

A particular case of the symmetry (34) relating the step evolution operators $\hat{u}_k$ and $\hat{u}_{K-k}$ and generalizing the TRS drives is considered in Appendix B. This case provides an example when the asymptotic fluctuation theorem implies finite-time counterpart, unlike the results in NESS [57].

As shown in the next section, the satisfaction of the finite-time fluctuation theorem does not imply the same in the asymptotic long-time limit even in the simplest possible example of a classical Markovian two-level system.

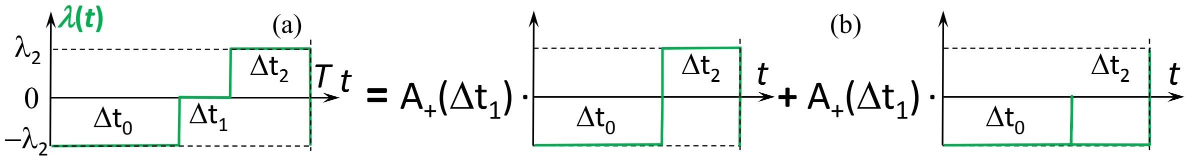

Figure 4. (a) Sketch of the cyclic three-step drive protocol $\lambda(t)$ (green line) and (b) its decomposition (41) into two-step cyclic and one-step non-cyclic drives with the coefficients $A_{\pm}(\Delta t_1)$ of the expansion (40) of the evolution operator $\hat{u}_1(\Delta t_1)$.

5. Two-level system

Two-level systems are special in several aspects. First, any rate matrix $\hat{\Gamma}$ in two-level systems satisfies LDB condition with certain energy difference $\beta(E_1 - E_0) \equiv \ln[\Gamma_{01}/\Gamma_{10}]$ normalized to temperature. Moreover, any probability distribution can be considered as thermal with a certain parameter $\beta(E_1 - E_0)$ possibly different from the above one ¶. In both cases the energy difference $E_1 - E_0$ might be not equal to the physical energy difference in non-equilibrium conditions, but as the two-level system has the only control parameter $2\lambda = \beta(E_1 - E_0) \equiv \ln[\Gamma_{01}/\Gamma_{10}]$ we will use it further. Second, there are only two drive symmetries of the kind of (34), TRS $\lambda(T-t) = \lambda(t)$ and anti-TRS drive $\lambda(T-t) = -\lambda(t)$. The difference between symmetric and anti-symmetric drives is subtle as the exchange of energies keeps the overall spectrum intact. However, one should take into account that the non-adiabatic exchange of energy levels affects the occupation probabilities $p_n(t)$. For example, if one prepares a two-level system in equilibrium with a certain ground E_0 and excited E_1 state energies and then suddenly exchange them ($\lambda(T/2-0) = -\lambda(T/2+0)$), the system would not be in the same equilibrium state and will decay to the new equilibrium after such quench perturbation.

To start with in this section we first go beyond symmetric and anti-symmetric drives mentioned above. As shown in the previous section any two-step drive, $K = 2$, is TRS and thus it leads to DFR (29) without any additional conditions (see, e.g., [68]). Therefore, we do one step beyond and provide an example of the simplest non-TRS drive, namely, three-step drive, $K = 3$, Fig. 4(a), and consider general conditions under which this drive satisfies both relations (15) and (17).

As follows from the calculations given in Appendix C, the necessary and sufficient condition for both DFRs (29) restrict the values of $\lambda(t)$ at the drive steps to be the following up to any permutation between steps

$$\lambda_0 = -\lambda_2, \quad \lambda_1 = 0\ . \tag{39}$$

The surprising thing here is that the above condition is independent not only of the zeroth time interval, but of *all* the time durations. One could understand this fact if for the generating function symmetry $G_q = G_{1-q}$ one needed to begin driving from the

¶ As one of consequences, in two-level systems it is possible to write fluctuation relations not only for thermodynamic quantities, but even for the finite-time average of the charge state [67] degeneracy point $\lambda = 0$ when both energies are equal in order to have nearly anti-TRS drive. However, for this, one has to make two other time intervals to be equal, which is not the case. Even more surprising thing is that the symmetry $G_q = G_{1-q}$ is valid for any permutations and time shifts of the drive.

The origin of this emerging symmetry is hidden in the structure of the evolution operator $\hat{u}_1(\Delta t_1)$ at $\lambda = 0$. Indeed, due to the equal values of both incoming rates $\Gamma_{01} = \Gamma_{10} \equiv |\gamma_{\min}|/2$ this evolution operator can be expanded into the superposition of the unity matrix $\hat{I}$ and the Pauli matrix σ_x reordering the energy levels E_n in the inverse order

$$\hat{u}_1(\Delta t_1) = A_+ \hat{I} + A_- \hat{\sigma}_x \ , \tag{40}$$

with $2A_\pm = 1 \pm e^{-|\gamma_{\min}|\Delta t_1}$. As a result, the generating function (32) splits into the sum of two-step cyclic and one-step acyclic drives corresponding to the first and second terms in r.h.s. of both following expressions, respectively (see Fig. 4 for details)

$$G_q(T) = \langle \mathbf{1}| e^{-2q\hat{e}_0} \hat{u}_2 e^{q\hat{e}_0} (A_+ \hat{I} + A_- \sigma_x) e^{q\hat{e}_0} |\mathbf{p}_{eq}(0)\rangle$$
$$= A_+ \langle \mathbf{1}| e^{-2q\hat{e}_0} \hat{u}_2 e^{2q\hat{e}_0} |\mathbf{p}_{eq}(0)\rangle + A_- \langle \mathbf{1}| e^{2q\hat{e}_0} |\mathbf{p}_{eq}(0)\rangle \ , \tag{41}$$
$$G_{1-q}(T) = \langle \mathbf{1}| e^{-q\hat{e}_0} (A_+ \hat{I} + A_- \sigma_x) e^{-q\hat{e}_0} \hat{u}_2 e^{2q\hat{e}_0} |\mathbf{p}_{eq}(0)\rangle$$
$$= A_+ \langle \mathbf{1}| e^{-2q\hat{e}_0} \hat{u}_2 e^{2q\hat{e}_0} |\mathbf{p}_{eq}(0)\rangle + A_- \langle \mathbf{1}| e^{2q\hat{e}_0} |\mathbf{p}_{eq}(0)\rangle \ . \tag{42}$$

This example opens the way to form non-TRS drives satisfying TRS versions (15, 17) of fluctuation theorems and motivates the studies in such simple models the first-passage time distribution [60, 59] and the analogy with multifractality [61] mentioned in the introduction.

Moreover, two-level systems allow one to find an explicit relation between the finite-time and asymptotic DFR (29). Indeed, we show below that if the asymptotic fluctuation theorem is valid $\Delta_q = \Delta_{1-q}$ for *two* drive protocols which differ only in the duration Δt_0 and $\Delta t_0'$ of the zeroth time interval, then its finite-time counterpart $G_q = G_{1-q}$ is also valid. Moreover in this case the asymptotic fluctuation theorem is valid $\Delta_q = \Delta_{1-q}$ for *any* Δt_0.

Surprisingly this statement also works in another direction: if asymptotic $\Delta_q = \Delta_{1-q}$ and finite-time $G_q = G_{1-q}$ fluctuation theorems are satisfied for a certain drive protocol, then they are valid for such protocol with *any* Δt_0.

The origin of this relations contains several ingredients. First one is the expression for the generating function G_q, Eq. (25), through the trace of the evolution operator (36), see the derivation in the previous section.

The second ingredient is that for a two-level system the validity of the symmetry $\Delta_q = \Delta_{1-q}$ is solely governed by the validity of the symmetry for the trace of the evolution operator

$$\mathrm{tr}\, \hat{U}_q(T) = \mathrm{tr}\, \hat{U}_{1-q}(T) \ . \tag{43}$$

Indeed, for any protocol and any classical Markovian N-level system the determinant of the evolution operator is q-independent and given by $\det \hat{U}_q(t) = e^{-\int_0^t \mathrm{tr}\, \hat{\Gamma}(t')dt'}$

(see Appendix A or [68] for details). In the two-level system the eigenvalues of $\hat{U}_q(T)$ are determined only by $\det \hat{U}_q(t)$ and $\operatorname{tr} \hat{U}_q(T)$, $2\varepsilon_q = \operatorname{tr} \hat{U}_q(T) + \sqrt{\left[\operatorname{tr} \hat{U}_q(T)\right]^2 - 4 \det \hat{U}_q(T)}$, thus Δ_q as the maximal eigenvalue among two is symmetric, $\Delta_q = \Delta_{1-q}$, if and only if Eq. (43) is satisfied.

The third and final ingredient for this calculation is the expression (37) for the step evolution operator similar to (40), where $-\gamma_{\min} = \Gamma_{01}(t_k) + \Gamma_{10}(t_k) > 0$ and $\hat{\delta u}_k$ is the constant matrix for two-level systems.

Combining ingredients (36, 37, 43) together one can express $G_q(T)$ via two $\operatorname{tr} \hat{U}_q(T, \Delta t_0)$ with different values Δt_0 and $\Delta t_0'$ of the zeroth time interval duration as follows

$$G_q(T) = \frac{\operatorname{tr} \hat{U}_q(T, \Delta t_0) - e^{-|\gamma_{\min}(0)|(\Delta t_0 - \Delta t_0')} \operatorname{tr} \hat{U}_q(T, \Delta t_0')}{1 - e^{-|\gamma_{\min}(0)|(\Delta t_0 - \Delta t_0')}} \,. \tag{44}$$

Analogously one can express $\operatorname{tr} \hat{U}_q(T, \Delta t_0'')$ through the same functions, see Appendix D.

However, such analysis cannot be repeated for a general classical Markovian N-level system. Indeed, as shown in the previous section the expression (36) is valid, while (43) is only necessary, but not sufficient condition for $\Delta_q = \Delta_{1-q}$ as not only determinant and trace govern the maximal eigenvalue of the matrix $\hat{U}_q(T)$. The expression (44) also cannot be written as, in general, the matrix structure of $\hat{\delta u}_k$ is time-dependent.

The only thing which one can derive is that the sufficient condition to have $G_q = G_{1-q}$ is the presence of the symmetry $\Delta_q = \Delta_{1-q}$ for N different zeroth time interval durations (leading to (43) for each of them). This sufficient condition comes from the fact that the matrix $\hat{\delta u}_k$ can be written as the sum of constant matrices with $N-1$ different exponentially decaying prefactors and, thus, one can derive expression for G_q analogous to (44), but it will include N traces $\operatorname{tr} \hat{U}_q(T)$ for the protocols with different Δt_0 in order to remove all $N-1$ exponentially decaying components of $\hat{\delta u}_k$.

This provides a hint that the symmetries both in finite-time fluctuation relations and in their periodic-NESS counterparts become more restrictive with increasing system degrees of freedom, but it cannot completely resolve the question about the relation between them.

6. Conclusion

To sum up, in this paper, the relations between finite-time (15) and infinite-time (17) fluctuation relations are considered. We are motivated to focus on the versions of these fluctuation theorems coinciding in their form with the ones for time-reversal symmetric drives as they provide the solid ground both for the straightforward calculations of first-passage-time distribution [59, 60] and for the unexpected analogy of the work statistics with the multifractality of the wavefunctions close to the Anderson localization transition [61].

In the general case of a classical Markovian N-level system, we derive the condition (34) with an arbitrary matrix $\hat{C}$ depending on q and on the protocol at one period, but not on the number of periods M to satisfy an infinite-time fluctuation theorem (17). Also we provide the sufficient condition (35) for the corresponding finite-time fluctuation theorem (15) posing additional restrictions on the initial distribution similarly to [57]. On the other hand, the particular case (B.1, B.3) of the above mentioned symmetries is considered in Appendix B and provides an example when the finite-time fluctuation theorem is less restrictive than its asymptotic counterpart.

In the particular case of a two-level system the explicit relation (44) between finite-time (15) and infinite-time (17) fluctuation relations is found. Its formulation reads in two ways: If the asymptotic fluctuation theorem is valid $\Delta_q = \Delta_{1-q}$ for *two* drive protocols which differ only in the duration Δt_0 and $\Delta t_0'$ of the zeroth time interval, then its finite-time counterpart $G_q = G_{1-q}$ is valid as well as the asymptotic one for such protocol with *any* Δt_0. If asymptotic $\Delta_q = \Delta_{1-q}$ and finite-time $G_q = G_{1-q}$ fluctuation theorems are satisfied for a certain drive protocol, then they are valid for such protocol with *any* Δt_0. Additionally, the class of drive protocols satisfying the above mentioned relations is extended from the time-reversal-(anti)symmetric ones and an example of the simplest non-time-reversal-(anti)symmetric drive is given.

7. Acknowledgements

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Appendix A. Rate equations and generating functions

In this Appendix section, we give detailed calculations of the probability distribution functions $P(X)$ of a certain stochastic quantity X and of the corresponding generating function G_q based on the rate equations (1). As in the case of the dissipated work the probability distribution $P(X)$ itself does not determine explicitly the system state $n(t)$ one have to generalize it to the n-resolved distribution function $|\mathbf{P}(X)\rangle = (P_0(X), \ldots, P_{N-1}(X))$, with the components defined as

$$P_n(X = x) = \langle \delta(X - x)\delta_{n,n(t)}\rangle \ . \tag{A.1}$$

The distribution function is given by the sum $P(X) = \langle \mathbf{1}|\mathbf{P}(X)\rangle \equiv \sum_n P_n(X)$.

In the special case of the work $X = W$, one can write rate equations for $P_n(W)$ explicitly [68]

$$\frac{d}{dt}|\mathbf{P}(W,t)\rangle = \hat{\Gamma}(t)\,|\mathbf{P}(W,t)\rangle - \frac{\partial}{\partial W}\left[\hat{\dot{W}}\,|\mathbf{P}(W,t)\rangle\right] \tag{A.2}$$

as the work rate in the certain state $\dot{W}_n \equiv \frac{dW}{dt}\big|_{n(t)=n} = \frac{\partial E_n}{\partial \lambda}\frac{d\lambda}{dt}$ (written in the matrix form $\dot{W}_{n,n'} \equiv \delta_{n,n'}\dot{W}_n$) is a deterministic function of the system state $n(t)$. As work performed on the system at time $t = 0$ is zero the initial condition for $|\mathbf{P}(W,t)\rangle$ reads as $|\mathbf{P}(W,0)\rangle = \delta(W)\,|\mathbf{p}(0)\rangle$. This analysis also works for any quantity X with the same property of $\dot{X}_n$.

In general, it is impossible to write the rate equation for $|\mathbf{P}(X)\rangle$ itself, but one can do it for the n-resolved generating function of the variable X defined as the Laplace transform of the latter

$$|\mathbf{G}_q\rangle = \int |\mathbf{P}(X)\rangle\, e^{-qX} dX \ , \quad G_{q,n} = \langle e^{-qX(t)}\delta_{n,n(t)}\rangle \ . \tag{A.3}$$

Indeed, considering the system state trajectory $\{n(t)\}$ as a set of jumps from n_{j-1} to n_j occurred at time instants $t_j^{(J)}$, $j = 1, N_J$, $t_j^{(J)} < t_{j+1}^{(J)}$, $t_0^{(J)} = 0$, $t_{N_J+1}^{(J)} = t$, Fig. 1, one can write the trajectory probability measure explicitly

$$P_{N_J}(t; n_0, t_0^{(J)}, n_1, t_1^{(J)}, \ldots, n_M, t_M^{(J)}) = \ p_{n_0}(t_0^{(J)}) e^{-\int_{t_0^{(J)}}^{t_1^{(J)}} \Gamma_{n_0,n_0}(t')dt'} \times \prod_{j=1}^{N_J} \Gamma_{n_j,n_{j-1}}(t_j^{(J)}) e^{-\int_{t_j^{(J)}}^{t_{j+1}^{(J)}} \Gamma_{n_j,n_j}(t')dt'} \tag{A.4}$$

which is the product of the probabilities $\exp\left[-\int_{t_j^{(J)}}^{t_{j+1}^{(J)}} \Gamma_{n_j,n_j}(t')dt'\right]$ to have *no jumps* in the system in the time interval $(t_j^{(J)}, t_{j+1}^{(J)})$ provided the system was in the state n_j at time instant $t_j^{(J)}$ and the conditional probabilities $\Gamma_{n_j,n_{j-1}}(t_j^{(J)})dt_j^{(J)}$ to have a jump from n_{j-1} to n_j in the time interval $(t_j^{(J)}, t_j^{(J)} + dt_j^{(J)})$ provided there was no jumps in the interval $(t_{j-1}^{(J)}, t_j^{(J)})$. As a result, the rate equations (1) can be easily derived from this expression with help of averaging over P_{N_J} of the definition of the probability distribution $p_n(t) = \left\langle \delta_{n,n(t)} \right\rangle$, see, e.g., [66].

To write the rate equation for the n-resolved generating function $G_{q,n} = \left\langle e^{-qX(t)}\delta_{n,n(t)} \right\rangle$ of the piecewise deterministic stochastic process [70] $X(t)$, one should average $e^{-qX(t)}\delta_{n,n(t)}$ over the same distribution (A.4). For this, one needs to write the expression for $X(t)$ at the same state trajectory

$$X(t) = \sum_j \left[\Delta X_{n_{j-1}\to n_j}(t_j^{(J)}) + \int_{t_{j-1}^{(J)}}^{t_j^{(J)}} \dot{X}_{n_{j-1}}(t')dt' \right] \ . \tag{A.5}$$

Equation (A.5) has both the deterministic contributions $\dot{X}_n(t)$ at fixed n and the stochastic jumps $\Delta X_{n\to n'}(t)$ due to the jumps in $n(t)$ (like for the total entropy production Δs_{tot}). These contributions enter the generating function expression just by modifying the rates

$$\Gamma_{n,n'}^{(q)}(t) = \Gamma_{n,n'}(t) e^{-q\Delta X_{n'\to n}(t)} \ , \quad \Gamma_{n,n}^{(q)}(t) = \Gamma_{n,n}(t) + q\dot{X}_n(t) \ . \tag{A.6}$$

Thus, with use of the standard trajectory representation of the Markov jump processes which is widely used in the full counting statistics (see, e.g., [66]), we derive the rate equations (21 for the generating function in the form of (1)

$$\frac{d}{dt}\,|\mathbf{G}_q(t)\rangle = \hat{\Gamma}^{(q)}(t)\,|\mathbf{G}_q(t)\rangle \ , \tag{A.7}$$

with the modified rates (A.6) and the initial condition $|\mathbf{G}_q(0)\rangle = |\mathbf{p}(0)\rangle$ provided $X(0) = 0$. Note that unlike Eq. (1) the latter equation does not conserve normalization condition as $\Gamma^{(q)}_{n,n} \neq \sum_{n' \neq n} \Gamma^{(q)}_{n',n}$.

For the quantities which depend only on the change of the system state $n(t)$ (like the heat Q or the environment entropy production Δs_m), only the incoming rates are modified by the exponential factor depending on the size of the corresponding jump $\Delta X_{n' \to n}(t)$

$$\Gamma^{(q)}_{n,n'}(t) = \Gamma_{n,n'}(t) e^{-q\Delta X_{n'\to n}(t)} \ . \tag{A.8}$$

Unlike this, for the quantities (like the work W) for which rate $\dot{X}_n(t)$ is a deterministic function of $n(t)$ only the escape rates should be modified

$$\Gamma^{(q)}_{n,n}(t) = \Gamma_{n,n}(t) + q\dot{X}_n(t) \ . \tag{A.9}$$

The probability distribution of X

$$P(X) = \frac{1}{2\pi i} \lim_{Q\to\infty} \int_{\chi - iQ}^{\chi + iQ} G_q(t) e^{qX} dq \ . \tag{A.10}$$

is given by the inverse Laplace transform of the generating function, where χ is greater than the real part of all singularities of $G_q(t)$ as a function of q and

$$G_q(t) = \langle \mathbf{1} | \mathbf{G}_q(t) \rangle \equiv \sum_n G_{q,n}(t) \ . \tag{A.11}$$

Note that the deterministic part $X_n(t) = \int^t \dot{X}_n(t)$ of the piecewise deterministic stochastic process (A.5) can be absorbed by the following transformation

$$\left| \tilde{\mathbf{G}}_q(t) \right\rangle = e^{q\hat{X}(t)} \left| \mathbf{G}_q(t) \right\rangle \ . \tag{A.12}$$

restoring a simple jump process with the jump size being the sum of two contributions $\Delta X_{n'\to n}(t) + (X_{n'}(t) - X_n(t))$. Here, $X_{n,n'}(t) \equiv \delta_{n,n'} X_n(t)$ and the l.h.s. satisfies the rate equations (A.7) with the rates replaced by

$$\tilde{\Gamma}^{(q)}_n = \Gamma_n \text{ and } \tilde{\Gamma}^{(q)}_{n,n'} = \Gamma_{n,n'} e^{-q[\Delta X_{n'\to n}(t) + X_{n'}(t) - X_n(t)]} \ . \tag{A.13}$$

The price paid for this simplification is the modification of the initial conditions

$$\left| \tilde{\mathbf{G}}_q(0) \right\rangle = e^{q\hat{X}(0)} \left| \mathbf{p}(0) \right\rangle \ . \tag{A.14}$$

The evolution operator $\hat{U}_q(t)$ entering the expression (25) for the generating function $G_q(MT)$ satisfies the same rate equations (21, A.7) as $|G_q(t)\rangle$. Thus, the measure of phase volume contraction of the system stochastic dynamics, namely, the determinant of the evolution operator $\det \hat{U}_q(t)$ satisfies the following rate equation

$$\frac{d}{dt} \det \hat{U}_q(t) = \operatorname{tr} \hat{\Gamma}^{(q)}(t) \det \hat{U}_q(t) \tag{A.15}$$

and does not depend on q as $\operatorname{tr} \hat{\Gamma}^{(q)}(t) = \operatorname{tr} \hat{\Gamma}(t) = \sum_n \Gamma_{n,n}(t)$

$$\det \hat{U}_q(t) = e^{-\int_0^t \operatorname{tr} \hat{\Gamma}(t') dt'} \equiv e^{-\tau(t)} \leq 1 \ . \tag{A.16}$$

The function $\tau(t)$ gives a certain rescaled "time" (analogous to the entropic time in Ref. [52]), which sets time of the fastest decay to unity. Note that the time-reversal transformation changing t by $t_{\max} - t$ changes the rescaled time $\tau(t)$ by $\tau(t_{\max}) - \tau(t)$ as well.

Appendix B. Example of symmetry (34)

The particular example of the symmetry (34) for the asymptotic fluctuation theorem (17) relating the step evolution operators $\hat{u}_k$ and $\hat{u}_{K-k}$ in time intervals Δt_k and Δt_{K-k} and generalizing the TRS drives can be written as follows

$$e^{q\hat{e}_k}\hat{u}_k e^{-q\hat{e}_k} = \hat{B}_q e^{q\hat{e}_{K-k}}\hat{u}_{K-k}e^{-q\hat{e}_{K-k}}\hat{B}_q^{-1} \ . \tag{B.1}$$

Here, $\hat{B}_q$ is a certain time-independent matrix. This symmetry corresponds to the following expression for the matrix $\hat{C} = e^{\hat{e}_0}\hat{u}_0 e^{-q\hat{e}_0}\hat{B}_q e^{q\hat{e}_0}$ from (34) if the matrix $e^{-q\hat{e}_0}\hat{B}_q e^{q\hat{e}_0}$ commute with $\hat{u}_0$

$$e^{-q\hat{e}_0}\hat{B}_q e^{q\hat{e}_0}\hat{u}_0 = \hat{u}_0 e^{-q\hat{e}_0}\hat{B}_q e^{q\hat{e}_0} \ . \tag{B.2}$$

Indeed,

$$\begin{aligned}\hat{U}^T_{1-q}(T) &= e^{\hat{e}_0}\hat{u}_0 e^{q(\hat{e}_1-\hat{e}_0)}\hat{u}_1\cdot\ldots\cdot\hat{u}_{K-1}e^{q(\hat{e}_0-\hat{e}_{K-1})}e^{-\hat{e}_0}\\ &= e^{\hat{e}_0}\hat{u}_0 e^{-q\hat{e}_0}e^{q\hat{e}_1}\hat{u}_1 e^{-q\hat{e}_1}\cdot\ldots\cdot e^{q\hat{e}_{K-1}}\hat{u}_{K-1}e^{-q\hat{e}_{K-1}}e^{(q-1)\hat{e}_0}\\ &= e^{\hat{e}_0}\hat{u}_0 e^{-q\hat{e}_0}\hat{B}_q e^{q\hat{e}_{K-1}}\hat{u}_{K-1}e^{-q\hat{e}_{K-1}}\cdot\ldots\cdot e^{q\hat{e}_1}\hat{u}_1 e^{-q\hat{e}_1}\hat{B}_q^{-1}e^{(q-1)\hat{e}_0}\\ &= e^{\hat{e}_0}\hat{u}_0(e^{-q\hat{e}_0}\hat{B}_q e^{q\hat{e}_0})\hat{U}_q(T)\hat{u}_0^{-1}(e^{-q\hat{e}_0}\hat{B}_q e^{q\hat{e}_0})^{-1}e^{-\hat{e}_0} = \hat{C}\hat{U}_q(T)\hat{C}^{-1} \ .\end{aligned}$$

In this case, the symmetry $G_q = G_{1-q}$ is fulfilled automatically as the commutation (B.2) leads to the common eigenbasis of both matrices $e^{-q\hat{e}_0}\hat{B}_q e^{q\hat{e}_0}$ and $\hat{u}_0$. Thus, the vectors $|\mathbf{p}_{eq}(0)\rangle$ and $\langle\mathbf{1}|$ are the right and left eigenvectors of $e^{-q\hat{e}_0}\hat{B}_q e^{q\hat{e}_0}$, respectively, with the same eigenvalue b

$$\left(e^{-q\hat{e}_0}\hat{B}_q e^{q\hat{e}_0}\right)|\mathbf{p}_{eq}(0)\rangle = b\,|\mathbf{p}_{eq}(0)\rangle \ , \quad \langle\mathbf{1}|\,e^{-q\hat{e}_0}\hat{B}_q e^{q\hat{e}_0} = \langle\mathbf{1}|\,b \tag{B.3}$$

and this matrix can be diminished in (32, 33) after the transformation (B.1). Note that the finite-time symmetry works even for lifted commutation relation (B.2) if Eq. (B.3) still holds. This hints that, in this concrete example, the periodic NESS fluctuation theorem is more restrictive on the drive than its finite-time counterpart.

As Eq. (B.1) works for all k and for general step evolution operators, the matrix $\hat{B}_q$ satisfies the following condition $\hat{B}_q^2 = \hat{I}$ and thus all eigenvalues, including b are 1 or -1. A reasonable example of the transformation $\hat{B}_q$ is the permutation of levels $E_n(t_k) = E_{P(n)}(t_{K-k})$ with $P(P(n)) \equiv n$, leading, e.g., to anti-symmetric drive when in the second half period all the levels E_n are put in the reversed order, $E_n(T-t) - E_m(T-t) = E_m(t) - E_n(t)$. As discussed in the main text, this level permutation does not change the system itself, but affects the dynamics of the occupancies $p_n(t)$ and thus leads to some non-trivial dissipated work.

To sum up this section, we provide a particular example of the symmetry (34) being probably just the permutation of energy levels, which demonstrate that the finite-time fluctuation theorem can be less restrictive than its asymptotic counterpart.

Appendix C. Three-step drive in two-level system

As mentioned in the main text for a two-level system, the only control parameter is $2\lambda(t) = \beta(E_1 - E_0)$. Omitting the unimportant global energy shift one can take

$E_0 = -E_1$ and write the energy matrix in the form of Pauli matrix $\beta\hat{E}(t) = \sigma_z\lambda(t)$. Then the free energy is $\beta F(t) = -\ln[2\cosh(\lambda(t)/2)]$, the equilibrium probability distribution vector $|\mathbf{p}_{eq}(t)\rangle = e^{-\sigma_z\lambda(t)}|\mathbf{1}\rangle$ and the matrix of the tunneling rates reads as

$$\hat{\Gamma}(t) = \gamma(t)\begin{pmatrix} -e^{\lambda(t)} & e^{-\lambda(t)} \\ e^{\lambda(t)} & -e^{-\lambda(t)} \end{pmatrix} = \gamma(t)(\sigma_x - \hat{I})e^{\sigma_z\lambda} \,. \quad \text{(C.1)}$$

The step evolution operator $\hat{u}_k = \exp[\hat{\Gamma}(t_k+0)\Delta t_k]$ can be written in a standard form (see, e.g., [68, 69])

$$\begin{aligned}\hat{u}_k &= \hat{I} + \frac{1-e^{-|\gamma_{\min}(t_k)|\Delta t_k}}{2\cosh\lambda_k}\left(\sigma_x - \hat{I}\right)e^{\sigma_z\lambda_k} \\ &= |\mathbf{p}_{eq}(t_k)\rangle\langle\mathbf{1}| + e^{-|\gamma_{\min}(t_k)|\Delta t_k}\frac{\left(\hat{I}-\sigma_x\right)e^{\sigma_z\lambda_k}}{2\cosh\lambda_k} \,, \end{aligned} \quad \text{(C.2)}$$

with $-\gamma_{\min}(t_k) = 2\gamma_k\cosh\lambda_k > 0$ and $|\mathbf{p}_{eq}(t_k)\rangle\langle\mathbf{1}| \equiv e^{-\sigma_z\lambda}\left(\hat{I}+\sigma_x\right)$. The last line in (C.2) confirms the general form (37), while the first line for $\lambda = 0$ goes to (40).

According to Eq. (27), the cumulative distribution function $\Delta_q(T)$ coincides with the logarithm of the maximal eigenvalue of $\hat{U}_q(T)$. For the two-level system, this eigenvalue can be explicitly written (see, e.g., [68, 69])

$$\varepsilon_q = \frac{\operatorname{tr}\hat{U}_q(T) + \sqrt{\left[\operatorname{tr}\hat{U}_q(T)\right]^2 - 4\det\hat{U}_q(T)}}{2}. \quad \text{(C.3)}$$

As follows from Eq. (A.16), the determinant $\det\hat{U}_q(T)$ does not depend on q. Thus using Eqs. (27), (36), and (C.3) one concludes that the analysis of $\operatorname{tr}\hat{U}_q(T)$ is enough for both DFRs (29).

Evaluating the trace of Eq. (26) one should keep only even powers of σ_x

$$\begin{aligned}\operatorname{tr}\hat{U}_q(T) = B + \sum_{k=0}^{2}\{\ & C_k\cosh\left[(\lambda_k-\lambda_{k-1})(2q-1)\right] \\ & - S_k\sinh\left[(\lambda_k-\lambda_{k-1})(2q-1)\right]\},\end{aligned} \quad \text{(C.4)}$$

with

$$B = \operatorname{tr}\left[\prod_{k=0}^{2}\left(\hat{I} - \frac{(1-e^{-|\gamma_{\min}(t_k)|\Delta t_k})e^{\sigma_z\lambda_k}}{2\cosh\lambda_k}\right)\right] \,, \quad \text{(C.5)}$$

$$C_k = (1+e^{-|\gamma_{\min}(t_{k+1})|\Delta t_{k+1}})\frac{(1-e^{-|\gamma_{\min}(t_k)|\Delta t_k})}{2\cosh\lambda_k}\frac{(1-e^{-|\gamma_{\min}(t_{k-1})|\Delta t_{k-1}})}{2\cosh\lambda_{k-1}} \quad \text{(C.6)}$$

$$S_k = 2\sinh\lambda_{k+1}\frac{(1-e^{-|\gamma_{\min}(0)|\Delta t_0})}{2\cosh\lambda_0}\frac{(1-e^{-|\gamma_{\min}(t_1)|\Delta t_1})}{2\cosh\lambda_1}\frac{(1-e^{-|\gamma_{\min}(t_2)|\Delta t_2})}{2\cosh\lambda_2} \quad \text{(C.7)}$$

Here, the indices k are considered modulo $K = 3$.

Thus, the symmetry $\operatorname{tr}\hat{U}_q(T) = \operatorname{tr}\hat{U}_{1-q}(T)$ is valid, if and only if, for any q

$$\sum_{k=0}^{2}\sinh\left[(\lambda_k-\lambda_{k-1})(2q-1)\right]\sinh\lambda_{k+1} = 0 \,. \quad \text{(C.8)}$$

Without loss of generality, let's consider $\lambda_0 < \lambda_1 < \lambda_2$ and send $q \to \infty$. Then one concludes that the coefficient $\sinh \lambda_1$ in front of the hyperbolic sine with the largest increment $\lambda_2 - \lambda_0 > \lambda_1 - \lambda_0, \lambda_2 - \lambda_1 > 0$ should go to zero, thus, $\lambda_1 = 0$. As a result, Eq. (C.8) reduces to

$$\sinh[\lambda_0]\sinh[\lambda_2(2q-1)] - \sinh[\lambda_2]\sinh[\lambda_0(2q-1)] = 0, \tag{C.9}$$

and leads to $\lambda_0 = -\lambda_2$. This completes the proof of Eq. (39).

Appendix D. Relations (36, 44) between G_q and $\operatorname{tr} U_q$

As the step evolution operator $\hat{u}_k(\Delta t_k)$ entering the expression (26) for the total evolution operator has the only non-negative eigenvalue 0 corresponding to the left $\langle \mathbf{1}|$ and right $|\mathbf{p}_{eq}\rangle$ eigenvectors, it can be represented in the form (37)

$$\hat{u}_k(\Delta t_k) = |\mathbf{p}_{eq}(t_k)\rangle \langle \mathbf{1}| + \hat{\delta u}_k(\Delta t_k) \ , \tag{D.1}$$

with the elements of the matrix $\hat{\delta u}_k(\Delta t_k)$ exponentially decaying with the time duration Δt_k. As a result,

$$\lim_{\Delta t_0 \to \infty} \hat{U}_q(T) = e^{q(\hat{e}_{K-1}-\hat{e}_0)}\hat{u}_{K-1}e^{q(\hat{e}_{K-2}-\hat{e}_{K-1})}\hat{u}_{K-2}\cdot\ldots\cdot e^{q(\hat{e}_0-\hat{e}_1)}|\mathbf{p}_{eq}(0)\rangle\langle\mathbf{1}| \tag{D.2}$$

and thus the trace of the latter $\lim_{\Delta t_0 \to \infty} \operatorname{tr} \hat{U}_q(T)$ coincides with the expression for $G_q(T)$ (36) and this concludes the derivation.

For the two-level system the expression (D.1) simplifies to (C.2) and thus

$$\operatorname{tr} U_q(T, \Delta t_0) = G_q + e^{-|\gamma_{\min}(0)|\Delta t_0} \operatorname{tr}\left[e^{q(\hat{e}_{K-1}-\hat{e}_0)}\hat{u}_{K-1}\cdot\ldots\cdot\hat{u}_1 e^{q(\hat{e}_0-\hat{e}_1)}\hat{\delta u}_0\right] . \tag{D.3}$$

Using the latter expression (D.3) for $\operatorname{tr} U_q(T, \Delta t_0)$ and $\operatorname{tr} U_q(T, \Delta t_0')$ and excluding the second terms from them, one comes to Eq. (44). The more general expression for $\operatorname{tr} U_q(T, \Delta t_0'')$ via $\operatorname{tr} U_q(T, \Delta t_0)$ and $\operatorname{tr} U_q(T, \Delta t_0')$ takes the form

$$\operatorname{tr} U_q(T, \Delta t_0'') = \frac{\left(e^{-|\gamma_{\min}(0)|\Delta t_0'} - e^{-|\gamma_{\min}(0)|\Delta t_0''}\right)\operatorname{tr}\hat{U}_q(T, \Delta t_0)}{e^{-|\gamma_{\min}(0)|\Delta t_0'} - e^{-|\gamma_{\min}(0)|\Delta t_0}} - \frac{\left(e^{-|\gamma_{\min}(0)|\Delta t_0} - e^{-|\gamma_{\min}(0)|\Delta t_0''}\right)\operatorname{tr}\hat{U}_q(T, \Delta t_0')}{e^{-|\gamma_{\min}(0)|\Delta t_0'} - e^{-|\gamma_{\min}(0)|\Delta t_0}} . \tag{D.4}$$

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