# NON-PERTURBATIVE $\beta$-FUNCTIONS VIA FEYNMAN GRAPHONS

ALI SHOJAEI-FARD

ABSTRACT. We show the existence of a new class of $\beta$-functions which can govern the running of strong coupling constants in gauge field theories.

CONTENTS



## 1. INTRODUCTION

The most difficult challenge in non-abelian gauge field theories with strong (bare or running) coupling constants such as Quantum Chromodynamics (QCD) is dealing with complicated infinities originated from Green's functions which encode quantum motions in these physical theories. Whenever a process involves energies of the order of (or lower than) $\Lambda_{\mathrm{QCD}}$, the gauge coupling becomes strong enough which invalid the applications of perturbative methods. The lattice (numerical) simulations can help us to determine some critical boundaries of temperatures around $170 \sim 180$ MeV where we can achieve quark-gluon plasma phase which is strongly coupled around the critical temperature. The lattice methods are not actually suitable for the study of real time dynamical processes. The AdS/CFT correspondence has already provided some new connections between non-perturbative aspects of non-abelian gauge filed theories and String Theory where the large $N$ limit of gauge theories should have a strongly coupled phase described perturbatively in terms of closed strings. [3, 6, 8, 14, 15]

In physical theories with strong coupling constants, it is already impossible to study the full behavior of quantum systems under perturbation series and therefore we need to apply non-perturbative methods such as numerical methods, Borel summation method, theory of instantons and lattice models. In addition, Green's functions, as the fundamental tools in Quantum Field Theory, enable us to organize a collection of all Feynman diagrams to all loop orders with a specified set of external edges. The self-similar nature of Green's functions can help us to study non-perturbative aspects in the context of fixed point equations of Green's functions. The resulting equations, which are known as Dyson–Schwinger equations, contain a collection of coupled integral equations depended on the coupling constant. In couplings more than or equal to 1, these equations behave non-perturbatively. In QCD with higher energies, we can expect the asymptotic freedom behavior which enables us to make computations via some advanced perturbative tools such as many loop techniques but in QCD with relatively lower energies, the story is so complicated. Search for

*Key words and phrases.* Non-perturbative Renormalization Group methods; Dyson–Schwinger equations; Feynman graphons.

Preprint for MPIM Library. Published version: https://www.worldscientific.com/doi/abs/10.1142/S0217732319501098.

the description of the phenomenology of running couplings in QCD, as a top priority in High Energy Physics, can improve our knowledge for the computation and the analysis of non-perturbative parameters. [1, 5, 6, 7, 8, 15, 16]

Thanks to the applications of the Connes–Kreimer renormalization Hopf algebra of Feynman diagrams to Quantum Field Theory together with other mathematical tools, we have a combinatorial reformulation for (non-linear) Dyson–Schwinger equations in the language of Hochschild cohomology theory. The linear version of these equations can be appeared in the limit of a vanishing $\beta$-function. [2, 4, 9, 11, 12, 20, 21, 22]. The unique solution of each non-linear equation DSE determines a free commutative non-cocommutative connected graded Hopf subalgebra of the renormalization Hopf algebra. If the $\beta$-function is vanishing, then Dyson–Schwinger equations can be reduced to a linear set of equations where the solutions of these equations can generate cocommutative Hopf subalgebras with the corresponding abelian dual Lie algebra structures. This mathematical approach to Dyson–Schwinger equations has already provided some new combinatorial and geometric tools for the computation of non-perturbative parameters. We can address the foundations of a differential Galois theory and a Tannakian formalism for the study of non-perturbative aspects of Quantum Field Theories [17] which led us to relate infinities of Dyson–Schwinger equations with the complexity of computability of intermediate algorithms in Theory of Computation [18]. In addition, recently we have built a new mathematical model for the description of convergence in sequences of Feynman diagrams via theory of graphons. Graphons have been introduced in infinite Combinatorics to study dense graphs as edge weighted graphs on a continuum of points. The rooted tree representations of Feynman diagrams in a given sequence of these graphs have been applied to associate a new class of dense graphs with respect to the graph limits of Feynman diagrams. This method provided a new description for solutions of Dyson–Schwinger equations which has led us to improve the formulation of a renormalization machinery for these non-perturbative type of equations [19].

In this work we plan to show a new application of graphons for the computation of physical parameters derived from non-perturbative parts of gauge field theories. We describe the non-perturbative version of the Connes–Kreimer renormalization group via Feynman graphons to obtain a new formulation for the $\beta$-functions at this level which are capable to govern the running of strong coupling constants.

## 2. Topological renormalization Hopf algebra of Feynman graphons

Graphons or bigraphons are bounded measurable (symmetric) functions defined on $\Omega \times \Omega$ with respect to a given probability space $\Omega$ such as $[0, 1]$. These objects have been introduced and applied in Combinatorics for the study of convergence notion for sequences of weighted finite graphs. Measure preserving maps such as $\rho$ allow us to change the label of graphons by $W^\rho(x, y) := W(\rho(x), \rho(y))$. The topology derived from the cut-norm can provide the notion of convergence. It is given by $\| W \|_\square := \sup_{S,T} | \int_{S\times T} W dxdy|$ where $S, T$ are measurable subsets of $[0, 1]$. This norm enables us to define the notion of distance between graphons given by

$$\delta_{\rm cut}(U, W) := \inf_{\rho,\tau}\{d_\square(U^\rho, W^\tau),\ d_\square(U^\rho, W^\tau) :=\| U^\rho - W^\tau \|_\square\} \tag{2.1}$$

such that the infimum is taken over all relabeling of graphons $U, W$. Two labeled graphons $V_1, V_2$ are called weakly equivalent iff there exists a graphon $U$ and Lebesgue measure-preserving maps $\rho, \tau : [0, 1] \longrightarrow [0, 1]$ such that $U^\rho = V_1$ and $U^\tau = V_2$ almost everywhere. We can check that for weakly equivalent labeled graphons $V_1, V_2$, $\delta_{\rm cut}(V_1, V_2) = 0$. Based on this equivalence relation, we can put all relabeled graphons with respect to a given graphon (which are actually weakly equivalent) in a unique equivalence class $[W] := \{W^\rho :\ \rho\}$ known as an unlabeled graphon class. All unlabeled graphon classes can be organized into a compact metric space with respect to the cut-distance topology. [13]

A Feynman graphon is an unlabeled graphon class $[W_\Gamma]$ which contains all labeled graphons corresponding to the rooted tree or forest representation of the finite Feynman diagram $\Gamma$ in a physical theory $\Phi$. We can say a sequence $\{\Gamma_n\}_{n\geq 0}$ of Feynman diagrams is convergent when $n$ goes to infinity, if the corresponding sequence $\{[W_{\Gamma_n}]\}_{n\geq 0}$ of unlabeled graphon classes is cut-distance convergent to the unlabeled graphon class $[W]$ when $n$ goes to infinity. The graphon class $[W]$ is actually the Feynman graphon corresponding to an infinite type graph called large Feynman diagram such as $X_{[W]}$. In this setting, the Feynman graphon corresponding to the unique solution of a given Dyson–Schwinger equation such as

$$\Gamma^{\bar{n}} = 1 + \sum_{\gamma, \mathrm{res}(\gamma)=\bar{n}} \frac{(\lambda g)^{|\gamma|}}{\mathrm{Sym}(\gamma)} B^+_\gamma (X^\gamma_{\mathcal{R}}) \tag{2.2}$$

can be determined as the convergent limit of the sequence of graphon classes of its partial formal sums with respect to the cut-distance topology. In the recursive equation (2.2), $\lambda g$ is the running coupling constant for $0 < \lambda \leq 1$, each $B^+_\gamma$ is the Hochschild closed one-cocycle with respect to the primitive graph $\gamma$ with external legs $\bar{n}$ and $X^\gamma_{\mathcal{R}}$ can be described as a monomial in superficially divergent Green's functions which dress the internal vertices and edges of $\gamma$. Thanks to this investigation, we can enrich the Connes–Kreimer renormalization Hopf algebra of Feynman diagrams to contain solutions of Dyson–Schwinger equations. The enriched space $H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi)$ is the topological Hopf algebra of Feynman diagrams equipped with the topology derived from cut-norm. The distance between finite or infinite Feynman diagrams can be defined via their corresponding unlabeled graphon classes namely, $d(\Gamma_1, \Gamma_2) := \delta_{\mathrm{cut}}([W_{\Gamma_1}], [W_{\Gamma_2}])$. [19]

**Lemma 2.1.** *The topological space of all Feynman graphons (corresponding to Feynman diagrams in a given physical theory $\Phi$) can be equipped with the Hopf algebra structure originated from the renormalization coproduct.*

*Proof.* Feynman graphons can play the role of generators of a free commutative unital algebra. The number of independent loops of Feynman diagrams can be considered to define the graduation parameter on Feynman graphons. For a given finite Feynman diagram $\Gamma$ with the corresponding unlabeled graphon $[W_\Gamma]$, define

$$\Delta_{\mathrm{graphon}}([W_\Gamma]) := \sum [W_\gamma] \otimes [W_{\Gamma/\gamma}] \tag{2.3}$$

such that the sum of unlabeled graphon classes is controlled by disjoint unions of 1PI superficially divergent subgraphs of $\Gamma$. For the unlabeled graphon class $[W_{\mathrm{DSE}}]$ corresponding to the large Feynman diagram $X_{\mathrm{DSE}}$ as the unique solution of the equation DSE, its coproduct can be determined as the convergent limit of the sequence $\{\Delta_{\mathrm{graphon}}([W_{Y_m}])\}_{m\geq 1}$ of the coproducts of the finite partial sums with respect to the cut-distance topology. This gives a connected graded unital counital commutative non-cocommutative bialgebra structure on Feynman graphons. The antipode is the direct result of the graduation parameter and the defined coproduct where we have $S_{\mathrm{graphon}}([W_\Gamma]) = -[W_\Gamma] - \sum S([W_\gamma])[W_{\Gamma/\gamma}]$. For the unlabeled graphon class $[W_{\mathrm{DSE}}]$, its antipode can be determined as the convergent limit of the sequence $\{S_{\mathrm{graphon}}([W_{Y_m}])\}_{m\geq 1}$ of antipodes of unlabeled graphons of finite partial sums $Y_m$ with respect to the cut-distance topology. Since partial sums are finite graphs, their corresponding graphon type antipodes $S_{\mathrm{graphon}}([W_{Y_m}])$ can be obtained inductively by the coproduct $\Delta_{\mathrm{graphon}}$.

The linear property of the coproduct and antipode together with the compactness of the cut-distance topology is enough to achieve the compatibility of this Hopf algebraic structure with the cut-distance topology. □

We use the notation $\mathcal{S}^{\Phi}_{\rm graphon}$ for this renormalization Hopf algebra of Feynman graphons. Now we want to show the existence of a filtration on solutions of fixed point equations which enables us to define a new generalization of the Connes–Kreimer renormalization group for the study of Dyson–Schwinger equations via Feynman graphons.

**Proposition 2.2.** *Feynman rules characters which live in the complex Lie group $\mathbb{G}^{\Phi}_{\rm graphon}(A_{\rm dr})$ of characters of the Hopf algebra $\mathcal{S}^{\Phi}_{\rm graphon}$ can filtrate solutions of Dyson–Schwinger equations.*

*Proof.* Consider the connected graded commutative non-cocommutative quasi-shuffle Hopf algebra $H_{\rm word}$ of words equipped with the product $\ominus_{\Theta}$ given by

$$au \ominus_{\Theta} bv := a(u \ominus_{\Theta} bv) + b(au \ominus_{\Theta} v) + \Theta(a,b)(u \ominus_{\Theta} v) \tag{2.4}$$

with respect to the Hoffman pairing $\Theta$ and the coproduct structure $\Delta_{\rm word}(w) = \sum_{vu=w} u \otimes v$ [10]. The grafting operator, which adds a letter to the first place of each word, determines a class of Hochschild one-cocycles. In addition, it enables us to embed the renormalization Hopf algebra of Feynman diagrams into the Hopf algebra of words via a morphism such as $\nu$ which sends each primitive Feynman graph $\gamma_n$ to a letter $a_n$. It is possible to modify $\nu$ to achieve a new homomorphism $\overline{\nu}$ which embeds the Hopf algebra $\mathcal{S}^{\Phi}_{\rm graphon}$ into the Hopf algebra of words.

Suppose $\mathcal{L}_{\rm word}$ be the Lie algebra corresponding to the Hopf algebra of words and define the decreasing sequence $\mathcal{L}_{\rm word} = \mathcal{L}_1 \geq \mathcal{L}_2 \geq \mathcal{L}_3 \geq ...$ such that $\mathcal{L}_{n+1}$ is generated by all objects $[x,y]$ with $x \in \mathcal{L}_{\rm word}$ and $y \in \mathcal{L}_n$. For letters $a_1, a_2, ..., \Theta(a_1,a_2), ...$, set $x_1, x_2, ..., \Theta(x_1,x_2), ... \in \mathcal{L}_{\rm word}/\mathcal{L}_2$. The linear invertible map $\nu$ can identify the duality between $H_{\rm word}$ and the filtered bialgebra $\mathcal{U}(\mathcal{L}_{\rm word})$ where we have

$$\nu(a_i) = [x_i], \ \ \nu(\Theta(a_i,a_j)) = [\Theta(x_i,x_j)], \ \ \nu(a_i a_j) = [x_i \otimes x_j], .... \tag{2.5}$$

The recursive nature of the (quasi-)shuffle product allows us to build a filtration algorithm by dealing with all words with length $k$ into the lexicographical order in terms of the concatenation commutator with respect to the Hall basis which generates words with the length $k-1$. We need to start with the maximal length of words and repeat the procedure for the full quasi-shuffle products of $k$ corresponding letters. The inverse of the embedding $\overline{\nu}$ allows us to modify this filtration on Feynman graphons in $\mathcal{S}^{\Phi}_{\rm graphon}$.

In addition, the morphism $\nu$ sends the renormalized Feynman rules character $\phi_r$ to a new character $\psi_r = \phi_r \circ \nu^{-1}$ with the property $\psi_r(u \ominus_{\Theta} v) = \psi_r(u).\psi_r(v)$ such that it can send words to the $L$-linear part of the log-expansion of the renormalized Green's functions. For a given Feynman diagram $\Gamma$ with the coradical degree $r_{\Gamma}$, we have $\phi_r(\Gamma) = \sum_{j=1}^{r_{\Gamma}} c_j^{\Gamma} L^j$ such that $c_j^{\Gamma} = c_1^{\otimes j} \tilde{\Delta}^{j-1}_{\rm FG}(\Gamma)$ while $c_1^{\otimes j} : H_{\rm FG}(\Phi) \otimes ...^{j \ \rm times} \otimes H_{\rm FG}(\Phi) \longrightarrow \mathbb{C}$ is a symmetric function depended on scattering angles. We can check that for any word $u$, $\psi_r(u) = \sum_{j=1}^{r_u} d_j^u L^j$ such that $d_j^u = c_j^{\nu^{-1}(u)}$.

If we apply the renormalized Feynman rules character $\phi_r$ to each Feynman diagram which contributes to a given Dyson–Schwinger equation with the general form (2.2), then we can obtain a polynomial in a suitable external scale parameter $L = {\rm log} S/S_0$ such that $S_0$ fixes a reference scale for the renormalization process. In addition, the morphism $\overline{\nu}$ is enough to relate graphon type of representation of Dyson–Schwinger equations with their word versions.

It is shown in [12] that $\psi_r$ maps the shuffle product $u_1 \ominus_{\Theta} ... \ominus_{\Theta} u_n$ to the $L^n$-term in the log-expansion which leads us to filtrate coefficients $X_n$ in the unique solution of each Dyson–Schwinger equation. As the result, the renormalized character $\tilde{\psi}_r := \tilde{\phi}_r \circ \overline{\nu}^{-1}$ can send the formal expansions $\sum_1^m u_{i_1} \ominus_{\Theta} ... \ominus_{\Theta} u_{i_k}$ of shuffle products of words corresponding to the partial sums $Y_m$ of $X_{\rm DSE}$ to a certain term in the expansion $\tilde{\phi}_r([W_{Y_m}]) = \sum_{j=1}^{r_{Y_m}} c_j^{Y_m} L^j$ such that $c_j^{Y_m} = c_1^{\otimes j} \tilde{\Delta}^{j-1}_{\rm graphon}([W_{Y_m}])$. Whenever $m$ tends to infinity, the sequence $\{c_j^{Y_m}\}_{m \geq 1}$ of coefficients converges to $c_j^{X_{\rm DSE}}$ (for each $j$) with respect to the cut-distance topology. Since Feynman rules characters are linear morphisms and the minimal subtraction map $R_{\rm ms}$ is continuous, the sequence $\{\tilde{\phi}_r([W_{Y_m}])\}_{m\geq 1}$ is convergent to $\tilde{\phi}_r([W_{\rm DSE}])$ when $n$ goes to infinity. Now if we apply Lemma 2.1 and the renormalized Feynman rules character $\tilde{\phi}_r$ to $X_{\rm DSE}$, then we can have $\tilde{\phi}_r([W_{\rm DSE}]) = \sum_{j=1}^{r_{X_{\rm DSE}}} c_j^{X_{\rm DSE}} L^j$ such that $c_j^{X_{\rm DSE}} = c_1^{\otimes j} \widetilde{\Delta}^{j-1}_{\rm graphon}([W_{\rm DSE}])$.

Set $\mathcal{S}^{\Phi,(i)}$ as the vector space generated by some Feynman graphons derived from Dyson–Schwinger equations such that these graphons are filtered in terms of the canonical filtration on their corresponding words. It means that the filtration $(i)$ is defined by applying $\overline{\nu}$ and $\tilde{\psi}_r$ while the associated words map to a similar term $i$ in the associated log-expansion. We have $\mathcal{S}^{\Phi,(0)} \subseteq \mathcal{S}^{\Phi,(1)} \subseteq ... \subseteq \mathcal{S}^{\Phi,(i)} \subseteq ... \subseteq \bigcup_{n\geq 0} \mathcal{S}^{\Phi,(n)} \subset \mathcal{S}^{\Phi}_{\rm graphon}$ as the resulting filtration on all Feynman graphons which contribute to solutions of Dyson–Schwinger equations and their partial sums. □

## 3. Non-perturbative Connes–Kreimer renormalization group via Feynman graphons

In this section we show that the topological Hopf algebra $\mathcal{S}^{\Phi}_{\rm graphon}$ can provide a new non-perturbative Renormalization Group for graph limits of sequences of Feynman diagrams.

Since $\mathcal{S}^{\Phi}_{\rm graphon}$ is graded via the number of independent loops of Feynman diagrams, the corresponding complex infinite dimensional Lie group $\mathbb{G}^{\Phi}_{\rm graphon}(\mathbb{C}) = {\rm Hom}(\mathcal{S}^{\Phi}_{\rm graphon}, \mathbb{C})$ can be described in terms of (algebraic) groups dual to free generated commutative algebras $\mathcal{S}^{\Phi}_i$ such that for all $i \geq 0:\ \Delta_{\rm graphon}(\mathcal{S}^{\Phi}_i) \subset \mathcal{S}^{\Phi}_i \otimes \mathcal{S}^{\Phi}_i,\ S_{\rm graphon}(\mathcal{S}^{\Phi}_i) \subset \mathcal{S}^{\Phi}_i$ and for all $i, j \geq 0$, there exists some $k > 0$ such that $\mathcal{S}^{\Phi}_i \cup \mathcal{S}^{\Phi}_j \subset \mathcal{S}^{\Phi}_k$.

**Lemma 3.1.** *The infinite dimensional complex Lie group $\mathbb{G}^{\Phi}_{\rm graphon}(\mathbb{C})$ of characters on Feynman graphons can be represented by the group ${\rm Diff}(V)$ of formal diffeomorphisms of $V$ tangent to the identity at $0 \in V$ for a given complex vector space $V$ with a basis labeled by running couplings of a given physical theory $\Phi$.*

*Proof.* We consider $g_0 = x + \sum_{n\geq 2} \alpha_n x^n$ as a representation of the bare coupling constant in terms of a power series in $x$ such that the coefficients $\alpha_{2n}$ are zero and the coefficients $\alpha_{2n+1}$ are finite linear combinations of products of Feynman graphons $[W_\Gamma]$ associated to finite Feynman diagrams such as $\Gamma$.

Set $H_{\rm diff}(V)$ as the Hopf algebra corresponding to the Lie group ${\rm Diff}(V)$. It has generators such as $f_n$ which contribute as coordinates of formal diffeomorphisms $\phi(x) = x + \sum_{n\geq 2} f_n(\phi) x^n$ such that $\phi(0) = 0,\ \phi'(0) = {\rm id}$. Its coproduct is given by $\Delta(f_n)(\phi_1 \otimes \phi_2) = f_n(\phi_2 \circ \phi_1)$.

We can define a unique algebra homomorphism $\Psi$ from $H_{\rm diff}(V)$ to the algebra generated by Feynman graphons $[W_\Gamma]$ which sends each $f_n$ to $\alpha_n$ and it has the property $(\Psi \otimes \Psi)\Delta\phi(x) = \Delta_{\rm graphon}\Psi(\phi(x))$.

For a given Dyson–Schwinger equation DSE with the corresponding Hopf subalgebra $H_{\rm DSE}$ and Lie subgroup $\mathbb{G}_{\rm DSE}(\mathbb{C})$, there exists a morphism $\Psi_{\rm DSE}$ of Hopf algebras from $H_{\rm diff}(V)$ to $H_{\rm DSE}$ with the corresponding group homomorphism $\hat{\Psi}_{\rm DSE}$ from $\mathbb{G}_{\rm DSE}(\mathbb{C})$ to ${\rm Diff}(V)$ [17]. Now by embedding $H_{\rm FG}(\Phi)$ inside $\mathcal{S}^{\Phi}_{\rm graphon}$, we can extend $\Psi$ to a new Hopf algebra homomorphism $\overline{\Psi} : H_{\rm diff}(V) \longrightarrow \mathcal{S}^{\Phi}_{\rm graphon}$ with the corresponding dual group homomorphism $\hat{\overline{\Psi}} : \mathbb{G}^{\Phi}_{\rm graphon}(\mathbb{C}) \longrightarrow {\rm Diff}(V)$. □

**Proposition 3.2.** *The pro-unipotent complex Lie group $\mathbb{G}^{\Phi}_{\rm graphon}(A_{\rm dr})$ encodes the renormalization machinery of Dyson–Schwinger equations under dimensional regularization and minimal subtraction scheme.*

*Proof.* We can describe unrenormalized regularized Feynman rules characters in $\mathbb{G}^{\Phi}_{\rm graphon}(\mathbb{C})$ in terms of the space of loops such as $\tilde{\gamma}_\mu$ on the infinitesimal punctured disk $\mathbf{\Delta}^*$ around the origin in the complex plane (identified by the regularization parameter) with values in $\mathbb{G}^{\Phi}_{\text{graphon}}(\mathbb{C})$. If we apply the Rota–Baxter property of $(A_{\text{dr}}, R_{\text{ms}})$, then we can achieve the unique Birkhoff factorization $(\tilde{\gamma}_-, \tilde{\gamma}_+)$ which can be modified for the level of Feynman rules characters $(\tilde{\phi}_-, \tilde{\phi}_+)$ such that $\tilde{\phi}^z([W_\Gamma]) := \phi^z(\Gamma)$. The twisted antipode $S^{\tilde{\phi}}_{R_{\text{ms}}}$ encodes the application of the minimal subtraction map in the BPHZ machinery. We can also define it for Feynman graphons on the basis of the coproduct $\Delta_{\text{graphon}}$ (Lemma 2.1) by $S^{\tilde{\phi}}_{R_{\text{ms}}}([W_\Gamma]) := -R_{\text{ms}}[\tilde{\phi}([W_\Gamma]) + \sum S^{\tilde{\phi}}_{R_{\text{ms}}}([W_\gamma])\tilde{\phi}([W_{\Gamma/\gamma}])]$. The pair $(A_{\text{dr}}, R_{\text{ms}})$ encodes dimensional regularization and minimal subtraction scheme where the idempotent linear map $R_{\text{ms}}$, which has the Rota–Baxter property, is the projection on the finite pole parts of Laurent series in $A_{\text{dr}}$. The existence of the Birkhoff factorization on $A_{\text{dr}}$ (i.e. $A_{dr} = A_- \oplus A_+$) shows us that the map $R_{\text{ms}}$, which is the projection on the subalgebra $A_-$, can be specified uniquely by its kernel $\text{Ker}(R_{\text{ms}}) = A_+$ [2]. It guarantees the continuity of the map $R_{\text{ms}}$ which means that the twisted antipode is also continuous.

For a given Dyson–Schwinger equation DSE with the corresponding sequence $\{Y_m\}_{m\geq 1}$ of partial sums, which is convergent to the unique solution $X_{\text{DSE}}$ with respect to the cut-distance topology [19], the sequence $\{S^{\tilde{\phi}}_{R_{\text{ms}}}([W_{Y_m}])\}_{m\geq 1}$ is also convergent where thanks to the continuity of the twisted antipode we have

$$S^{\tilde{\phi}}_{R_{\text{ms}}}([W_{\text{DSE}}]) = \lim_{m\to\infty} S^{\tilde{\phi}}_{R_{\text{ms}}}([W_{Y_m}]) = \lim_{m\to\infty} \sum_{i=1}^{m} S^{\tilde{\phi}}_{R_{\text{ms}}}([W_{X_i}])$$

$$= \lim_{m\to\infty} \sum_{i=1}^{m} -R_{\text{ms}}(\tilde{\phi}([W_{X_i}]) - R_{\text{ms}}(\sum S^{\tilde{\phi}}_{R_{\text{ms}}}([W_\gamma])\tilde{\phi}([W_{X_i/\gamma}]))). \tag{3.1}$$

The expression $S^{\tilde{\phi}}_{R_{\text{ms}}}([W_{\text{DSE}}])$ is a graphon representation for the counterterm generated from the renormalization of $X_{\text{DSE}}$. In addition, the sequence $\{S^{\tilde{\phi}}_{R_{\text{ms}}} * \tilde{\phi}([W_{Y_m}])\}_{m\geq 1}$ of Feynman graphons is also convergent where we have

$$S^{\tilde{\phi}}_{R_{\text{ms}}} * \tilde{\phi}([W_{\text{DSE}}]) = \lim_{m\to\infty} S^{\tilde{\phi}}_{R_{\text{ms}}} * \tilde{\phi}([W_{Y_m}]) = \lim_{m\to\infty} \sum_{i=1}^{m} S^{\tilde{\phi}}_{R_{\text{ms}}} * \tilde{\phi}([W_{X_i}]). \tag{3.2}$$

The expression $S^{\tilde{\phi}}_{R_{\text{ms}}} * \tilde{\phi}([W_{\text{DSE}}])$ is a graphon representation for the renormalized value generated from the renormalization of $X_{\text{DSE}}$. □

**Corollary 3.3.** *The graphon representation of Feynman diagrams determines a non-perturbative Renormalization Group machinery for solutions of Dyson–Schwinger equations under running couplings $\lambda g$.*

*Proof.* The filtration defined by Proposition 2.2 allows us to define a new one-parameter group $\{\hat{\theta}_t\}_t$ of automorphisms on $\mathbb{G}^{\Phi}_{\text{graphon}}(A_{\text{dr}})$ with the infinitesimal generator $\frac{d}{dt}|_{t=0}\hat{\theta}_t = \hat{Y}$ such that $\hat{Y}$ sends each Feynman graphon $[W_\Gamma] \in \mathcal{S}^{\Phi,(n)}$ to its corresponding filtration rank $n_{[W_\Gamma]} = n$. In other words, $\hat{Y}([W_\Gamma]) := n[W_\Gamma]$, for each real number $t$, $\hat{\theta}_t([W_\Gamma]) = e^{nt}[W_\Gamma]$, and for each Feynman rules character $\tilde{\phi}$, $< \hat{\theta}_t(\tilde{\phi}), [W_\Gamma] >:=< \tilde{\phi}, \hat{\theta}_t([W_\Gamma]) >$. For the loop $\tilde{\gamma}_\mu$ which encodes the regularized unrenormalized Feynman rules character $\tilde{\phi}$, one should have $\tilde{\gamma}_{e^t\mu}(z) = \hat{\theta}_{tz}(\tilde{\gamma}_\mu(z))$ which leads us to show that the limit $\tilde{\gamma}_-(z)\hat{\theta}_{tz}(\tilde{\gamma}_-(z)^{-1})$ is convergent (when $z$ goes to 0) to a character such as $\tilde{F}_t$ in $\mathbb{G}^{\Phi}_{\text{graphon}}(A_{\text{dr}})$ such that for each Feynman graphon $[W_\Gamma]$, $\tilde{F}_t([W_\Gamma])$ is a polynomial in $t$. The minus part of the Birkhoff factorization of the loop $\tilde{\gamma}_\mu$ is determined via the twisted antipode which is continuous. As the output, we obtain the one-parameter topological subgroup $\{\tilde{F}_t\}_t$ of $\mathbb{G}^{\Phi}_{\text{graphon}}(A_{\text{dr}})$ (with respect to the topology of simple convergence) such that for each real value $t$, $\tilde{\gamma}_{e^t\mu^+}(0) = \tilde{F}_t\tilde{\gamma}_{\mu^+}(0)$.

For a given Dyson–Schwinger equation DSE with the corresponding sequence $\{Y_m\}_{m\geq 1}$ of partial sums, which is convergent to the unique solution $X_{\rm DSE} = \sum_{n\geq 0}(\lambda g)^n X_n$ with respect to the cut-distance topology [19], the sequence $\{\tilde{F}_t([W_{Y_m}])\}_{m\geq 1}$ is convergent to $\tilde{F}_t([W_{\rm DSE}])$ for each $t$. □

The running of coupling constants in QCD have been experimentally studied usually in the preturbation domain [3, 6, 7]. The Gluonic self-coupling and color confinement make so complications for the study of the non-abelian QCD coupling at small momentum transfer where having a concrete description from non-perturbative effective couplings can be addressed as one important progress in this setting [1, 5, 8, 15, 16, 20]. Thanks to the graphon representation of Feynman diagrams now we have a new possibility to deal with strong running copulings and Dyson–Schwinger equations at this non-perturbative setting where we can show that the running of strong coupling constants can be governed via $\beta$-functions generated by the non-perturbative Renormalization Group $\{\tilde{F}_t\}_t$. In other words, the graphon model renormalization program of Dyson–Schwinger equations (Proposition 3.2) and the filtration of these non-perturbative equations (Proposition 2.2) allow us to apply the infinitesimal generator of the Renormalization Group $\{\tilde{F}_t\}_t$ (Corollary 3.3) as a new parameter for the study of non-perturbative running couplings.

**Corollary 3.4.** *For the given loop $\tilde{\gamma}_\mu$ with values in $\mathbb{G}^{\Phi}_{\rm graphon}(\mathbb{C})$ which encodes the regularized Feynman rules characters acting on solutions of Dyson–Schwinger equations, there exists a $\beta$-function $\tilde{\beta}$ which controls the running of strong couplings.*

*Proof.* Thanks to Proposition 3.2, Corollary 3.3 and Theorem 1.58 in [4], the negative component of the Birkhoff factorization of the loop $\tilde{\gamma}_\mu$ determines the unique element $\tilde{\beta}$ in $\mathfrak{g}^{\Phi}_{\rm graphon}(\mathbb{C})$ which can be presented in terms of the time ordered exponential by

$$\tilde{\gamma}_-(z) = T\exp(\frac{-1}{z}\int_0^\infty \hat{\theta}_{-t}(\tilde{\beta})dt). \tag{3.3}$$

If we work on the geometric setting for the study of Dyson–Schiwnger equations built in [16, 17, 18] and also apply their graphon representation given in [19], then it is possible to show that $\tilde{\beta}$ can determine a unique class $\omega_{\tilde{\beta}}$ of flat equi-singular $\mathbb{G}^{\Phi}_{\rm graphon}(\mathbb{C})$-connections on the trivial principal bundle $P^0_{\rm graphon} := \mathbf{\Delta} \times \mathbb{C}^* - \pi^{-1}(\{0\}) \times \mathbb{G}^{\Phi}_{\rm graphon}(\mathbb{C})$ derived from the regularization process. This correspondence can be presented in terms of the differential equation $\mathbf{D}\tilde{\gamma}_\mu = \omega_{\tilde{\beta}}$ such that

$$\tilde{\gamma}_\mu(z,v) = T\exp(\frac{-1}{z}\int_0^v u^{\hat{Y}}(\tilde{\beta})\frac{du}{u}) \tag{3.4}$$

where $u = tv,\ t \in [0,1]$ and $u^{\hat{Y}}$ is the action of $\mathbb{G}_m$ on $\mathbb{G}^{\Phi}_{\rm graphon}(\mathbb{C})$.

Consider an equation DSE (in the running coupling $\lambda g$) with the unique solution $X_{\rm DSE} = \sum_{n\geq 0}(\lambda g)^n X_n$ and the corresponding sequence $\{Y_m\}_{m\geq 1}$ of partial sums. For each $m$, $Y_m := X_0 + (\lambda g)^1 X_1 + ... + (\lambda g)^m X_m$ is the disjoint union of finite number of Feynman diagrams where if we apply Corollary 3.3, then we can observe that the limit of the holomorphic function

$$z < \tilde{\gamma}_-(z)^{-1} \otimes \tilde{\gamma}_-(z)^{-1}, (S_{\rm graphon} \otimes \hat{Y})\Delta_{\rm graphon}([W_{Y_m}]) > \tag{3.5}$$

is $\frac{d}{dt}\tilde{F}_t|_{t=0}([W_{Y_m}])$ when $z$ tends to 0 while $\hat{Y}(\tilde{\gamma}_-(\infty)) = 0$. The continuity of the coproduct and antipode of the topological Hopf algebra $\mathcal{S}^{\Phi}_{\rm graphon}$ with respect to the cut-distance topology enable us to show that whenever $m$ goes to infinity the limit of the holomorphic function

$$z < \tilde{\gamma}_-(z)^{-1} \otimes \tilde{\gamma}_-(z)^{-1}, (S_{\rm graphon} \otimes \hat{Y})\Delta_{\rm graphon}([W_{\rm DSE}]) > \tag{3.6}$$

is $\frac{d}{dt}\tilde{F}_t|_{t=0}([W_{\rm DSE}])$ while $z$ tends to $0$. It means that

$$< \tilde{\gamma}_-(z)\hat{Y}(\tilde{\gamma}_-(z)^{-1}), [W_{\rm DSE}] >= \frac{1}{z}\frac{d}{dt}\tilde{F}_t|_{t=0}([W_{\rm DSE}]) \quad (3.7)$$

which leads us to the equation $\hat{Y}(\tilde{\gamma}_-(z)^{-1}) = \frac{1}{z}\tilde{\gamma}_-(z)^{-1}\frac{d}{dt}\tilde{F}_t|_{t=0}$ for all Feynman graphons. The infinitesimal generator $\frac{d}{dt}\tilde{F}_t|_{t=0}$ of the non-perturbative Renormalization Group $\{\tilde{F}_t\}_t$ is our promising $\beta$-function. It is actually the unique element in the complex Lie algebra $\mathfrak{g}^{\Phi}_{\rm graphon}(\mathbb{C})$ which controls the residue of $\tilde{\gamma}_\mu$. □


## ACKNOWLEDGMENTS

The author is grateful to Max Planck Institute for Mathematics for the support and hospitality.


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MAX PLANCK INSTITUTE FOR MATHEMATICS, VIVATSGASSE 7, 53111 BONN, GERMANY.

ORCID ID: 0000-0002-6418-3227

*E-mail address*: shojaeifa@yahoo.com shojaei-fard@mpim-bonn.mpg.de