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Supersymmetric Field Theories from Twisted Vector Bundles

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Abstract: We give a description of the delocalized twisted cohomology of an orbifold and the Chern character of a twisted vector bundle in terms of supersymmetric Euclidean field theories. This includes the construction of a twist functor for 1|1-dimensional EFTs from the data of a gerbe with connection.

1. Introduction

In this paper, we explore a twisted version of the Stolz–Teichner program on the use of supersymmetric Euclidean field theories (EFTs) as geometric cocycles for cohomology theories [26]. We focus on twisted 1|1 and 0|1-dimensional EFTs over an orbifold $\mathfrak{X}$; the corresponding cohomology theories are (twisted) K-theory and delocalized de Rham cohomology. One of our main goals is to describe the Chern character

$$\mathrm{ch}\colon K^{\alpha}(\mathfrak{X}) \to H^{\mathrm{ev}}_{\mathrm{deloc}}(\mathfrak{X}, \alpha)$$

of a twisted vector bundle in terms of dimensional reduction of field theories. (For compact $\mathfrak{X}$, this Chern character map provides an isomorphism after complexification; thus, delocalized cohomology, which we recall below, is a stronger invariant than regular de Rham cohomology.) Here, the twist is $\alpha \in H^3(\mathfrak{X}; \mathbb{Z})$. Thus, on the field theory side, our first task is to construct from α a Euclidean twist functor (or anomaly)

$$T \in 1|1\text{-ETw}(\mathfrak{X})$$

for 1|1-EFTs over $\mathfrak{X}$ and describe its dimensional reduction $T' \in 0|1\text{-ETw}(\Lambda\mathfrak{X})$, which is a twist over the inertia orbifold $\Lambda\mathfrak{X}$. It will turn out, as expected, that T'-twisted field theories model the delocalized cohomology group $H^{\mathrm{ev}}_{\mathrm{deloc}}(\mathfrak{X}, \alpha)$. Next, we construct, from the data of an α-twisted vector bundle $\mathfrak{V}$ on $\mathfrak{X}$, a T-twisted 1|1-EFT, and show that its dimensional reduction, which is T'-twisted, corresponds to $\mathrm{ch}(\mathfrak{V})$.

1.1. Field theories and twisted cohomology. In this paper, we use the Stolz–Teichner framework of geometric field theories laid out in [26], which draws on the functorial approach to quantum field theory of Segal, Atiyah and many others. A supersymmetric Euclidean (quantum) field theory of dimension $d|\delta$ over an orbifold $\mathfrak{X}$ is a symmetric monoidal functor

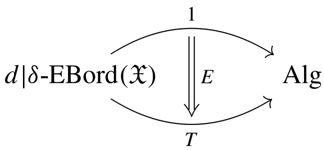
$$E \in \mathrm{Fun}^{\otimes}_{\mathrm{SM}}(d|\delta\text{-EBord}(\mathfrak{X}), \mathrm{Vect})$$

between a bordism category and the category Vect of complex super vector spaces. Roughly speaking, the bordism category in question has closed $(d-1)|\delta$-dimensional supermanifolds as objects and $d|\delta$-dimensional bordism between them as morphisms; all supermanifolds are equipped with a Euclidean structure (which boils down to a flat Riemannian metric in the purely bosonic case $\delta = 0$) and a smooth map to $\mathfrak{X}$. Thus, E can be thought of as a family of field theories parametrized by $\mathfrak{X}$. Field theories can be pulled back along maps $\mathfrak{Y} \to \mathfrak{X}$. The subscript "SM" above indicates that we require the assignment E to be smooth, in the sense that it sends smooth families of objects and morphisms to smooth families. To make precise sense of this, we promote $d|\delta$-EBord(X) and Vect to internal categories in symmetric monoidal stacks over the site SM of supermanifolds, and E to a functor of internal categories.

Many interesting constructions do not quite produce a field theory as defined above, but rather an "anomalous" or twisted theory [12,13,21, etc.]. In our framework, those are defined as follows. We write

$$d|\delta\text{-ETw}(\mathfrak{X}) = \mathrm{Fun}^{\otimes}_{\mathrm{SM}}(d|\delta\text{-EBord}(\mathfrak{X}), \mathrm{Alg})$$

for the groupoid of $d|\delta$-dimensional Euclidean twists over $\mathfrak{X}$. Here Alg is the internal category of (bundles of) algebras, bimodules, and bimodule maps. Finally, given $T \in d|\delta$-ETw$(\mathfrak{X})$, a twisted field theory is a natural transformation E



from the trivial twist 1 (which maps everything to $\mathbb{C}$) to T. We write $d|\delta$-EBord$^T(\mathfrak{X})$ for the groupoid of T-twisted Euclidean field theories over $\mathfrak{X}$. (See [26] for the complete definitions, including more details on the categorical and supergeometry aspects.)

A conjecture of Stolz and Teichner [25] states that 2|1-EFTs provide geometric cocycles for the cohomology theory TMF of topological modular forms, in the sense that, for any manifold X,

$$2|1\text{-EFT}^n(X)/\text{concordance} \cong \mathrm{TMF}^n(X).$$

Here, two field theories E_0, E_1 are said to be concordant if there exists $E \in d|\delta$-EFT$(X \times \mathbb{R})$ such that $E_i \cong E|_{X\times\{i\}}$. (Among other difficulties, a solution to the conjecture certainly requires that we refine the definitions and pass to fully extended geometric field theories.) In this paper, we focus on the 1|1 and 0|1-dimensional cases, where an analogue of that conjecture states that the relevant cohomology theories are topological K-theory and de Rham cohomology [15,16]. When we replace the background manifold X by an orbifold $\mathfrak{X}$, it is natural to ask what kind of information about twisted equivariant cohomology such field theories capture—but the orbifold perspective is important to deal with twists, even if $\mathfrak{X}$ is equivalent to a manifold.

We begin our study with a classification of $0|1$-dimensional twists for EFTs over an orbifold (Sect. 2). For a particular $T_\alpha \in 0|1\text{-ETw}(\Lambda\mathfrak{X})$, concordance classes of twisted EFTs over the inertia $\Lambda\mathfrak{X}$ (the orbifold of "constant loops") are in natural bijection with the delocalized twisted cohomology $H^*_{\text{deloc}}(\mathfrak{X}, \alpha)$. Then, turning to $1|1$-dimensional considerations, we construct $T_{\tilde{\mathfrak{X}}} \in 1|1\text{-ETw}(\mathfrak{X})$ taking as input $\alpha \in H^3(\mathfrak{X}; \mathbb{Z})$, or, rather, a $(\mathbb{C}^\times\text{-})$gerbe with connection $\tilde{\mathfrak{X}} \to \mathfrak{X}$ representing that class (Sect. 4). This is an extension of the transgression construction for gerbes [8] in the sense that it produces, in particular, a line bundle on the stack $\mathfrak{K}(\mathfrak{X})$ of supercircles over $\mathfrak{X}$; that stack is a super analogue of $L\mathfrak{X}/\!\!/ \operatorname{Diff}^+(S^1)$, and the line bundle we obtain is a super analogue of the usual transgression of the gerbe.

It is now reasonable to conjecture that

$$1|1\text{-EFT}^{T_{\tilde{\mathfrak{X}}}}(\mathfrak{X})/\text{concordance} \cong K^\alpha(\mathfrak{X}), \tag{1.1}$$

but this question is open even in the case where $\mathfrak{X}$ is a manifold and $T_{\tilde{\mathfrak{X}}}$ is trivial, so we will not dwell on it here. Instead, we will demonstrate the meaningfulness of our construction by associating a twisted field theory $E_{\mathfrak{V}}$ to any $\tilde{\mathfrak{X}}$-twisted vector bundle $\mathfrak{V}$, and identifying its dimensional reduction. Here again, the partition function of $E_{\mathfrak{V}}$ is the super counterpart of a classical construction, namely the trace of the holonomy, which in this case is not a function but rather a section of the transgression of $\tilde{\mathfrak{X}}$.

1.2. Dimensional reduction and the Chern character. Dimensional reduction is, intuitively, the assignment of a $(d-1)$-dimensional theory to a d-dimensional theory induced by the functor of bordism categories $S^1 \times —\colon (d-1)\text{-Bord} \to d\text{-Bord}$. For field theories over an orbifold, the action of the circle group $\mathbb{T}$ on the inertia $\Lambda\mathfrak{X}$ can be used to refine this to (partial) assignments

$$1|1\text{-ETw}(\mathfrak{X}) \to 0|1\text{-ETw}(\Lambda\mathfrak{X}), \qquad 1|1\text{-EFT}^T(\mathfrak{X}) \to 0|1\text{-EFT}^{T'}(\Lambda\mathfrak{X}).$$

Our dimensional reduction procedure was developed in [23] and is recalled in Sect. 5.1. It is given by the pull-push operation along functors between certain variants of the corresponding Euclidean bordism categories

$$0|1\text{-EBord}(\Lambda\mathfrak{X}) \leftarrow 0|1\text{-EBord}^{\mathbb{T}}(\Lambda\mathfrak{X}) \to 1|1\text{-EBord}(\mathfrak{X}). \tag{1.2}$$

The lack of direct map from left to right is due to certain subtleties concerning Euclidean supergeometry, as explained in the reference above.

The results of this paper can be summarized in the following statement.

Theorem 1.3. *Let $\mathfrak{X}$ be an orbifold. To any gerbe with connection $\tilde{\mathfrak{X}}$ and $\tilde{\mathfrak{X}}$-twisted vector bundle $\mathfrak{V}$ over $\mathfrak{X}$, correspond*

$$T_{\tilde{\mathfrak{X}}} \in 1|1\text{-ETw}(\mathfrak{X}), \quad E_{\mathfrak{V}} \in 1|1\text{-EFT}^{T_{\tilde{\mathfrak{X}}}}(\mathfrak{X})$$

such that the diagram

$$\begin{array}{ccccc} & \overset{E}{\nearrow} & 1|1\text{-EFT}^{T_{\tilde{\mathfrak{X}}}}(\mathfrak{X}) & \xrightarrow{\text{red}} & 0|1\text{-EFT}^{T'_{\tilde{\mathfrak{X}}}}(\Lambda\mathfrak{X}) \\ \operatorname{Vect}^{\tilde{\mathfrak{X}}}(\mathfrak{X}) & & \vdots & & \downarrow \\ & \searrow & K^\alpha(\mathfrak{X}) & \xrightarrow{\text{ch}} & H^{\text{ev}}_{\text{deloc}}(\mathfrak{X}, \alpha) \end{array}$$

commutes. Here, $\alpha \in H^3(\mathfrak{X}; \mathbb{Z})$ *is the Dixmier–Douady class of* $\tilde{\mathfrak{X}}$ *and* $T'_{\tilde{\mathfrak{X}}}$ *is the dimensional reduction of* $T_{\tilde{\mathfrak{X}}}$.

Since the right vertical map is a bijection on concordance classes (Theorem 2.11), this gives a geometric interpretation of the twisted orbifold Chern character.

This theorem generalizes results of Han [14] and Dumitrescu [10] for the untwisted, non-equivariant case. In a different direction, we point out that Berwick-Evans and Han [6] extended that story to the equivariant case for Lie group actions. We also point out that, in the case of a global quotient orbifold $X /\!\!/ G$ and twist coming from a central extension of G, Berwick-Evans [5] obtained a somewhat different field-theoretic interpretation of $K^\alpha(X /\!\!/ G) \otimes \mathbb{C}$. In fact, he also found a description of TMF $\otimes\, \mathbb{C}$ in terms of "simple" 2|1-EFTs, and it would be interesting to investigate if these are obtained, in our language, as the dimensional reduction of full-blown 2|1-EFTs.

1.3. Notation and conventions. We generally follow Deligne and Morgan's [9] treatment of supermanifolds. We also use Dumitrescu's [11] notion of super parallel transport (for connections), with the difference that we always transport along the *left*-invariant vector field $D = \partial_\theta - \theta\partial_t$ of $\mathbb{R}^{1|1}$. For the notion of Euclidean structures (in dimension 1|1 and 0|1), see [23, appendix B].

Most manipulations in this paper happen in the bicategory of stacks (Grothendieck fibrations satisfying descent) over the site of supermanifolds (see e.g. Behrend and Xu [3] for details). Every Lie groupoid presents a stack, and we use, concretely, the stack of torsors as a model. An orbifold is a stack presented by a proper étale Lie groupoid. We fix, once and for all, an étale Lie groupoid presentation for our orbifold $\mathfrak{X}$,

$$s, t\colon X_1 \rightrightarrows X_0.$$

This determines presentations

$$\hat{X}_1 \rightrightarrows \hat{X}_0, \quad \Pi T X_1 \rightrightarrows \Pi T X_0, \quad \Pi T \hat{X}_1 \rightrightarrows \Pi T \hat{X}_0,$$

of $\Lambda\mathfrak{X}$ (the inertia orbifold), $\Pi T\mathfrak{X}$ (the stack of maps $\mathbb{R}^{0|1} \to \mathfrak{X}$), and $\Pi T\Lambda\mathfrak{X}$ respectively. There are obvious maps

$$\mathfrak{X} \xrightarrow{i} \Lambda\mathfrak{X} \xrightarrow{p} \mathfrak{X}, \quad \Pi T\mathfrak{X} \xrightarrow{i} \Pi T\Lambda\mathfrak{X} \xrightarrow{p} \Pi T\mathfrak{X}$$

that we often leave implicit.

We fix also a gerbe with connection $\tilde{\mathfrak{X}} \to \mathfrak{X}$ and, when needed, an $\tilde{\mathfrak{X}}$-twisted vector bundle $\mathfrak{V}$; those are assumed to come with presentations as well, with the notation introduced in "Appendix B" section. Note that the chosen presentation $X_1 \rightrightarrows X_0$ of $\mathfrak{X}$ must be such that $\tilde{\mathfrak{X}}$ admits a presentation as a central extension $L \to X_1$, which is a nontrivial condition. For instance, when $\mathfrak{X}$ is just a manifold, we need, in general, to choose as presentation the Čech groupoid $\coprod_{i,j} U_i \cap U_j \rightrightarrows \coprod_i U_i$ of some open cover.

2. Twisted 0|1-EFTs and de Rham Cohomology

In this section, we extend in two directions the results of Hohnhold et al. [16] on the relation between 0-dimensional supersymmetric field theories over a manifold and de Rham cohomology. First, we replace the target manifold by an orbifold, and, second, we provide a classification of twists. This provides, in particular, a field-theoretic description of the delocalized twisted de Rham cohomology of an orbifold, which is isomorphic, via the Chern character, to complexified twisted K-theory [1,27].

We denote by $\mathfrak{B}(\mathfrak{X})$ the stack of fiberwise connected bordisms in $0|1\text{-EBord}(\mathfrak{X})$, which can be described concretely as

$$\mathfrak{B}(\mathfrak{X}) = \underline{\mathrm{Fun}}_{\mathrm{SM}}(\mathbb{R}^{0|1}, \mathfrak{X})/\!\!/\mathrm{Isom}(\mathbb{R}^{0|1}).$$

When $\mathfrak{X}$ is a manifold, the mapping stack $\underline{\mathrm{Fun}}_{\mathrm{SM}}(\mathbb{R}^{0|1}, \mathfrak{X})$ is represented by the parity-reversed tangent bundle, so we will in general write $\Pi T\mathfrak{X} = \underline{\mathrm{Fun}}_{\mathrm{SM}}(\mathbb{R}^{0|1}, \mathfrak{X})$ for the stack of superpoints. If the stack $\mathfrak{X}$ admits a Lie groupoid presentation $X_1 \rightrightarrows X_0$, then $\Pi T\mathfrak{X}$ can be presented by the Lie groupoid $\Pi T X_1 \rightrightarrows \Pi T X_0$; in particular, if $\mathfrak{X}$ is an orbifold, $\Pi T\mathfrak{X}$ is again an orbifold.

We define the groupoid of Euclidean $0|1$-twists over $\mathfrak{X}$ to be

$$0|1\text{-ETw}(\mathfrak{X}) = \mathrm{Fun}_{\mathrm{SM}}(\mathfrak{B}(\mathfrak{X}), \mathrm{Vect})$$

and, for each $T \in 0|1\text{-ETw}(\mathfrak{X})$, the corresponding set of T-twisted topological respectively Euclidean field theories over $\mathfrak{X}$ to be the set of global sections of T:

$$0|1\text{-EFT}^T(\mathfrak{X}) = C^\infty(\mathfrak{B}(\mathfrak{X}), T).$$

In these definitions, Vect can be the stack of real or complex super vector bundles, but ultimately we are interested in the complex case.

We recall the construction, in Hohnhold et al. [16, definition 6.2], of the twist

$$T_1 \colon \mathfrak{B}(\mathrm{pt}) = \mathrm{pt}/\!\!/\mathrm{Isom}(\mathbb{R}^{0|1}) \to \mathrm{Vect}.$$

This functor is entirely specified by the requirement that the point pt maps to the odd complex line $\Pi\mathbb{C}$, and by a group homomorphism $\mathrm{Isom}(\mathbb{R}^{0|1}) \to \mathrm{GL}(0|1) \cong \mathbb{C}^\times$, which we take to be the projection onto $\mathbb{Z}/2 = \{\pm 1\}$. We set $T_n = T_1^{\otimes n}$, and use the same notation for the pullback of those line bundles to $\mathfrak{B}(\mathfrak{X})$.

2.1. Superconnections and twists. Following Quillen [20], we define a superconnection $\mathbb{A}$ on a $\mathbb{Z}/2$-graded complex vector bundle $V \to X$ to be an odd operator (with respect to the total $\mathbb{Z}/2$-grading) on $\Omega^*(X; V)$ satisfying the Leibniz rule

$$\mathbb{A}(\omega f) = (d\omega)f + (-1)^{|\omega|}\omega\mathbb{A}f. \tag{2.1}$$

Here, $\omega \in \Omega^*(X)$ and $f \in \Omega^*(X; V)$. It follows that $\mathbb{A}$ is entirely determined by its restriction to $\Omega^0(X; V)$; denoting by A_i, $i \geq 0$, the component $\Omega^0(X; V) \to \Omega^i(X; V)$, we find that A_1 is an affine (even) connection and all other A_i are $\Omega^0(X)$-linear odd homomorphisms. The even operator $\mathbb{A}^2 \colon \Omega^*(X; V) \to \Omega^*(X; V)$ is $\Omega^*(X)$-linear, and is called the curvature of $\mathbb{A}$. In particular, a flat superconnection is a differential on $\Omega^*(X; V)$.

Now, let $V_0, V_1 \to X$ be complex super vector bundles and $\mathbb{A}_i$, $i = 0, 1$, superconnections. Then there exists a superconnection $\mathbb{A}$ on the homomorphism bundle $\mathrm{Hom}(V_0, V_1) \to X$, characterized by

$$(\mathbb{A}\Phi)f = \mathbb{A}_1(\Phi f) - (-1)^{|\Phi|}\Phi(\mathbb{A}_0 f)$$

for any section Φ of $\Omega^*(X; \mathrm{Hom}(V_0, V_1))$ of parity $|\Phi|$ and $f \in \Omega^*(X; V_0)$. We define $\mathrm{Vect}^{\mathbb{A}}$ to be the prestack on Man whose objects over X are vector bundles with superconnection $(V, \mathbb{A})$, and morphisms $(V_0, \mathbb{A}_0) \to (V_1, \mathbb{A}_1)$ are sections $\Phi \in \Omega^*(X; \mathrm{Hom}(V_0, V_1))$ of even total degree satisfying $\mathbb{A}(\Phi) = 0$. This turns out to be a stack.

There is a nice interpretation of superconnections in terms of Euclidean supergeometry. Consider the pullback bundle $\pi^* V \to \Pi T X$ along $\pi : \Pi T X \to X$. Its sections on an open $U \subset X$ are given by $\Omega^*(U) \otimes_{C^\infty(U)} C^\infty(U; V) = \Omega(U; V)$, and to say that a given odd, fiberwise linear vector field $\mathbb{A}$ on $\pi^* V$ is π-related to the de Rham vector field d on the base is precisely the same as saying that Eq. (2.1) holds. Thus a superconnection on V gives $\pi^* V$ the structure of an $\mathrm{Isom}(\mathbb{R}^{1|1})$-equivariant vector bundle over $\Pi T X$, where the action on the base is via the projection $\mathrm{Isom}(\mathbb{R}^{1|1}) \to \mathrm{Isom}(\mathbb{R}^{0|1})$ and the identification $\Pi T X = \underline{\mathrm{SM}}(\mathbb{R}^{0|1}, X)$. The superconnection is flat if and only if this action factors through $\mathrm{Isom}(\mathbb{R}^{0|1})$. There is also a converse statement.

Theorem 2.2. *The stack map* $\mathrm{Vect}^{\mathbb{A}} \to \mathrm{Vect}(\Pi T — /\!\!/ \mathrm{Isom}(\mathbb{R}^{1|1}))$ *defined above is an equivalence. The same is true for the map* $\mathrm{Vect}^{\mathbb{A}\flat} \to \mathrm{Vect}(\Pi T — /\!\!/ \mathrm{Isom}(\mathbb{R}^{0|1}))$.

As usual, we extend the above definitions by saying that a vector bundle with superconnection on a stack $\mathfrak{X}$ is a fibered functor $V : \mathfrak{X} \to \mathrm{Vect}^{\mathbb{A}}$; it is flat if it takes values in the substack $\mathrm{Vect}^{\mathbb{A}\flat}$ of flat superconnections. With this in place, we can return to our discussion of twisted field theories.

Proposition 2.3. *For $\mathfrak{X}$ a differentiable stack, there is a natural equivalence of groupoids*

$$\mathrm{Vect}^{\mathbb{A}\flat}(\mathfrak{X}) \to 0|1\text{-ETw}(\mathfrak{X}).$$

Proof. There exists a bisimplicial manifold $\{\Pi T X_j \times \mathrm{Isom}(\mathbb{R}^{0|1})^{\times i}\}_{i,j \geq 0}$ whose vertical structure maps give nerves of Lie groupoids presenting $\Pi T \mathfrak{X} \times \mathrm{Isom}(\mathbb{R}^{0|1})^{\times i}$ and whose horizontal structure maps give nerves of presentations of $\Pi T X_j /\!\!/ \mathrm{Isom}(\mathbb{R}^{0|1})$. Applying Vect, we get a double cosimplicial groupoid

$$\begin{array}{ccccc} \vdots & & \vdots & & \\ \uparrow\uparrow\uparrow & & \uparrow\uparrow\uparrow & & \\ \mathrm{Vect}(\Pi T X_1) & \rightrightarrows & \mathrm{Vect}(\Pi T X_1 \times \mathrm{Isom}(\mathbb{R}^{0|1})) & \substack{\to\\\to\\\to} & \cdots \\ \uparrow\uparrow & & \uparrow\uparrow & & \\ \mathrm{Vect}(\Pi T X_0) & \rightrightarrows & \mathrm{Vect}(\Pi T X_0 \times \mathrm{Isom}(\mathbb{R}^{0|1})) & \substack{\to\\\to\\\to} & \cdots \end{array} \tag{2.4}$$

Now we calculate the (homotopy) limit of this diagram in two different ways. Taking the limit of the columns and then the limit of the resulting cosimplicial groupoid, we get, by proposition 8 of [23],

$$\begin{aligned} &\mathrm{holim}\left(\mathrm{Fun}_{\mathrm{SM}}(\Pi T \mathfrak{X}, \mathrm{Vect}) \rightrightarrows \mathrm{Fun}_{\mathrm{SM}}(\Pi T \mathfrak{X} \times \mathrm{Isom}(\mathbb{R}^{0|1}), \mathrm{Vect}) \substack{\to\\\to\\\to} \cdots\right) \\ &\qquad\qquad \cong \mathrm{Fun}_{\mathrm{SM}}(\Pi T \mathfrak{X} /\!\!/ \mathrm{Isom}(\mathbb{R}^{0|1}), \mathrm{Vect}) = 0|1\text{-ETw}(\mathfrak{X}). \end{aligned}$$

On the other hand the limit of each row is equivalent to $\mathrm{Vect}(\Pi T X_i /\!\!/ \mathrm{Isom}(\mathbb{R}^{0|1}))$, and the stack map $\mathrm{Vect}^{\mathbb{A}\flat} \to \mathrm{Vect}(\Pi T — /\!\!/ \mathrm{Isom}(\mathbb{R}^{0|1}))$ of Theorem 2.2 gives us a levelwise equivalence of simplicial groupoids

$$\mathrm{Vect}^{\mathbb{A}\flat}(X_\bullet) \to \mathrm{Vect}(\Pi T X_\bullet /\!\!/ \mathrm{Isom}(\mathbb{R}^{0|1})).$$

Taking limits, we get an equivalence $\mathrm{Vect}^{\mathbb{A}\flat}(\mathfrak{X}) \to 0|1\text{-ETw}(\mathfrak{X})$. □

Remark 2.5. We can drop the flatness condition by considering vector bundles on $\Pi T\mathfrak{X}/\!\!/ \mathrm{Isom}(\mathbb{R}^{1|1})$.

2.2. Concordance of flat sections. The goal of this section is to identify concordance classes of twisted 0|1-EFTs. This is an extension of the well-known fact that closed differential forms are concordant through closed forms if and only if they are cohomologous; the extension takes place in two orthogonal directions: we replace manifolds with differentiable stacks and the trivial flat line bundle with an arbitrary flat superconnection. Fix a differentiable stack $\mathfrak{X}$ with presentation $X_1 \rightrightarrows X_0$ and let $T \in 0|1\text{-ETw}(\mathfrak{X})$ be the twist associated to the flat superconnection $(V, \mathbb{A})$ on $\mathfrak{X}$.

Proposition 2.6. *There are natural bijections*

$$0|1\text{-EFT}^{T}(\mathfrak{X}) \cong \{\mathbb{A}\text{-}\textit{closed even forms with values in } V\},$$
$$0|1\text{-EFT}^{T\otimes T_1}(\mathfrak{X}) \cong \{\mathbb{A}\text{-}\textit{closed odd forms with values in } V\}.$$

Proof. The vector bundle $T\colon \mathfrak{B}(\mathfrak{X}) \to \mathrm{Vect}$ determines a sheaf Γ_T on $\mathfrak{B}(\mathfrak{X})$, assigning to an object $f\colon S \to \mathfrak{B}(\mathfrak{X})$ the complex vector space of sections of f^*T. The bundle T is specified by a coherent family of objects in the double cosimplicial groupoid (2.4), representing an object in the limit of that diagram, and a global section is specified by a coherent family of sections.

Similarly, the superconnection $\mathbb{A}$ determines a sheaf $\Gamma^*_{\mathbb{A}}$ on $\mathfrak{X}$ whose sections over $f\colon S \to \mathfrak{X}$ are the super vector space of forms in $\Omega^*(S, f^*V)$ annihilated by $\mathbb{A}$. Global sections of $\Gamma^*_{\mathbb{A}}$ are the super vector space

$$\Gamma^*_{\mathbb{A}}(\mathfrak{X}) = \lim\left(\Gamma^*_{\mathbb{A}}(X_0) \rightrightarrows \Gamma^*_{\mathbb{A}}(X_1)\right).$$

Now, the data of $(V, \mathbb{A})$ is determined, by hypothesis, by the same coherent family of objects in (2.4) as T. Suppose we are given a coherent family of sections. Individually, the bottom row of (2.4) specifies an element of $\Omega^*(X_0, V)$ which is invariant under the $\mathrm{Isom}(\mathbb{R}^{0|1}) \cong \mathbb{R}^{0|1} \rtimes \mathbb{Z}/2$-action; this means it is even and closed, i.e., a section of $\Gamma^0_{\mathbb{A}}(X_0)$. Similarly, the second row by itself specifies a section of $\Gamma^0_{\mathbb{A}}(X_1)$, and the coherence conditions involving vertical maps say these two things determine a section of $\Gamma^0_{\mathbb{A}}(\mathfrak{X})$. The correspondence between sections of T and $\Gamma^0_{\mathbb{A}}$ is clearly bijective.

Replacing T with $T \otimes T_1$ in the above argument amounts to replacing V with its parity reversal ΠV (cf. Hohnhold et al. [16, proposition 6.3]). □

Recall that a flat superconnection defines a differential on the space of forms with values in the corresponding vector bundle.

Proposition 2.7. *Concordance classes of EFTs are in bijection with cohomology classes:*

$$0|1\text{-EFT}^{T\otimes T_n}[\mathfrak{X}] \cong H^{\bar{n}}(\Omega^*(\mathfrak{X}, V), \mathbb{A}).$$

Proof. By naturality of the correspondences in the previous proposition, it suffices to show that

$$\Gamma^{\bar{n}}_{\mathbb{A}}(\mathfrak{X})/\text{concordance} \cong H^{\bar{n}}(\Omega^*(\mathfrak{X}, V), \mathbb{A}).$$

Suppose, first, that the closed forms $\omega_0, \omega_1 \in \Omega^*(\mathfrak{X}, V)$ are cohomologous, i.e., $\omega_1 - \omega_0 = \mathbb{A}\alpha$. Then

$$\omega = \omega_0 + \mathbb{A}(t\alpha) \in \Omega^*(\mathfrak{X} \times \mathbb{R}, V)$$

satisfies $i_j^*\omega = \omega_j$, $j = 0, 1$. (Here, we use the same notation for an object over $\mathfrak{X}$ and its pullback via $\mathrm{pr}_1 : \mathfrak{X} \times \mathbb{R} \to \mathfrak{X}$; as usual, t is the coordinate on $\mathbb{R}$.)

Conversely, suppose we are given a closed form $\omega \in \Omega^*(\mathfrak{X} \times \mathbb{R}, V)$ with $\omega_j = i_j^*\omega$, $j = 0, 1$. We need to find a form $\alpha \in \Omega^*(\mathfrak{X}, V)$ such that $\omega_1 - \omega_0 = \mathbb{A}\alpha$. Schematically, it will be $\alpha = -\int_{\mathfrak{X}\times[0,1]/\mathfrak{X}} \omega$. More precisely, we need to define $\alpha_f \in \Omega^*(S, f^*V)$ for each S-point $f : S \to \mathfrak{X}$. That will be given by the fiberwise integral

$$\alpha_f = -\int_{S\times[0,1]/S} (f \times \mathrm{id})^*\omega,$$

which is clearly natural in S. Notice that the vector bundle in which ω takes values comes with a canonical trivialization along the $\mathbb{R}$-direction, so the integral makes sense.

Now, define operators $\mathbb{A}_f = f^*\mathbb{A}\otimes 1$ and $d_{\mathbb{R}} = 1\otimes d$ on $\Omega^*(S\times\mathbb{R}, V) \cong \Omega^*(S, V)\otimes \Omega^*(\mathbb{R})$. From the derivation property of $\mathbb{A}$, it follows that $(f \times \mathrm{id})^*\mathbb{A} = \mathbb{A}_f + d_{\mathbb{R}}$. Then, writing $\omega_f = (f \times \mathrm{id})^*\omega$, we have

$$\begin{aligned} f^*\mathbb{A}(\alpha_f) &= -f^*\mathbb{A}\int_{S\times[0,1]/S} \omega_f = -\int_{S\times[0,1]/S} \mathbb{A}_f\omega_f = \int_{S\times[0,1]/S} d_{\mathbb{R}}\omega_f \\ &= f^*\omega_1 - f^*\omega_0. \end{aligned}$$

Thus $\omega_1 - \omega_0 = \mathbb{A}\alpha$. □

Applying the theorem to the trivial twist T, we get the following.

Corollary 2.8. *For any differentiable stack $\mathfrak{X}$, there is a natural bijection*

$$0|1\text{-EFT}^{T_n}(\mathfrak{X}) \cong \Omega^{\bar{n}}_{\mathrm{cl}}(\mathfrak{X})$$

between T_n-twisted $0|1$-EFTs over $\mathfrak{X}$ and closed differential forms of parity $\bar{n}$ on $\mathfrak{X}$. If $\mathfrak{X}$ is an orbifold, passing to concordance classes gives an isomorphism with $\mathbb{Z}/2$-graded delocalized cohomology

$$0|1\text{-EFT}^{T_n}[\Lambda\mathfrak{X}] \cong H^{\bar{n}}_{\mathrm{deloc}}(\mathfrak{X}).$$

Remark 2.9. Replacing $\mathfrak{B}(\mathfrak{X})$ with the stack of connected $0|1$-dimensional manifolds over the orbifold $\mathfrak{X}$,

$$\mathfrak{B}_{\mathrm{top}}(\mathfrak{X}) = \underline{\mathrm{Fun}}_{\mathrm{SM}}(\mathbb{R}^{0|1}, \mathfrak{X})/\!\!/\mathrm{Diff}(\mathbb{R}^{0|1}),$$

we arrive at the notion of *topological* twists and (twisted) field theories. The basic twist T_1 lifts in a natural way to $\mathfrak{B}_{\mathrm{top}}(\mathfrak{X})$, and in this case T_n in fact depends on n, and not just on its parity. In an entirely analogous way to Hohnhold et al. [16], one can show that

$$0|1\text{-TFT}^{T_n}(\mathfrak{X}) \cong \Omega^n_{\mathrm{cl}}(\mathfrak{X}), \quad 0|1\text{-TFT}^{T_n}[\Lambda\mathfrak{X}] \cong H^n_{\mathrm{deloc}}(\mathfrak{X}),$$

where the latter identification requires the assumption that $\mathfrak{X}$ is an orbifold.

2.3. Twisted de Rham cohomology for orbifolds. In this section, we review the construction of delocalized twisted de Rham cohomology for orbifolds due to Tu and Xu [27], and show that it can, in fact, be interpreted as concordance classes of suitably twisted

0|1-EFTs. In view of Proposition 2.7, this is not necessarily surprising; the point here is to give explicit descriptions allowing us to show, in Sect. 5, how the relevant twist arises, in a natural way, from dimensional reduction.

Let $\mathfrak{X}$ be an orbifold and $\tilde{\mathfrak{X}}$ a gerbe with Dixmier–Douady class $\alpha \in H^3(\mathfrak{X}, \mathbb{Z})$, both with presentations as in "Appendix B" section. Then the ($\mathbb{Z}/2$-graded) delocalized twisted cohomology groups $H^*_{\text{deloc}}(\mathfrak{X}, \alpha)$ are defined to be the cohomology of the complex

$$(\Omega^*(\Lambda\mathfrak{X}, L'), \nabla' + \Omega \wedge \cdot). \tag{2.10}$$

Here, Ω^* stands for the $\mathbb{Z}/2$-graded de Rham complex, $\Lambda\mathfrak{X}$ is the inertia orbifold, Ω is the 3-curvature of $\tilde{\mathfrak{X}}$ pulled back to $\Lambda\mathfrak{X}$, and (L', ∇') is a line bundle with flat connection we will describe below. (This differs from the definition of Tu and Xu [27, section 3.3] in that we perform the usual trick to convert between $\mathbb{Z}/2$-graded and 2-periodic $\mathbb{Z}$-graded complexes, and we have chosen a more convenient constant in front of Ω, which produces an isomorphic chain complex. We also remark that changing the gerbe with connective structure representing the class α produces a noncanonically isomorphic complex; a specific isomorphism between the complexes depends on the choice of isomorphism between the gerbes.)

The line bundle L' on the inertia groupoid $\hat{X}_1 \rightrightarrows \hat{X}_0$ is as follows:

(1) the underlying line bundle L' is the restriction of L to $\hat{X}_0 \subset X_1$, with the restricted connection ∇';
(2) the isomorphisms $s^*L' \to t^*L'$ over $\hat{X}_1$ are described fiberwise by the composition

$$L'_{(x,g)} = L_g \to L_f \otimes L_g \otimes L_{f^{-1}} \to L_{fgf^{-1}} = L'_{(x',g')},$$

where $f \in X_1$ induces a morphism $f: (x, g) \to (x', g')$ in $\hat{X}_1 \rightrightarrows \hat{X}_0$, and we used the canonical map $\mathbb{C} \to L_f \otimes L_{f^{-1}}$.

It is an exercise to check that ∇' is invariant, and flat (provided our central extension admits a curving, which we always assume).

Now, the operator $\nabla' + \Omega \wedge \cdot$ on $\Omega^*(\Lambda\mathfrak{X}, L')$ is a flat superconnection on $L' \in \text{Vect}(\Lambda\mathfrak{X})$ (since ∇' is flat, $d\Omega = 0$ and $\Omega \wedge \Omega = 0$), and therefore gives rise to a twist $T_\alpha: \mathfrak{B}(\Lambda\mathfrak{X}) \to \text{Vect}$. Combining Proposition 2.7 with the main result of Tu and Xu [27], we obtain a field-theoretic interpretation of complexified twisted K-theory. (The compactness assumption can be dropped by using field theories and de Rham cohomology with compact support.)

Theorem 2.11. *For every compact orbifold $\mathfrak{X}$ and $\alpha \in H^3(\mathfrak{X}, \mathbb{Z})$, there are natural bijections*

$$0|1\text{-EFT}^{T_\alpha}[\Lambda\mathfrak{X}] \cong H^{\text{ev}}_{\text{deloc}}(\mathfrak{X}, \alpha) \cong K^\alpha(\mathfrak{X}) \otimes \mathbb{C},$$
$$0|1\text{-EFT}^{T_\alpha \otimes T_1}[\Lambda\mathfrak{X}] \cong H^{\text{odd}}_{\text{deloc}}(\mathfrak{X}, \alpha) \cong K^{1+\alpha}(\mathfrak{X}) \otimes \mathbb{C}.$$

Remark 2.12. All objects indexed by α actually depend, up to noncanonical isomorphism, on the choice of a gerbe representative and its connective structure. This abuse of language is standard in the literature.

To finish this section, we rephrase the description of (L', ∇') in terms of a Deligne 2-cocycle (h, A, B) on $X_1 \rightrightarrows X_0$ representing $\tilde{\mathfrak{X}}$ (see (4.8)–(4.10) for our notation).

Then L' is topologically trivial and the connection ∇' is $d + A|_{\hat{X}_0}$; flatness is due to the fact that $dA|_{\hat{X}_0} = (t^*B - s^*B)|_{\hat{X}_0} = 0$ since $s = t$ on $\hat{X}_0$. To describe the isomorphism $s^*L' \to t^*L'$, we use as input $h \in C^\infty(X_2, \mathbb{C}^\times)$, and we just need to specify a $\mathbb{C}^\times$-valued function H on $\hat{X}_1$. Let $v = (g, z) \in X_1 \times \mathbb{C}$ and $\tilde{f} = (f, w) \in X_1 \times \mathbb{C}$. Then $\tilde{f}^{-1} = (f^{-1}, w^{-1}h^{-1}(f, f^{-1}))$, and we find that $\tilde{f}v = (fg, zwh(f, g))$ and

$$\begin{aligned}\tilde{f}v\tilde{f}^{-1} &= (fgf^{-1}, zwh(f, g)w^{-1}h(f, f^{-1})h(fg, f^{-1}))\\ &= (g', zh(f, g)h^{-1}(f, f^{-1})h(fg, f^{-1})).\end{aligned}$$

Thus, using the cocycle condition (4.8) for the triple (g', f, f^{-1}), we get

$$\begin{aligned}H\left(g \xrightarrow{f} g'\right) &= h(f, g)h^{-1}(f, f^{-1})h(fg, f^{-1})\\ &= \frac{h(f, g)}{h(fgf^{-1}, f)}.\end{aligned} \tag{2.13}$$

3. Torsors and Bordisms Over an Orbifold

In this section, we provide a manageable model of the bordism category $1|1\text{-EBord}(\mathfrak{X})$ as a category internal to symmetric monoidal stacks. This will also fix notation used in the remainder of the paper.

3.1. Basic definitions. We start recalling the construction of Euclidean bordism categories over a manifold X, then note that it immediately generalizes to the case of bordisms over a stack $\mathfrak{X}$, and finally recast the result in the language of torsors for a given Lie groupoid presentation of $\mathfrak{X}$.

Given integers $d, \delta \geq 0$ (subject to certain conditions) and a manifold X, Stolz and Teichner [26] construct a bordism category $d|\delta\text{-EBord}(X)$, which we briefly review here. It is a category internal to the category of symmetric monoidal stacks; that is, it is given by symmetric monoidal stacks $d|\delta\text{-EBord}(X)_i$, $i = 0, 1$, called the stack of objects and the stack of morphisms respectively, together with functors

$$\begin{array}{c} d|\delta\text{-EBord}(X)_1 \times^{s,t}_{d|\delta\text{-EBord}(X)_0} d|\delta\text{-EBord}(X)_1 \\ \downarrow c \\ d|\delta\text{-EBord}(X)_1 \\ \downarrow s \quad \uparrow u \quad \downarrow t \\ d|\delta\text{-EBord}(X)_0, \end{array}$$

standing for composition, source, target and unit, satisfying the expected conditions up to prescribed natural transformations (associator and left and right unitors, similar to the data of a bicategory). In the stack of objects $d|\delta\text{-EBord}(X)_0$, an object lying over S is given by the following collection of data:

(1) a submersion $Y \to S$ with $d|\delta$-dimensional fibers and Euclidean structure (in the sense of [26, section 4.2] or, more succinctly, [23, appendix B]);
(2) a map $f : Y \to X$;

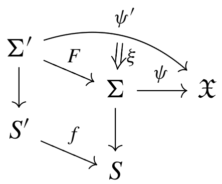
(3) a submersion $Y^c \to S$ with $(d-1|\delta)$-dimensional fibers, fiberwise embedding $Y^c \to Y$, and a decomposition $Y \backslash Y^c = Y^+ \amalg Y^-$.

The S-family Y^c is called the *core*. A morphism in the stack of objects is given by a germ (around the cores) of $(G, \mathbb{M})$-isometries respecting the maps to X. Thus, $Y^{\pm}$ should be thought as germs of collar neighborhoods of the core; they are a technical device needed, among other things, to define the composition functor c. In the stack of morphism $d|\delta\text{-EBord}(X)_1$, an object lying over S is given by the following collection of data:

(1) a submersion $\Sigma \to S$ with $d|\delta$-dimensional fibers and Euclidean structure;
(2) a map $f\colon \Sigma \to X$;
(3) objects $(Y_{\text{in}}, Y^c_{\text{in}}, Y^{\pm}_{\text{in}})$, $(Y_{\text{out}}, Y^c_{\text{out}}, Y^{\pm}_{\text{out}})$ of $d|\delta\text{-EBord}(X)_0$;
(4) isometries $Y_{\text{in}} \to \Sigma$ and $Y_{\text{out}} \to \Sigma$ respecting the maps to X.

The maps of item (4) are "parametrizations of the boundary", and are subject to certain conditions formalizing this idea. A morphism in the stack of morphisms is given by (1) isomorphisms in the object stack between the respective incoming and outgoing boundaries and (2) an isometry between the Σ's (or, more precisely, germs of isometries around their cores, that is, the region between Y^c_{in} and Y^c_{out}), respecting the maps to X and the parametrizations of the boundaries. The symmetric monoidal structures in the stacks of objects and morphisms are given by fiberwise disjoint union.

Now, turning to aspects more specific to this paper, we note that is easy to extend the above definition of bordism category to the case where X is replaced by a "generalized manifold", or stack $\mathfrak{X}$: an object in $d|\delta\text{-EBord}(\mathfrak{X})_1$ is given by an object in $d|\delta\text{-EBord}(\text{pt})_1$ together with an object of $\mathfrak{X}_\Sigma$ (which, by the Yoneda lemma, corresponds to a map $\psi\colon \Sigma \to \mathfrak{X}$ in the realm of generalized manifolds) and the corresponding boundary information, which we will not detail here. A morphism over $f\colon S' \to S$ in the stack of bordisms is determined by a fiberwise isometry $F\colon \Sigma' \to \Sigma$ covering f (and suitably compatible with the boundary information) together with a morphism ξ between objects of $\mathfrak{X}_{\Sigma'}$ as indicated in the diagram below.



(3.1)

Remark 3.2. The bordisms-over-stacks point of view is very natural from the perspective of geometric structures. The treatment of rigid geometries in [26], and in particular Euclidean structures, can be interpreted as the definition of a stack of atlases. Then, letting $\mathfrak{X}$ denote the stack of Euclidean atlases, we recover $1|1$-EBord from the plain topological bordism category as $1|1\text{-Bord}(\mathfrak{X})$. We will further develop this idea elsewhere.

Finally, we assume $\mathfrak{X}$ is a differentiable stack with Lie groupoid presentation $X = (X_1 \rightrightarrows X_0)$, and recall a convenient way to describe maps $\Sigma \to \mathfrak{X}$, namely as X-torsors. (See Behrend and Xu [3, section 2.4] for a full account of the theory of torsors.) An X-torsor over Σ is given by

(1) a submersion $\pi: U \to \Sigma$,
(2) an anchor map $\psi_0: U \to X_0$, and
(3) an action map $\mu: U \times_{X_0} X_1 \to U$.

The conditions required of μ make it equivalent to the data of a map ψ_1 such that

$$\begin{array}{ccc} U \times_\Sigma U & \xrightarrow{\psi_1} & X_1 \\ \downarrow\downarrow & & \downarrow\downarrow \\ U & \xrightarrow{\psi_0} & X_0 \end{array}$$

is an internal functor satisfying the condition that

$$(\mathrm{pr}_1, \psi_1)\colon U \times_\Sigma U \to U \times^{\psi_0, t}_{X_0} X_1$$

is a diffeomorphism; in that case, μ can be recovered as the inverse to the above followed by projection onto the second factor. We will often denote the torsor simply by ψ, and write (Σ, ψ) for a bordism in $1|1\text{-EBord}(\mathfrak{X})$.

A morphism of torsors is an equivariant map between the corresponding U's. Thus, given a second bordism (Σ', ψ') equipped with an X-torsor, which, more specifically, is given by the data

$$\Sigma' \to S', \quad \pi'\colon U' \to \Sigma', \quad \psi'_0, \quad \mu',$$

a morphism $(F, \lambda)\colon (\Sigma, \psi) \to (\Sigma', \psi')$ in $1|1\text{-EBord}(\mathfrak{X})_1$ covering $f\colon S \to S'$ is determined by a fiberwise isometry $F\colon \Sigma \to \Sigma'$ covering f and compatible with the boundary data, together with an equivariant map $\lambda\colon U \to U'$ covering F and compatible with the anchor maps: $\psi'_0 \circ \lambda = \psi_0$ (this, again, is taken up to a suitable germ equivalence relation). When $\pi' = \pi$, the datum of λ is equivalent to an internal natural transformation between the internal functors determined by ψ, ψ'; namely we set $\Lambda\colon U \to X_1$ to be the composition

$$U \xrightarrow{(\mathrm{id},\lambda)} U \times_S U \xrightarrow[\mu'\text{-action}]{\cong} U \times^{\psi'_0, t}_{X_0} X_1 \xrightarrow{\mathrm{pr}_2} X_1.$$

The stack of objects $1|1\text{-EBord}(\mathfrak{X})_0$ has an analogous description. To fix our notation, which closely follows the previous discussion, an object here is given by (1) an S-family $Y \to S$ of $1|1$-manifolds, (2) a codimension 1 family of submanifolds $Y^c \subset Y$, called the core, (3) fiberwise Euclidean structures on the pair (Y, Y^c), (4) a decomposition $Y \backslash Y^c = Y^+ \amalg Y^-$, and (5) an X-torsor

$$\pi\colon U \to Y, \quad \psi_0\colon U \to X_0, \quad \mu\colon U \times_{X_0} X_1 \to U$$

on Y. We typically write (Y, ψ) for such an object. A morphism $(Y, \psi) \to (Y', \psi')$ is given by the germ (around Y^c) of an isometry $F\colon Y \to Y'$ together with the germ (around $\pi^{-1}(Y^c)$) of an equivariant map $\lambda_0\colon U \to U'$.

3.2. Skeletons. To get an intuitive understanding of bordism categories over a stack $\mathfrak{X}$, avoiding torsors, we can think as follows. First, fix a Lie groupoid $X_1 \rightrightarrows X_0$ presentation of $\mathfrak{X}$. Then some bordisms and isometries between them can be represented by pictures like the following.

$$\begin{array}{ccc} \Sigma & \xrightarrow{\psi} & X_0 \\ \downarrow & & \\ S & & \end{array} \qquad \begin{array}{ccccc} \Sigma' & \xrightarrow{F} & \Sigma & \xrightarrow{\xi} & X_1 \\ \downarrow & & \downarrow & & \\ S' & \longrightarrow & S & & \end{array} \tag{3.3}$$

In fact, the ψ and ξ above relate to their counterparts in (3.1) by postcomposition with the atlas $X_0 \to \mathfrak{X}$ respectively whiskering with the natural transformation between the two maps $X_1 \rightrightarrows \mathfrak{X}$. Not every bordism is of this form, but in a fully extended framework it is intuitively clear that every bordism can be expressed as a composition of those; it is also not hard to conceive relations between those basic building blocks. The notion of skeletons, which we introduce now, is essentially a way of dealing systematically with these generators and relations in our case of interest.

From now on, we assume that X is an orbifold groupoid, so that the submersions π underlying all X-torsors are étale. We then define a *skeleton* of a fiberwise connected S-family $(Y, \psi) \in 1|1\text{-EBord}(\mathfrak{X})_0$ to be given by a map $\iota\colon S \times \mathbb{R}^{0|1} \to U$ such that the composition $\pi \circ \iota$ gives a Euclidean parametrization of the core Y_c. In general, a skeleton is given by a skeleton for each connected component.

A skeleton of a fiberwise connected S-family $(\Sigma, \psi) \in 1|1\text{-EBord}(\mathfrak{X})_1$ is given by skeletons for the incoming and outgoing boundary components, together with the following:

(1) A collection I_i, $0 \le i \le n$, of S-families of superintervals, as defined in "Appendix A" section. We denote the inclusion of the outgoing and incoming boundary components by

$$S \times \mathbb{R}^{0|1} \overset{\iota_i^{\text{out}}}{\hookrightarrow} S \times \mathbb{R}^{1|1} \overset{\iota_i^{\text{in}}}{\hookleftarrow} S \times \mathbb{R}^{0|1}.$$

It is not required that these intervals have strictly positive length.

(2) For each i, an embedding $I_i \hookrightarrow U$, by which we mean a Euclidean map from a neighborhood of the "core" $[\iota_i^{\text{out}}, \iota_i^{\text{in}}] \subset S \times \mathbb{R}^{1|1}$ of I_i into U.

Here and in what follows, we write

$$b_i\colon S \times \mathbb{R}^{0|1} \xrightarrow{\iota_i^{\text{out}}} S \times \mathbb{R}^{1|1} \hookrightarrow U,$$
$$a_i\colon S \times \mathbb{R}^{0|1} \xrightarrow{\iota_i^{\text{in}}} S \times \mathbb{R}^{1|1} \hookrightarrow U.$$

We require the following conditions of the above data:

(1) for each $1 \le i < n$, the maps $\pi \circ b_i$ and $\pi \circ a_{i+1}\colon S \times \mathbb{R}^{0|1} \to \Sigma$ coincide;
(2) if Σ is a family of supercircles, then we also have $\pi \circ b_n = \pi \circ a_0$;
(3) if Σ has boundary, then the maps a_0 and b_n, together with the parametrization of the boundary, induce skeletons on each boundary component; we require that those agree with the initially given skeletons.

These conditions mean, intuitively, that the superintervals $I_i \hookrightarrow U \xrightarrow{\pi} \Sigma$ prescribe an expression of Σ as a composition of shorter pieces (right elbows and length zero left elbows) in $1|1$-EBord, together with expressions of each of these pieces in the form (3.3).

We will use the shorthand notation $I = \{I_i\}$ to refer to the skeleton, and (Σ, ψ, I) to refer to a bordism with a choice of skeleton.

Notice that b_{i-1} and a_i are isomorphic in the groupoid of $(S \times \mathbb{R}^{0|1})$-points of $U \times_\Sigma U \rightrightarrows U$, since their images in Σ agree; we denote by $j_i : S \times \mathbb{R}^{0|1} \to U \times_\Sigma U$ the unique morphism $b_{i-1} \to a_i$, that is, the unique map such that

$$\mathrm{pr}_1 \circ j_i = b_{i-1}, \quad \mathrm{pr}_2 \circ j_i = a_i. \tag{3.4}$$

We denote by $1|1\text{-EBord}(\mathfrak{X})_i^{\mathrm{skel}}$, $i = 0, 1$, the variants of $1|1\text{-EBord}(\mathfrak{X})_i$ where each bordism and boundary component comes with a choice of skeleton; morphisms in these stacks are just morphisms in the old variants, after forgetting the skeleton. There is a canonical choice of skeleton on the composition of bordisms with skeleton. With this observation, we obtain an internal category $1|1\text{-EBord}(\mathfrak{X})^{\mathrm{skel}}$.

Proposition 3.5. *The forgetful map* $1|1\text{-EBord}(\mathfrak{X})^{\mathrm{skel}} \to 1|1\text{-EBord}(\mathfrak{X})$ *is a levelwise equivalence.*

This is clear, since all spaces of skeletons are contractible. It is also clear that $1|1\text{-EBord}(\mathfrak{X})$ does not depend on the choice of a Lie groupoid presentation for $\mathfrak{X}$, since it only makes reference to torsors over it. On the other hand, the definition of $1|1\text{-EBord}(\mathfrak{X})^{\mathrm{skel}}$ does make explicit reference to the groupoid $X_1 \rightrightarrows X_0$, so the notation is slightly abusive. This is harmless, as shown by the previous proposition.

Remark 3.6. Evidently, we can form a pullback of $(\Sigma, \psi, I) \in 1|1\text{-EBord}(\mathfrak{X})_1^{\mathrm{skel}}$ via a map $f : S' \to S$ by simply choosing a cartesian morphism $\lambda : (\Sigma', \psi') \to (\Sigma, \psi)$ covering f in $1|1\text{-EBord}(\mathfrak{X})_1$ and any skeleton I' for (Σ', ψ'). However, we note that there is a canonical choice to be made: we ask that I, I' have the same indexing set and

$$\begin{array}{ccc} I'_i & \longrightarrow & I_i \\ \downarrow & & \downarrow \\ U' & \xrightarrow{\lambda} & U \end{array}$$

is cartesian for all i. We denote that skeleton by $\lambda^* I$, the endpoints a'_i by $\lambda^* a_i$, etc.

We call the collection of maps $I_i \to U \xrightarrow{\pi} \Sigma$ the associated triangulation of the skeleton I. Triangulations such that the diagrams

$$\begin{array}{ccc} I'_i & \longrightarrow & I_i \\ \downarrow & & \downarrow \\ \Sigma' & \xrightarrow{F} & \Sigma \end{array}$$

are cartesian will be called *compatible*.

Finally, suppose we have $(\Sigma, \psi) \in 1|1\text{-EBord}(\mathfrak{X})_1$ and

$$I = \{I_i\}_{i \in \mathcal{I}}, \quad I' = \{I'_i\}_{i \in \mathcal{I}'}$$

two skeletons. Then we say that I' is refinement of I if there is a surjective map $r : \mathcal{I}' \to \mathcal{I}$, such that, for each $i \in \mathcal{I}$, $r^{-1}(i)$ indexes a collection $I'_{i_1}, \dots, I'_{i_n} \subset U$ where $b'_{i_k} = a'_{i_{k+1}}$ for each $1 \leq k < n$ and $a_{i_1} = a'_i, b_{i_n} = b'_i$; in words, the I'_{i_k}, $1 \leq k \leq n$, are adjacent subintervals whose concatenation is precisely I_i. We denote by $R^I_{I'} : (\psi, I') \to (\psi, I)$ the morphism having the identity as its underlying torsor map.

3.3. The globular subcategory; the superpath stack. Denote by

$$1|1\text{-EBord}(\mathfrak{X})_0^{\text{glob}} \subset 1|1\text{-EBord}(\mathfrak{X})_0^{\text{skel}}$$

the sub-prestack with the same objects but containing only those morphisms $(F, \lambda)\colon (Y, \psi) \to (Y', \psi')$ such that the diagram

$$\begin{array}{ccc} S \times \mathbb{R}^{0|1} & \xrightarrow{f\times\text{id}} & S' \times \mathbb{R}^{0|1} \\ \downarrow\iota & & \downarrow\iota' \\ U & \xrightarrow{\quad\lambda\quad} & U' \end{array} \tag{3.7}$$

commutes, where $f\colon S \to S'$ is the map F lies over. Denote by $1|1\text{-EBord}(\mathfrak{X})_1^{\text{glob}} \subset 1|1\text{-EBord}(\mathfrak{X})_1^{\text{skel}}$ the sub-prestack containing only those morphisms that map into $1|1$ -EBord$(\mathfrak{X})_0^{\text{glob}}$ via the source and target functors. These two objects fit together into a "globular" internal category $1|1\text{-EBord}(\mathfrak{X})^{\text{glob}}$, which can be thought of as a smooth bicategory.

For each test manifold S, we obtain from each of the above variants a category

$$1|1\text{-EBord}(\mathfrak{X})_S^{\text{glob}}, \qquad 1|1\text{-EBord}(\mathfrak{X})_S^{\text{skel}}$$

internal to symmetric monoidal groupoids. Those internal categories are fibrant in the sense of Shulman [22], and they clearly determine the same symmetric monoidal bicategory. Thus, the inclusion $1|1\text{-EBord}(\mathfrak{X})^{\text{glob}} \to 1|1\text{-EBord}(\mathfrak{X})^{\text{skel}}$ ought to be considered as an equivalence of internal categories. Since we do not know of a comprehensive theory of internal categories to quote from, we will leave this as an informal statement.

Our construction of twists and twisted field theories below will be based on the globular variant. This provides some slight simplifications and allows us to focus on the more conceptual side of the discussion. More specifically, in order to extend our construction of the twist functor T in Sect. 4 from $1|1\text{-EBord}(\mathfrak{X})^{\text{glob}}$ to $1|1\text{-EBord}(\mathfrak{X})^{\text{skel}}$, it is necessary (and sufficient) to choose a stable trivialization of the gerbe $\tilde{\mathfrak{X}}$ (passing, if needed, to a finer Lie groupoid presentation of $\mathfrak{X}$), as the reader unsatisfied with the argument of the previous paragraph should be able to verify.

3.4. Some examples. We give here an essentially complete description of $1|1\text{-EBord} = 1|1\text{-EBord}(\text{pt})$. Examples of objects and morphisms in $1|1\text{-EBord}(\mathfrak{X})$ can then be constructed by pulling back to a general base space S and choosing a torsor representing a map to $\mathfrak{X}$.

The object stack $1|1\text{-EBord}_0$ contains an object sp given by the manifold $Y = \mathbb{R}^{1|1}$ with core $Y^c = \mathbb{R}^{0|1}$ and neighborhoods $Y^+ = \mathbb{R}^{1|1}_{>0}$, $Y^- = \mathbb{R}^{1|1}_{<0}$. There is a similar (but nonisomorphic) object $\overline{\text{sp}}$ with Y^+ and Y^- interchanged. Both of them have $\mathbb{Z}/2$ as automorphism group, generated by the flip fl: $\mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$ which acts by -1 on odd functions.

Objects in $1|1\text{-EBord}_1$ are given by fiberwise disjoint unions of one of four kinds of basic bordisms: superintervals, left and right elbows, and supercircles. The basic building blocks are as follows.

(1) The left elbow of length 0, $L_0 \in 1|1\text{-EBord}(\overline{\text{sp}} \amalg \text{sp}, \emptyset)$, has $\mathbb{R}^{1|1}$ as underlying manifold. The boundary parametrization $L_0 \leftarrow \overline{\text{sp}} \amalg \text{sp}$, in terms of the underlying manifolds, is the map id $\amalg$ id.

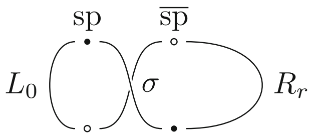


Fig. 1. A supercircle of length r.

(2) The right elbows $R_r \in 1|1\text{-EBord}(\emptyset, \text{sp} \amalg \overline{\text{sp}})$ form a $\mathbb{R}^{1|1}_{>0}$-family, parametrizing the "super length" r. The underlying family of Euclidean manifolds is $\mathbb{R}^{1|1}_{>0} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}_{>0}$, and the parametrization of the boundary is

$$\text{id} \amalg T_r \colon (\mathbb{R}^{1|1}_{>0} \times \mathbb{R}^{1|1}) \amalg (\mathbb{R}^{1|1}_{>0} \times \mathbb{R}^{1|1}) \to \mathbb{R}^{1|1}_{>0} \times \mathbb{R}^{1|1},$$

where $T_r \colon (r, s) \mapsto (r, r \cdot s)$ is the translation on affine Euclidean space $\mathbb{R}^{1|1}$ by the amount specified by the $\mathbb{R}^{1|1}_{>0}$ parameter.

(3) Any isomorphism $F \colon Y \to Y'$ in $1|1\text{-EBord}_0$ leads to a "thin" bordism $F \in 1|1\text{-EBord}_1$, having Y' as underlying manifold, $F \colon Y \to Y'$ as incoming parametrization, and $\text{id}_{Y'}$ as outgoing parametrization.

Some isomorphisms in $1|1\text{-EBord}_1$ are listed below. They restrict to the identity on the boundaries. Isomorphisms (1) and (4) are given by the obvious identification of the underlying manifold of each bordism, and (2) and (3) by a flip.

(1) $\text{fl}^2 \cong \text{id}_{\text{sp}}$
(2) $L_0 \circ (\text{fl} \amalg \text{fl}) \cong L_0$
(3) $(\text{fl} \amalg \text{fl}) \circ R_r \cong R_{\text{fl}(r)}$
(4) $R_{r_1 \cdot r_2} \cong R_{r_1} \circ (\text{id}_{\text{sp}} \amalg L_0 \amalg \text{id}_{\overline{\text{sp}}}) \circ R_{r_2}$.

The last of them is an isomorphisms of $(\mathbb{R}^{1|1}_{>0} \times \mathbb{R}^{1|1}_{>0})$-families, and r_1, r_2 indicate the coordinate function of each factor; more formally, the R's indicate pullbacks of R_r along the multiplication respectively projection maps $\mathbb{R}^{1|1}_{>0} \times \mathbb{R}^{1|1}_{>0} \to \mathbb{R}^{1|1}_{>0}$. Similar bordisms $\bar{L}$, $\bar{R}$, etc., are obtained by reversing the roles of sp and $\overline{\text{sp}}$. They satisfy analogous relations to the above, and there are also isomorphisms

$$L \circ \sigma \cong \bar{L}, \qquad \sigma \circ R \cong \bar{R}.$$

A family $I_r \in 1|1\text{-EBord}(\text{sp}, \text{sp})$ of intervals of length r is obtained by composing L_0 and R_r along the common $\overline{\text{sp}}$ boundary:

$$I_r = (\text{id}_{\text{sp}} \amalg L_0) \circ (R_r \amalg \text{id}_{\text{sp}}).$$

Similarly, a family $K_r \in 1|1\text{-EBord}(\emptyset, \emptyset)$ of supercircles of length r is obtained from elbows and the braiding isomorphism in the way indicated in Fig. 1. There are stacks $\mathfrak{P}$ and $\mathfrak{K}$ of superintervals respectively supercircles. More generally, we write

$$\mathfrak{K}(\mathfrak{X}), \mathfrak{P}(\mathfrak{X}) \hookrightarrow 1|1\text{-EBord}(\mathfrak{X})^{\text{glob}}_1$$

for the stacks of supercircles respectively superintervals over $\mathfrak{X}$.

Remark 3.8. Hohnhold et al. [15, theorem 6.42] provide generators and relations for a variant of 1|1-EBord (in their case, unoriented and satisfying a positivity condition, but, more importantly, not extended up to include isometries of bordisms). Since our goal is to give some examples of field theories, and not a classification, we will be satisfied with a somewhat informal approach to constructing functors between internal categories. In fact, we will explain our constructions in detail on superintervals, and we will be less explicit about extending fibered functors on $\mathfrak{P}(\mathfrak{X})$ to full-blown internal functors. Such details are usually easy to guess, and compatibility with relations (1)–(4) above will be easy to verify.

4. Twists for 1|1-EFTs from Gerbes with Connection

Let $\mathfrak{X}$ be an orbifold and $\tilde{\mathfrak{X}} \to \mathfrak{X}$ be a gerbe with connective structure, presented by a Lie groupoid $X = (X_1 \rightrightarrows X_0)$ respectively a central extension having $L \to X_1$ as the underlying line bundle with connection, as in "Appendix B" section. The goal of this section is to associate to $\tilde{\mathfrak{X}}$ a Euclidean 1|1-twist

$$T = T_{\tilde{\mathfrak{X}}} \in 1|1\text{-ETw}(\mathfrak{X}) = \text{Fun}^{\otimes}_{\text{SM}}(1|1\text{-EBord}(\mathfrak{X})^{\text{glob}}, \text{Alg}).$$

This construction is drastically simplified by the fact that it takes values in the subcategory $B\text{Line} \hookrightarrow \text{Alg}$, where Line denotes the symmetric monoidal stack of complex line bundles and BLine the internal category having trivial stack of objects and Line as stack of morphisms. In other words, the only relevant algebra in this construction is the monoidal unit $\mathbb{C} \in \text{Alg}$, and the only relevant modules are the invertible ones.

4.1. Construction of the twist functor. At the level of object stacks, there is no work to do. We just need to describe a map $1|1\text{-EBord}(\mathfrak{X})^{\text{glob}}_1 \to \text{Line}$ of symmetric monoidal stacks, which by abuse of notation we still call T, together with natural isomorphisms μ and ϵ, the compositor and unitor (cf. [26, defintion 2.18]). We start discussing the underlying fibered functor T.

Fix a fiberwise connected S-family (Σ, ψ, I) of bordisms with skeleton. Our goal is to describe a line bundle $T(\Sigma, \psi, I)$ over S. Recall (3.4) that the skeleton I determines a collection of maps $j_i : S \times \mathbb{R}^{0|1} \to U \times_\Sigma U$. We set

$$T(\Sigma, \psi, I) = \bigotimes\nolimits_{1 \le i \le n} L_{j_i}.$$

Here and in what follows, we write, for any map $f : S \times \mathbb{R}^{0|1} \to U \times_\Sigma U$,

$$L_f = (\psi_1 \circ f)^* L|_{S \times 0}.$$

As to morphisms, we initially consider two cases.

(1) $\lambda : (\Sigma, \psi, I) \to (\Sigma', \psi', I')$ is a refinement of skeletons, i.e., the underlying torsors are equal, the torsor map λ is the identity, and I is a refinement of I'.
(2) $\lambda : (\Sigma, \psi, I) \to (\Sigma', \psi', I')$ covers a map $f : S \to S'$ and I and $\lambda^* I'$ are compatible skeletons. This means that the endpoints of the superintervals $I_i, \lambda^* I'_i \subset U$ are (uniquely) isomorphic in the groupoid $U \times_K U \rightrightarrows U$, and we denote by $\tilde{a}_i, \tilde{b}_i \in (U \times_K U)_{S \times \mathbb{R}^{0|1}}$ the corresponding morphisms $a_i \to \lambda^* a'_i$, $b_i \to \lambda^* b'_i$. Note that $\tilde{a}_i, \tilde{b}_i$ are the endpoints of the superintervals

$$J_i = I_i \times_K \lambda^* I'_i \subset U \times_K U.$$

In situation (1), the line bundles $T(\Sigma, \psi, I)$ and $T(\Sigma, \psi, I')$ only differ by the addition of canonically trivial tensor factors, and $T(\lambda)$ is the natural identification. Situation (2) is more interesting. We denote by $\mathrm{SP}_i : L_{\tilde{a}_i} \to L_{\tilde{b}_i}$ the super parallel transport of $\psi_1^* L$ along J_i, and by

$$h_i : L_{\tilde{a}_i} \otimes L_{j_i} \to L_{\lambda^* j_i'} \otimes L_{\tilde{b}_{i-1}}$$

the gerbe multiplication map, or any other map obtained by adjunction. Then we finally consider the composition

$$\begin{aligned} L_{\tilde{a}_0}^\vee &\xrightarrow{\mathrm{SP}_0^\vee} L_{\tilde{b}_0}^\vee \xrightarrow{h_1} L_{\lambda^* j_1'} \otimes L_{\tilde{a}_1}^\vee \otimes L_{j_1}^\vee \xrightarrow{\mathrm{SP}_1^\vee} L_{\lambda^* j_1'} \otimes L_{\tilde{b}_1}^\vee \otimes L_{j_1}^\vee \xrightarrow{h_2} \\ &\xrightarrow{h_2} L_{\lambda^* j_1'} \otimes L_{\lambda^* j_2'} \otimes L_{\tilde{a}_2}^\vee \otimes L_{j_2}^\vee \otimes L_{j_1}^\vee \xrightarrow{\mathrm{SP}_2^\vee} \dots \to \\ &\xrightarrow{\mathrm{SP}_n^\vee} \Big(\bigotimes\nolimits_{1 \le i \le n} L_{\lambda^* j_i'}\Big) \otimes L_{\tilde{b}_n}^\vee \otimes \Big(\bigotimes\nolimits_{1 \le i \le n} L_{j_i}^\vee\Big). \end{aligned} \tag{4.1}$$

Since we are working with globular bordisms, $\tilde{a}_0$ and $\tilde{b}_n$ are identities, so $L_{\tilde{a}_0}$, $L_{\tilde{b}_n}$ are trivial and we let

$$T(\lambda) : T(\Sigma, \psi, I) = \bigotimes_{1 \le i \le n} L_{j_i} \to \bigotimes_{1 \le i \le n} L_{\lambda^* j_i'} = \lambda^* T(\Sigma', \psi', I') \tag{4.2}$$

be adjoint to the above.

Proposition 4.3. *The prescriptions above uniquely determine a symmetric monoidal fibered functor* $T : 1|1\text{-EBord}(\mathfrak{X})_1^{\mathrm{glob}} \to \mathrm{Vect}$.

Proof. Initially, we will assume we can pick compatible refinements for (families of) triangulations of bordisms whenever needed, and explain at the end of the proof how to deal with the fact that such refinements do not always exist.

Fix a morphism $\lambda : (\Sigma, \psi, I) \to (\Sigma', \psi', I')$ in $1|1\text{-EBord}(\mathfrak{X})^{\mathrm{glob}}$ and compatible refinements for the triangulations of Σ and Σ'. This yields refinements $\bar{I}$, $\bar{I}'$ of I respectively I'. We can then express λ as the composition

$$\begin{array}{ccccccc} (\psi, I) & \xleftarrow{R_I^{\bar{I}}} & (\psi, \bar{I}) & \xrightarrow{\bar{\lambda}} & (\psi', \bar{I}') & \xrightarrow{R_{I'}^{\bar{I}'}} & (\psi', I') \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S & = & S & \xrightarrow{f} & S' & = & S' \end{array}$$

where $\bar{\lambda}$ is the morphism in $1|1\text{-EBord}(\mathfrak{X})_1^{\mathrm{glob}}$ corresponding to the same torsor map as λ, but relating objects with different skeletons. This fixes $T(\lambda)$.

Of course, we need to check that taking this as a definition for $L(\lambda)$ is consistent, that is, independent on the choice of $\bar{I}$ and $\bar{I}'$. Verifying this in the case when all triangulations involved admit compatible refinements boils down to checking that if the original triangulations of Σ and Σ' were already compatible, applying formula (4.2) would give

$$T(\lambda) \circ T(R_I^{\bar{I}}) = T(R_{I'}^{\bar{I}'}) \circ T(\bar{\lambda}).$$

But this is easy to see. When calculating $T(\bar{\lambda})$, each appearance of SP_i in (4.1) gets replaced by a composition

$$\mathrm{SP}_{i_k}^\vee \circ h_{i_k} \circ \dots \circ \mathrm{SP}_{i_1}^\vee \circ h_{i_1} \circ \mathrm{SP}_{i_0}^\vee,$$

where SP_{i_j}, $0 \le j \le k$, denotes parallel transport along a subdivision $J_{i_j} \subset J_i$ and the h_{i_j} are tautological identifications involving $L_{i_j} = \mathbb{C}$. Thus our claim follows from compatibility of super parallel transport with gluing of superintervals.

Next, we verify that L respects compositions of isometries, at least when compatible refinements can be chosen. So let us fix composable morphisms as in the diagram below.

$$\begin{array}{ccccc} (\Sigma, \psi, I) & \xrightarrow{\lambda} & (\Sigma', \psi', I') & \xrightarrow{\lambda'} & (\Sigma'', \psi'', I'') \\ \downarrow & & \downarrow & & \downarrow \\ S & \xrightarrow{f} & S' & \xrightarrow{f'} & S'' \end{array}$$

We can assume the all skeletons are compatible. Using (4.1) and the structure maps of the gerbe, we see that (up to braiding),

$$T(\lambda) \otimes f^*T(\lambda') = T(\lambda'') \otimes \mathrm{Id}_{f^*T(\Sigma')} .$$

It follows that

$$T(\lambda'') = T(\lambda') \circ T(\lambda).$$

Next, suppose we have a morphism $\lambda\colon (\psi, I) \to (\psi', I')$ where compatible refinements of the underlying triangulations fail to exist. Since every morphism in 1|1 -EBord$(\mathfrak{X})_1^{\mathrm{skel}}$ can be expressed as the composition of a morphism covering the identity and a morphism involving pullback skeletons, it suffices to consider the case when λ covers id: $S \to S$. Write c_i for the common value of $\pi \circ b_{i-1} = \pi \circ a_i : S \times \mathbb{R}^{0|1} \to \Sigma$, and define c_i' similarly. Then the nonexistence of a common refinement means that there is a pair $c_i, \lambda^* c_i' : S \times \mathbb{R}^{0|1} \to \Sigma$ "crossing over" one another; more precisely, in a Euclidean local chart, neither $(c_i)_{\mathrm{red}} \le (\lambda^* c_i')_{\mathrm{red}}$ nor the opposite holds. It suffices to define $T(\lambda)$ in a small neighborhood S_p of each point $p \in S$ where that happens; assuming, for simplicity, that the triangulations $\{c_j\}$, $\{\lambda^* c_j'\}$ are identical except for the problematic index i, it suffices to analyze the situation in a small neighborhood in Σ of the point $x = c_i(p) = \lambda^* c_i'(p)$. Then we can choose $d^1 : S_p \times \mathbb{R}^{0|1} \to \Sigma|_{S_p}$ sufficiently close to c_i satisfying either $d(p) < c_i(p), \lambda^* c_i'(p)$ or the opposite inequality. Denote by $I^1, I^{1\prime}$ the modifications of I, I' obtained by replacing $c_i, \lambda^* c_i'$ with d^1. Then of course I and I^1 admit a common refinement, and so do I' and $I^{1\prime}$; moreover, I^1 and $I^{1\prime}$ are based on the same triangulation of Σ. We have a commutative square

$$\begin{array}{ccc} (\Sigma, \psi, I^1) & \xrightarrow{\lambda^1} & (\Sigma', \psi', I^{1\prime}) \\ \downarrow & & \downarrow \\ (\Sigma, \psi, I) & \xrightarrow{\lambda} & (\Sigma', \psi', I'), \end{array} \tag{4.4}$$

and this stipulates the value of T on $\lambda\colon (\Sigma, \psi, I) \to (\Sigma', \psi', I')$; here, the unlabeled arrows refer to morphisms whose underlying torsor maps are the identity. We need to see why this is independent on the choice of d^1. Suppose we repeat the above procedure using a different choice $d^2 : S_p \times \mathbb{R}^{0|1} \to \Sigma|_{S_p}$; to compare them, we can use a third $d^3 : S_p \times \mathbb{R}^{0|1} \to \Sigma|_{S_p}$ (restricting, perhaps, to a smaller neighborhood S_p) which stays away from both d^1 and d^2. Thus, we can assume without loss of generality that d^1 and d^2 stay away from one another. We have a commutative diagram

$$\begin{array}{ccccccc} & & (\Sigma,\psi,I^1) & \xrightarrow{\lambda^1} & (\Sigma',\psi',I^{1\prime}) & & \\ & \swarrow & \downarrow & & \downarrow & \searrow & \\ (\Sigma,\psi,I) & & & & & & (\Sigma',\psi',I') \\ & \nwarrow & \downarrow & & \downarrow & \nearrow & \\ & & (\Sigma,\psi,I^2) & \xrightarrow{\lambda^2} & (\Sigma',\psi',I^{2\prime}) & & \end{array}$$

where the skeletons appearing in each triangle and the in middle square admit common refinements, and it follows that $T(\lambda)$, as prescribed by (4.4), is independent on the choice of d^1. Similarly, we can reduce the verification that T respects composition of morphisms to the case where all triangulations involved admit compatible refinements.

So far, we have defined T on fiberwise connected families of bordisms. Since the symmetric monoidal structure of $1|1\text{-EBord}(\mathfrak{X})_1^{\text{glob}}$ is free, we are done. □

The functor T is compatible with composition of bordisms in an obvious way, and we will not spell out the definition of the compositor and unitor promoting T to a functor of internal categories.

4.2. On the choice of presentations. We need to argue that T, as constructed above, depends only on the gerbe with connection $\tilde{\mathfrak{X}} \to \mathfrak{X}$, and not on the chosen Lie groupoid presentations for $\mathfrak{X}$ and $\tilde{\mathfrak{X}}$. To formulate this statement more precisely, we introduce some notation. Recall that Proposition 3.5 justified the lack of reference to $X_1 \rightrightarrows X_0$ in the notations $1|1\text{-EBord}(\mathfrak{X})^{\text{skel}}$ and $1|1\text{-EBord}(\mathfrak{X})^{\text{glob}}$. In this subsection, we must be explicit about the choices of presentations, so we will write

$$1|1\text{-EBord}(X_\bullet) = 1|1\text{-EBord}(\mathfrak{X})^{\text{glob}}.$$

Then, what we have achieved in Sect. 4.1 is the construction, from a Lie groupoid $X_1 \rightrightarrows X_0$ and a central extension $L \to X_1$, of a functor of internal categories

$$T_L \colon 1|1\text{-EBord}(X_\bullet) \to \text{Alg}.$$

Suppose now that $X_1' \rightrightarrows X_0'$ and the central extension $L' \to X_1'$ provide a second presentation of $\mathfrak{X}$ and the gerbe $\tilde{\mathfrak{X}}$. By [3, proposition 4.15], there exists a Lie groupoid $X_1'' \rightrightarrows X_0''$ and a central extension $L'' \to X_1''$ together with a Morita morphism $X_\bullet'' \to X_\bullet$ and a Morita morphism of central extensions $L'' \to L$, as well as similar data for $L' \to X_1' \rightrightarrows X_0'$. These Morita morphisms are uniquely determined, up to unique natural isomorphism, by the requirement that they induce the identity of $\tilde{\mathfrak{X}} \to \mathfrak{X}$ [3, section 2.6].

We now have a diagram as follows.

$$\begin{array}{ccccc} & & 1|1\text{-EBord}(X_\bullet'') & & \\ & \swarrow & \downarrow{\scriptstyle T_{L''}} & \searrow & \\ 1|1\text{-EBord}(X_\bullet') & & & & 1|1\text{-EBord}(X_\bullet) \\ & {\scriptstyle T_{L'}}\searrow & \downarrow & \swarrow{\scriptstyle T_L} & \\ & & \text{Alg} & & \end{array} \tag{4.5}$$

The claim that T_L determines a twist functor $T = T_{\tilde{\mathfrak{X}}}$ depending only on the gerbe $\tilde{\mathfrak{X}}$ over $\mathfrak{X}$ is formalized by the following statement.

Proposition 4.6. *In the above situation, there exist canonical* 2*-morphisms making* (4.5) *commute.*

The proof is just a verification that the construction of T_L is natural with respect to internal functors between Lie groupoids (of which Morita morphisms are particular cases), so we will omit further details.

Remark 4.7. Not every gerbe $\tilde{\mathfrak{X}}$ over $\mathfrak{X}$ is necessarily presentable as a central extension of a given presentation $X_1 \rightrightarrows X_0$; this condition is equivalent to $\tilde{\mathfrak{X}}$ admitting a trivialization when restricted to X_0 [3, proposition 4.12]. Thus, given $\tilde{\mathfrak{X}}$, we need to pick a sufficiently fine groupoid presentation of $\mathfrak{X}$. In this paper, we never let $\tilde{\mathfrak{X}}$ vary or make structural statements about the groupoid of twists $1|1$-ETw($\mathfrak{X}$), so we are allowed to fix, once and for all, a Lie groupoid $X_1 \rightrightarrows X_0$ suitable for $\tilde{\mathfrak{X}}$.

4.3. The restriction to $\mathfrak{K}(\mathfrak{X})$. We denote by $\mathfrak{K}(\mathfrak{X})$ the substack of closed and connected bordisms in $1|1$-EBord$(\mathfrak{X})^{\mathrm{glob}}$. The twist functor $T_{\tilde{\mathfrak{X}}}$ determines, by restriction, a line bundle Q on $\mathfrak{K}(\mathfrak{X})$. Our goal in this section is to give a detailed description of Q, in terms of a Čech cocycle for Deligne cohomology representing the gerbe $\tilde{\mathfrak{X}} \to \mathfrak{X}$.

Let us start fixing some notation. The orbifold $\mathfrak{X}$ will be presented, as before, by an étale Lie groupoid $X_1 \rightrightarrows X_0$, and the gerbe $\tilde{\mathfrak{X}} \to \mathfrak{X}$ will be presented by a Čech 2-cocycle for groupoid cohomology with coefficients in the Deligne complex $\mathbb{C}^\times(3)$,

$$\mathbb{C}^\times \xrightarrow{d\log} \Omega^1 \to \Omega^2.$$

More explicitly, this cocycle is given by a triple

$$(h, A, B) \in C^\infty(X_2, \mathbb{C}^\times) \times \Omega^1_{\mathbb{C}}(X_1) \times \Omega^2_{\mathbb{C}}(X_0)$$

satisfying the cocycle conditions

$$h(a,b)h(a,bc)^{-1}h(ab,c)h(b,c)^{-1} = 1 \text{ in } C^\infty(X_3, \mathbb{C}^\times), \tag{4.8}$$

$$\mathrm{pr}_2^* A + \mathrm{pr}_1^* A - c^* A = d\log h \text{ in } \Omega^1(X_2), \tag{4.9}$$

$$t^* B - s^* B = dA \text{ in } \Omega^2(X_1), \tag{4.10}$$

where $X_n = X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is the space of sequences of n composable morphisms.

To an object $(\psi, I) \in \mathfrak{K}(\mathfrak{X})_S$ as above the fibered functor

$$Q = T_{\tilde{\mathfrak{X}}}|_{\mathfrak{K}(\mathfrak{X})} \colon \mathfrak{K}(\mathfrak{X}) \to \mathrm{Vect}$$

assigns the trivial line bundle over S; the interesting discussion, of course, concerns morphisms. Fix a second object $(\psi', I') \in \mathfrak{K}(\mathfrak{X})_{S'}$ and a morphism $\lambda\colon (\psi, I) \to (\psi', I')$ over $f\colon S \to S'$. To that Q assigns a linear map between the corresponding lines, which we identify with a function $Q(\lambda)\colon S \to \mathbb{C}^\times$. We consider the two special cases of Sect. 4.1, using the notation fixed there.

Proposition 4.11. *If* λ *is a refinement of skeletons, then* $Q(\lambda) = 1$. *If the skeletons of* K, K' *are compatible, then* $Q(\lambda) \in C^\infty(S, \mathbb{C}^\times)$ *is given by*

$$Q(\lambda) = \exp\left(\sum_{1\le i\le n} \int_{J_i} \mathrm{vol}_D \langle D, \psi_1^* A\rangle\right) \prod_{1\le i\le n} \left.\frac{\psi_2^* h(\tilde{a}_i, j_i)}{\psi_2^* h(\lambda^* j_i', \tilde{b}_{i-1})}\right|_{S\times\{0\}}. \tag{4.12}$$

Here, $D \in C^\infty(TU)$ is a choice of Euclidean vector field for the Euclidean structure induced by $\pi : U \to K$, and vol_D the corresponding volume form (cf. "Appendix A" section). Moreover, $\psi_2 : U \times_K U \times_K U \to X_2$ denotes the map induced by ψ_1.

Proof. The first claim is obvious. As to the second claim, each h_i in (4.1) contributes one factor in the product, and each SP_i contributes one term in the summation. In fact, we see easily from Proposition A.1 that super parallel transport on along a superinterval J with respect to the connection form $d - A$ is given by $\exp(\int_J \mathrm{vol}_D \langle D, A\rangle)$. All terms of the from $h(f, f^{-1})$ introduced by identifications $L_f^\vee \cong L_{f^{-1}}$ cancel out. □

Remark 4.13. From the data of a Čech-cocycle presentation of a gerbe with connection, Lupercio and Uribe [17] constructed a line bundle with connection on the loop orbifold $L\mathfrak{X}$. Our construction incorporates a super analogue of this transgression procedure: compare (4.12) with Lupercio and Uribe's definition 4.2. Our proof that T is a functor (or, rather, its purely bosonic analog) provides a more geometric explanation for Lupercio and Uribe's calculations with Deligne Čech cocycles.

Remark 4.14. The usual transgression of gerbes produces, in fact, a $\mathrm{Diff}^+(S^1)$-equivariant line bundle on the loop space [8, proposition 6.2.3]. Our construction gives a line bundle on the moduli stack of supercircles over $\mathfrak{X}$, and not just on a "super loop space", so the super analogue of $\mathrm{Diff}^+(S^1)$-equivariance is automatically built into our discussion.

5. Dimensional Reduction of Twists

In this section, we show that dimensional reduction of the $1|1$-twists from Sect. 4 recovers the $0|1$-twists described in Sect. 2.3. So we start with the twist functor

$$T_{\tilde{\mathfrak{X}}} \in 1|1\text{-ETw}(\mathfrak{X})$$

associated to a gerbe with connection $\tilde{\mathfrak{X}}$, and describe its pullback to $0|1\text{-EBord}^{\mathbb{T}}(\Lambda\mathfrak{X})$ by the functor in (1.2). As we will see below, that corresponds to the data of a line bundle with superconnection on $\Lambda\mathfrak{X}$. We will then find that it is flat, and hence, by Proposition 2.3, descends to a line bundle $T'_{\tilde{\mathfrak{X}}}$ on $\mathfrak{B}(\Lambda\mathfrak{X}) = \Pi T\Lambda\mathfrak{X} /\!\!/ \mathrm{Isom}(\mathbb{R}^{0|1})$, or, in other words, a $0|1$-dimensional Euclidean twist over $\Lambda\mathfrak{X}$. We call $T'_{\tilde{\mathfrak{X}}}$ the dimensional reduction of $T_{\tilde{\mathfrak{X}}}$.

Theorem 5.1. *Let $\mathfrak{X}$ be an orbifold, $\tilde{\mathfrak{X}}$ a gerbe with connection, and $\alpha \in H^3(\mathfrak{X}; \mathbb{Z})$ its Dixmier–Douady class. Then the twist $T'_{\tilde{\mathfrak{X}}} \in 0|1\text{-ETw}(\Lambda\mathfrak{X})$ obtained by dimensional reduction of the twist $T_{\tilde{\mathfrak{X}}} \in 1|1\text{-ETw}(\mathfrak{X})$ is isomorphic to the twist T_α from Theorem 2.11.*

This means, in particular, that

$$0|1\text{-EFT}^{T'_{\tilde{\mathfrak{X}}} \otimes T_i}[\Lambda\mathfrak{X}] \cong K^{i+\alpha}(\mathfrak{X}) \otimes \mathbb{C},$$

and suggests that $T_{\tilde{\mathfrak{X}}}$ is the correct $1|1$-twist to represent α-twisted K-theory in the sense of (1.1).

The proof of the theorem (and the flatness claim necessary to state it) will occupy the remainder of this section. Before getting started, we record a technical lemma.

Lemma 5.2. *Let X be an ordinary manifold,* $\mathrm{ev}\colon \Pi TX \times \mathbb{R}^{0|1} \to X$ *be evaluation map, and write $\tilde{\omega}$ for the function on ΠTX corresponding to the differential form $\omega \in \Omega^n(X)$. Then we have*

$$\langle (\partial_\theta)^{\wedge n}, \mathrm{ev}^*\omega\rangle = \pm n!(\tilde{\omega} + \theta \widetilde{d\omega}),$$

where the sign $\pm$ is -1 if $n \equiv 2, 3$ mod 4 and $+1$ otherwise.

Proof. It suffices to prove the lemma for $\omega = f_0 df_1 \ldots df_n$, where $f_i \in C^\infty(X)$. The action $\mu\colon \Pi TX \times \mathbb{R}^{0|1} \to \Pi TX$ is given by the formula

$$\mu^\sharp\colon \tilde{\omega} \mapsto \tilde{\omega} + \theta D\tilde{\omega},$$

where D denotes the de Rham vector field on ΠTX. Thus

$$\begin{aligned}\mathrm{ev}^*\omega &= (f_0 + \theta Df_0) \prod_{1\le i\le n} d(f_i + \theta Df_i)\\ &= (f_0 + \theta Df_0) \prod_{1\le i\le n} (df_i + d\theta Df_i + \theta dDf_i).\end{aligned}$$

Writing $F_i = df_i + d\theta Df_i + \theta dDf_i$, we have

$$i_{\partial_\theta}\mathrm{ev}^*\omega = (-1)^{i-1} \sum_{1\le i\le n} (f_0 + \theta Df_0) F_1 \ldots F_{i-1} Df_i F_{i+1} \ldots F_n.$$

Contracting an expression like the one under the summation sign with ∂_θ produces as many new terms as there are F_i's, and, in each of those, one of the F_i's get converted into a Df_i. Note also that commuting i_{∂_θ} with any F_i or Df_i produces a minus sign. Iterating this process, we find

$$\begin{aligned}(i_{\partial_\theta})^n \mathrm{ev}^*\omega &= (-1)^{0+1+\cdots+n-1} n!(f_0 + \theta Df_0) Df_1 \ldots Df_n\\ &= \pm n!(\tilde{\omega} + \theta\widetilde{d\omega}).\end{aligned}$$

□

Corollary 5.3. *For D the de Rham vector field on ΠTX, $\mu\colon \Pi TX \times \mathbb{R}^{0|1} \to \Pi TX$ the action map, $\pi : \Pi TX \to X$ the projection, and the remaining notation as in the lemma,*

$$\mu^*\langle D^{\wedge n}, \pi^*\omega\rangle = \pm n!(\tilde{\omega} + \theta\widetilde{d\omega}).$$

5.1. Review of dimensional reduction. In this subsection we recall the main points about our dimensional reduction procedure, also fixing the notation. See [23] for the complete story.

It is sufficient to describe the effect of the internal functors (1.2) on the corresponding stacks of closed and connected bordisms, which we denote

$$\mathfrak{B}(\Lambda\mathfrak{X}) \xleftarrow{\mathcal{L}} \mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X}) \xrightarrow{\mathcal{R}} \mathfrak{K}(\mathfrak{X}). \tag{5.4}$$

The left stack was already introduced in Sect. 2, and is given by

$$\mathfrak{B}(\Lambda\mathfrak{X}) = \underline{\mathrm{Fun}}_{\mathrm{SM}}(\mathbb{R}^{0|1}, \Lambda\mathfrak{X}) /\!\!/ \mathrm{Isom}(\mathbb{R}^{0|1}).$$

The middle stack is defined as

$$\mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X}) = \underline{\mathrm{Fun}}_{\mathrm{SM}}(\mathbb{R}^{0|1}, \Lambda\mathfrak{X})/\!\!/\mathrm{Isom}(\mathbb{R}^{1|1}),$$

and the map $\mathcal{L}$ is induced by the group homomorphism

$$\mathrm{Isom}(\mathbb{R}^{1|1}) = \mathbb{R}^{1|1} \rtimes \mathbb{Z}/2 \to \mathbb{R}^{0|1} \rtimes \mathbb{Z}/2 = \mathrm{Isom}(\mathbb{R}^{0|1}).$$

Next we turn to a brief description of the map $\mathcal{R}$, focusing on our case of interest. We fix an étale Lie groupoid presentation $X_1 \rightrightarrows X_0$ for $\mathfrak{X}$, so that we also get presentations

$$\Pi T X_1 \rightrightarrows \Pi T X_0, \quad \hat{X}_1 \rightrightarrows \hat{X}_0, \quad \Pi T \hat{X}_1 \rightrightarrows \Pi T \hat{X}_0,$$

of $\Pi T\mathfrak{X}$, $\Lambda\mathfrak{X}$ and $\Pi T \Lambda\mathfrak{X}$ respectively. Then we have an atlas

$$\check{x} \colon \Pi T \hat{X}_0 \to \mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X}).$$

Now, we want to describe the $\Pi T \hat{X}_0$-family classified by the map $\mathcal{R} \circ \check{x}$, which we will denote $K_{\check{x}}$. Chasing through the construction of [23], we find that

$$K_{\check{x}} = (K, \psi \colon K \to \mathfrak{X}, I) \in \mathfrak{K}(\mathfrak{X})$$

is given, in the language of torsors, by the following data:

(1) the trivial family $K = \Pi T \hat{X}_0 \times \mathbb{T}^{1|1}$ of length 1 supercircles, together with the standard covering $U = \Pi T \hat{X}_0 \times \mathbb{R}^{1|1} \to K$,
(2) the map $\psi_0 \colon U \to X_0$ given by the composition

$$U = \Pi T \hat{X}_0 \times \mathbb{R}^{1|1} \to \Pi T \hat{X}_0 \times \mathbb{R}^{0|1} \xrightarrow{\mathrm{ev}} \hat{X}_0 \overset{p}{\twoheadrightarrow} X_0,$$

(3) the map $\psi_1 \colon U \times_K U \to X_1$ which, over the component of points differing by n units, is the n-fold iterate of

$$\alpha \colon \Pi T \hat{X}_0 \times \mathbb{R}^{1|1} \to \Pi T \hat{X}_0 \times \mathbb{R}^{0|1} \xrightarrow{\mathrm{ev}} \hat{X}_0 \overset{i}{\hookrightarrow} X_1,$$

(4) a skeleton we may choose to be $\Pi T \hat{X}_0 \times [0, 1] \subset U$.

Remark 5.5. To arrive at the above description of $K_{\check{x}}$, it is helpful to note that $\mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X})$ admits a more geometrical formulation (cf. [23, section 3.2]), where the map $\check{x}$ classifies the $\Pi T \hat{X}_0$-family of gadgets given by

(1) the trivial family of Euclidean 0|1-manifolds $\Sigma = \Pi T \hat{X}_0 \times \mathbb{R}^{0|1} \to \Pi T \hat{X}_0$,
(2) the trivial $\mathbb{T}$-bundle $P = \Pi T \hat{X}_0 \times \mathbb{T}^{1|1} \to \Sigma$ with the standard principal connection, and
(3) the map $\Sigma \to \Lambda\mathfrak{X}$ given by the composition

$$\Sigma = \Pi T \hat{X}_0 \times \mathbb{R}^{0|1} \xrightarrow{\mathrm{ev}} \hat{X}_0 \xrightarrow{\hat{x}} \Lambda\mathfrak{X},$$

where $\hat{x}$ is the versal family for $\Lambda\mathfrak{X}$.

In this picture, the map $\mathcal{L}$ simply forgets P. The subtle aspect about $\mathcal{R}$ is that the map $\psi \colon K \to \mathfrak{X}$ is not simply the composition

$$K = P \to \Sigma \to \Lambda\mathfrak{X} \to \mathfrak{X},$$

but, rather, is given by a descent construction using the canonical automorphism of the inertia $\Lambda\mathfrak{X}$.

Eventually, we will need to understand the action of $\mathbb{R}^{1|1} \subset \mathrm{Isom}(\mathbb{R}^{1|1})$ on $K_{\check{x}}$ by rotations. More precisely, the natural isomorphism between the stack maps

$$\Pi T\hat{X}_0 \times \mathbb{R}^{1|1} \underset{\mu}{\overset{\mathrm{pr}}{\rightrightarrows}} \Pi T\hat{X}_0 \xrightarrow{\check{x}} \mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X})$$

leads to an isomorphism

$$\mu^* K_{\check{x}} \cong \mathrm{pr}^* K_{\check{x}} \text{ over } \Pi T\hat{X}_0 \times \mathbb{R}^{1|1}.$$

This will be described later (cf. Fig. 2), but we would like to notice two useful facts now. First, $\mathcal{R}$ has image in the substack

$$\mathfrak{K}_1(\mathfrak{X}) \cong \underline{\mathrm{Fun}}_{\mathrm{SM}}(\mathbb{T}^{1|1}, \mathfrak{X}) /\!\!/ \mathrm{Isom}(\mathbb{T}^{1|1})$$

of length 1 supercircles, so that the $\mathbb{R}^{1|1}$-action comes from the action on $\mathbb{T}^{1|1}$ by rotations. Second, an expression of $K_{\check{x}}$ as a composition of more basic bordisms (left and right elbows and braiding) is obtained as follows.

Let $R_{\check{x}}^{[0,r]}$ be the $S = (\Pi T\hat{X}_0 \times \mathbb{R}^{1|1}_{>0})$-family of bordisms such that

(1) its image in $1|1\text{-EBord}_1$ is simply the pullback of the $\mathbb{R}^{1|1}_{>0}$-family R_r described in Sect. 3.4;
(2) the X-torsor over $\Sigma = \Pi T\hat{X}_0 \times \mathbb{R}^{1|1}_{>0} \times \mathbb{R}^{1|1}$ is given by the trivial covering

$$U = \Pi T\hat{X}_0 \times \mathbb{R}^{1|1}_{>0} \times \mathbb{R}^{1|1} \to \Sigma,$$

the map

$$\psi_0 \colon U = \Pi T\hat{X}_0 \times \mathbb{R}^{1|1}_{>0} \times \mathbb{R}^{1|1} \xrightarrow{\mathrm{pr}} \Pi T\hat{X}_0 \times \mathbb{R}^{0|1} \xrightarrow{\mathrm{ev}} \hat{X}_0 \overset{p}{\twoheadrightarrow} X_0,$$

and the obvious $\psi_1 \colon U \times_\Sigma U = U \to X_1$;
(3) the skeleton is given by the family $[0, r] \subset \Sigma$ of superintervals, where r denotes the standard coordinate function of the $\mathbb{R}^{1|1}_{>0}$ factor of S.

For future reference, note that there are obvious variants $R_{\check{x}}^{[r,1]}$, $R_{\check{x}}^{[r,1+r]}$ corresponding to different choices of skeletons. Also, we write $R_{\check{x}}^{[0,1]}$ for the restriction of $R_{\check{x}}^{[0,r]}$ to $\Pi T\hat{X}_0 \times \{1\}$. This description also fixes $\Pi T\hat{X}_0$-families $\mathrm{sp}_{\check{x}}, \overline{\mathrm{sp}}_{\check{x}} \in 1|1\text{-EBord}(\mathfrak{X})_0$ such that $R_{\check{x}}^{[0,1]}$ is a bordism $\emptyset \to \mathrm{sp}_{\check{x}} \amalg \overline{\mathrm{sp}}_{\check{x}}$, and therefore also a left elbow $L_{\check{x}} \colon \overline{\mathrm{sp}}_{\check{x}} \amalg \mathrm{sp}_{\check{x}} \to \emptyset$. With this notation, we can finally write

$$K_{\check{x}} \cong L_{\check{x}} \circ \sigma_{\mathrm{sp}_{\check{x}}, \overline{\mathrm{sp}}_{\check{x}}} \circ R_{\check{x}}^{[0,1]}.$$

5.2. The underlying line bundle. Our goal for the remainder of this proof is to understand the restriction of the line bundle Q from Sect. 4.3, which we call

$$Q' \colon \Pi T\Lambda\mathfrak{X} /\!\!/ \mathrm{Isom}(\mathbb{R}^{1|1}) = \mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X}) \xrightarrow{\mathcal{R}} \mathfrak{K}(\mathfrak{X}) \xrightarrow{Q} \mathrm{Vect}.$$

Thus, Q' is identified with line bundle with superconnection on $\Lambda\mathfrak{X}$ (cf. Remark 2.5).

Let us not worry about the superconnection for now and simply describe the underlying line bundle. Thus our goal is to describe the line bundle on $\Pi T\hat{X}_0$ (which we still call Q') induced by the atlas $\check{x}$ and the isomorphism H between its two pullbacks via the source and target maps $\Pi T\hat{X}_1 \rightrightarrows \Pi T\hat{X}_0$.

As a warm-up, pick a point $(x \in X_0, g: x \to x) \in \Pi T\hat{X}_0$. It determines a point of $\mathfrak{K}(\mathfrak{X})$ consisting of the length 1 constant superpath x in $\mathfrak{X}$ with its endpoints glued together via g. Thus $Q'_{(x,g)} = L_g$. From this we already conclude that $Q' \to \Pi T\hat{X}_0$ is trivial, since we assume the same of L. Now fix a second point $(x', g') \in \Pi T\hat{X}_0$ and a compatible morphism $f: x \to x'$; meaning that $g' = fgf^{-1}$. This gives rise to a morphism $\bar{f}$ in $\mathfrak{K}(\mathfrak{X})$ which, in the same vein as above, we can describe as being the constant f. From (4.12), we conclude that the identification $H(f): Q'_{(x,g)} \to Q'_{(x',g')}$ is through

$$H(f) = L(\bar{f}) = \frac{h(f,g)}{h(g',f)} = \frac{h(f,g)}{h(fgf^{-1},f)},$$

which agrees with (2.13).

Since we worked above with *points*, we have only shown that the restriction of Q' to $\Lambda\mathfrak{X} \hookrightarrow \Pi T\Lambda\mathfrak{X}$ agrees with the L' from Sect. 2.3. We want a slightly stronger statement, namely we want to identify Q' with the a pullback of L' via $\Pi T\Lambda\mathfrak{X} \to \Lambda\mathfrak{X}$. To do this, we need to fully describe the isomorphism H, which we see as a function $H: \Pi T\hat{X}_1 \to \mathbb{C}^\times$. Consider again the versal family $\check{x}: \Pi T\hat{X}_0 \to \mathfrak{B}^{\mathbb{T}}(\mathfrak{X})$. Then H is determined by the composition

$$\begin{array}{ccccc} & & \Pi T\hat{X}_0 & & \\ & \overset{s}{\nearrow} & & \overset{\check{x}}{\searrow} & \\ \Pi T\hat{X}_1 & & \Downarrow & & \mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X}) \xrightarrow{\mathcal{R}} \mathfrak{K}(\mathfrak{X}) \xrightarrow{U} \text{Vect}. \\ & \underset{t}{\searrow} & & \underset{\check{x}}{\nearrow} & \\ & & \Pi T\hat{X}_0 & & \end{array}$$

From (4.12) and the above description of $K_{\check{x}} = \mathcal{R} \circ \check{x}$, we see that

$$H = \exp\left(\int_{[0,1]} \text{vol}_D \langle D, \beta^* A\rangle\right) \frac{h(f,g)}{h(fgf^{-1},f)},$$

where β is the composition

$$U = \Pi T\hat{X}_1 \times \mathbb{R}^{1|1} \to \Pi T\hat{X}_1 \times \mathbb{R}^{0|1} \xrightarrow{\text{ev}} \hat{X}_1 \xrightarrow{p} X_1,$$

the integral is fiberwise over $\Pi T\hat{X}_1$, and f, g are similar to the paragraph above (except now they stand for $\hat{X}_1$-points of X_1 instead of pt-points). From Lemma 5.2, we have

$$\int_{[0,1]} \text{vol}_D \langle D, \text{ev}^* A\rangle = \int_{[0,1]} \text{vol}_D\ \tilde{A} + \theta\widetilde{dA} = \widetilde{dA} = t^*\tilde{B} - s^*\tilde{B},$$

so that

$$H = \frac{t^* \exp(\tilde{B})}{s^* \exp(\tilde{B})} \frac{h(f,g)}{h(fgf^{-1},f)}.$$

Seeing $\exp(-\tilde{B})\colon \Pi T\hat{X}_0 \to \mathbb{C}^\times$ as an isomorphism

$$\exp(-\tilde{B})\colon Q' \to \pi^* L', \tag{5.6}$$

of trivial line bundles, H gets identified with the defining datum (2.13) of L', as desired.

5.3. The superconnection. A superconnection on the line bundle $L'\colon \Lambda\mathfrak{X} \to$ Vect consists of a superconnection $\mathbb{A}$ on the underlying line bundle $L' \to \hat{X}_0$ whose two pullbacks over $\hat{X}_1$ are identified with one another through the isomorphism $s^*L' \to t^*L'$. Since we just want to describe the superconnection on $\hat{X}_0$ (which we know a priori to be invariant), it suffices to look at the versal family

$$\check{x}\colon \Pi T\hat{X}_0 \to \mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X}) = \Pi T\Lambda\mathfrak{X}/\!\!/\mathrm{Isom}(\mathbb{R}^{1|1})$$

and its image in $\mathfrak{K}(\mathfrak{X})$; nothing here will involve $\hat{X}_1$. Now, the superconnection we are seeking to describe is geometrically encoded by the $\mathbb{R}^{1|1}$-action on $\Pi T\hat{X}_0$ and the line bundle Q' over it. More specifically, the operator $\mathbb{A}$ is the infinitesimal generator associated to the vector field $D = \partial_\theta - \theta\partial_t$.

Thus, we need to understand the action of $\mathbb{R}^{1|1}$ by rotations of the $\Pi T\hat{X}_0$-family $K_{\check{x}}$, i.e., the isomorphism

$$\mu^* K_{\check{x}} \to \mathrm{pr}^* K_{\check{x}} \text{ over } \Pi T\hat{X}_0 \times \mathbb{R}^{1|1}, \tag{5.7}$$

and its image under $T_{\tilde{\mathfrak{X}}}$, where $\mu, \mathrm{pr}\colon \Pi T\hat{X}_0 \times \mathbb{R}^{1|1} \to \Pi T\hat{X}_0$ are the action respectively projection maps.

The isomorphism (5.7) can be expressed as a composition of simpler steps as indicated in Fig. 2. There, the left elbows $\subset\!\!\vdots$ always represent the bordism $L_{\check{x}}$, or, more precisely, its pullback via $\mathrm{pr}\colon \Pi T\hat{X}_0 \times \mathbb{R}^{1|1} \to \Pi T\hat{X}_0$; the right elbows $\vdots\!\!\supset_{[a,b]}$ represent bordisms $R_{\check{x}}^{[a,b]}$ with the indicated skeleton $[a,b]$; straight and crossing lines denote appropriate identities and braidings or, more precisely, their avatars as thin bordisms. Thus, for instance, the second picture represents the composition

$$L_{\check{x}} \circ \sigma_{\mathrm{sp}_{\check{x}} \amalg \overline{\mathrm{sp}}_{\check{x}}} \circ (\mathrm{Id}_{\overline{\mathrm{sp}}_{\check{x}}} \amalg L_{\check{x}} \amalg \mathrm{Id}_{\mathrm{sp}_{\check{x}}}) \circ (R_{\check{x}}^{[0,r]} \amalg R_{\check{x}}^{[r,1]})$$

(leaving implicit, as usual, pullbacks via projection maps). The isometries between successive pictures are the obvious ones. For example, the second isomorphism is the semigroup property of right elbows and the fourth uses the symmetry condition of the braiding. Now, the image of each of the intermediate steps under $T_{\tilde{\mathfrak{X}}}$ is a canonically trivial line bundle (over $\Pi T\hat{X}_0 \times \mathbb{R}^{1|1}$), and, with the exception of the fifth step, the corresponding isomorphism of line bundles is the identity. In fact, under the canonical trivilizations of all line bundles in question, we have an identity

$$T_{\tilde{\mathfrak{X}}}(\mu^* K_{\check{x}} \to \mathrm{pr}^* K_{\check{x}}) = T_{\tilde{\mathfrak{X}}}(R_{\check{x}}^{[1,1+r]} \to R_{\check{x}}^{[0,r]}).$$

The right-hand side, once identified with a $\mathbb{C}$-valued function on $\Pi T\hat{X}_0$, can be calculated from (4.12) and is given by

$$\exp\left(-\int_{[0,r]} \mathrm{vol}_D \langle D, \mathrm{ev}^* A\rangle\right) = \exp\int_{[0,r]} \mathrm{vol}_D\, -\tilde{A} = \exp(\theta\tilde{A}) = 1 + \theta\tilde{A}.$$

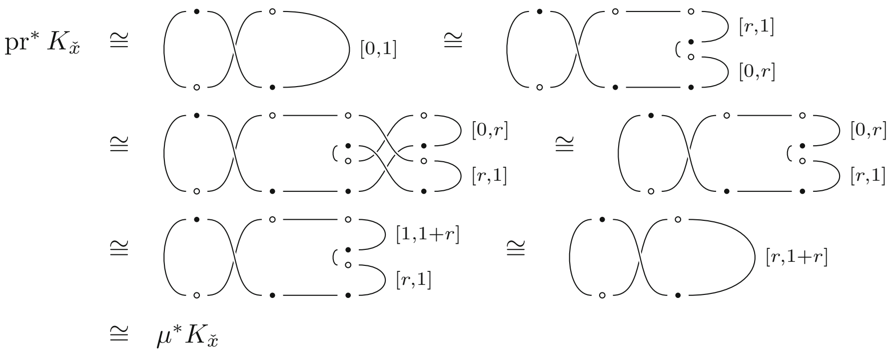


Fig. 2. The map (5.7), step by step.

Here, the second equality uses Corollary 5.3 and the fact that $\widetilde{dA}$ vanishes, the integral being fibered over $\Pi T\hat{X}_0$.

Thus, the $\mathbb{R}^{1|1}$-action on $Q' = \Pi T\hat{X}_0 \times \mathbb{C}$, expressed as an algebra homomorphism

$$C^\infty(\Pi T\hat{X}_0 \times \mathbb{C}) \to C^\infty(\Pi T\hat{X}_0 \times \mathbb{C} \times \mathbb{R}^{1|1}),$$

is characterized by

$$\tilde{\omega} \in C^\infty(\Pi T\hat{X}_0) \cong \Omega^*(\hat{X}_0) \mapsto \tilde{\omega} + \theta\widetilde{d\omega}, \quad z \in C^\infty(\mathbb{C}) \mapsto (1 + \theta\tilde{A})z,$$

and the infinitesimal action sends

$$\tilde{\omega}z \mapsto (\partial_\theta - \theta\partial_t)((\tilde{\omega} + \theta\widetilde{d\omega})(1 + \theta\tilde{A})z) = (\widetilde{d\omega} + \tilde{A}\tilde{\omega})z = (D_d + \tilde{A})\tilde{\omega}z.$$

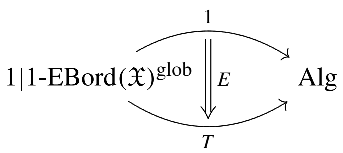
The superconnection corresponding to this odd, fiberwise linear vector field on the total space of Q' is given by the formula $d + A$ and, applying the gauge transformation (5.6), we find that the superconnection on L' is given by the formula

$$\mathbb{A} = d + A + dB = d + A + \Omega,$$

which agrees with (2.10).

6. 1|1-EFTs and the Chern Character of Twisted Vector Bundles

Fix an orbifold $\mathfrak{X}$, a gerbe with connection $\tilde{\mathfrak{X}}$ and an $\tilde{\mathfrak{X}}$-twisted vector bundle with connection $\mathfrak{V}$, with the usual notation of "Appendix B" section. In this section, we associate to $\mathfrak{V}$ a $1|1$-dimensional $T = T_{\tilde{\mathfrak{X}}}$-twisted field theory $E = E_{\mathfrak{V}}$



and show that its dimensional reduction provides a geometric interpretation of the twisted orbifold Chern character.

If $\tilde{\mathfrak{X}}$ is trivial, and thus V is just a usual vector bundle over $\mathfrak{X}$, the basic idea behind the construction of the field theory $E = E_V \in 1|1\text{-EFT}(\mathfrak{X})$ is the following. To a posivitely oriented superpoint of $\mathfrak{X}$, specified by a map to the atlas $X \to \mathfrak{X}$,

$$\mathbb{R}^{0|1} \xrightarrow{x} X \to \mathfrak{X},$$

we assign the vector space $V_{x(0)}$. To a superinterval Σ as in (3.3), we assign the super parallel transport along ψ^*V. Orientation-reversed manifolds map to the dual vector spaces and maps, and the image of elbows is determined by the duality pairing. We need to check that this is consistent with isometries between bordisms, in particular those of the form (3.3). But, indeed, the data of the vector bundle V on $\mathfrak{X}$ induces a superconnection-preserving bundle map $\psi'^*V \to (\psi \circ F)^*V$; this ensures that E_V is an internal functor.

Now, if V is twisted by a nontrivial gerbe, then ψ'^*V and $(\psi \circ F)^*V$ only become isomorphic after tensoring with an appropriate line bundle, namely ξ^*L. As we will see, this deviation from functoriality is expressed by the fact that E_V is a twisted field theory, and the relevant twist is the $T_{\tilde{\mathfrak{X}}}$ from Sect. 4.

Remark 6.1. The natural transformation E is not invertible, so we must choose between the lax or oplax variants. We make the choice that better fits with our conventions for twisted vector bundles. Now, restricting to the moduli stack of closed, connected bordisms $\mathfrak{K}(\mathfrak{X})$, the twist T gives us a line bundle, and we made the convention that E maps $1 \to T$, and not the opposite, so that its partition function (i.e., the restriction $E|_{\mathfrak{K}(\mathfrak{X})}$) determines a section of $T|_{\mathfrak{K}(\mathfrak{X})}$, and not of its dual.

6.1. Construction of the twisted field theory. Unraveling the definition of natural transformation between internal functors (see [26, section 5.1] for a detailed explanation), we see that, at the level of object stacks, E determines a symmetric monoidal fibered functor

$$E\colon 1|1\text{-EBord}(\mathfrak{X})_0^{\text{glob}} \to \text{Alg}_1.$$

More specifically, E assigns to an S-family Y in the domain a left $T(Y)$-module $E(Y)$; thus, by construction of T, $E(Y)$ is nothing but a vector bundle over S. When

$$Y = (Y \to S, \psi, \iota\colon S \times \mathbb{R}^{0|1} \to U)$$

is a positively oriented superpoint, we set

$$E(Y) = V_\iota,$$

where, for any $f\colon S \times \mathbb{R}^{0|1} \to U$, we write

$$V_f = (\psi_0 \circ f)^* V_{|S\times 0}.$$

If Y is negatively oriented, then $E(Y)$ is the dual of the above. To a morphism $\lambda\colon Y \to Y'$ over $f\colon S \to S'$, we assign the obvious identification

$$E(\lambda)\colon V_\iota \to f^*V_{\iota'},$$

which makes sense in view of condition (3.7). All the remaining data is determined, uniquely up to unique isomorphism, by the symmetric monoidal requirement for E.

At the level of morphism stacks, E assigns, to each S-family of bordisms Σ from Y_0 to Y_1, a map of $T(Y_1)$-modules; using the fact that the only algebra in sight is $\mathbb{C}$, we find that $E(\Sigma)$ is a linear map

$$E(\Sigma)\colon E(Y_1) \to T(\Sigma) \otimes E(Y_0)$$

of vector bundles over S. Now there are several kinds of bordisms to consider. For the sake of clarity, we focus on the superpath stack $\mathfrak{P}(\mathfrak{X}) \hookrightarrow 1|1\text{-EBord}(\mathfrak{X})_1^{\text{glob}}$.

Fix $\Sigma \in \mathfrak{P}(\mathfrak{X})$, and recall our usual notation fixed in Sects. 3.2 and 4.1. Instead of describing $E(\Sigma)$, it is more convenient to describe its inverse

$$E^{-1}(\Sigma)\colon \left(\bigotimes\nolimits_{1\le i\le n} L_{j_i}\right) \otimes V_{a_0} \to V_{b_n},$$

which we set to be the composition

$$E^{-1}(\Sigma) = (\mathrm{id}_n \otimes E_n) \circ \cdots \circ (\mathrm{id}_1 \otimes E_1) \circ (\mathrm{id}_0 \otimes E_0)$$

where id_k denotes the identity map of $\bigotimes_{k<i\le n} L_{j_i}$, $E_0 = \mathrm{SP}_0$, and E_i is the composition

$$E_i : L_{j_i} \otimes V_{b_{i-1}} \xrightarrow{m_{j_i, b_{i-1}}} V_{a_i} \xrightarrow{\mathrm{SP}_i} V_{b_i} \tag{6.2}$$

for $1 \le i \le n$. Here, SP_i denotes the super parallel transport of $\psi_0^* V$ along I_i.

Finally, there are certain conditions on the above data that need to be verified. It is clear that $Y \mapsto E(Y)$ is a fibered functor, so it remains to verify that $\Sigma \mapsto E(\Sigma)$ determines a natural transformation between appropriate fibered functors, and moreover that this is compatible with compositions of bordisms and identity bordisms, that is, commutativity of diagrams (3.5) and (3.6) in [18].

Proposition 6.3. *The assignment $\Sigma \mapsto E(\Sigma)$ respects compositions of intervals. That is, given bordisms*

$$Y_0 \xrightarrow{\Sigma_1} Y_1 \xrightarrow{\Sigma_2} Y_2$$

in $\mathfrak{P}(\mathfrak{X})$ and writing $\Sigma = \Sigma_2 \amalg_{Y_1} \Sigma_1$, the diagrams

$$\begin{array}{ccc} T(\Sigma_2)\otimes T(\Sigma_1)\otimes E(Y_0) & \xrightarrow{E^{-1}(\Sigma_1)} & T(\Sigma_2)\otimes E(Y_1) \\ \downarrow{\scriptstyle \mu_{\Sigma_2,\Sigma_1}} & & \downarrow{\scriptstyle E^{-1}(\Sigma_2)} \\ T(\Sigma)\otimes E(Y_0') & \xrightarrow{E^{-1}(\Sigma)} & E(Y_2) \end{array} \qquad \begin{array}{ccc} T(\mathrm{Id}_{Y_0})\otimes E(Y_0) & \xrightarrow{E^{-1}(\mathrm{Id}_{Y_0})} & E(Y_0) \\ \downarrow{\scriptstyle \epsilon_{Y_0}} & \nearrow & \\ \mathrm{Id}_{T(Y_0)}\otimes E(Y_0) & & \end{array}$$

commute.

Proof. This is clear from the definitions. □

Proposition 6.4. *The assignment $\Sigma \mapsto E(\Sigma)$ is natural in $\Sigma \in \mathfrak{P}(\mathfrak{X})$, that is, for each morphism $\lambda\colon \Sigma \to \Sigma'$ lying over $f\colon S \to S'$, the diagram*

$$\begin{array}{ccc} T(\Sigma)\otimes E(Y_0) & \xrightarrow{E^{-1}(\Sigma)} & E(Y_1) \\ \downarrow{\scriptstyle T(\lambda)\otimes E(\lambda_0)} & & \downarrow{\scriptstyle E(\lambda_1)} \\ T(\Sigma')\otimes E(Y_0') & \xrightarrow{E^{-1}(\Sigma')} & E(Y_1') \end{array} \tag{6.5}$$

commutes.

Here, $\lambda_i : Y_i \to Y_i'$ denotes the image of λ in $1|1\text{-EBord}(\mathfrak{X})_0$ via the source and target functors for $i = 0, 1$ respectively.

Proof. If λ is a refinement of skeletons, the claim follows immediately from the gluing property of super parallel transport. Next, let us consider the case where the skeletons of Σ, Σ' are compatible and the base map $f : S \to S'$ is the identity. For simplicity, we may assume that the skeletons comprise three intervals I_0, I_1, I_2, and that the first and last of them have length zero. The general case can be deduced by induction, using also the compatibility with compositions.

In this particular situation, (6.5) corresponds to the outer square in the following diagram.

$$
\begin{array}{ccccccc}
L_{j_2} \otimes L_{j_1} \otimes V_{b_0} & \xrightarrow{m} & L_{j_2} \otimes V_{a_1} & \xrightarrow{\mathrm{SP}_V} & L_{j_2} \otimes V_{b_1} & \xrightarrow{m} & V_{a_2} \\
h\uparrow & (a) & m\uparrow & (d) & m\uparrow & & \Vert \\
L_{j_2} \otimes L^\vee_{\tilde{a}_1} \otimes L_{j_1'} \otimes V_{b_0} & \xrightarrow{m} & L_{j_2} \otimes L^\vee_{\tilde{a}_1} \otimes V_{a_1'} & \xrightarrow{\mathrm{SP}_L \otimes \mathrm{SP}_V} & L_{j_2} \otimes L^\vee_{\tilde{b}_1} \otimes V_{b_1'} & & \Vert \\
\downarrow \mathrm{SP}_L & (b) & \downarrow \mathrm{SP}_L & & \Big\downarrow & & \Vert \\
L_{j_2} \otimes L^\vee_{\tilde{b}_1} \otimes L_{j_1'} \otimes V_{b_0} & \xrightarrow{m} & L_{j_2} \otimes L^\vee_{\tilde{b}_1} \otimes V_{a_1'} & (e) & \Big\downarrow h & (f) & \Vert \\
\downarrow h & (c) & \downarrow h & & \Big\downarrow & & \Vert \\
L_{j_2'} \otimes L_{j_1'} \otimes V_{b_0} & \xrightarrow{m} & L_{j_2'} \otimes V_{a_1'} & \xrightarrow{\mathrm{SP}_V} & L_{j_2'} \otimes V_{b_1'} & \xrightarrow{m} & V_{a_2}
\end{array}
$$

Here, each morphism is tensored with an appropriate identity, which we leave implicit. We notice that each of the inner diagrams commutes: (a) and (f) due to the compatibility between h and m (cf. (B.1)), (b), (c) and (e) because the maps in question act independently on the various tensor factors, and (d) due to the compatibility between the connections of L and V. Thus the outer square commutes, as claimed. □

Finally, we briefly explain how to extend E from $\mathfrak{P}(\mathfrak{X})$ to the whole of $1|1\text{-EBord}(\mathfrak{X})$. First, we require the so-called spin-statistics relation, that is, that the flip of $1|1\text{-EBord}(\mathfrak{X})$ maps to the grading involution of Vect. Second, recall that in Stolz and Teichner's definition, the bordism category does not admit duals, since a length zero right elbow R_0 is not allowed; this is done because they want to allow field theories with infinite dimensional state spaces. In our example, we could introduce those additional morphisms R_0, and then E would be uniquely determined by the above prescriptions, the requirement that duals map to duals, and the symmetric monoidal property. Concretely, suppose that Σ is a family of length zero left elbows, that is

$$\Sigma : \bar{Y} \amalg Y \to \emptyset,$$

where Y denotes an S-family of positive superpoints with a choice of skeleton, say $\iota : S \times \mathbb{R}^{0|1} \to U$ and $\bar{Y}$ denotes its orientation reversal, with the same underlying skeleton. Then we define $E(\bar{Y}) = V_i^\vee$ and

$$E(\Sigma) : V_\iota^\vee \otimes V_\iota \to \mathbb{C}_S$$

to be the evaluation pairing. The image of other kinds of bordisms is determined similarly.

Remark 6.6. A trivialization of $L \to X_1$ allows us to extend, in a fairly obvious way, the construction above to $1|1\text{-EBord}(\mathfrak{X})^{\mathrm{skel}}$.

6.2. Dimensional reduction and the Chern character. In this subsection, we finish the proof of Theorem 1.3, by showing that the diagram indeed commutes. So our goal is to study the dimensional reduction of the twisted field theory E associated to the $\tilde{\mathfrak{X}}$-twisted vector bundle $\mathfrak{V}$. Let us denote it by

$$E' \in 0|1\text{-EFT}^{T'_{\tilde{\mathfrak{X}}}}(\Lambda\mathfrak{X}),$$

and recall, from Proposition 2.6 and Theorem 5.1 that E' determines and is completely determined by an even, closed form $\omega \in \Omega^*(\Lambda\mathfrak{X}; L')$. The underlying form on the atlas $\hat{X}_0$, which by abuse of notation we still denote ω, corresponds to the section

$$E'(\mathcal{L} \circ \check{x}) \in C^\infty(\Pi T\hat{X}_0; Q')$$

under the isomorphism (5.6) and usual identification $C^\infty(\Pi T\hat{X}_0) \cong \Omega^*(\hat{X}_0)$.

Proposition 6.7. $\omega = \mathrm{ch}(\mathfrak{V}) \in \Omega^*(\Lambda\mathfrak{X}; L')$.

Proof. It suffices to verify that the underlying forms on $\hat{X}_0$ agree. Our dimensional reduction procedure dictates that $E'(\mathcal{L} \circ \check{x}) = E(K_{\check{x}})$, where $K_{\check{x}} \in \mathfrak{K}(\mathfrak{X})$ is the special $\Pi T\hat{X}_0$-family from Sect. 5.1. By (6.2), $E(K_{\check{x}})$ is obtained from the linear map

$$\pi^*V \xrightarrow{\mathrm{SP}^{-1}} \pi^*V \xrightarrow{m^{-1}} Q' \otimes \pi^*V,$$

by taking supertrace in the $\mathrm{End}(\pi^*V)$ component. (For clarity, we are leaving pullbacks by p and i implicit.) Here, SP denotes the super parallel transport along the superinterval

$$\Pi T\hat{X}_0 \times [0,1] \subset \Pi T\hat{X}_0 \times \mathbb{R}^{1|1} \xrightarrow{\mathrm{ev}} \hat{X}_0,$$

which Dumitrescu [10] identified with $\exp(-\nabla_V^2)$. Thus, the above homomorphism of vector bundles on $\Pi T\hat{X}_0$ is given by

$$m^{-1} \circ \exp(\nabla_V^2) \in C^\infty(\Pi T\hat{X}_0; \mathrm{Hom}(\pi^*V, Q' \otimes \pi^*V)).$$

Using (5.6), our chosen identification $\exp(-\tilde{B})\colon Q' \to \pi^*L'$, we get

$$\omega = \mathrm{str}(m^{-1} \circ \exp(\nabla_V^2 - B)) \in \Omega^*(\hat{X}_0; L'),$$

which agrees with the definition (B.2) of $\mathrm{ch}(\mathfrak{V})$. □

Appendix A. A Primitive Integration Theory on $\mathbb{R}^{1|1}$

Integration of compactly supported sections of the Berezinian line bundle is relatively simple to define [9]. The notion of domains with boundary requires more care, as shown by the following paradox, known as Rudakov's example: on $\mathbb{R}^{1|2}$ with coordinates t, θ_1, θ_2, we consider a function u with $\partial_{\theta_1} u = \partial_{\theta_2} u = 0$. Then $\int_{[0,1]\times\mathbb{R}^{0|2}} [dtd\theta]\, u = 0$, but performing the change of coordinates $t = t' + \theta_1\theta_2$ we get

$$\int_{[0,1]\times\mathbb{R}^{0|2}} [dt'd\theta]\, u(t' + \theta_1\theta_2, \theta_1, \theta_2) = \int_{[0,1]\times\mathbb{R}^{0|2}} [dtd\theta]\, u + \theta_1\theta_2\partial_t u = u(1) - u(0).$$

It turns out that the correct notion of boundary of a domain U in a supermanifold X is a codimension $1|0$ submanifold $K \hookrightarrow X$ whose reduced manifold is the boundary of

$|U|$. With this proviso, an integration theory featuring the expected Stokes formula still exists [4], and we would like to describe it concretely in a very special case.

Given $a, b\colon S \to \mathbb{R}^{1|1}$, we define the superinterval $[b, a] \subset S \times \mathbb{R}^{1|1}$ to be the domain with boundary prescribed by the embeddings

$$i_a\colon S \times \mathbb{R}^{0|1} \hookrightarrow S \times \mathbb{R}^{1|1} \xrightarrow{a\cdot} S \times \mathbb{R}^{1|1},$$
$$i_b\colon S \times \mathbb{R}^{0|1} \hookrightarrow S \times \mathbb{R}^{1|1} \xrightarrow{b\cdot} S \times \mathbb{R}^{1|1}.$$

We think of i_a as the incoming and i_b as the outgoing boundary components. To be consistent with the usual definition of $1|1$-EBord, we will to assume that, modulo nilpotents, $a \geq b$ (cf. Hohnhold et al. [15, definition 6.41]).

The fiberwise Berezin integral of a function $u = f + \theta g \in C^\infty(S \times \mathbb{R}^{1|1})$ on $[b, a]$ will be denoted $\int_{[b,a]}[dtd\theta]\, u$. Now, notice that we can always find primitives with respect to the Euclidean vector field $D = \partial_\theta - \theta\partial_t$. In fact, if $G \in C^\infty(S \times \mathbb{R})$ satisfies $\partial_t G = g$, then

$$u = D(\theta f - G).$$

It is also clear that any two primitives differ by a constant. We have a fundamental theorem of calculus.

Proposition A.1. *Given $u, v \in C^\infty(S\times\mathbb{R}^{1|1})$ with $u = (\partial_\theta - \theta\partial_t)v$ and $a, b\colon S \to \mathbb{R}^{1|1}$, with $a \geq b$ modulo nilpotents, we have*

$$\int_{[b,a]} [dtd\theta]\, u = v(b) - v(a).$$

To clarify the meaning of the right-hand side, when using $a, b\colon S \to \mathbb{R}^{1|1}$, etc., as arguments to a function, we implicitly identify them with maps $S \to S \times \mathbb{R}^{1|1}$, to avoid convoluted notation like $v(\mathrm{id}_S, b)$.

Proof. Using partitions of unity, it suffices to prove the analogous statement for the half-unbounded interval $[b, +\infty]$, namely

$$\int_{[b,+\infty]} [dtd\theta]\, u = v(b),$$

assuming u and v are compactly supported. Writing $u = f + \theta g$, we have $v = \theta f - G$ with G the compactly supported primitive of g. Thus,

$$v(b) = b_1 f(b_0) - G(b_0),$$

where b_0, b_1 are the components of b. On the other hand the embedding $i_b\colon S \times \mathbb{R}^{0|1} \to S \times \mathbb{R}^{1|1}$ corresponding to the outgoing boundary of $[b, +\infty]$ is expressed, on T-points, as

$$(s, \theta) \mapsto (s, b_0 + b_1\theta, \theta + b_1).$$

Thus, the domain of integration is picked out by the equation $t \geq b_0 + b_1\theta$. Performing the change of coordinates $t' = t - b_0 - b_1\theta$, whose Berezinian is 1, we get

$$\int_{t \geq b_0 + b_1\theta} [dtd\theta]\, u = \int_{t' \geq 0} [dt'd\theta]\, f(t' + b_0 + b_1\theta) + \theta g(t' + b_0 + b_1\theta)$$
$$= \int_{t' \geq 0} [dt'd\theta]\, b_1\theta\partial_t f(t' + b_0) + \theta g(t' + b_0)$$
$$= b_1 f(b_0) - G(b_0).$$

□

As we noticed in the proof, translations on $\mathbb{R}^{1|1}$ preserve the canonical section $[dtd\theta]$ of the Berezinian line; the flips $\theta \mapsto -\theta$ of course do not. Thus, an abstract Euclidean $1|1$-manifold X does not come with a canonical section of $\mathrm{Ber}(\Omega^1_X)$, but the choice of a Euclidean vector field D fixes a section, which we denote vol_D. We can then restate the proposition in a coordinate-free way as follows: for any S-family of superintervals $[b, a]$ with a choice of Euclidean vector field D,

$$\int_{[b,a]} \mathrm{vol}_D\, Du = u(b) - u(a).$$

Appendix B. Gerbes, Twisted Vector Bundles and Chern Forms

A central extension of the Lie groupoid $X_1 \rightrightarrows X_0$ is given by

(1) a complex line bundle $L \to X_1$ with connection ∇_L,
(2) a (connection-preserving) isomorphism $h\colon \mathrm{pr}_2^* L \otimes \mathrm{pr}_1^* L \to c^*L$ over the space $X_2 = X_1 \times_{X_0} X_1$ of pairs of composable morphisms,
(3) a form $B \in \Omega^2(X_0)$ (called curving).

In more friendly notation, for composable (S-points) $f, g \in X_1$, h is an operation

$$h_{f,g}\colon L_f \otimes L_g \to L_{f \circ g}.$$

The multiplication h must satisfy the natural associativity condition, and the curvature of L the equation $\nabla_L^2 = t^*B - s^*B$. Note that dB is invariant, and therefore determines a form $\Omega \in \Omega^3(\mathfrak{X})$, called 3-curvature. Also, there are canonical isomorphisms $L|_{X_0} \cong \mathbb{C}$ and $L_{f^{-1}} \cong L_f^\vee$. For better legibility, we will typically use $L_f^\vee$ instead of $L_{f^{-1}}$.

There is an appropriate notion of Morita equivalence for central extensions [3, section 4.3]. Then, just like differentiable stacks are Lie groupoids up to Morita equivalence, a gerbe with connection $\tilde{\mathfrak{X}}$ over $\mathfrak{X}$ can be defined as a central extension up to Morita equivalence. Gerbes over an orbifold $\mathfrak{X}$ are classified by classes in $H^3(\mathfrak{X}; \mathbb{Z})$, and $[\Omega]$ is the image in de Rham cohomology.

If $L \to X_1 \rightrightarrows X_0$ is a presentation of the gerbe $\tilde{\mathfrak{X}}$, then an $\tilde{\mathfrak{X}}$-twisted vector bundle $\mathfrak{V}$ is presented by

(1) a (complex, super) vector bundle $V \to X_0$ with connection ∇_V and
(2) an isomorphism $m\colon L \otimes s^*V \to t^*V$ of vector bundles with connection over X_1 (where the domain is endowed with the tensor product connection $\nabla_L \otimes 1 + 1 \otimes \nabla_V$)

satisfying certain natural conditions, namely the commutativity of the following diagrams, where x, y, z and $f\colon x \to y$, $g\colon y \to z$ denote generic (S-)points of X_0 respectively X_1.

$$\begin{array}{ccc} L_g \otimes L_f \otimes V_x & \xrightarrow{\mathrm{id} \otimes m_{f,x}} & L_g \otimes V_y \\ {\scriptstyle h_{g,f} \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle m_{g,y}} \\ L_{g \circ f} \otimes V_x & \xrightarrow{m_{g\circ f,x}} & V_z \end{array} \qquad \begin{array}{ccc} L_{\mathrm{id}_x} \otimes V_x & \xrightarrow{m_{\mathrm{id}_x,x}} & V_x \\ \downarrow & \nearrow {\scriptstyle \cdot} & \\ \mathbb{C} \otimes V_x & & \end{array} \tag{B.1}$$

The Chern character form of $\mathfrak{V}$ is calculated from the above presentation as follows:

$$\mathrm{ch}(\mathfrak{V}) = \mathrm{str}(i^* m^{-1} \circ p^* \exp(\nabla_V^2 - B)) \in \Omega^{\mathrm{ev}}(\Lambda \mathfrak{X}; L'). \tag{B.2}$$

(Recall that we write $i\colon \hat{X}_0 \to X_1$ for the inclusion and $p\colon \hat{X}_0 \to X_0$ for the map $s|_{\hat{X}_0} = t|_{\hat{X}_0}$.) Let us describe the underlying $L' = L|_{\hat{X}_0}$-valued differential form on $\hat{X}_0$ in more detail. The isomorphism $m\colon L \otimes s^*V \to t^*V$ gives us an identification

$$\exp(\nabla_L^2) s^* \exp(\nabla_V^2) = t^* \exp(\nabla_V^2).$$

Using the fact that $\nabla_L^2 = t^*B - s^*B$, we get

$$s^*(\exp(\nabla_V^2 - B)) = t^*(\exp(\nabla_V^2 - B)),$$

so this defines an $\mathrm{End}(p^*V)$-valued form on $\hat{X}_0$. Now, i^*m is an isomorphism $i^*L \otimes p^*V \to p^*V$, and the form $\mathrm{ch}(\mathfrak{V})$ is obtained by composing the coefficients of $\exp(\nabla_V^2 - B)$ with

$$\mathrm{End}(p^*V) \xrightarrow{i^*m^{-1}} i^*L \otimes \mathrm{End}(p^*V) \xrightarrow{\mathrm{id} \otimes \mathrm{str}} i^*L.$$

Proposition B.3. *This $L|_{\hat{X}_0}$-valued form defines an even, closed element in the complex* (2.10).

Proof. This is easy to check directly. It also follows from Theorem 5.1 and Proposition 6.7, since we know a priori that the form ω in the statement of that proposition is even and closed with respect to the relevant differential. □

Remark B.4. If $X_1 \rightrightarrows X_0$ is the groupoid of a finite group action on a manifold and the central extension is trivial, then $\mathfrak{V}$ is just the data of an equivariant vector bundle. In this case, $\mathrm{ch}(\mathfrak{V})$ represents the equivariant Chern character of Baum and Connes [2]. If $X_1 \rightrightarrows X_0$ is Morita equivalent to a manifold, then $\mathfrak{V}$ is what is traditionally called a twisted vector bundle, and $\mathrm{ch}(\mathfrak{V})$ agrees with the definition of Bouwknegt et al. [7], Park [19], and others.

Remark B.5. Finite-dimensional twisted vector bundles only exist when the twisting gerbe represents a torsion class. Thus, it would be interesting to allow a more general target category and investigate 1|1-EFTs twisted by non-torsion classes.

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