RECURRENCE RELATIONS FOR POLYNOMIALS OBTAINED BY ARITHMETIC FUNCTIONS

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Abstract. Families of polynomials associated to arithmetic functions $g(n)$ are studied. The case $g(n) = \sigma(n)$, the divisor sum, dictates the non-vanishing of the Fourier coefficients of powers of the Dedekind eta function. The polynomials $P_n^g(X)$ are defined by n-term recurrence relations. For the case that $g(x)$ is a polynomial of degree d, we prove that at most a $d+2$ term recurrence relation is needed. For the special case $g(x) = x$ we obtain explicit formulas and results.

1. Introduction

Let $q = e^{2\pi i\tau}$ and τ be in the upper complex half-plane. We consider

$$\exp\left(X\sum_{n\geq 1} g(n)\frac{q^n}{n}\right) = \sum_{n\geq 0} P_n^g(X)q^n,$$

where $g : \mathbb{N} \to \mathbb{N}$ is an arithmetic function normalized such that $g(1) = 1$. This implies a recursive definition of the polynomials $P_n^g(X)$:

$$P_n^g(X) = \frac{X}{n}\sum_{k=1}^{n} g(k)P_{n-k}^g(X) \qquad \text{for all} \qquad n \geq 1, \tag{1}$$

where $P_0^g(X) := 1$.

Let for the moment $g(n) = \sigma(n) = \sum_{t|n} t$ be the sum of all divisors of n. Then we obtain the powers of the Euler product

$$\prod_{n\geq 1}(1-q^n)^{-X} = \exp\left(X\sum_{n\geq 1}\sigma(n)\frac{q^n}{n}\right).$$

This is essentially an identity involving powers of the Dedekind eta function $\eta(\tau) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ (see [On03] for more details). Hence, the roots of $P_n^{\sigma}(X)$ dictate the vanishing properties of the n-th Fourier coefficients of powers of the Dedekind eta function (see [HNR17, HLN18, HNR18] for more details and the connection to the Lehmer conjecture [Le47]).

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Explicit formulas for $P_n^g(X)$ in the case $g(n)=1$ for all $n \in \mathbb{N}$ are given by

$$\exp\left(X\sum_{n\geq 1}\frac{q^n}{n}\right) = (1-q)^{-X} = \sum_{n\geq 0}\binom{-X}{n}(-q)^n.$$

Hence,

$$P_n^g(X) = \frac{1}{n!}X(X+1)\cdots(X+n-1) = \frac{1}{n!}\sum_{k=1}^{n} S_{n,k}X^k \tag{2}$$

for $n \geq 1$ and $P_0^g(X) = 1$. Here, the coefficients $S_{n,k}$ are the Stirling numbers of the first kind.

The aim of this paper is to study the polynomials $P_n^g(X)$ for some interesting functions $g(n)$ particularly for $g(n) = n$. In this case the coefficients of X^k of the polynomials $n!P_n^g(X)$ are the so-called Lah numbers $\frac{n!}{k!}\binom{n-1}{k-1}$ ([Ai07]) and the polynomials are related to the generalized Laguerre polynomials [Do16]

$$L_n^{(\alpha)}(X) = \sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-X)^k}{k!} \qquad (\alpha > -1). \tag{3}$$

In the above case, we show that the sequence of the polynomials $P_n^g(X)$ satisfies a three term recurrence relation. More generally, we show that if $g(x)$ is a polynomial of degree d, then $(P_n^g(X))_{n\geq 0}$ satisfies a recurrence relation of order at most $d+2$. From the properties of orthogonal polynomials [Do16], we can also derive that in the case $g(x) = x$ the roots are all simple and real. Putting $\widetilde{P}_n^g(X) = X^{-1}P_n^g(X)$, we show that $\widetilde{P}_n^g(X)$ is irreducible in this case.

2. Recurrence relation for the case when $g(x)$ is a polynomial

In this section, we prove the following.

Theorem 1. *Let $g(x)$ be a polynomial of degree d. For $0 \leq m, j \leq d$ define*

$$\begin{aligned} b_m &= \sum_{\mu=0}^{m}(-1)^{\mu}\binom{m}{\mu}g(-\mu), \\ c_j &= (-1)^j\left(b_0\binom{d}{j} + \sum_{\iota=1}^{d-j}\iota\, b_{\iota}\binom{d-\iota}{j}\right). \end{aligned}$$

Then the polynomials $P_n^g(X)$ satisfy the following $d+2$ term recurrence relations with $P_0^g(X) = 1$ and

$$P_{n+1}^g(X) = \frac{1}{n+1}\sum_{j=0}^{\min\{n,d\}}\left(c_jX + (-1)^j\binom{d+1}{j+1}(n-j)\right)P_{n-j}^g(X).$$

For the proof, we want to express our polynomial $g(x)$ in terms of the basis

$$\{1, x, x(x+1), x(x+1)(x+2)/2!, \ldots, x(x+1)(x+2)\cdots(x+d)/d!, \ldots\}.$$

Hence, we first show how to determine the coefficients in terms of this basis. These are exactly the b_m as stated in the theorem.

Lemma 1. *Let $Q(x) = \sum_{m=0}^{d} b_m \frac{x(x+1)\cdots(x+m-1)}{m!}$. Then the coefficients b_m for $0 \le m \le d$ can be recovered as $b_m = \sum_{\mu=0}^{m} (-1)^{\mu} \binom{m}{\mu} Q(-\mu)$.*

We give a proof of this lemma at the end of this section.

Proof of Theorem 1. We have,

$$\frac{g(n)}{n} = \frac{b_0}{n} + \sum_{m=1}^{d} b_m \frac{(n+1)\cdots(n+m-1)}{(m-1)!}.$$

Then

$$\begin{aligned} X\sum_{n\ge1} g(n)\frac{q^n}{n} &= X\left(\sum_{n\ge1}\frac{b_0}{n} + \sum_{m=1}^{d} b_m \frac{(n+1)\cdots(n+m-1)}{(m-1)!}\right) q^n \\ &= Xb_0\sum_{n\ge1}\frac{q^n}{n} + X\sum_{m=1}^{d} b_m \sum_{n\ge1}\frac{(n+1)\cdots(n+m-1)}{(m-1)!} q^n \end{aligned}$$

for $|q| < 1$. The first inner sum is $-\log(1-q)$. The generic inner sum for $m \ge 1$ is

$$\begin{aligned} \sum_{n\ge1}\frac{(n+1)\cdots(n+m-1)}{(m-1)!} q^n &= \frac{1}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial q^{m-1}}\left(\frac{1}{1-q}\right)\right) - 1 \\ &= \frac{1}{(1-q)^m} - 1. \end{aligned}$$

Our exponential generating function is therefore

$$\begin{aligned} F(q,X) &:= \exp\left(X\sum_{n\ge1} g(n)\frac{q^n}{n}\right) \\ &= \exp\left(-b_0X\log(1-q) + X\sum_{m=1}^{d} b_m\left(\frac{1}{(1-q)^m} - 1\right)\right). \end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{\partial}{\partial q} F(q,X) \\
= \;& F(q,X)\frac{\partial}{\partial q}\left(-b_0 X\log(1-q)+X\sum_{m=1}^{d} b_m\left(\frac{1}{(1-q)^m}-1\right)\right) \\
= \;& XF(q,X)\left(\frac{b_0}{1-q}+\sum_{m=1}^{d}\frac{m\,b_m}{(1-q)^{m+1}}\right).
\end{aligned}$$

Thus,

$$(1-q)^{d+1}\frac{\partial}{\partial q}F(q,X) = XF(q,X)\left(b_0(1-q)^d+\sum_{m=1}^{d} m\,b_m(1-q)^{d-m}\right),$$

or

$$\begin{aligned}
&\left(\sum_{j=0}^{d+1}(-1)^j\binom{d+1}{j}q^j\right)\times\left(\sum_{n\geq 1} n\,P_n^g(X)q^{n-1}\right) \\
= \;& X\left(\sum_{n\geq 0}P_n^g(X)q^n\right)\left(b_0(1-q)^d+\sum_{m=1}^{d} m\,b_m(1-q)^{d-m}\right).
\end{aligned}$$

Now

$$b_0(1-q)^d+\sum_{m=1}^{d} m\,b_m(1-q)^{d-m}=\sum_{j=0}^{d} c_j q^j,$$

where

$$c_j=(-1)^j b_0\binom{d}{j}+\sum_{m=1}^{d-j} m\,b_m(-1)^j\binom{d-m}{j}.$$

Thus,

$$\begin{aligned}
&\left(\sum_{j=0}^{d+1}(-1)^j\binom{d+1}{j}q^j\right)\times\left(\sum_{n\geq 1} n\,P_n^g(X)q^{n-1}\right) \\
= \;& X\left(\sum_{n\geq 0}P_n^g(X)q^n\right)\left(\sum_{j=0}^{d} c_j q^j\right).
\end{aligned}$$

Expanding and identifying the coefficients of q^n on left and right-hand sides above we get

$$\sum_{j=0}^{\min\{d,n\}+1}(-1)^j\binom{d+1}{j}(n+1-j)P_{n+1-j}^g(X)=X\sum_{j=0}^{\min\{d,n\}} c_j P_{n-j}^g(X).$$

Hence, we obtain

$$P_{n+1}^{g}(X) = \frac{1}{n+1} \sum_{j=0}^{\min\{n,d\}} \left(c_j X + (-1)^j \binom{d+1}{j+1} (n-j) \right) P_{n-j}^{g}(X).$$

□

Example 1. We give an explicit two term recurrence relation for $(P_n^g(X))_{n\geq 0}$ when $g(n) = 1$.

For $g(n) = 1$ we have in the preceding notation $d = 0$, $b_0 = 1$, and $c_0 = 1$. This leads to $P_0^g(X) = 1$ and

$$P_{n+1}^{g}(X) = \frac{1}{n+1}(X+n) P_n^g(X)$$

for all $n \geq 0$, which yields exactly the polynomials (2) from the Introduction.

Example 2. We give an explicit three term recurrence relation for $(P_n^g(X))_{n\geq 0}$ when $g(n) = n$.

For $g(n) = n$ we have in the preceding notation $d = 1$, $b_0 = 0$, $b_1 = 1$, $c_0 = 1$, and $c_1 = 0$. This leads to $P_0^g(X) = 1$, $P_1^g(X) = XP_0^g(X)$, and

$$P_{n+1}^{g}(X) = \frac{1}{n+1}\big((2n+X)P_n^g(X) - (n-1)P_{n-1}^g(X)\big)$$

for all $n \geq 1$.

Proof of Lemma 1. We obtain for $x = -N$ and $0 \leq N \leq d$:

$$\begin{aligned}
& \sum_{m=0}^{d}\sum_{\mu=0}^{m} (-1)^{\mu} \binom{m}{\mu} Q(-\mu) \frac{-N(-N+1)\cdots(-N+m-1)}{m!} \\
= \; & \sum_{m=0}^{N}\sum_{\mu=0}^{m} (-1)^{\mu} \binom{m}{\mu} Q(-\mu)(-1)^m \binom{N}{m} \\
= \; & \sum_{\mu=0}^{N}\sum_{m=\mu}^{N} (-1)^{m+\mu} \frac{N!}{\mu!\,(m-\mu)!\,(N-m)!} Q(-\mu) \\
= \; & \sum_{\mu=0}^{N}\sum_{m=0}^{N-\mu} (-1)^{m} \frac{N!}{\mu! m!\,(N-m-\mu)} Q(-\mu) \\
= \; & \sum_{\mu=0}^{N}\sum_{m=0}^{N-\mu} (-1)^{m} \binom{N-\mu}{m}\binom{N}{\mu} Q(-\mu) = \sum_{\mu=0}^{N} \delta_{N,\mu} \binom{N}{\mu} Q(-\mu) \\
= \; & Q(-N).
\end{aligned}$$

□

3. Irreducibility of $\widetilde{P}_n^g(X)$ when $g(n) = n$

In this case $P_n(X) := P_n^g(X)$ has the following closed form (compare with [HN18]):

$$(4)\quad P_n(X) = \frac{1}{n!}\sum_{k=1}^{n} \frac{n!}{k!}\binom{n-1}{k-1} X^k = X \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{X^k}{(k+1)!} = X\widetilde{P}_n(X).$$

The coefficients of the polynomial $n!P_n(X)$ are the so-called Lah numbers $\frac{n!}{k!}\binom{n-1}{k-1}$ (compare with e. g. [Ai07]).

Theorem 2. *$\widetilde{P}_n(X)$ is irreducible for all $n \geq 1$.*

Theorem 3. *For $g(n) = n$ all the roots of the polynomials $P_n(X) = P_n^g(X)$ are simple and non-positive real numbers. Additionally the negative roots are interlacing from n to $n+1$.*

By interlacing we mean that if $s_{n,1} < \ldots < s_{n,n-1} < s_{n,n} = 0$ are the roots of $P_n(X)$ then $s_{n+1,1} < s_{n,1} < s_{n+1,2} < s_{n,2} < \ldots < s_{n+1,n-1} < s_{n,n-1} < s_{n+1,n} < 0$.

For the proofs of the preceding two theorems the important thing to notice is the following lemma.

Lemma 2. *For $n \geq 1$*

$$P_n(X) = \frac{X}{n} L_{n-1}^{(1)}(-X)$$

where $L_{n-1}^{(1)}(X) = \sum_{k=0}^{n-1} \binom{(n-1)+1}{(n-1)-k} \frac{(-X)^k}{k!}$ is the $n-1$st generalized Laguerre polynomial with $\alpha = 1$ (compare with (3) form the Introduction).

Proof. From the explicit formula (4) we obtain

$$\begin{aligned} P_n(X) &= X \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{X^k}{(k+1)!} = X \sum_{k=0}^{n-1} \frac{(n-1)!}{k!\,(n-1-k)!} \frac{X^k}{(k+1)!} \\ &= \frac{X}{n} \sum_{k=0}^{n-1} \frac{n!}{(k+1)!\,(n-1-k)!} \frac{X^k}{k!} = \frac{X}{n} \sum_{k=0}^{n-1} \binom{n}{n-1-k} \frac{X^k}{k!} \\ &= \frac{X}{n} L_{n-1}^{(1)}(-X). \end{aligned}$$

□

Theorem 2 now follows from the irreducibility of $L_{n-1}^{(1)}(X)$ shown by Schur in [Sc31].

Theorem 3 follows from the respective properties of the family of orthogonal polynomials $L_{n-1}^{(1)}(X)$ (see e. g. [Do16]).

Remark 1. Other proofs of Theorem 3 are also possible: We could show directly that the polynomials $\widetilde{P}_n(X)$ form a system of orthogonal polynomials with respect to the measure on $(-\infty, 0]$ with Lebesgue density $x \mapsto (-x)\, e^x$.

Or we could use the recurrence relation from Example 2 from Section 2 and apply [WY05, Theorem 1].

4. Partial results when $g(n) = n^2$

In this case $P_n(X) := P_n^g(X)$ has the following closed form (compare [HN18]):

$$P_n(X) = X \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \binom{n+k}{2k+1} X^k = X \widetilde{P}_n(X). \tag{5}$$

Note that

$$\widetilde{P}_n(X) = \sum_{k=0}^{n-1} A_{n,k} \frac{X^k}{(k+1)!}, \quad A_{n,k} \in \mathbb{Z}. \tag{6}$$

In our case $A_{n,k} = \binom{n+k}{2k+1}$.

Remark 2. There is a theorem of Schur [Sc29, Satz IV] and a generalization of Allen and Filaseta [AF03, Theorem 2] concerning the irreducibility of polynomials of the form (6). But both require $|A_{n,0}| = 1$ whereas in our case (5) we have $A_{n,0} = n$. Schur used [Sc29, Satz IV] to prove the irreducibility of the generalized Laguerre polynomials $L_{n-1}^{(1)}(X)$ in [Sc31].

Proposition 1. *If $n-1$ is prime then $\widetilde{P}_n(X)$ with $A_{n,k} = \binom{n+k}{2k+1}$, $k = 0, \dots, n-1$, as in (5) is irreducible.*

Proof. If we multiply with $n!$ the polynomial $n!\widetilde{P}_n(X)$ has the integer coefficients $\frac{n!}{(k+1)!} A_{n,k}$. Hence, $n-1 \mid \frac{n!}{(k+1)!} A_{n,k}$ for $k = 0, \dots, n-3$ and $n-1 \mid \frac{n!}{(n-1)!} A_{n,n-2} = (2n-2)\, n$. If $n-1$ is prime then $(n-1)^2$ does not divide $n!n = \frac{n!}{1!} A_{n,0}$. So, the criterion of Eisenstein yields in this case that $\widetilde{P}_n(X)$ is irreducible. □

Remark 3. Note that $n-1$ prime is the only case where Eisenstein's criterion applies to the polynomials (5) since we need a prime p that divides in particular $\frac{n!}{(n-1)!} A_{n,n-2} = 2(n-1)\, n$. In case $p \mid n$, obviously $p^2 \mid n!n = n!A_{n,0}$. In case $p \mid n-1$ but $p \neq n-1$, then also $p^2 \mid n^2 (n-1) \frac{(n-2)!}{p!} p! = n!A_{n,0}$.

We used a computer to check that the above polynomials are irreducible for all $n \leq 100$.

Proposition 2. *The polynomials $P_n(X)$ satisfy a four term recurrence relation with*

$$P_0(X) = 1,\ P_1(X) = XP_0(X),\ P_2(X) = \frac{1}{2}\left((X+3)\, P_1(X) + XP_0(X)\right),$$

and

$$P_{n+1}(X) = \frac{1}{n+1}\left(3(X+n)\, P_n(X) + (X-3n+3)\, P_{n-1}(X) + (n-2)\, P_{n-2}(X)\right),$$

for $n \geq 2$.

Proof. In the notation of Theorem 1 for $g(n) = n^2$ we have $d = 2$, $b_0 = 0$, $b_1 = -1$, $b_2 = 2$, $c_0 = 3$, $c_1 = 1$, and $c_2 = 0$. Using Theorem 1 this leads to the stated four term recurrence relation. $\square$

5. A necessary condition for a three term recurrence relation formula in the case of arbitrary $g(n)$

We are going to prove a necessary condition that the $P_n^g(X)$ are a family of orthogonal polynomials and hence fulfill a three term recurrence relation of the form

$$XP_n^g(X) = \alpha_n P_{n+1}^g(X) + \beta_n P_n^g(X) + \gamma_n P_{n-1}^g(X) \tag{7}$$

for some α_n, β_n, and γ_n. From the proof of the criterion the quite remarkable fact can be observed that it does not depend on n in the sense that if the criterion is *not* fulfilled also (7) cannot be fulfilled for *any* $n \geq 2$.

Proposition 3. *Let* $g(n)$ *be an arbitrary arithmetic function with* $g(1) = 1$. *A necessary condition that the* $P_n^g(X)$ *are a family of orthogonal polynomials is* $g(2)^3 - 2g(2)\,g(3) + g(4) = 0$.

Corollary 1. *For* $g(n) = n^\ell$, $\ell \neq 0, 1$, *the polynomials* $P_n^g(X)$ *do* not *satisfy a three term recurrence relation of the form (7) and hence they are* not *a system of orthogonal polynomials.*

Remark 4. This allows us to complement the result of Proposition 2 for $g(n) = n^2$ since $2 = \ell \neq 0, 1$.

Of course we also could just have checked directly that $g(2)^3 - 2g(2)\,g(3) + g(4) = 64 - 72 + 16 = 8 \neq 0$.

Proof of Corollary 1. For $g(x) = x^\ell$, $x > 0$, we obtain $g''(x) = (\ell - 1)\,\ell x^{\ell-2}$ for the second derivative which is positive for $\ell > 1$ or $\ell < 0$ and negative for $0 < \ell < 1$. This implies $g(x)$ is strictly concave up for $\ell > 1$ or $\ell < 0$ and strictly concave down for $0 < \ell < 1$. Hence $3^\ell < \left(2^\ell + 4^\ell\right)/2$ in case $\ell > 1$ or $\ell < 0$ and $3^\ell > \left(2^\ell + 4^\ell\right)/2$ in case $0 < \ell < 1$. However we only need that we do not have equality in these cases. Hence $g(2)^3 - 2g(2)\,g(3) + g(4) = 2^{3\ell} - 2^{\ell+1}3^\ell + 4^\ell = 2^\ell\left(4^\ell - 2 \cdot 3^\ell + 2^\ell\right) \neq 0$ for $\ell \neq 0, 1$. $\square$

In the proof of Proposition 3 we need explicitly the four highest coefficients of $P_n^g(X)$. This will be done separately in the following.

Lemma 3. *Let $g(n)$ be an arbitrary arithmetic function with $g(1)=1$. Let $P_n^g(X)=\sum_{k=0}^n A_{n,k}X^k$. Then*

$$\begin{aligned}
A_{n,n} &= \frac{1}{n!}, \quad n\geq 0,\\
A_{n,n-1} &= \frac{1}{n!}g(2)\binom{n}{2}, \quad n\geq 1,\\
A_{n,n-2} &= \frac{1}{n!}\left(3g(2)^2\binom{n}{4}+2g(3)\binom{n}{3}\right), \quad n\geq 2,\\
A_{n,n-3} &= \frac{1}{n!}\left(15g(2)^3\binom{n}{6}+20g(2)g(3)\binom{n}{5}+6g(4)\binom{n}{4}\right), \quad n\geq 3.
\end{aligned}$$

where a binomial coefficient $\binom{n}{k}$ is assumed to be 0 in case $n<k$.

We give a proof at the end of this section and first turn to the proof of Proposition 3.

Proof of Proposition 3. We have

$$(n+1)!P_{n+1}^g(X)-n!XP_n^g(X)=\sum_{k=1}^{n+1}\left((n+1)!A_{n+1,k}-n!A_{n,k-1}\right)X^k.$$

With the previous lemma for $k=n+1$ in particular, we get that

$$(n+1)!A_{n+1,n+1}-n!A_{n,n}=0.$$

For $k=n,n-1,n-2$ we obtain

$$(n+1)!A_{n+1,n}-n!A_{n,n-1}=g(2)\binom{n+1}{2}-g(2)\binom{n}{2}=g(2)\,n, \tag{8}$$

$$\begin{aligned}
&(n+1)!A_{n+1,n-1}-n!A_{n,n-2}\\
&= 3g(2)^2\binom{n+1}{4}+2g(3)\binom{n+1}{3}-3g(2)^2\binom{n}{4}-2g(3)\binom{n}{3}\\
&= 3g(2)^2\binom{n}{3}+2g(3)\binom{n}{2},
\end{aligned}$$

$$\begin{aligned}
&(n+1)!A_{n+1,n-2}-n!A_{n,n-3}\\
&= 15g(2)^3\binom{n+1}{6}+20g(2)g(3)\binom{n+1}{5}+6g(4)\binom{n+1}{4}\\
&\quad -15g(2)^3\binom{n}{6}-20g(2)g(3)\binom{n}{5}-6g(4)\binom{n}{4}\\
&= 15g(2)^3\binom{n}{5}+20g(2)g(3)\binom{n}{4}+6g(4)\binom{n}{3}.
\end{aligned}$$

From (8) we observe that the coefficient of X^n of

$$(n+1)!P_{n+1}^g(X)-n!XP_n^g(X)-n!g(2)\,nP_n^g(X)$$

equals 0. Since the degree of $P_n^g(X)$ is n the coefficient of X^{n+1} remains 0.

For the coefficients of X^{n-1} and X^{n-2} we obtain

$$\begin{aligned}
& (n+1)!A_{n+1,n-1} - n!A_{n,n-2} - n!g(2)\,nA_{n,n-1} \\
= \; & 3g(2)^2\binom{n}{3} + 2g(3)\binom{n}{2} - g(2)^2 n\binom{n}{2} = 2\left(g(3) - g(2)^2\right)\binom{n}{2}, \\
& (n+1)!A_{n+1,n-2} - n!A_{n,n-3} - n!g(2)\,nA_{n,n-2} \\
= \; & 15g(2)^3\binom{n}{5} + 20g(2)\,g(3)\binom{n}{4} + 6g(4)\binom{n}{3} \\
& - g(2)\,n\left(3g(2)^2\binom{n}{4} + 2g(3)\binom{n}{3}\right) \\
= \; & g(2)^3\left(15\binom{n}{5} - 3n\binom{n}{4}\right) + g(2)\,g(3)\left(20\binom{n}{4} - 2n\binom{n}{3}\right) \\
& + 6g(4)\binom{n}{3} \\
= \; & -12g(2)^3\binom{n}{4} + g(2)\,g(3)\,(3n-15)\binom{n}{3} + 6g(4)\binom{n}{3}.
\end{aligned}$$

Going back we observe that the coefficient of X^{n-1} of

$$\begin{aligned}
& (n+1)!P_{n+1}^g(X) - n!XP_n^g(X) - n!g(2)\,nP_n^g(X) \\
& - (n-1)!2\left(g(3) - g(2)^2\right)\binom{n}{2}P_{n-1}^g(X)
\end{aligned}$$

equals 0.

For the coefficient of X^{n-2} we obtain now

$$\begin{aligned}
& (n+1)!A_{n+1,n-2} - n!A_{n,n-3} - n!g(2)\,nA_{n,n-2} \\
& - (n-1)!2\left(g(3) - g(2)^2\right)\binom{n}{2}A_{n-1,n-2} \\
= \; & -12g(2)^3\binom{n}{4} + g(2)\,g(3)\,(3n-15)\binom{n}{3} + 6g(4)\binom{n}{3} \\
& - 2\left(g(3) - g(2)^2\right)\binom{n}{2}g(2)\binom{n-1}{2} \\
= \; & g(2)^3\left(-12\binom{n}{4} + 2\binom{n}{2}\binom{n-1}{2}\right) \\
& + g(2)\,g(3)\left((3n-15)\binom{n}{3} - 2\binom{n}{2}\binom{n-1}{2}\right) + 6g(4)\binom{n}{3} \\
= \; & 6\left(g(2)^3 - 2g(2)\,g(3) + g(4)\right)\binom{n}{3}.
\end{aligned}$$

This means the polynomial

$$\begin{aligned}&(n+1)!P_{n+1}^{g}(X)-n!XP_{n}^{g}(X)-n!g(2)\,nP_{n}^{g}(X)\\&-(n-1)!2\left(g(3)-g(2)^{2}\right)\binom{n}{2}P_{n-1}^{g}(X)\end{aligned}$$

can only be 0 if $g(2)^{3}-2g(2)\,g(3)+g(4)=0$. □

Example 3. For $g(n)=\sigma(n)$, we obtain $27-24+7=10\neq 0$.

Proof of Lemma 3. From the recursive definition (1) we obtain

$$\begin{aligned}\sum_{m=0}^{n}A_{n,m}X^{m}&=P_{n}^{g}(X)=\frac{X}{n}\sum_{k=1}^{n}g(k)\,P_{n-k}^{g}(X)\\&=\frac{1}{n}\sum_{k=1}^{n}g(k)\sum_{m=0}^{n-k}A_{n-k,m}X^{m+1}\\&=\frac{1}{n}\sum_{m=1}^{n}\sum_{k=1}^{n-m+1}g(k)\,A_{n-k,m-1}X^{m}.\end{aligned}$$

In particular $A_{n,n}=\frac{1}{n}A_{n-1,n-1}=\frac{1}{n!}$. (This follows by induction. In the cases of the other coefficients below also induction is applied.) Further,

$$\begin{aligned}A_{n,n-1}&=\frac{1}{n}\left(A_{n-1,n-2}+g(2)\,A_{n-2,n-2}\right)=\frac{1}{n}\left(A_{n-1,n-2}+\frac{g(2)}{(n-2)!}\right)\\&=\frac{1}{n!}\left((n-1)!A_{n-1,n-2}+g(2)\,(n-1)\right)=\frac{g(2)}{n!}\binom{n}{2},\end{aligned}$$

$$\begin{aligned}A_{n,n-2}&=\frac{1}{n}\left(A_{n-1,n-3}+g(2)\,A_{n-2,n-3}+g(3)\,A_{n-3,n-3}\right)\\&=\frac{1}{n}\left(A_{n-1,n-3}+\frac{g(2)^{2}}{(n-2)!}\binom{n-2}{2}+\frac{g(3)}{(n-3)!}\right)\\&=\frac{1}{n!}\left((n-1)!A_{n-1,n-3}+3g(2)^{2}\binom{n-1}{3}+2g(3)\binom{n-1}{2}\right)\\&=\frac{1}{n!}\left(3g(2)^{2}\binom{n}{4}+2g(3)\binom{n}{3}\right),\end{aligned}$$

$$\begin{aligned}
A_{n,n-3} &= \frac{1}{n}\left(A_{n-1,n-4} + g(2)\,A_{n-2,n-4} + g(3)\,A_{n-3,n-4} + g(4)\,A_{n-4,n-4}\right) \\
&= \frac{1}{n}\left(A_{n-1,n-4} + \frac{g(2)}{(n-2)!}\left(3g(2)^2\binom{n-2}{4} + 2g(3)\binom{n-2}{3}\right)\right. \\
&\quad \left. + g(3)\,\frac{g(2)}{(n-3)!}\binom{n-3}{2} + \frac{g(4)}{(n-4)!}\right) \\
&= \frac{1}{n!}\left((n-1)!A_{n-1,n-4} + 15g(2)^3\binom{n-1}{5} + 8g(2)\,g(3)\binom{n-1}{4}\right. \\
&\quad \left. + 12g(2)\,g(3)\binom{n-1}{4} + 6g(4)\binom{n-1}{3}\right) \\
&= \frac{1}{n!}\left(15g(2)^3\binom{n}{6} + 20g(2)\,g(3)\binom{n}{5} + 6g(4)\binom{n}{4}\right).
\end{aligned}$$

□

6. Open Question

We proved that the polynomials $P_n^g(X)$ attached to $g(n) = n$ are irreducible and have real roots. Moreover, the roots interlace. Actually we were able to identify these polynomials with certain Laguerre polynomials which satisfy a three term recurrence relation. As indicated in the introduction, our main interest are the roots of $P_n^\sigma(X)$, since these roots dictate the vanishing properties of powers the Dedekind eta function [HNR17, HLN18], including the Lehmer conjecture [Le47]. Since $n \leq \sigma(n) \leq n^2$ properties of $P_n^\sigma(X)$ are expected to be deduced from the polynomials attached to $g(n) = n$ and $g(n) = n^2$ (as for example the size of the coefficients of $P_n^\sigma(X)$). We have for $g(n) = n^2$:

$$\widetilde{P}_n(X) = \sum_{k=0}^{n-1} A_{n,k}\frac{X^k}{(k+1)!}, \qquad \text{where } A_{n,k} = \binom{n+k}{2k+1} \in \mathbb{Z}. \tag{9}$$

We have shown in this paper that these polynomials satisfy a four term recursion and that they are not orthogonal. Numerical calculations for $n \leq 100$ indicate that the polynomials are also irreducible. For $n \leq 100$ we have checked that they have real roots and that the roots interlace. We end with this observation and ask the question of whether these facts hold for general n.

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