



Metric systolicity and two-dimensional Artin groups

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Abstract

We introduce the notion of metrically systolic simplicial complexes. We study geometric and large-scale properties of such complexes and of groups acting on them geometrically. We show that all two-dimensional Artin groups act geometrically on metrically systolic complexes. As direct corollaries we obtain new results on two-dimensional Artin groups and all their finitely presented subgroups: we prove that the Conjugacy Problem is solvable, and that the Dehn function is quadratic. We also show several large-scale features of finitely presented subgroups of two-dimensional Artin groups, lying background for further studies concerning their quasi-isometric rigidity.

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1 Introduction

Artin groups are among most intensively studied classes of groups in Geometric Group Theory. Conjecturally, they possess nice geometric, topological, algebraic, and algorithmic properties, but most of such features are established only for rather restricted subclasses. Even in the case of two-dimensional Artin groups such basic questions as solvability of the Conjugacy Problem or the form of the Dehn function have remained open. One, conjectural, approach to many questions concerning Artin groups is showing that they act geometrically on CAT(0) spaces. Such results were established only for a number of rather limited subclasses of Artin groups, for: right-angled Artin groups (RAAGs) [12]; certain classes of 2-dimensional Artin groups [3,7]; Artin groups of finite type with three generators [9]; 3-dimensional Artin groups of type FC [4]; spherical Artin groups of type A_4 and B_4 [8]; 6-strand braid group [30]. Another method of treating Artin groups is finding other non-positive-curvature-like structures describing them. Such approach was successfully carried out e.g. in [1,2,5,42,43]. In [33] the authors undertake similar path showing that Artin groups of large type are systolic, that is, simplicially non-positively curved. This allowed to prove many new results about such groups. In the current article we exhibit a non-positive-curvature-like structure of all two-dimensional Artin groups and all their finitely presented subgroups, and conclude a number of new algorithmic, and large-scale geometric results for those groups.

As the main tool we introduce a new notion of *metrically systolic* simplicial complex. Roughly speaking, a simply connected flag simplicial complex with a piecewise Euclidean metric on its 2-skeleton is metrically systolic if all essential loops in links of vertices have (angle) length at least 2π (see Sect. 2 for details). This definition may be treated as a metric analogue of the definition of *systolic complex* (see e.g. [17,29,33,35]). Our main tool for exploring features of metrically systolic complexes is the use of disc diagrams. It allows us to prove the following results about metrically systolic complexes and groups acting on them geometrically, that is, *metrically systolic* groups.

Theorem 1.1 *Let X be a metrically systolic complex, and let G be a metrically systolic group. Then the following properties hold.*

- (1) *Every loop in X bounds a CAT(0) disc diagram (see Theorems 2.6 and 2.8 in the text).*
- (2) *The Dehn functions of X and G are quadratic (see Corollary 2.7).*

- (3) *Finitely presented subgroups of G are metrically systolic (see Theorem 3.1).*
- (4) *If G is torsion-free and g^m is conjugated to g^n only when $g^n = g^m$, for every $g \in G$, then the Conjugacy Problem is solvable in G (see Theorem 3.6).*
- (5) *X and G have constant filling radius for 2-spherical cycles (see Theorem 3.7 and Corollary 3.8).*
- (6) *Morse Lemma for 2-dimensional quasi-discs in X (see Theorem 3.9).*

We believe that metrically systolic complexes deserve further extensive studies on their own; see a list of open questions in Sect. 7. Geometrically, metric systolicity enables us to formalize a weaker notion of non-positively curved space where one only requires every minimal filling disc of a 1-cycle to be non-positively curved. This naturally arises by examining the geometry of 2-dimensional Artin groups. It is interesting to compare this with the work of Petrunin and Stadler [44], where (roughly speaking) they showed any minimal disc in a $CAT(0)$ space is $CAT(0)$. Thus it is natural to wonder whether one can set up this weaker notion in a more analytical way and apply it to natural classes of examples.

In the current paper we focus on the use of metric systolicity in the context of Artin groups. To this end, starting with the standard Cayley complex for a 2-dimensional Artin group G , we modify it to obtain a metrically systolic G -complex. Therefore, we conclude the following.

Theorem 1.2 (Theorem 6.1) *Two-dimensional Artin groups are metrically systolic.*

We refer to the next section for an intuitive explanation of the construction of the complex, as well as comparison with our previous work on constructing systolic complexes for large-type Artin groups from [33].

Direct consequences of Theorems 1.1 and 1.2 are new results on 2-dimensional Artin groups and their subgroups listed in Corollary 1.3. Let us note that even if 2-dimensional Artin groups were $CAT(0)$, this, a priori, would not say anything about their finitely presented subgroups—this suggests an important advantage of metric systolicity. Moreover, by Brady and Crisp [3], there are 2-dimensional Artin groups which can not act nicely on 2-dimensional $CAT(0)$ complexes. On the other hand, metric systolicity enables us to stay in the 2-dimensional world—one need to study only $CAT(0)$ disc diagrams. This will be convenient for our further work in [34] concerning quasi-isometries of 2-dimensional Artin groups.

Corollary 1.3 *Let G be a finitely presented subgroup of a 2-dimensional Artin group. Then:*

- (1) *G has quadratic Dehn function and, in particular, solvable Word Problem;*
- (2) *G has solvable Conjugacy Problem;*
- (3) *G has constant filling radius for 2-spherical cycles;*
- (4) *Morse Lemma for two-dimensional quasi-discs in G holds.*

Dehn function, Word Problem, and Conjugacy Problem are among the most basic notions explored in the context of any group. Still, little was known about them for 2-dimensional Artin groups and their finitely presented subgroups prior to our work.

As far as we know there have been no general results concerning Dehn function for 2-dimensional Artin groups before. Chermak [18] proved the Word Problem is

solvable for 2-dimensional Artin groups, but no general statement of this type have been known for finitely presented subgroups.

Solvability of the Conjugacy Problem for 2-dimensional Artin groups and their finitely presented subgroups follows directly from Theorem 1.1 (4). It is so because 2-dimensional Artin groups are torsion-free by [13], and their cyclic subgroups are undistorted (see Theorem 1.4 below). Prior to our result solvability of the Conjugacy Problem was established only for a few particular subclasses of Artin groups: braid groups [27], finite type Artin groups [10,15,16,23], large-type Artin groups [1,2], triangle-free Artin groups [43], 3-dimensional Artin groups of type FC [4], certain 2-dimensional Artin groups with 3 generators [3], some Artin groups of Euclidean type [11,24,25,38,39], RAAG’s [14,32,45,47]. In particular, the question about solvability of the Conjugacy Problem has been open for the class of 2-dimensional Artin groups.

Assertions (3) and (4) from Corollary 1.3 could be derived without referring to metric systolicity. However, for the proof of the strong form of (3), as presented in Theorem 3.7 in the text, the use of metric systolicity is very convenient. This result, in turn, is a crucial ingredient in the proof of the Morse Lemma for two-dimensional quasi-discs (see the proof of Theorem 3.9). The latter is an important large-scale feature of metrically systolic complexes, groups, and of 2-dimensional Artin groups.

The metrically systolic complexes constructed in Theorem 1.2, as well as the large-scale features mentioned above, will play fundamental role in the study of quasi-isometric invariants of 2-dimensional Artin groups in our subsequent work [34]. For applications in [34] we need another result, presented in the following theorem. It does not rely on metric systolicity, and follows from known facts, but it seems that it is not present in the literature.

Theorem 1.4 (Theorem 7.7 and Corollary 7.8) *Let A_Γ be a 2-dimensional Artin group. Then*

- (1) *every abelian subgroup of A_Γ is quasi-isometrically embedded;*
- (2) *nontrivial solvable subgroups are either \mathbb{Z} or virtually \mathbb{Z}^2 .*

Comments on the proof of Theorem 1.2. Here we present a rough idea of the construction of metrically systolic complexes for two dimensional Artin groups.

Let Γ be a finite simple graph with each of its edges labeled by an integer ≥ 2 . An Artin group with defining graph Γ , denoted A_Γ , is given by the following presentation. The generators of A_Γ are in one to one correspondence with vertices of Γ , and there is a relation of the form $\underbrace{aba \cdots}_n = \underbrace{bab \cdots}_n$ whenever two vertices a and b are connected

by an edge labeled by n .

An Artin group is of *dimension d* if it has cohomological dimension d . By Charney and Davis [13], A_Γ has dimension ≤ 2 if and only if for any triangle $\Delta \subset \Gamma$ with its sides labeled by p, q, r , we have $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$. In particular, the class of all *large-type* Artin groups, where the label of each edge in Γ is ≥ 3 , is properly contained in the class of Artin groups of dimension ≤ 2 .

Let A_Γ be an Artin group of dimension ≤ 2 and let X_Γ^* be the presentation complex of A_Γ . A natural way to metrize X_Γ^* is to declare each 2-cell in X_Γ^* is a regular polygon in the Euclidean plane. However, if we take 2-cells Π_1 and Π_2 (say, two n -gons) such

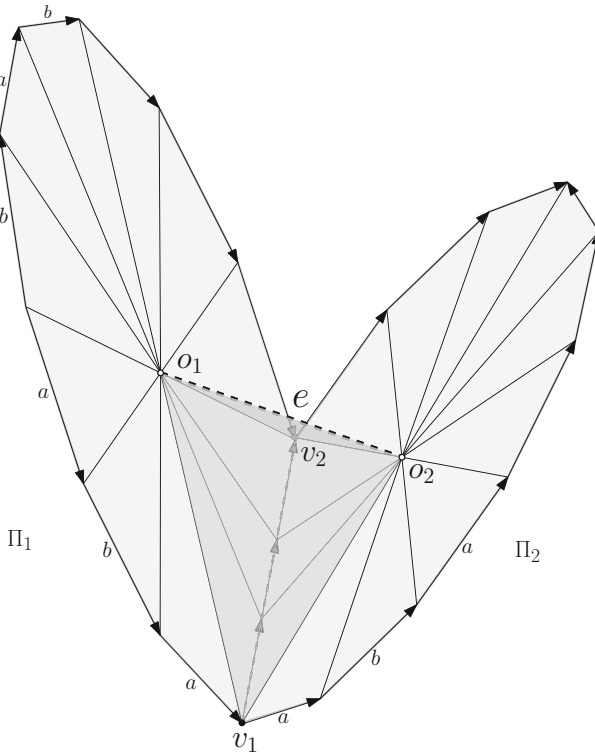


Fig. 1 Adding the edge e and the triangles $\Delta_1 = o_1o_2v_1$, and $\Delta_2 = o_1o_2v_2$ (in the universal cover of X_Γ^*)

that $P = \Pi_1 \cap \Pi_2$ is a path with ≥ 2 edges, then any interior vertex of P is non-positively curved. Let o_i be the center of Π_i and let the two endpoints of P be v_1 and v_2 . Let K be the region in $\Pi_1 \cup \Pi_2$ bounded by the 4-gon whose vertices are o_1, o_2, v_1 and v_2 . Those positively curved corner points are contained in K . Now we add a new edge e between o_1 and o_2 and add two new triangles $\{\Delta_i\}_{i=1}^2$ such that the three sides of Δ_i are $e, \overline{o_1v_i}$ and $\overline{o_2v_i}$; see Fig. 1.

Geometrically, one can think of K as a configuration sitting inside the Euclidean 3-space \mathbb{E}^3 . Then positively curved points in K give rise to corners in the configuration. Now we use the polyhedron bounded by $K \cup \Delta_1 \cup \Delta_2$ to fill these corners. Combinatorially, one can think of $\Delta_1 \cup \Delta_2$ as a replacement of K to get rid of positively curved points in the disc diagram.

Now we decide the length of e . From the geometric viewpoint, e should be shorter if P is longer. From the combinatorial viewpoint, we would like o_i to be flat after we replace K by $\Delta_1 \cup \Delta_2$. Thus $\angle_{o_1}(v_1, o_2) = \angle_{o_1}(o_2, v_2) = \frac{|P|}{4n} 2\pi$ ($|P|$ is the number of edges in P), which determines the length of e .

Pick a triangle $\Delta \subset \Gamma$, then Δ gives rise to three 2-cells arranged in a cyclic fashion around a vertex v . The condition on two dimensionality of A_Γ implies v is already non-positively curved in such configuration, so we do not apply any modifications here.

The main difference between the construction in [33] and the one in this paper is that the former is purely combinatorial, while the current one uses both the metric and combinatorial structure. Thus the method in this paper has more flexibility and applies to a much larger class of Artin groups. Moreover, the structure of flat points in the disc diagrams is more convenient for our later use in [34]. However, since we are now outside the purely combinatorial setting, some results from [33]—e.g. biautomaticity—are much harder to obtain.

Organization of the paper. In Sect. 2 we define metrically systolic complexes and prove their fundamental property—every cycle can be filled by a $CAT(0)$ disc diagram. In Sect. 3, we prove the rest of properties in Theorem 1.1, using $CAT(0)$ disc diagrams as a basic tool. In Sect. 4, we construct the metrically systolic complexes for 2-generated Artin groups. In Sect. 5, we study the local properties of these complexes with two purposes. First we show the complexes in Sect. 4 are indeed metrically systolic. Second we show that there are no local obstructions to metric systolicity if we glue these complexes together under certain conditions. In Sect. 6 we glue the complexes for dihedral Artin groups to construct the metrically systolic complexes for any 2-dimensional Artin groups, and prove Theorem 1.2. In Sect. 7, we prove Theorem 1.4 and leave some open questions about metrically systolic groups and complexes.

2 Metrically systolic complexes

In this section we introduce the notion of metrically systolic complex. Then we show its most important feature, used later extensively for proving other properties of metrically systolic complexes and groups. The feature is the existence of $CAT(0)$ disc diagrams filling any cycle inside the complex; see Theorems 2.6 and 2.8. The proofs presented in Sect. 2.2 go by modifying any given disc diagram to a $CAT(0)$ one by performing a finite sequence of local “moves”. As an immediate consequence we obtain the quadratic Dehn function in Corollary 2.7.

2.1 Definition

Let X be a flag simplicial complex with its two-skeleton $X^{(2)}$ equipped with a metric d in which every 2-simplex (triangle) is isometric to a Euclidean triangle. For a vertex $v \in X$ its *link*, denoted $lk(v, X^{(1)})$, is the full subcomplex (subgraph) of $X^{(1)}$ spanned by all vertices adjacent to v . Every link is equipped with an *angular metric*, defined as follows. For an edge $\overline{v_1 v_2}$, we define the *angular length* of this edge to be the angle $\angle_v(v_1, v_2)$ with the apex v . This turns the link into a metric graph, and the angular metric, which we denote by d_{\angle} , is the path metric of this metric graph (note that a priori we do not know whether $\angle_v(v_1, v_2) = d_{\angle}(v_1, v_2)$ for adjacent vertices v_1 and v_2). The *angular length* of a path ω in the link, which we denote by $\text{length}_{\angle}(\omega)$, is the summation of angular lengths of edges in this path. In this paper we assume that the following weak form of triangle inequality holds for angular length in X : for each $v \in X$ and every three pairwise adjacent vertices v_1, v_2, v_3 in the link of v we have

that $\angle_v(v_1, v_3) \leq \angle_v(v_1, v_2) + \angle_v(v_2, v_3)$. Then we call X (with metric d) a *metric simplicial complex*.

Remark 2.1 Note that we allow that the inequality becomes equality—intuitively it corresponds to degenerate 2-simplices in a link, which corresponds to degenerate 3-simplices in X .

For $k = 4, 5, 6, \dots$, a simple k -cycle C in a simplicial complex is *2-full* if there is no edge connecting any two vertices in C having a common neighbor in C .

Definition 2.2 (*Metrically systolic complexes and groups*) A link in a metric simplicial complex is 2π -*large* if every 2-full simple cycle in the link has angular length at least 2π . A metric simplicial complex X is *locally 2π -large* if every its link is 2π -large. A simply connected locally 2π -large metric complex is called a *metrically systolic complex*. *Metrically systolic groups* are groups acting geometrically by isometries on metrically systolic complexes.

Remark 2.3 A *systolic complex*, that is, a connected simply connected flag simplicial complex for which all full cycles in links consist of at least six edges is metrically systolic when equipped with the metric in which all triangles are Euclidean triangles with edges of unit lengths. For more on systolic complexes see e.g. [17,19,26,29,33,35,36,49].

2.2 CAT(0) disc diagrams

A standard reference for singular disc diagrams (or van Kampen diagrams) is [37, Chapter V]. Our approach is close to the ones from [17, Section 5] and [35, Section 1]. The material is rather standard, however we need a precise description of diagram modifications for further use.

A *singular disc* D is a simplicial complex isomorphic to a finite connected and simply connected subcomplex of a triangulation of the plane. There is the (obvious) *boundary cycle* for D , that is, a map from a triangulation of 1-sphere (circle) to the boundary of D , which is injective on edges. For a cycle C in a simplicial complex X , a *singular disc diagram for C* is a simplicial map $f : D \rightarrow X$ from a singular disc D , which maps the boundary cycle of D onto C ; see Fig. 2 (left). By the relative simplicial approximation theorem [50], for every cycle in a simply connected simplicial complex there exists a singular disc diagram (cf. also van Kampen's lemma e.g. in [37, pp. 150–151]). Below we describe how to obtain singular disc diagrams with some additional properties, by modifying a given one.

A singular disc diagram is called *nondegenerate* if it is injective on all simplices. It is *reduced* if distinct adjacent triangles (i.e., triangles sharing an edge) are mapped into distinct triangles. The *area* of a singular disc diagram is the number of 2-simplices (triangles) in the underlying singular disc. A singular disc diagram for a cycle C in X is *minimal* if it has the minimal area among singular disc diagrams for C in X . For a metric simplicial complex X and a nondegenerate singular disc diagram $f : D \rightarrow X$ we equip D with a metric in which $f|_\sigma$ is an isometry onto its image, for every simplex σ in D . Then, f is a *CAT(0) singular disc diagram* if D is CAT(0), that is, if the angular

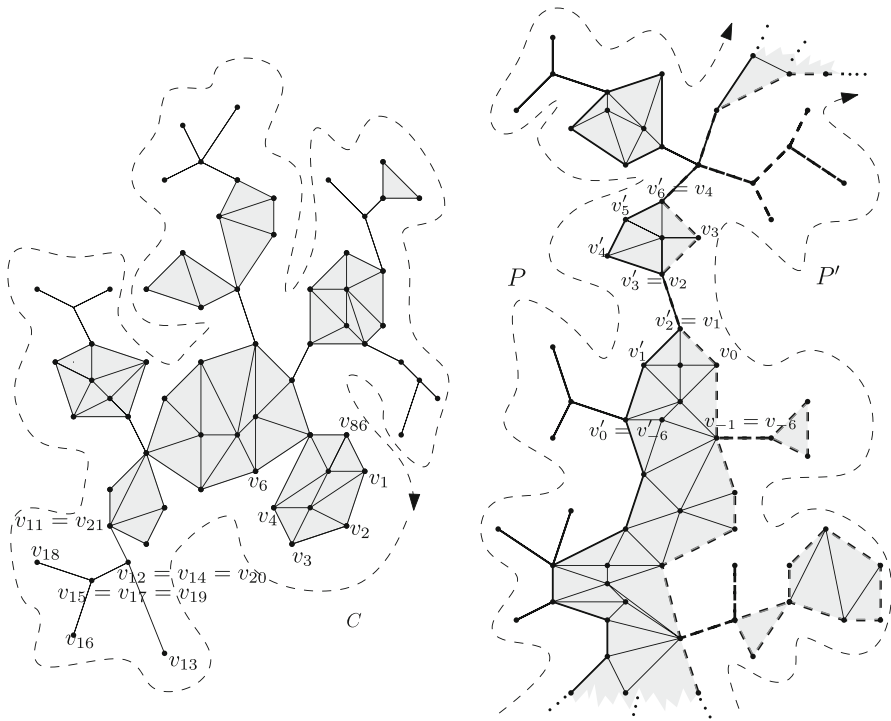


Fig. 2 A singular disc with the boundary cycle $C = (v_1, v_2, \dots, v_{86})$ (left), and a singular strip for a pair $P = (\dots, v_{-1}, v_0, v_1, v_2, \dots)$ (solid thick), $P' = (\dots, v'_{-1}, v'_0, v'_1, v'_2, \dots)$ (dashed thick)

length of every link in D being a cycle (that is, the link of an interior vertex in D) is at least 2π .

Parallely to singular disc diagrams one may consider a related notion of singular strip diagrams. A *singular strip* S is a simplicial complex isomorphic to an infinite connected and simply connected subcomplex of a triangulation of the plane whose complement has two infinite components. The two infinite paths being boundaries of those components are called the *boundary paths* of S . Having two infinite paths P, P' in X , a *singular strip diagram for the pair P, P'* is a simplicial map $f : S \rightarrow X$ from a singular strip S into X mapping boundary paths of S onto, respectively, P and P' ; see Fig. 2 (right). A *nondegenerate, reduced* or *CAT(0) singular strip diagram* is defined analogously as the corresponding singular disc diagram.

Having a singular disc diagram $f : D \rightarrow X$ for a cycle C in X we describe a way of producing a new singular disc diagram $f' : D' \rightarrow X$ for C , with some additional properties (see e.g. Theorem 2.6 below). In order to do this we need elementary operations—*moves*—described below.

A-move: Assume there exist pairwise adjacent vertices u, v, w not bounding a triangle in D , that is, there are vertices v_1, \dots, v_k in the region in D bounded by edges between u, v, w . The new singular disc D' is obtained from D by removing all the vertices v_i (and hence also edges containing them); see Fig. 3 (at the top). The new map

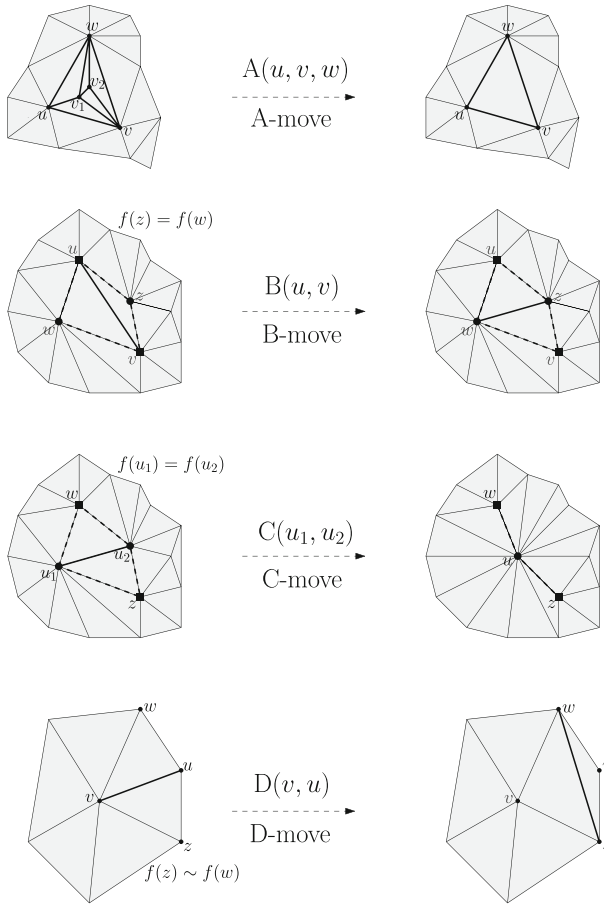


Fig. 3 Moves

$f': D' \rightarrow X$ is defined as $f'(x) = f(x)$, for all vertices x in D' , and then extended simplicially. Such modification is called the *A-move on u, v, w* and is denoted by $A(u, v, w)$.

For the next moves we assume that the situation as above does not happen, that is, each triple of pairwise adjacent vertices defines a triangle in D . In particular it means that for each internal edge \overline{uv} in D there are exactly two vertices w, z each adjacent to both u and v .

B-move: Assume there are two triangles uvw and uvz such that $f(w) = f(z)$. The new singular disc D' is obtained from D by removing the edge \overline{uv} and adding an edge \overline{wz} ; see Fig. 3. By our assumptions D' is a simplicial singular disc. The new map $f': D' \rightarrow X$ is defined as $f'(x) := f(x)$, for all vertices x in D' , and then extended simplicially. Such modification is called *B-move on u, v* and is denoted by $B(u, v)$.

C-move: Assume there is an edge $\overline{u_1u_2}$ such that $f(u_1) = f(u_2)$. Such edge need to be internal, so that there are two triangles u_1u_2w and u_1u_2z containing the edge. The

new singular disc D' is obtained from D by removing u_1, u_2 (and all edges containing them), and then adding a new vertex u adjacent to all vertices (of D except u_1, u_2) that are adjacent in D to u_1 or u_2 ; see Fig. 3. By our assumptions D' is a simplicial singular disc. The new map $f': D' \rightarrow X$ is defined as $f'(x) := f(x)$, for all vertices $x \neq u$ in D' , and $f'(u) := f(u_1) = f(u_2)$, and then extended simplicially. Such modification is called *C-move on u_1, u_2* and is denoted by $C(u_1, u_2)$.

D-move: Assume there is a vertex v in D with the link being a cycle (that v is an internal vertex), and such that for a vertex u adjacent to w, z in the link the vertices $f(w)$ and $f(z)$ are adjacent (we write $f(w) \sim f(z)$). Then the new singular diagram D' is obtained from D by removing the edge \overline{uv} and adding an edge \overline{wz} ; see Fig. 3 (bottom). The new map $f': D' \rightarrow X$ is defined as $f'(x) := f(x)$, for all vertices x in D' , and then extended simplicially. Such modification is called *D-move on v, u* and is denoted by $D(v, u)$.

The following lemma is essentially the same as [17, Lemma 5.1] and [35, Lemma 1.6]. Although in the latter two only simple cycles are considered, the general case follows by decomposing a given cycle into simple pieces. We omit the straightforward proof.

Lemma 2.4 *Let $f: D \rightarrow X$ be a singular disc diagram for a cycle C in a simplicial complex X . Then by applying A-moves, B-moves, and C-moves the diagram may be modified to a nondegenerate reduced singular disc diagram for C . In particular, any minimal singular disc diagram for C is nondegenerate and reduced.*

The main technical tool for dealing with metrically systolic complexes are CAT(0) singular disc diagrams. Their existence is established in the following theorem. It is an analogue of a result for systolic complexes obtained in [17, pp. 159–161] and [35, Lemma 1.7]. The proof is also an analogue of the systolic case proof. Before the theorem we prove a useful lemma.

Lemma 2.5 *Let $f: D \rightarrow X$ be a singular disc diagram into a metrically systolic complex X . Suppose that there is an interior vertex v in D whose link is a cycle C of angular length less than 2π . Then, by performing a finite number of A-, and D-moves we may find a singular disc diagram $f': D' \rightarrow X$ such that D' is a union of the full subcomplex of D spanned by all vertices of D except v , and triangles with vertices in C , and the map f' agrees with f on all vertices of D' and on all edges coming from D .*

Proof We proceed by induction on the combinatorial length of C . If this length is 3 then we perform A-move. Assume that C consists of at least 4 edges. Denote $C = (v_1, v_2, \dots, v_k)$. Then $\overline{f}(C) = (\overline{f}(v_1), \overline{f}(v_2), \dots, \overline{f}(v_k))$ is a cycle in X of angular length less than 2π . There is $2 < l \leq k$ such that $C' = (\overline{f}(v_1), \overline{f}(v_2), \dots, \overline{f}(v_l))$ is a simple cycle. This is a cycle in the link of $\overline{f}(v)$ of angular length less than 2π . If $l = 3$ then $\overline{f}(v_1)$ and $\overline{f}(v_3)$ are adjacent. If $l > 3$ then, by metric systolicity, C' is not 2-full. This means that there exists a vertex, say $\overline{f}(v_2)$, such that its neighbors in C' —in our case $\overline{f}(v_1)$ and $\overline{f}(v_3)$ —are adjacent. Hence we may perform D-move $D(v, v_2)$, to obtain a new singular disc diagram $\overline{f}': \overline{D}' \rightarrow X$. Furthermore, $\angle_v(v_1, v_3)$

in \overline{D}' is at most $\angle_v(v_1, v_2) + \angle_v(v_2, v_3)$ in \overline{D} , so that the angular length of the link of v in \overline{D}' , being the cycle (v_1, v_3, \dots, v_k) , is less than 2π . By the inductive assumption we obtain the desired diagram $f': D' \rightarrow X$. \square

Theorem 2.6 (CAT(0) disc diagram) *Let $f: D \rightarrow X$ be a singular disc diagram for a cycle C in a metrically systolic complex X . By performing a finite number of A-, B-, C-, D-moves the diagram may be modified to a CAT(0) nondegenerate reduced singular disc diagram $f': D' \rightarrow X$ for C . Furthermore:*

- (1) *f' does not use any new vertices in the sense that there is an injective map i from the vertex set of D' to the vertex set of D such that $f = f' \circ i$ on the vertex set of D ;*
- (2) *the number of 2-simplices in D' is at most the number of 2-simplices in D ;*
- (3) *any minimal singular disc diagram for C is CAT(0) nondegenerate and reduced.*

Proof We proceed with successive diagrams $\overline{f}: \overline{D} \rightarrow X$, starting from $\overline{f} := f$ depending on the following cases.

Case 1: A-move, B-move, or C-move may be performed. Then the new diagram $\overline{f}': \overline{D}' \rightarrow X$ is obtained by performing the corresponding move.

Case 2: No A-move, B-move, or C-move may be performed and there exists an internal vertex v whose link is a cycle $C = (v_1, v_2, \dots, v_k)$ of angular length less than 2π (in the metric induced from X). Then, by Lemma 2.5 there exists a singular disc diagram $\overline{f}': \overline{D}' \rightarrow X$, where \overline{D}' is obtained from \overline{D} by replacing the star of v with a disc without internal vertices, and \overline{f}' coincides with \overline{f} on all vertices except v .

Case 3: We are not in situations from Case 1 or Case 2. Then the diagram $\overline{f}: \overline{D} \rightarrow X$ is a CAT(0) nondegenerate reduced singular disc diagram for C .

After performing modifications as in Case 2, the area of the diagram decreases. Proceeding as in Case 1, that is performing A-moves, B-moves, or C-moves eventually decreases the area of the diagram. It is so because A-move and C-move decrease the area, and after performing B-move we are in position to perform A-move or C-move. Hence eventually we end up in Case 3.

Assertions (1), (2) and (3) follow immediately from the construction. \square

Corollary 2.7 *The Dehn function of a metrically systolic complex or group is at most quadratic.*

For further applications (e.g. in [34]) we will need singular disc diagrams with some further features (see Theorem 2.8 below). To construct them we have to consider other types of moves: E-moves and F-moves described below. Again, starting from a singular disc diagram $f: D \rightarrow X$ into a metrically systolic complex X we construct a new diagram $f': D' \rightarrow X$. For the new moves we assume that we are in the situation when no A-, B-, or C-move may be performed, and there is an interior vertex v and two vertices w, z in its link such that $f(w) = f(z)$. Observe that then w and z are not adjacent.

E-move: Assume that there does not exist a vertex different than v and adjacent to both w, z . We assume furthermore that the angular lengths of two paths between w and z in

the link of v are strictly smaller than 2π . The new disc diagram $f': D' \rightarrow X$ is obtained as follows. First we construct an intermediate singular disc D'' by “collapsing” vertices w, z to a single vertex x , that is, we remove w, z , and introduce a new vertex x adjacent to all vertices that were adjacent in D to w or z ; see Fig. 4 (top). Furthermore, we add two “copies” v', v'' of the vertex v , adjacent to vertices in two paths of the link of v , and to x . A singular disc diagram $f'': D'' \rightarrow X$ is defined by setting $f''(x) = f(w)$, $f''(v') = f''(v'') = f(v)$, and f'' agrees with f otherwise. Observe that the angular lengths of links of v' and v'' are strictly smaller than 2π . Hence, by double application of Lemma 2.5 we find a desired singular disc diagram $f': D' \rightarrow X$ with the two links filled without internal vertices.

F-move: Assume that there exists a vertex u different than v and adjacent to both w, z . We first construct a singular disc diagram $f'': D'' \rightarrow X$ by joining w and z by an edge, removing edges from v “crossing” the new edge \overline{wz} and adding a copy v' of v adjacent to vertices in the original link of v not adjacent to v anymore; see Fig. 4 (bottom). In D'' there is a triangle wzu , and performing the A-move $A(u, w, z)$ we obtain the desired singular disc diagram $f': D' \rightarrow X$.

Theorem 2.8 (CAT(0) disc diagram II) *Let $f: D \rightarrow X$ be a singular disc diagram for a cycle C in a metrically systolic complex X . By performing a finite number of A-, B-, C-, D-, E-, F-moves the diagram may be modified to a CAT(0) nondegenerate reduced singular disc diagram $f': D' \rightarrow X$ for C satisfying the following property. For every flat vertex $v \in D'$ the restriction $f|_{\text{St}(v)}$ is injective. Furthermore:*

- (1) f' does not use any new vertices in the sense that there is an injective map i from the vertex set of D' to the vertex set of D such that $f = f' \circ i$ on the vertex set of D ;
- (2) the number of 2-simplices in D' is at most the number of 2-simplices in D ;
- (3) any minimal singular disc diagram for C is such.

Proof By Theorem 2.6, using finitely many A-, B-, C-, D-moves we may modify f to a CAT(0) nondegenerate reduced singular disc diagram \overline{f} . Moreover, we may reach the situation when no A-, B-, C-move is possible. If for every flat vertex v the restriction $\overline{f}|_{\text{St}(v)}$ is injective then we are done with $f' = \overline{f}$. If not, we are in a position to perform an E-move or an F-move. Both decrease the area.

Applying iteratively the above procedure we finally obtain the desired singular disc diagram $f': D' \rightarrow X$. Assertions (1), (2), and (3) follow directly from the construction. \square

Remark 2.9 Observe that the assertion of the lemma is not true if the vertex is not flat—the star of such vertex could be mapped onto the simplicial cone over a wedge of two cycles.

Remark 2.10 We could reduce the number of moves for proving Theorem 2.6 or Theorem 2.8 by allowing singular discs to be non-simplicial, as e.g. in [35, proof of Lemma 1.6]. We decided to stay in the realm of simplicial complexes.

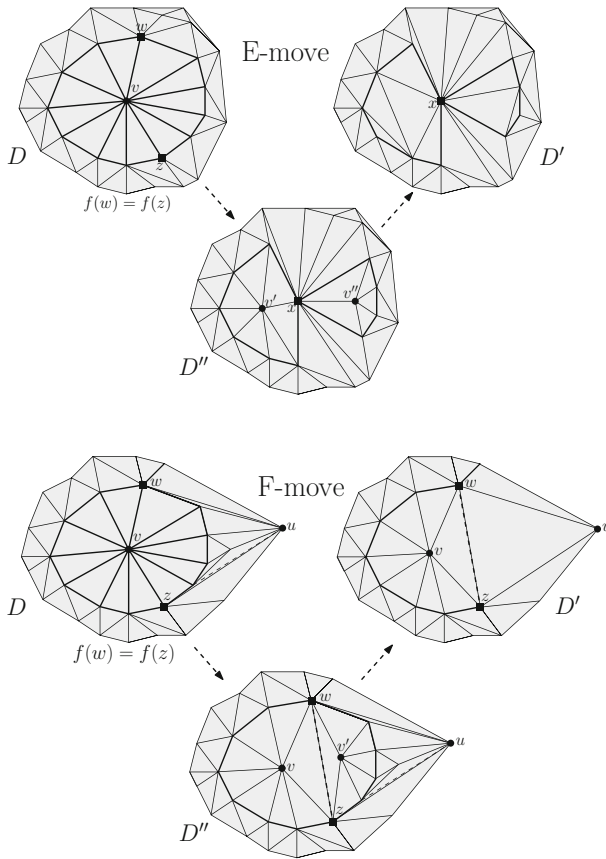


Fig. 4 E-move and F-move

3 Properties of metrically systolic complexes and groups

In this section we prove several properties of metrically systolic complexes and groups. In particular, such properties hold for two-dimensional Artin groups, and—as explained in Sect. 3.1 below—for all their finitely presented subgroups.

3.1 Finitely presented subgroups

In this section we show that being metrically systolic for groups is inherited by taking finitely presented subgroups. It follows that all subsequent features (and the quadratic isoperimetric inequality established above) of metrically systolic groups are valid also for all their finitely presented subgroups. In particular, they hold for all finitely presented subgroups of two-dimensional Artin groups.

Theorem 3.1 *Finitely presented subgroups of metrically systolic groups are metrically systolic.*

Proof In view of [31, Theorem 1.1] (compare also [49, Corollary 5.8]) it is enough to show that the class of locally 2π -large complexes is closed under taking covers and full subcomplexes.

Let $\tilde{X} \rightarrow X$ be a cover of a locally 2π -large complex X . Then links in \tilde{X} are combinatorially isomorphic to links in X . It follows that such links equipped with a metric induced by the isomorphism are 2π -large. Such metric on links is the angular metric coming from the metric on \tilde{X} induced by the covering. Therefore, \tilde{X} is metrically systolic.

Let \tilde{X} be a full subcomplex of a metrically systolic complex X , equipped with a subcomplex metric. Let C be a 2-full simple cycle in the link of a vertex of \tilde{X} . By fullness of \tilde{X} , C is 2-full in X , hence its angular length is at least 2π . Therefore, the angular length of C in \tilde{X} is at least 2π as well. It follows that \tilde{X} is locally 2π -large. \square

3.2 Solvability of the Conjugacy Problem

In this section we show that the Conjugacy Problem is solvable for torsion-free metrically systolic groups satisfying some additional technical assumption; see Theorem 3.6. The proof is a typical argument for showing solvability of the Conjugacy Problem in the non-positive curvature setting; see e.g. [6, pp. 445–446]

Below, and in further parts of the article we use the following convention concerning quasi-isometries.

Definition 3.2 Assume $K, L > 1$. A (K, L) -quasi-isometric embedding is a map $f: (X, d_X) \rightarrow (Y, d_Y)$ between metric spaces such that

$$K^{-1}d_X(x, y) - L \leq d_Y(f(x), f(y)) \leq Kd_X(x, y) + L,$$

for all $x, y \in X$.

A (K, L) -quasi isometry is a (K, L) -quasi-isometric embedding $f: X \rightarrow Y$ having an L -coarse inverse $\tilde{f}: Y \rightarrow X$, that is, a (K, L) -quasi-isometric embedding such that $d_X(x, \tilde{f} \circ f(x)) \leq L$ for all $x \in X$, and $d_Y(y, f \circ \tilde{f}(y)) \leq L$ for all $y \in Y$.

For the rest of the section let G be a torsion-free group acting geometrically on a metrically systolic complex X . We will use here the induced metric d in the one-skeleton of X . By scaling the metric we may assume that all edges have length at most 1. Let S be a finite (symmetrized) generating set for G , and let $\Gamma := \text{Cay}(G, S)$ be the corresponding Cayley graph. Let d_S be the word metric on G and (the 0-skeleton of) Γ , and $|g|_S = d_S(1_G, g)$.

The following two lemmas are standard but we formulate them for the purpose of refereeing to constants appearing later. The first one is just the Milnor-Schwarz lemma.

Lemma 3.3 *There exist $K_1, L_1 > 1$ such that for every vertex $x \in X$ the orbit map $\Phi: (G, d_S) \rightarrow X: g \mapsto gx$ is a (K_1, L_1) -quasi-isometry, and for every vertex $v \in X$ there exists $g \in G$ such that $d(v, \Phi(g)) \leq L_1$.*

Let D be a planar CAT(0) 2-complex constructed from triangles isometric to triangles in X . Let δ be a CAT(0) geodesic between two given vertices v, u in D . A path δ'

in the 1-skeleton of D is *approximating* the geodesic δ if δ' is contained in the union of all edges and triangles intersecting δ , and δ' is the shortest path with this property. The following is a consequence of e.g. [6, Proposition I.7.31].

Lemma 3.4 *There exist constants $K_2, L_2 > 1$ depending only on the geometry of X (in fact, on the set of isometry types of triangles in X) such that $K_2^{-1}|\delta'| - L_2 \leq |\delta| \leq K_2|\delta'| + L_2$.*

Let $K := \max\{K_1, K_2\}$ and $L = \max\{L_1, L_2\}$. In particular, it means that the assertions of Lemmas 3.3 and 3.4 hold when the corresponding constants K_i and L_i are replaced by K and L .

Lemma 3.5 *Let $g, h \in G$ be conjugate elements, such that, for every vertex $v \in X$, the shortest path between v and gv consists of at least 4 edges. Then there exists an element $p \in G$, conjugating them, that is, $g = php^{-1}$, and such that $|p|_S \leq A$, where A is a constant depending only on $|g|_S$ and $|h|_S$ (and on the action of G on X).*

Proof For every generator $s \in S$, choose a geodesic 1-skeleton path $q(s)$ in X , between x and sx . Let p be an element conjugating g and h . We will show that starting with p we may find a conjugator p' with $|p'|_S \leq A$, where A is a constant depending only on $|g|_S$ and $|h|_S$.

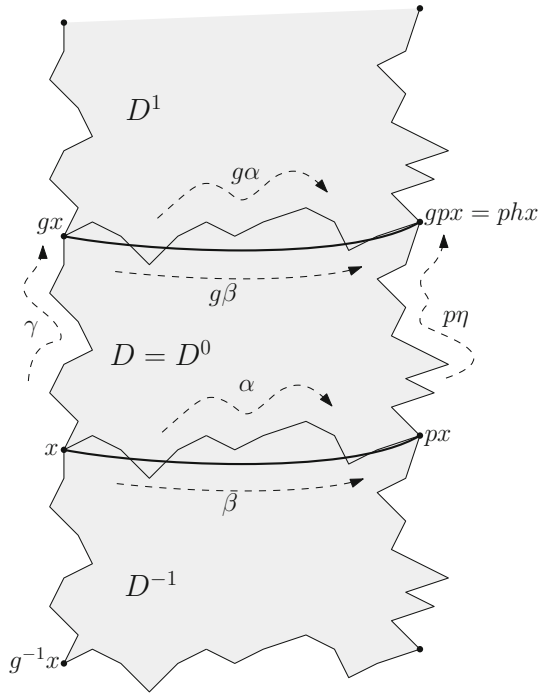
Let α_S, γ_S , and η_S be geodesics in Γ between 1_G and, respectively p, g , and h . Let $s_1^p \cdots s_a^p, s_1^g \cdots s_b^g$, and $s_1^h \cdots s_c^h$ be words in S defined by these geodesics. Let α be the concatenation of paths $q(s_1^p), s_1^p q(s_2^p), s_1^p s_2^p q(s_3^p) \dots, s_1^p \cdots s_{a-1}^p q(s_a^p)$. Similarly, let γ be the concatenation of paths $q(s_1^g), s_1^g q(s_2^g), s_1^g s_2^g q(s_3^g) \dots, s_1^g \cdots s_{b-1}^g q(s_b^g)$, and let η be the concatenation of paths $q(s_1^h), s_1^h q(s_2^h), \dots, s_1^h \cdots s_{c-1}^h q(s_c^h)$. Consider the cycle C based at x , being the concatenation of (in this order) $\gamma, g\alpha, p\eta$, and α ; see Fig. 5.

By Lemma 3.3, there exist constants E_1 and F_1 depending only on $|g|_S$ and $|h|_S$ (and the action of G on X) such that $|\gamma| \leq E_1, |\eta| \leq F_1$, where $|\cdot|$ denotes the d -length. In what follows we will consider constants depending on E_1, F_1 , and K, L leading, eventually, to a constant A as in the statement of the lemma.

Let $f: D \rightarrow X$ be a singular disc diagram for the cycle C . We create a singular strip diagram $\bar{f}: \bar{D} \rightarrow X$ as follows. For every $n \in \mathbb{Z}$ let D^n be a copy of D , and let f^n be the simplicial map such that $f^n(v) := g^n f(v)$, for every vertex $v \in D$ —here we identify D^n with D . In particular $f^0 = f$. Next, for every n , we identify the copy of the path $g\alpha$ in D^n with the copy of the path α in D^{n+1} . This way we obtain a singular strip $\bar{D} = \bigcup_{n \in \mathbb{Z}} D^n$. We define the map \bar{f} as the union of maps f^n , for all n . This way we obtain the singular strip diagram $\bar{f}: \bar{D} \rightarrow X$ for the pair of paths $\bar{\gamma}, \bar{p}\eta$, where $\bar{\gamma}$ is the concatenation of paths $g^n \gamma$, and $\bar{p}\eta$ is the concatenation of paths $g^n p\eta$, for all $n \in \mathbb{Z}$; see Fig. 5. Observe that there is a $\langle g \rangle$ -action on $\bar{D}: g^n D^m = D^{n+m}$, and that the map \bar{f} is equivariant with respect to this action and the $\langle g \rangle$ -action on X .

For every $m \neq n$, and for each triple of pairwise adjacent vertices v_1, v_2, v_3 in \bar{D} , the A-moves $A(g^m v_1, g^m v_2, g^m v_3)$ and $A(g^n v_1, g^n v_2, g^n v_3)$ may be performed independently, since the shortest path between $g^m v_i$ and $g^n v_j$ has at least 3 edges. Similarly, B-moves, C-moves, and D-moves may be performed independently for

Fig. 5 Scheme for proving Lemma 3.5



distinct translates of the defining vertices. Thus, we may define an *equivariant A-move* on u, v, w as the modification consisting of A-moves $A(g^n u, g^n v, g^n w)$, for all n . Similarly we define *equivariant B-move*, *equivariant C-move*, and *equivariant D-move*. As an equivariant analogue of Theorem 2.6 we claim that by performing a finite number of equivariant moves the singular strip diagram $\bar{f} : \bar{D} \rightarrow X$ may be modified to a CAT(0) nondegenerate reduced singular strip diagram $\bar{f}' : \bar{D}' \rightarrow X$ for the pair $\bar{\gamma}, \bar{p}\eta$.

Let β' be the CAT(0) geodesic in \bar{D}' with endpoints x, px (that is, their preimages in \bar{D}'). Let $d_{\bar{D}'}$ denote the CAT(0) distance in \bar{D}' . Since $d_{\bar{D}'}(x, gx) \leq |\gamma| \leq E_1$, and $d_{\bar{D}'}(x, hx) \leq |\eta| \leq F_1$, by the CAT(0) geometry, and the $\langle g \rangle$ -invariance of \bar{D}' , we have

$$d_{\bar{D}'}(y, gy) \leq \max\{E_1, F_1\} =: E,$$

for every point $y \in \beta'$.

Let $\beta = (v_0 = x, v_1, \dots, v_r = px)$ be a path in the 1-skeleton of \bar{D}' with endpoints x, px (that is, their preimages in \bar{D}') approximating the CAT(0) geodesic in \bar{D}' between x and px . Then $g\beta$ approximates $g\beta'$, and hence, for every vertex v of β , we have

$$d_{\bar{D}'}(v, gv) \leq d_{\bar{D}'}(v, y) + d_{\bar{D}'}(y, gy) + d_{\bar{D}'}(gy, gv) \leq 1 + E + 1, \tag{1}$$

where $y \in \beta'$ is a point closest to v . Using Lemma 3.4 we get

$$d(v, gv) \leq K(E + 2) + L. \tag{2}$$

For every $i \in \{1, \dots, r - 1\}$ we find $g_i \in G$ such that $d(v_i, g_i x) \leq L$ (see Lemma 3.3). Additionally, we set $g_0 = 1_G$ and $g_r = p$. Then, by Lemma 3.3, for every $i = 0, 1, \dots, r - 1$ we have

$$\begin{aligned} d_S(g_i, g_{i+1}) &= d_S(gg_i, gg_{i+1}) \leq Kd(g_i x, g_{i+1} x) + KL \\ &\leq K(d(g_i x, v_i) + d(v_i, v_{i+1}) + d(v_{i+1}, g_{i+1} x)) + KL \\ &\leq K(L + 1 + L) + KL =: \bar{L}. \end{aligned} \tag{3}$$

By (1) and (2), we have

$$\begin{aligned} d_S(g_i, gg_i) &\leq Kd(g_i x, gg_i x) + KL \\ &\leq K(d(g_i x, v_i) + d(v_i, gv_i) + d(gv_i, gg_i x)) + KL \\ &\leq K(L + E + 2 + L) + KL =: \hat{L}. \end{aligned} \tag{4}$$

For every i we choose a d_S -geodesic between g_i and g_{i+1} . Let β_S be their concatenation. This is a path in Γ connecting 1_G and p . By (3) and (4), for every $a \in \beta_S$, we have

$$d_S(a, ga) \leq d_S(a, g_i) + d_S(g_i, gg_i) + d_S(gg_i, ga) \leq 2\bar{L} + \hat{L} =: \tilde{L},$$

where g_i is the closest to a among g_i 's.

Now consider the quadrilateral Q in Γ formed by paths $\gamma_S, \beta_S, p\eta_S, g\beta_S$. For every vertex v on β_S pick a geodesic γ_v between $v \in \beta_S$ and $gv \in g\beta_S$. There are at most $A := |\tilde{S}|^{\tilde{L}+1}$ different up to G -action on Γ paths of length less than \tilde{L} . Hence if $|\beta_S|_S > A$ then there are two vertices $v, v' \in \beta_S$ such that the two paths γ_v and $\gamma_{v'}$ are the same up to G . Cutting Q along such paths and gluing together we obtain a quadrilateral Q' formed by paths $\gamma_S, \beta'_S, p\eta_S, g\beta'_S$, and such that again $d_S(a, ga) \leq \tilde{L}$, for all $a \in \beta'_S$. This way we construct a quadrilateral Q'' consisting of paths $\gamma_S, \beta''_S, p\eta_S, g\beta''_S$, with $|\beta''_S|_S \leq A$. Hence we obtain an element $p' \in G$ conjugating g and h , with $|p'|_S \leq A$. \square

Theorem 3.6 *Let G be a torsion-free metrically systolic group such that for every element $g \neq 1_G$ of G if g^n and g^m are conjugated then $n = m$. Then the Conjugacy Problem is solvable for G .*

Proof Suppose G acts geometrically on a metrically systolic complex X . Let $g = php^{-1}$. By the assumption on conjugates of g , we may find n such that the displacement of g^n is as large as in Lemma 3.5. Note that n does not depend on g , it only depends on the number of elements in the orbit of G contained in a ball of X of given size. Clearly $g^n = ph^n p^{-1}$. By Lemma 3.5 the displacement of p is bounded by value depending only on displacements of g , and h , and the action of G on X . Hence there

is a bound on the number of possible p 's. Note that this number is of the same order as the number of words we need to search in the $CAT(0)$ case. \square

3.3 Spherical fillings

The following result is a direct analogue of [36, Theorem 9.2] and [26, Theorem 2.4] concerning systolic complexes.

Theorem 3.7 *Let X be a metrically systolic complex and $f: S \rightarrow X$ be a simplicial map from a triangulation of the two-sphere. Then f can be extended to a simplicial map $F: B \rightarrow X$, where B is a triangulation of a 3-ball such that $\partial B = S$ and B has no internal vertices.*

Proof The proof is a direct analogue of the proof of [26, Theorem 2.4]. It goes by the induction on the area (number of triangles) of S . If the area is 4 (the smallest possible) then S is the 2-skeleton of the tetrahedron and the result follows by flagness of X . For larger area we consider the two following subcases.

Case 1: S is not flag. Then we proceed exactly as in the proof of [26, Theorem 2.4]: we decompose S into two discs along an “empty” triangle, create two spheres of smaller area and use the induction assumption.

Case 2: S is flag. Since the 2-sphere does not admit a metric of non-positive curvature there exists a vertex v in S whose link, a cycle C , has angular length less than 2π . We have the decomposition $S = D_1 \cup D_2$, where D_1 is the star of v and D_2 is the complement of the interior of D_1 . By Lemma 2.5 the cycle $f|_C$ has a singular disc diagram D with no internal vertices. Let B_1 be the simplicial cone over D with apex v , and let $F_1: B_1 \rightarrow X$ be the simplicial map such that $F_1(u) = f(u)$, for all vertices u (it is well defined by flagness of X). Then $S_2 = D_2 \cup D$ is a simplicial sphere of area smaller than the one of S . Let $f_2: S_2 \rightarrow X$ be the simplicial map coinciding on vertices with f . Applying the inductive assumption we extend it to $F_2: B_2 \rightarrow X$, where B_2 is a triangulation of the ball with no internal vertices satisfying $\partial B_2 = S_2$. Finally we put $B = B_1 \cup B_2$ and $F = F_1 \cup F_2$. \square

Januszkiewicz–Świątkowski introduced in [36] the notion of *constant filling radius for k -spherical cycles*, shortly S^kFRC . This is a coarse feature of metric spaces saying, roughly, that in large scale every k -sphere has a filling within its uniform neighbourhood. A direct consequence of Theorem 3.7 is the following.

Corollary 3.8 *Metrically systolic complexes and groups are S^2FRC , that is, they have constant filling radius for 2-spherical cycles.*

3.4 Morse Lemma for 2-dimensional quasi-discs

In this section we prove a Morse Lemma for 2-dimensional quasi-discs. It states, roughly speaking, that, for a given cycle C in a metrically systolic complex, a quasi-isometrically embedded disc diagram is contained in an a -neighbourhood of any other singular disc diagram for C , with a independent of the size of the disc.

We use the combinatorial metric on simplicial complexes. In particular, the distance between adjacent vertices is 1. Let $B(R, v)$ denote the (*combinatorial*) ball of radius R centered at v , that is the full subcomplex of a simplicial complex spanned by all vertices at distance at most R from v . Similarly, the *sphere* $S(R, v)$ is the full subcomplex spanned by all vertices at distance R from v . Let $T(r, R; v)$ denote the *tube* (annulus) of radii r, R around v , that is, the full subcomplex spanned by all vertices u such that $r \leq d(v, u) \leq R$. Observe that for (L, A) -quasi-isometry f we have $f(T(r, R; v)) \subseteq T(L^{-1}r - A, LR + A; f(v))$. Recall that the *systolic plane*, denoted \mathbb{E}_Δ^2 , is the triangulation of the Euclidean plane by regular triangles.

Theorem 3.9 (Morse Lemma for 2-dimensional quasi-discs) *Let D be a combinatorial ball in the systolic plane \mathbb{E}_Δ^2 . Let $f : D \rightarrow X$ be a disc diagram for a cycle C in X being an (L, A) -quasi-isometric embedding. Let $g : D' \rightarrow X$ be a singular disc diagram for C . Then $\text{im}(f) \subseteq N_a(\text{im}(g))$, where $a > 0$ is a constant depending only on L and A .*

Proof There exist constants $L' \geq L$ and $A' \geq A$ depending only on L, A such that $f : D \rightarrow f(D)$ is an (L', A') -quasi-isometry, and there is an (L', A') -quasi-isometry $\tilde{f} : f(D) \rightarrow D^{(0)}$ such that $\tilde{f} \circ f$ and $f \circ \tilde{f}$ are A' -close to identities. Let $K \geq \max\{L', A', 3\}$. We will further work with K instead of L, A —this will make the computations easier. In particular (L', A') —quasi-isometries are (K, K) -quasi-isometries. We claim that $a = K^{20}$ satisfies the assertion of the lemma.

We proceed by contradiction. Suppose there is a vertex $v \in \text{im}(f) \setminus N_a(\text{im}(g))$. Then clearly $d(v, C) > a$. Let $v' := f(\tilde{f}(v))$. Then $d(v, v') \leq K$.

Let $X_1 = \text{span}(f(D) \cap B(K^{12}; v))$, and let $X_2 = \text{span}(g(D') \cup (f(D) - B(K^8, v)))$.

Let $\alpha = (v_0, v_1, \dots, v_k)$ be a cycle in $S(K^{10}, \tilde{f}(v))$ being a generator of $H_1(S(K^{10}, \tilde{f}(v)); \mathbb{Z})$. Observe that then α represents also a generator of $H_1(T(K^5, K^{15}, \tilde{f}(v)); \mathbb{Z})$. Let $f(\alpha) = (f(v_0), f(v_1), \dots, f(v_k))$ be the cycle (possibly with $f(v_i) = f(v_j)$ for some $i \neq j$) being the image of α . Observe that, by $d(v, v') \leq K$ and $K \geq 3$, we have

$$\begin{aligned} f(\alpha) &\subseteq T(K^{-1}K^{10} - K, K^{11} + K, v') \\ &\subseteq T(K^{-1}K^{10} - 2K, K^{11} + 2K, v) \subseteq T(K^8, K^{12}; v). \end{aligned}$$

Claim. The cycle $f(\alpha)$ is not null-homologous inside $T(K^8, K^{12}; v) \cap X_1$.

To prove the claim suppose, by contradiction, that $f(\alpha)$ is null-homologous in $T(K^8, K^{12}; v) \cap X_1$. Then there exists a simplicial map

$$h : T \rightarrow T(K^8, K^{12}; v) \cap X_1$$

from a simplicial 2-complex T to $T(K^8, K^{12}; v) \cap X_1$ sending the boundary cycle to $f(\alpha)$. We define a map $\tilde{f} \circ h : T \rightarrow D$ as follows. For every vertex $u \in T$ we send it to $\tilde{f} \circ h(u)$. An edge \overline{uw} is sent to a geodesic between $\tilde{f}(u)$ and $\tilde{f}(w)$. A triangle uwz is sent to a singular disc in D bounded by the chosen geodesic between images

of vertices. Since

$$\begin{aligned}\bar{f}(T(K^8, K^{12}; v) \cap X_1) &\subseteq T(K^7 - K, K^{13} + K, \bar{f}(v)) \\ &\subseteq T(K^6, K^{14}, \bar{f}(v)),\end{aligned}$$

and since the image of every edge has diameter at most K , and similarly the image of every triangle has diameter at most K , we have that the image of $\bar{f} \circ h$ is contained in $T(K^6 - K, K^{14} + K, \bar{f}(v))$. Furthermore, for every i , we have $d(v_i, \bar{f}(f(v_i))) \leq K$, and $d(\bar{f}(f(v_i)), \bar{f}(f(v_{i+1}))) \leq K$. Therefore, there exists a homotopy between α and the image of $f(\alpha)$ by $\bar{f} \circ h$ within the $2K$ -neighborhood of α . It follows that α is null-homologous within $T(K^5, K^{15}; \bar{f}(v))$ - contradiction concluding the proof of the claim.

Let Y be a simplicial complex homeomorphic to an annulus (tube) in \mathbb{E}^2 with the inner boundary cycle isomorphic to the boundary cycle C of D , and admitting a simplicial retraction on C . Observe that the boundary cycle of D' is also C . Let $\bar{D} = D \cup_C Y$ be the complex obtained by gluing D and Y along C . Similarly, let $\bar{D}' = D' \cup_C Y$. Both, \bar{D} and \bar{D}' are non-singular discs, with isomorphic boundaries C' —the other boundary cycle of Y . Consider a triangulated sphere $S := \bar{D} \cup_{C'} \bar{D}'$ obtained by the identification of the boundaries, and the map $\psi: S \rightarrow X$ being the union of maps f, g , and the retraction maps sending copies of Y to their internal cycles C . By Theorem 3.7 there exists a simplicial extension of ψ to a three-ball without internal vertices. Hence $[\psi] = 0$ in $H_2(X_1 \cup X_2; \mathbb{Z})$.

On the other hand the 1-cycle α is null-homotopic inside $B(K^{10}, \bar{f}(v)) \subseteq D$. Hence there exists a simplicial disc $D_1 \subseteq B(K^{10}, \bar{f}(v))$ providing the homotopy. Similarly, there is a disc $D_2 \subseteq D - B(K^{10}, \bar{f}(v)) \cup_C Y \cup_{C'} \bar{D}'$ with boundary equal α . Observe that $\psi(D_1) \subseteq X_1$, $\psi(D_2) \subseteq X_2$, and $\psi(\alpha) \subseteq X_1 \cap X_2$. Therefore, in the Mayer-Vietoris sequence for the pair X_1, X_2 the boundary map

$$H_2(X_1 \cup X_2; \mathbb{Z}) \rightarrow H_1(X_1 \cap X_2; \mathbb{Z})$$

sends $[\psi]$ to the nontrivial element represented by α . Hence the contradiction concluding the proof of the lemma. \square

Remark 3.10 In fact, a more general version of Lemma 3.9 could be proved following the same lines. Namely, we could require that $f: D \rightarrow X$ is a disc diagram being a quasi-isometry such that D is quasi-isometric to a ball in \mathbb{E}_Δ^2 , rather than being the ball itself. Since the original statement allows technically much simpler proof, and it is the version that we subsequently use in [34], we decided to formulate it this way.

4 The complexes for 2-generated Artin groups

In this section, we focus on 2-generated Artin groups. We construct metric simplicial complexes for them by modifying their Cayley complexes (see the “comments on the proof” section in the Introduction for an intuitive explanation). Later in Sect. 5 we will show these metric simplicial complexes are metrically systolic, and in Sect. 6 we will

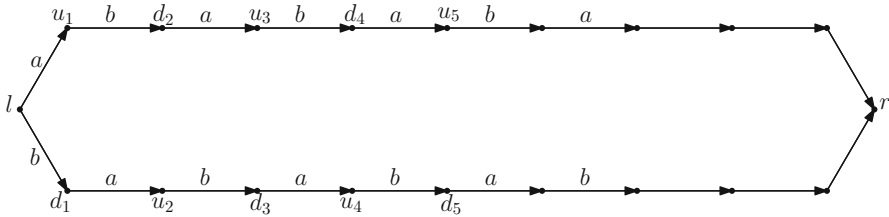


Fig. 6 Precell Π^*

glue them together to form metrically systolic complexes for general two-dimensional Artin groups.

4.1 Precells in the presentation complex

Let DA_n be the 2-generator Artin group presented by $\langle a, b \mid \underbrace{aba \cdots}_n = \underbrace{bab \cdots}_n \rangle$.

Let P_n be the standard presentation complex for DA_n . Namely the 1-skeleton of P_n is the wedge of two oriented circles, one labeled a and one labeled b . Then we attach the boundary of a closed 2-cell C to the 1-skeleton with respect to the relator of DA_n . Let $C \rightarrow P_n$ be the attaching map. Let X_n^* be the universal cover of P_n . Then any lift of the map $C \rightarrow P_n$ to $C \rightarrow X_n^*$ is an embedding (cf. [33, Corollary 3.3]). These embedded discs in X_n^* are called *precells*. Figure 6 depicts a precell Π^* . X_n^* is a union of copies of Π^* 's. We pull back the labeling and orientation of edges in P_n to obtain labeling and orientation of edges in X_n^* . We label the vertices of Π^* as in Fig. 6. The vertices l and r are called the *left tip* and the *right tip* of Π^* . The boundary $\partial\Pi^*$ is made of two paths. The one starting at l , going along $\underbrace{aba \cdots}_n$ (resp. $\underbrace{bab \cdots}_n$), and ending at r is called the *upper half* (resp. *lower half*) of $\partial\Pi^*$. The orientation of edges inside one half is consistent, thus each half has an orientation. We summarize several basic properties of how these precells intersect each other. See [33, Section 3.1] for proofs of these properties.

Lemma 4.1 *Let Π_1^* and Π_2^* be two different precells in X_n^* . Then*

- (1) *either $\Pi_1^* \cap \Pi_2^* = \emptyset$, or $\Pi_1^* \cap \Pi_2^*$ is connected;*
- (2) *if $\Pi_1^* \cap \Pi_2^* \neq \emptyset$, $\Pi_1^* \cap \Pi_2^*$ is properly contained in the upper half or in the lower half of Π_1^* (and Π_2^*);*
- (3) *if $\Pi_1^* \cap \Pi_2^*$ contains at least one edge, then one end point of $\Pi_1^* \cap \Pi_2^*$ is a tip of Π_1^* , and another end point of $\Pi_1^* \cap \Pi_2^*$ is a tip of Π_2^* , moreover, among these two tips, one is a left tip and one is a right tip.*

Lemma 4.2 *Suppose there are three precells Π_1^* , Π_2^* and Π_3^* such that $\Pi_1^* \cap \Pi_2^*$ is a nontrivial path P_1 in the upper half of Π_2^* , and $\Pi_3^* \cap \Pi_2^*$ is a nontrivial path P_3 in the lower half of Π_2^* . Then $\Pi_1^* \cap \Pi_3^*$ is either empty or one point.*

Corollary 4.3 *Let Π_1^* and Π_2^* be two different precells in X_n^* . If $\Pi_1^* \cap \Pi_2^*$ contains at least one edge, and $\Pi_3^* \cap \Pi_2^* = \Pi_1^* \cap \Pi_2^*$, then $\Pi_3^* = \Pi_1^*$.*

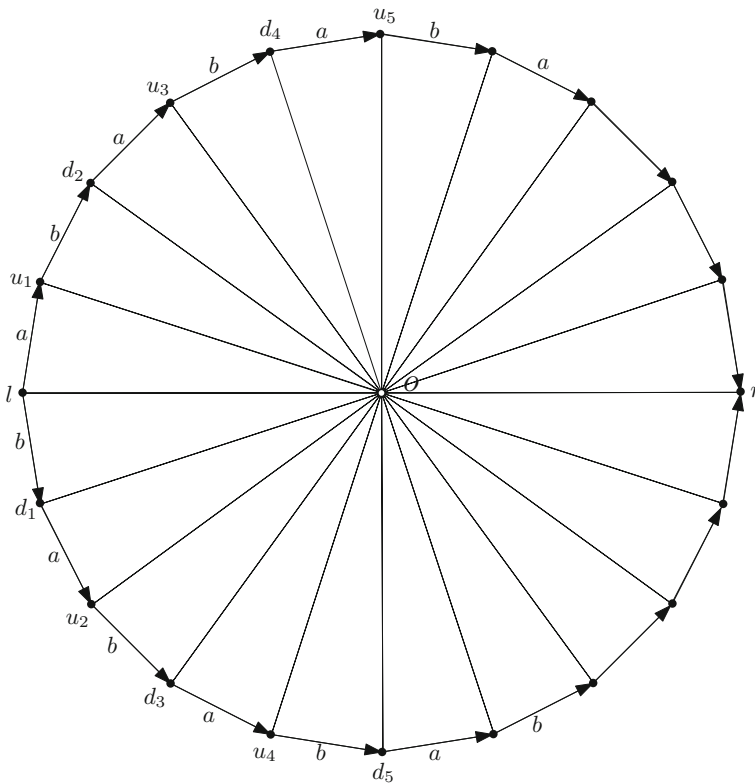


Fig. 7 Cell Π

Proof We apply Lemma 4.1 (3) to $\Pi_3^* \cap \Pi_2^*$ and $\Pi_1^* \cap \Pi_2^*$ to deduce that either Π_1^* and Π_3^* have the same left tip, or they have the same right tip. Thus $\Pi_1^* = \Pi_3^*$. \square

4.2 Subdividing and systolizing the presentation complex

We subdivide each precell in X_n^* as in Fig. 7 to obtain a simplicial complex X_n^Δ . A cell of X_n^Δ is defined to be a subdivided precell, and we use the symbol Π for a cell. The original vertices of X_n^* in X_n^Δ are called the *real vertices*, and the new vertices of X_n^Δ after subdivision are called *interior vertices*. The interior vertex in a cell Π is denoted o as in Fig. 7. (Here and further we use the convention that the real vertices are drawn as solid points and the interior vertices as circles.)

Let Λ be the collection of all unordered pairs of cells of X_n^Δ such that their intersection contains at least two edges (these intersections are connected by Lemma 4.1). For each $(\Pi_1, \Pi_2) \in \Lambda$, we add an edge between the interior vertex of Π_1 and the interior vertex of Π_2 (cf. Fig. 1). Denote the resulting complex by X'_n . It is clear that DA_n acts on X'_n . Let X_n be the flag completion of X'_n . Then X_n is the simplicial complex we will work with.

Now we give an alternative, but more detailed definition of X'_n . Pick a base cell Π in X_n^Δ such that $\ell \in \Pi$ coincides with the identity element of DA_n . Let Λ_0 be the collection of pairs of the form $(\Pi, u_i^{-1}\Pi)$, $(\Pi, d_i^{-1}\Pi)$ for $i = 1, \dots, n - 2$ (here each vertex of Π can be identified as an element of DA_n , and $u_i^{-1}\Pi$ means the image of Π under the action of u_i^{-1}). Then the following is proved in [33, Section 3.1].

- Lemma 4.4** (1) $\Lambda_0 \subset \Lambda$.
- (2) *Different elements in Λ_0 are in different DA_n -orbits.*
- (3) *Every DA_n -orbit in Λ contains an element from Λ_0 .*

For each $1 \leq i \leq n - 2$, we add an edge between $o \in \Pi$ and $u_i^{-1}o \in u_i^{-1}\Pi$, and an edge between $o \in \Pi$ and $d_i^{-1}o \in d_i^{-1}\Pi$. Then we use the action of DA_n to add more edges in the equivariant way. The resulting complex is exactly X'_n , by Lemma 4.4.

Definition 4.5 We assign lengths to edges of X_n . Edges between a real vertex and an interior vertex have length 1. Edges between two real vertices have length equal to the distance between two adjacent vertices in a regular $(2n)$ -gon with radius 1.

Now we assign lengths to edges between two interior vertices. First define a function $\phi : [0, \pi) \rightarrow \mathbb{R}$ as follows. Let $\Delta(ABC)$ be a Euclidean isosceles triangle with length of AB and AC equal to 1, and $\angle_A(B, C) = \alpha$. Then $\phi(\alpha)$ is defined to be the length of BC . For $1 \leq i \leq n - 2$, let e_i be the edge between o and $u_i^{-1}o$ (or o and $d_i^{-1}o$). Then the length of e_i is defined to be $\phi(\frac{i}{2n}2\pi)$. Now we use the DA_n action to define the length of edges between interior vertices in an equivariant way.

Note that $\Pi \cap u_i^{-1}\Pi$ and $\Pi \cap d_i^{-1}\Pi$ have $n - i$ edges. Thus we have the following observation by using the DA_n -action and Lemma 4.4.

Lemma 4.6 *Suppose $\Pi_1 \cap \Pi_2$ has m edges for $m \geq 2$. Let $o_i \in \Pi_i$ be the interior vertex for $i = 1, 2$. Then there is an edge between o_i and o_j in X_n whose length is $\phi(\frac{n-m}{2n}2\pi)$.*

Lemma 4.7 *The lengths of the three sides of each triangle in $X_n^{(1)}$ satisfy the strict triangle inequality. Thus each 2-simplex of X_n can be metrized as a non-degenerate Euclidean triangle whose three sides have length equal to the assigned length of the corresponding edges.*

Proof We only prove the case when this triangle is made of three interior vertices $\{o_i \in \Pi_i\}_{i=1}^3$. The other cases are already clear from the construction. By Lemma 4.2, $\Pi_1 \cap \Pi_2$ and $\Pi_1 \cap \Pi_3$ are contained in the same half (say upper half) of Π_1 , otherwise $\Pi_2 \cap \Pi_3$ is at most one vertex, which contradicts that o_2 and o_3 are joined by an edge. We assume without loss of generality that Π_1 is the base cell Π . By Lemma 4.1 (3), each of Π_2 and Π_3 contains exactly one tip of Π_1 . We first consider the case when Π_2 contains the left tip of Π_1 and Π_3 contains the right tip of Π . Suppose $\Pi_2 \cap \Pi_1$ (resp. $\Pi_3 \cap \Pi_1$) contains m_2 (resp. m_3) edges. Then by Lemma 4.1 (3), $\Pi_2 \cap \Pi_3$ contains $m_2 + m_3 - n$ edges. By Lemma 4.6, $\text{length}(\overline{o_1o_2}) = \phi(\frac{n-m_2}{2n}2\pi)$, $\text{length}(\overline{o_1o_3}) = \phi(\frac{n-m_3}{2n}2\pi)$, and $\text{length}(\overline{o_2o_3}) = \phi(\frac{n-(m_2+m_3-n)}{2n}2\pi)$. Note that $\pi > \frac{n-(m_2+m_3-n)}{2n}2\pi = \frac{n-m_2}{2n}2\pi + \frac{n-m_3}{2n}2\pi$, thus we can place o_2, o_1, o_3 consecutively in

the unit circle such that they span a Euclidean triangle with side lengths as required. Next we consider the case that both Π_2 and Π_3 contains the left tip of Π_1 . We assume without loss of generality that $\Pi_1 \cap \Pi_2 \subsetneq \Pi_1 \cap \Pi_3$. Then, by Corollary 4.1 (3), the left tip of Π_3 is contained in $\Pi_2 \cap \Pi_3$. Thus we can repeat the argument in the previous case with Π_1 replaced by Π_3 . The case when both Π_2 and Π_3 contain the right tip of Π_1 can be handled similarly. \square

From now on, we think of each 2-simplex of X_n as a Euclidean triangle with the required side lengths. If three vertices x_1, x_2 and x_3 span a 2-simplex in X_n , then we use $\angle_{x_1}(x_2, x_3)$ to denote the angle at x_1 of the associated Euclidean triangle.

5 The link of X_n

In this section we study links of vertices in the complex X_n defined in the previous section.

Choose a vertex $v \in X_n$, let Λ_v be the link $lk(v, X_n^{(1)})$ of v in X_n , i.e. Λ_v is the full subgraph of $X_n^{(1)}$ spanned by vertices which are adjacent to v . For an edge $\overline{v_1 v_2} \subset \Lambda_v$, we define the *angular length* of this edge to be $\angle_v(v_1, v_2)$. This makes Λ_v a metric graph. We define angular metric on Λ_v in the same way as in Sect. 2.1 and use the notation from over there.

The main result of the section is the following proposition.

Proposition 5.1 *Let v be a vertex of X_n .*

- (1) *The angular lengths of the three sides of each triangle in Λ_v satisfy the triangle inequality.*
- (2) *Let σ be a simple cycle in Λ_v which is 2-full. Then $\text{length}_{\angle}(\sigma) \geq 2\pi$.*

We caution the reader that each edge in Λ_v has an angular length, and has a length as defined in the previous section. Here we mostly work with angular length, but will switch to length occasionally. In this section we study the structure of Λ_v with respect to the angular metric.

The proof of Proposition 5.1 is divided into two cases: the case of a real vertex v is treated in Sect. 5.1 and the case of an interior vertex v is treated in Sect. 5.2. In each case we first describe precisely the combinatorial and metric structure of the link and then we study in details angular lengths of simple cycles in the link.

5.1 Link of a real vertex

The main purpose of this section is to prove Proposition 5.1 for a real vertex v .

Since the links of any two real vertices are isomorphic as metric graphs with the angular metric, we can assume without loss of generality that v is the vertex l in the base cell Π (cf. Fig. 7).

In the following proof, we will assume $u_0 = d_0 = \ell$ and $u_n = d_n = r$. Recall that each edge of X_n which belongs to X_n^* has an orientation and is labeled by one of the generators a and b . We will first establish a sequence of lemmas towards the proof of Proposition 5.1.

The vertices of Λ_v can be divided into two classes.

- (1) Real vertices a^i, a^o, b^i and b^o , where a^i and a^o are the vertices in Λ_v which correspond to the incoming and outgoing a -edge containing v (b^i and b^o are defined similarly).
- (2) Interior vertices. There is a 1-1 correspondence between such vertices and cells in X_n that contain ℓ . Thus the interior vertices of Λ_v are of form $w^{-1}o$ where w is a vertex of $\partial\Pi$ (recall that we have identified vertices of X_n^* with group elements of DA_n , and ℓ is identified with the identity element of DA_n , so $w^{-1}o$ means the image of o under the action of w^{-1}). More precisely, interior vertices of Λ_v are $\{\ell^{-1}o, r^{-1}o, d_1^{-1}o, d_2^{-1}o, \dots, d_{n-1}^{-1}o, u_1^{-1}o, u_2^{-1}o, \dots, u_{n-1}^{-1}o\}$.

The edges of Λ_v can be divided into two classes.

- (1) Edges between a real vertex and an interior vertex. These are exactly the edges of Λ_v which are in X_n^Δ , and they are called edges of *type I*.
- (2) Edges between two interior vertices. These are exactly the edges of Λ_v which are not in X_n^Δ , and they are called edges of *type II*.

Note that there do not exist edges of Λ_v which are between two real vertices.

Now we characterize all edges of type I. See Fig. 8 below for a picture of Λ_v with only edges of type I shown.

- Lemma 5.2** (1) *The collection of vertices in Λ_v which are connected to b^i (resp. a^i) by an edge of type I is exactly $\{d_1^{-1}o, d_2^{-1}o, \dots, d_n^{-1}o\}$ (resp. $\{u_1^{-1}o, u_2^{-1}o, \dots, u_n^{-1}o\}$).*
- (2) *The collection of vertices in Λ_v which are connected to a^o (resp. b^o) by an edge of type I is exactly $\{d_0^{-1}o, d_1^{-1}o, \dots, d_{n-1}^{-1}o\}$ (resp. $\{u_0^{-1}o, u_1^{-1}o, \dots, u_{n-1}^{-1}o\}$).*
- (3) *Each edge of type I has angular length equal to $\frac{n-1}{4n}2\pi$.*

Proof If a vertex in Λ_v is adjacent to b^i , then this vertex must be an interior vertex, hence is of form $w^{-1}o$ for a vertex $w \in \partial\Pi$. Note that if there is a vertex $w' \in \partial\Pi$ such that there is a b -edge pointing from w' to w , then by applying the action of w^{-1} to the triangle $\Delta(w'wo)$, we know $w^{-1}o$ and b^i are adjacent. We can reverse this argument to show that $w^{-1}o$ and b^i are adjacent, then there is a b -edge in $\partial\Pi$ terminating at w . It follows that b^i is connected to $w^{-1}o$ if and only if $u = d_i^{-1}$ for $1 \leq i \leq n$. Thus the part of (1) concerning b^i follows. We can analyze vertices to b^o, a^i and a^o in a similar way. Thus (1) and (2) follow. Note that the angular length of each edge of type I is equal to half of the interior angle of a regular $2n$ -gon. Thus (3) follows. \square

- Lemma 5.3** (1) *There is an edge of type II between $d_i^{-1}o$ and $d_j^{-1}o$ if and only if $1 \leq |j - i| \leq n - 2$.*
- (2) *There is an edge of type II between $u_i^{-1}o$ and $u_j^{-1}o$ if and only if $1 \leq |j - i| \leq n - 2$.*
- (3) *If $1 \leq i \leq n - 1$ and $1 \leq j \leq n - 1$, then there is no edge between $d_i^{-1}o$ and $u_j^{-1}o$.*
- (4) *Suppose $0 \leq i < j \leq n$ and $j - i \leq n - 2$. Then the edge between $d_i^{-1}o$ and $d_j^{-1}o$ has angular length $= \frac{j-i}{2n}2\pi$.*

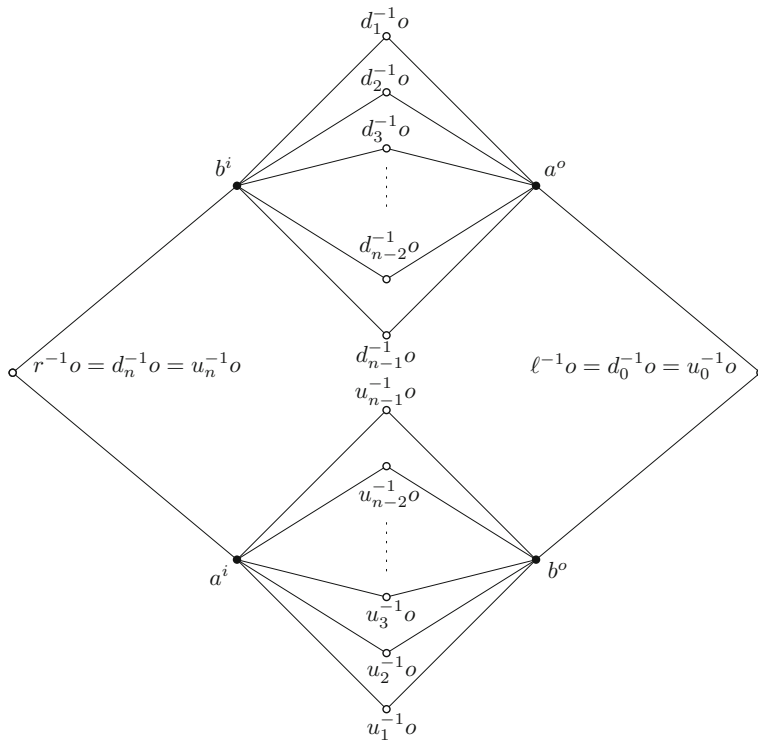


Fig. 8 Edges of type I in the link of a real vertex

(5) Suppose $0 \leq i < j \leq n$ and $j - i \leq n - 2$. Then the edge between $u_i^{-1}o$ and $u_j^{-1}o$ has angular length $= \frac{j-i}{2n} 2\pi$.

Proof First we claim the number of edges in $d_i^{-1}\Pi \cap d_j^{-1}\Pi$ equals to $n - (|j - i|)$. We assume without loss of generality that $i < j$. Then the number of edges in $d_i^{-1}\Pi \cap d_j^{-1}\Pi$ equals the number of edges in $\Pi \cap d_i d_j^{-1}\Pi$. By direct computation, we know $d_i d_j^{-1} = u_{j-i}^{-1}$ or d_{j-i}^{-1} . Moreover, the number of edges in $\Pi \cap d_k^{-1}\Pi$ (or $\Pi \cap u_k^{-1}\Pi$) equals $n - k$ for any $1 \leq k \leq n - 1$. Thus the claim follows.

There is an edge of type II between $d_i^{-1}o$ and $d_j^{-1}o$ if and only if $d_i^{-1}\Pi \cap d_j^{-1}\Pi$ has at least two edges, thus (1) follows from the claim. (4) follows the claim and Lemma 4.6. (2) and (5) can be proved in a similar way. To see (3), note that $d_i^{-1}\Pi \cap \Pi$ (resp. $u_i^{-1}\Pi \cap \Pi$) is contained in the upper half (resp. lower half) of Π . Thus (3) follows from Lemma 4.2. \square

Corollary 5.4 (1) The angular lengths of the three sides of any triangle in Λ_v satisfy the triangle inequality.

(2) Let Δ be a 3-simplex in X_n which contains a real vertex. Then there exists a (possibly degenerate) 3-simplex Δ' in the Euclidean 3-space such that there is a simplicial isomorphism $\Delta \rightarrow \Delta'$ which preserves the lengths of edges.

Proof Let Δ be a triangle in Λ_v . Since no two real vertices in Λ_v are adjacent, Δ either has two interior vertices, or three interior vertices. In the former case, since the angular length of any edge of type II is at most $\frac{n-2}{2n}2\pi$ (Lemma 5.3), it is less than the summation of the angular length of two edges of type I (Lemma 5.2), we consequently deduce that the triangle inequality holds. Moreover, (2) holds by triangle inequality and that the summation of the angular length of two edges of type I in Δ is $< \pi$. In the latter case, by Lemma 5.3 (3), the three vertices of Δ are either of form $d_i^{-1}o, d_j^{-1}o, d_k^{-1}o$, or of form $u_i^{-1}o, u_j^{-1}o, u_k^{-1}o$. By Lemma 5.3 (4), $\angle_v(d_i^{-1}o, d_j^{-1}o) + \angle_v(d_j^{-1}o, d_k^{-1}o) = \angle_v(d_i^{-1}o, d_k^{-1}o)$ when $i < j < k$. A similar equality holds with d replaced by u . Thus (1) and (2) follow. \square

We record a simple graph theoretic observation for later use.

Definition 5.5 A simple graph Γ is a *tree of cliques* if there are complete subgraphs $\{\Delta_i\}_{i=1}^k$ such that

- (1) $\Gamma = \cup_{i=1}^k \Delta_i$;
- (2) for each $1 < m \leq k$, $(\cup_{i=1}^{m-1} \Delta_i) \cap \Delta_m$ is a complete subgraph.

Lemma 5.6 *Let Γ be a tree of cliques. Then the following hold.*

- (1) Any simple n -cycle for $n \geq 4$ in Γ is not 2-full.
- (2) If Γ is a metric graph such that the three sides of each of its triangle satisfy the triangle inequality then, for any edge $\overline{w_1w_2} \subset \Gamma$, the length of $\overline{w_1w_2}$ is bounded above by the length of any edge path from w_1 to w_2 .

Proof For (1), we induct on the number k in Definition 5.5. Let $\omega \subset \Gamma$ be a simple n -cycle. If $\omega \subset \cup_{i=1}^{k-1} \Delta_i$, then ω is not 2-full by induction. Now we assume $\omega \not\subset \cup_{i=1}^{k-1} \Delta_i$. Then there must be an edge $e \subset \omega$ such that $e \not\subset \cup_{i=1}^{k-1} \Delta_i$. Let s, t be two vertices of e . By Definition 5.5 (1), $e \subset \Delta_k$. Hence $\{s, t\} \subset \Delta_k$. If $\{s, t\} \subset \cup_{i=1}^{k-1} \Delta_i$, then by Definition 5.5 (2) and the assumption that Γ is simple, we know $e \subset (\cup_{i=1}^{k-1} \Delta_i) \cap \Delta_k$, which is a contradiction. So at least one of $\{s, t\}$ is not contained in $\cup_{i=1}^{k-1} \Delta_i$. Now we assume $s \in \Delta_k \setminus (\cup_{i=1}^{k-1} \Delta_i)$. Let t_1 and t_2 be two vertices in ω that are adjacent to s . Since $n \geq 4$, t_1 and t_2 have combinatorial distance ≥ 2 in ω . By Definition 5.5 (1), the edge $\overline{t_1s}$ is contained in one of the Δ_i . Thus we must have $\overline{t_1s} \subset \Delta_k$. In particular $t_1 \in \Delta_k$. Similarly, $t_2 \in \Delta_k$. Thus there is an edge between t_1 and t_2 , and ω is not 2-full.

For (2), we can assume without loss of generality that $\overline{w_1w_2}$ together with another given edge path from w_1 to w_2 form a simple cycle. Thus it suffices to show that for any simple cycle $\omega \subset \Gamma$, the length of an edge $e \in \omega$ is bounded above by the summation of the lengths of other edges in ω . Let n be the number of edges in ω . We induct on n . The case $n = 3$ follows from the assumption. The case $n \geq 4$ follows from the induction assumption and from the fact that ω is not 2-full. \square

Let Λ_v^+ be the full subgraph of Λ_v spanned by $\{b^i, a^o, d_0^{-1}o, d_1^{-1}o, \dots, d_n^{-1}o\}$. Let Λ_v^- be the full subgraph of Λ_v spanned by $\{b^o, a^i, u_0^{-1}o, u_1^{-1}o, \dots, u_n^{-1}o\}$.

Lemma 5.7 *Each of Λ_v^+ and Λ_v^- is a tree of cliques.*

Proof We define the following sets of vertices of Λ_v^+ .

- (1) $V_1 = \{b^i, d_n^{-1}o, d_{n-1}^{-1}o, \dots, d_2^{-1}o\}$;
- (2) $V_2 = \{b^i, d_{n-1}^{-1}o, d_{n-2}^{-1}o, \dots, d_1^{-1}o\}$;
- (3) $V_3 = \{a^o, d_{n-1}^{-1}o, d_{n-2}^{-1}o, \dots, d_1^{-1}o\}$;
- (4) $V_4 = \{a^o, d_{n-2}^{-1}o, d_{n-3}^{-1}o, \dots, d_0^{-1}o\}$.

By Lemmas 5.2 and 5.3, each V_i spans a complete subgraph, which we denote by Δ_i . Moreover, $\Lambda_v^+ = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$. Definition 5.5 (2) can be verified directly. Thus Λ_v^+ is a tree of cliques. Similarly, Λ_v^- is a tree of cliques. □

Lemma 5.8 *Let $\sigma \subset \Lambda_v$ be a simple cycle such that $\sigma \not\subset \Lambda_v^+$ and $\sigma \not\subset \Lambda_v^-$. Then $\ell^{-1}o \in \sigma$ and $r^{-1}o \in \sigma$. Consequently, if σ is 2-full simple n -cycle in Λ_v for $n \geq 4$, then $\ell^{-1}o \in \sigma$ and $r^{-1}o \in \sigma$.*

Proof It follows from Lemmas 5.2 and 5.3 (3) that there are no edges between a vertex in $\Lambda_v^+ \setminus \{\ell^{-1}o, r^{-1}o\}$ and a vertex in $\Lambda_v^- \setminus \{\ell^{-1}o, r^{-1}o\}$. Thus vertices of $\Lambda_v^+ \setminus \{\ell^{-1}o, r^{-1}o\}$ and vertices of $\Lambda_v^- \setminus \{\ell^{-1}o, r^{-1}o\}$ are in two different connected components of $\Lambda_v \setminus \{\ell^{-1}o, r^{-1}o\}$. Since σ is a simple cycle, it follows that at least one of the following three situations happens: (1) $\sigma \subset \Lambda_v^+$; (2) $\sigma \subset \Lambda_v^-$; (3) $r^{-1}o \in \sigma$ and $\ell^{-1}o \in \sigma$. Thus the first statement follows. Lemmas 5.7 and 5.6 imply that (1) and (2) are not possible, thus the second statement follows. □

Lemma 5.9 *Any edge path in Λ_v from $r^{-1}o$ to $\ell^{-1}o$ has angular length $\geq \pi$.*

Proof Let ω be an edge path from $r^{-1}o$ to $\ell^{-1}o$. Since vertices of $\Lambda_v^+ \setminus \{\ell^{-1}o, r^{-1}o\}$ and vertices of $\Lambda_v^- \setminus \{\ell^{-1}o, r^{-1}o\}$ are in different components of $\Lambda_v \setminus \{\ell^{-1}o, r^{-1}o\}$, there is a sub-path $\omega' \subset \omega$ traveling from $r^{-1}o$ to $\ell^{-1}o$ such that $\omega' \subset \Lambda_v^+$ or $\omega' \subset \Lambda_v^-$. So it suffices to show any edge path ω in Λ_v^+ or Λ_v^- from $r^{-1}o$ to $\ell^{-1}o$ has angular length $\geq \pi$. We only prove the case $\omega \subset \Lambda_v^+$ since the other case is similar. Note that ω has to pass through at least one vertex in $\{d_i^{-1}o\}_{i=1}^{n-1}$, so we can divide into the following four cases.

Case 1: If there exists $1 < k < n - 1$ such that $d_k^{-1}o \in \omega$, then Lemma 5.6 (2), Lemma 5.7 and Lemma 5.3 imply that $\text{length}_\angle(\omega) \geq \angle_v(d_n^{-1}o, d_k^{-1}o) + \angle_v(d_k^{-1}o, d_0^{-1}o) = \frac{n-k}{2n}2\pi + \frac{k}{2n}2\pi = \pi$.

Case 2: If both $d_1^{-1}o$ and $d_{n-1}^{-1}o$ are in ω , then Lemma 5.6 (2), Lemmas 5.7 and 5.3 imply that $\text{length}_\angle(\omega) \geq \angle_v(d_n^{-1}o, d_{n-1}^{-1}o) + \angle_v(d_{n-1}^{-1}o, d_1^{-1}o) + \angle_v(d_1^{-1}o, d_0^{-1}o) = \frac{1}{2n}2\pi + \frac{n-2}{2n}2\pi + \frac{1}{2n}2\pi = \pi$.

Case 3: Suppose among $\{d_i^{-1}o\}_{i=1}^{n-1}$, only $d_1^{-1}o$ is inside ω . Then we must have $b^i \in \omega$ (since there has to be a vertex in ω which is adjacent to $r^{-1}o$). Thus $\text{length}_\angle(\omega) \geq \angle_v(d_n^{-1}o, b^i) + \angle_v(b^i, d_1^{-1}o) + \angle_v(d_1^{-1}o, d_0^{-1}o) = \frac{n-1}{4n}2\pi + \frac{n-1}{4n}2\pi + \frac{1}{2n}2\pi = \pi$.

Case 4: Suppose among $\{d_i^{-1}o\}_{i=1}^{n-1}$, only $d_{n-1}^{-1}o$ is inside ω . This can be dealt in the same way as the previous case. □

Proof of Proposition 5.1 (for real vertices) Proposition 5.1 (1) follows from Corollary 5.4 and (2) follows from Lemma 5.8 and Lemma 5.9. □

The following lemma will be used in Sect. 6.

- Lemma 5.10** (1) $d_{\angle}(a^i, b^i) = d_{\angle}(a^i, b^o) = d_{\angle}(a^o, b^i) = d_{\angle}(a^o, b^o) = \frac{n-1}{2n}2\pi$.
 (2) $d_{\angle}(a^i, a^o) = d_{\angle}(b^i, b^o) = \pi$.

Recall that d_{\angle} denotes the angular metric on Λ_v .

Proof Note that all edges of type II are between two interior vertices, and there are no edges between real vertices. Thus to travel from one real vertex to another real vertex in Λ_v , one has to go through at least two edges of type I. Then (1) follows from Lemma 5.2 (3). Now we prove (2). Still, traveling from a^i to a^o has to go through at least two edges of type I. However, one readily verifies that only two edges of type I do not bring one from a^i to a^o . So we need at least one other edge. By Lemmas 5.2 and 5.3, an edge in Λ_v has angular length at least $\frac{1}{2n}2\pi$. Thus $d_{\angle}(a^i, a^o) \geq \pi$. On the other hand, the distance π can be realized by $a^o \rightarrow d_{n-1}^{-1}o \rightarrow r^{-1}o \rightarrow a^i$. Thus $d_{\angle}(a^i, a^o) = \pi$. Similarly, we obtain $d_{\angle}(b^i, b^o) = \pi$. \square

It is natural to ask when an edge path in Λ_v from $r^{-1}o$ to $\ell^{-1}o$ has angular length exactly π . We record the following simple observation about such edge paths. The following will be crucial for applications in [34].

Lemma 5.11 *Suppose v is real and ω is an edge path in Λ_v from $r^{-1}o$ to $\ell^{-1}o$ of angular length π . Then either $\omega \subset \Lambda_v^+$ or $\omega \subset \Lambda_v^-$. If $\omega \subset \Lambda_v^+$, then the following are the only possibilities for ω :*

- (1) $\omega = r^{-1}o \rightarrow b^i \rightarrow d_1^{-1}o \rightarrow d_0^{-1}o$;
- (2) $\omega = d_n^{-1}o \rightarrow d_{n-1}^{-1}o \rightarrow a^o \rightarrow \ell^{-1}o$;
- (3) $\omega = d_{i_1}^{-1}o \rightarrow d_{i_2}^{-1}o \rightarrow \dots \rightarrow d_{i_k}^{-1}o$, where $n = i_1 > i_2 > \dots > i_k = 0$.

A similar statement holds for $\omega \subset \Lambda_v^-$.

Proof Note that ω is embedded, otherwise we can pass to a shorter path from $r^{-1}o$ to $\ell^{-1}o$, which contradicts Lemma 5.9. The statement $\omega \subset \Lambda_v^+$ or $\omega \subset \Lambda_v^-$ follows from the fact that there are no edges between a vertex in $\Lambda_v^+ \setminus \{r^{-1}o, \ell^{-1}o\}$ and a vertex in $\Lambda_v^- \setminus \{r^{-1}o, \ell^{-1}o\}$. Now we assume $\omega \subset \Lambda_v^+$.

If ω does not contain any real vertices, then we are in case (3), by Lemma 5.3 (4). If ω contains a real vertex, then it contains at least two edges of type I. Note that the angular length of ω with two edges of type I removed is $\pi - \frac{n-1}{2n}2\pi = \frac{1}{2n}2\pi$, which equals to the smallest possible angular length of edges in Λ_v . Thus we are in cases (1) or (2). \square

5.2 Link of an interior vertex

In this section we prove Proposition 5.1 for an interior vertex v .

We assume without loss of generality that v is the interior vertex o of the base cell Π . Moreover, we assume $v \in X_n$ for $n \geq 3$, since the $n = 2$ case is clear. Vertices of Λ_v can be divided into the following two classes.

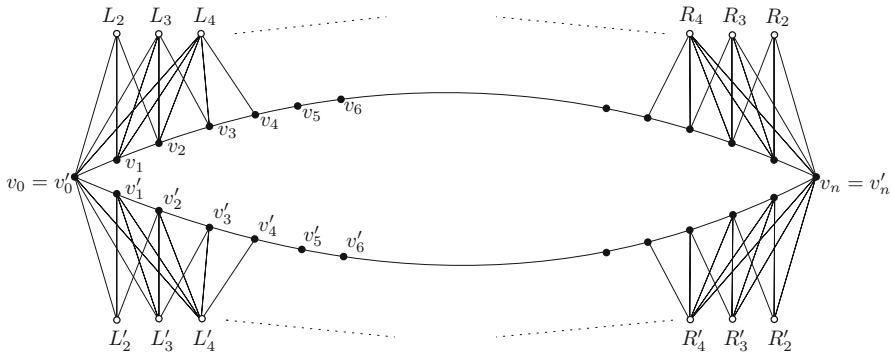


Fig. 9 The link of an interior vertex

- (1) Real vertices. These are the vertices in $\partial\Pi$.
- (2) Interior vertices. They are the interior vertices of some cell Π' such that $\Pi' \cap \Pi$ contains at least two edges.

For the convenience of the proof, we name the vertices in $\partial\Pi$ differently in this section. The vertices in the upper half (resp. lower half) of $\partial\Pi$ are called v_0, v_1, \dots, v_n (resp. v'_0, v'_1, \dots, v'_n) from left to right. Note that $v_0 = v'_0$ and $v_n = v'_n$.

Let \mathcal{P} be the collection of subcomplexes of $\partial\Pi$ such that

- (1) they are homeomorphic to the unit interval $[0, 1]$;
- (2) each of them has m edges where $2 \leq m \leq n - 1$;
- (3) each of them is contained in a half of $\partial\Pi$, and has nontrivial intersection with $\{\ell, r\} \subset \partial\Pi$.

By Lemma 4.1 (3), for each interior vertex of Λ_v , the intersection of the cell containing this interior vertex and Π is an element in \mathcal{P} . This actually induces a one to one correspondence between interior vertices of Λ_v and elements of \mathcal{P} by Corollary 4.3. Thus we can name the interior vertices of Λ_v as follows. If the intersection of the cell which contains this interior vertex and Π is a path in the upper half (resp. lower half) of $\partial\Pi$ that starts at ℓ and has i edges, then we denote this interior vertex by L_i (resp. L'_i). If the intersection of the cell which contains this interior vertex and Π is a path in the upper half (resp. lower half) of $\partial\Pi$ that ends at r and has i edges, then we denote this interior vertex by R_i (resp. R'_i). Note that i is ranging from 2 to $n - 1$; see Fig. 9. Let Π_{L_i} be the cell that contains L_i . We define $\Pi_{L'_i}, \Pi_{R_i}$ and $\Pi_{R'_i}$ similarly.

Now we characterize edges in Λ_v . They are divided into three classes.

- (1) *Edges of type I.* They are edges between real vertices of Λ_v . Hence they are exactly edges in $\partial\Pi$. Each of them has angular length $= \frac{1}{2n} 2\pi$.
- (2) *Edges of type II.* They are edges between a real vertex and an interior vertex, and they are characterized by Lemma 5.12 below.
- (3) *Edges of type III.* They are edges between interior vertices of Λ_v , and they are characterized by Lemma 5.13 below.

We refer to Fig. 9 for a picture of Λ_v . Edges of type I and some edges of type II are drawn, but edges of type III are not drawn in the picture.

- Lemma 5.12** (1) *The collection of vertices in $\partial\Pi$ that are adjacent to L_i (resp. L'_i) is $\{v_0, v_1, \dots, v_i\}$ (resp. $\{v'_0, v'_1, \dots, v'_i\}$).*
 (2) *The collection of vertices in $\partial\Pi$ that are adjacent to R_i (resp. R'_i) is $\{v_n, v_{n-1}, \dots, v_{n-i}\}$ (resp. $\{v'_n, v'_{n-1}, \dots, v'_{n-i}\}$).*
 (3) *The angular length of any edge between L_i and a real vertex of Λ_v is $\frac{i}{4n}2\pi$. The same holds with L_i replaced by L'_i, R_i and R'_i .*

Proof Note that $\{v_0, v_1, \dots, v_i\}$ are the vertices of $\partial\Pi_{L_i} \cap \partial\Pi$. Thus the part of (1) concerning L_i holds. We can prove the rest of (1), as well as (2), in a similar way. For (3), pick v_m with $0 \leq m \leq i$, then $\angle_{v_m}(L_i, o) = \frac{n-i}{2n}2\pi$. Since $\Delta(v_m L_i o)$ is an isosceles triangle with v_m being the apex, (3) follows. \square

- Lemma 5.13** (1) *L_i and L_j (or R_i and R_j, L'_i and L'_j, R'_i and R'_j) are connected by an edge in Λ_v if and only if $|j - i| \leq n - 2$. Moreover, the length of this edge is $\phi(\frac{|j-i|}{2n}2\pi)$ (see Definition 4.5 for ϕ).*
 (2) *L_i and R_j (or L'_i and R'_j) are connected by an edge in Λ_v if and only if $i + j - n \geq 2$. Moreover, the length of this edge is $\phi(\frac{2n-i-j}{2n}2\pi)$.*
 (3) *L_i is not adjacent to any L'_j or R'_j . R_i is not adjacent to any L'_j or R'_j .*

Note that claims (1) and (2) concern the length, not the angular length of the edge.

Proof We prove (1). Suppose without loss of generality that $i < j$. By Lemma 4.1 (3), the number of edges in $\Pi_{L_i} \cap \Pi_{L_j}$ is $n - (j - i)$. Thus L_i and L_j are adjacent if and only if $n - (j - i) \geq 2$. Now the length formula in (1) follows from Lemma 4.6. Other parts of (1) can be proved in a similar way. (2) can be deduced in a similar way by noting that the number of edges in $\Pi_{L_i} \cap \Pi_{R_j}$ is $i + j - n$. (3) follows from Lemma 4.2. \square

Corollary 5.14 *The angular lengths of the three sides of each triangle in Λ_v satisfy the triangle inequality.*

Proof The case when the triangle contains a real vertex follows from Corollary 5.4 (2) (consider the 3-simplex of X_n spanned by this triangle and v). Now we assume the triangle has no real vertices.

Case 1: the three vertices of the triangle are L_i, L_j and L_k with $i < j < k$. By Lemma 4.6, the length of $\overline{oL_i}$ is $\phi(\frac{n-i}{2n}2\pi)$. By Lemma 5.13 (1), the length of $\overline{L_i L_j}$ is $\phi(\frac{j-i}{2n}2\pi)$. Since $\frac{n-j}{2n}2\pi + \frac{j-i}{2n}2\pi = \frac{n-i}{2n}2\pi$, we can arrange L_i, L_j, L_k, o in the unit circle as in Fig. 10 left such that the distance between any two points in $\{L_i, L_j, L_k, o\}$ in the Euclidean plane equal to the length of the edge between them in X_n . In particular, $\angle_o(L_i, L_j) + \angle_o(L_j, L_k) = \angle_o(L_i, L_k)$.

Case 2: the three vertices of the triangle are L_i, L_j and R_k with $i < j$. By Lemma 5.13 (2), $\frac{2n-i-k}{2n}2\pi < \pi$ and the length of $\overline{L_i R_k}$ is $\phi(\frac{2n-i-k}{2n}2\pi)$. Thus we can arrange L_i, L_j, o, R_k as in Fig. 10 right and argue as before. Then other cases are similar. \square

Let Λ_v^+ be the full subgraph of Λ_v spanned by

$$\{v_0, v_1, \dots, v_n\} \cup \{L_2, L_3, \dots, L_{n-1}\} \cup \{R_2, R_3, \dots, R_{n-1}\}.$$

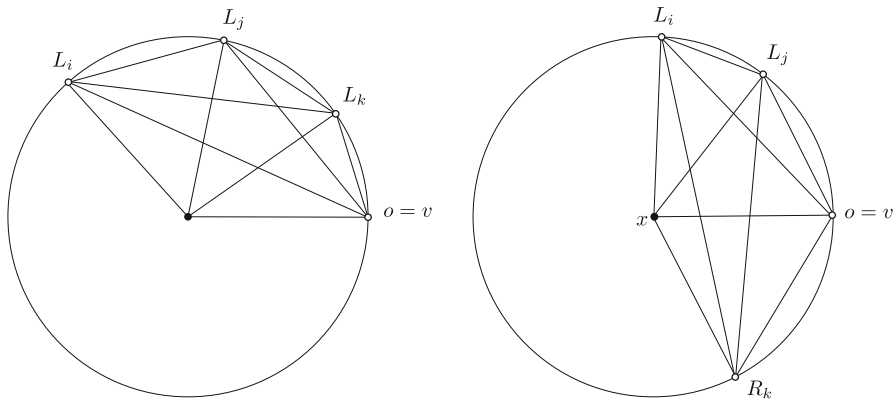


Fig. 10 Proof of Corollary 5.14: Case 1 on the left, Case 2 on the right

Let Λ_v^- be the full subgraph of Λ_v spanned by

$$\{v'_0, v'_1, \dots, v'_n\} \cup \{L'_2, L'_3, \dots, L'_{n-1}\} \cup \{R'_2, R'_3, \dots, R'_{n-1}\}.$$

Lemma 5.15 *Each of Λ_v^+ and Λ_v^- is a tree of cliques.*

Proof We only consider Λ_v^+ since Λ_v^- is similar. We define a sequence of collections of vertices of Λ_v^+ as follows. Let $S_1 = \{L_2, \dots, L_{n-1}\}$, $S_{n-1} = \{R_{n-1}, \dots, R_2\}$, and $S_i = \{L_{i+1}, \dots, L_{n-1}, R_{n-1}, \dots, R_{n-i+1}\}$ for $2 \leq i \leq n-2$. By Lemma 5.13, each S_i spans a clique. Moreover, any pair of adjacent interior vertices in Λ_v are contained in at least one of the S_i .

For $1 \leq i \leq n$, let $K_i = \{v_i, v_{i-1}\}$. For $1 \leq i \leq 2n-2$, $V_i = K_{\lfloor i/2 \rfloor + 1} \cup S_{\lceil i/2 \rceil}$ (e.g. $V_1 = K_1 \cup S_1$, $V_2 = K_2 \cup S_1$, $V_3 = K_2 \cup S_2$, $V_4 = K_3 \cup S_2, \dots, V_{2n-3} = K_{n-1} \cup S_{n-1}$, $V_{2n-2} = K_n \cup S_{n-1}$). By Lemma 5.12, each V_i spans a clique Δ_i . Moreover, any edge of Λ_v^+ is contained in at least one of the Δ_i . Thus $\Lambda_v^+ = \bigcup_{i=1}^{2n-2} \Delta_i$. Note that for each $2 \leq i \leq 2n-2$, $\Delta_i \setminus \Delta_{i-1}$ has exactly one vertex, and this vertex is not contained in $\bigcup_{m=1}^{i-1} \Delta_m$. Thus Definition 5.5 (2) holds, hence Λ_v^+ is a tree of cliques. \square

Lemma 5.16 *Let $\sigma \subset \Lambda_v$ be a simple cycle such that $\sigma \not\subseteq \Lambda_v^+$ and $\sigma \not\subseteq \Lambda_v^-$. Then $v_0 \in \sigma$ and $v_n \in \sigma$. Consequently, if σ is a 2-full simple n -cycle in Λ_v for $n \geq 4$, then $v_0 \in \sigma$ and $v_n \in \sigma$.*

Proof It follows from Lemmas 5.12 and 5.13 (3) that there are no edges between a vertex in $\Lambda_v^+ \setminus \{v_0, v_n\}$ and a vertex in $\Lambda_v^- \setminus \{v_0, v_n\}$. Based on Lemma 5.15, the rest of the proof is identical to the proof of Lemma 5.8. \square

Lemma 5.17 *Any edge path in Λ_v from v_0 to v_n has angular length $\geq \pi$.*

Proof Let $\omega \subset \Lambda_v$ be an edge path from v_0 to v_n . As in the proof of Lemma 5.9, we only consider the case $\omega \subset \Lambda_v^+$.

Case 1: There are two adjacent vertices of ω such that one is L_j and another is R_k . Note that v_0 is adjacent to L_j , and R_k is adjacent to v_n . As in the proof of

Lemma 5.14, we arrange L_j, o and R_k consecutively in a unit circle such that the Euclidean distance between any of two points in $\{L_j, o, R_k\}$ equals to the length of the edge in X_n between these two points. Consequently $\angle_v(L_j, R_k) = \angle_o^E(L_j, R_k)$, where \angle^E denotes the angle in the Euclidean plane. We refer to Fig. 10 right. Recall that in such an arrangement, $\angle_x^E(L_j, o) = \frac{n-j}{2n}2\pi$, $\angle_x^E(o, R_k) = \frac{n-k}{2n}2\pi$ and $\angle_x^E(L_j, R_k) = \frac{2n-k-j}{2n}2\pi < \pi$. By Lemma 5.12, $\angle_v(v_0, L_j) = \frac{j}{4n}2\pi$, thus $\angle_v(v_0, L_j) = \angle_{L_j}^E(x, o)$. Similarly, $\angle_v(R_k, v_n) = \angle_{R_k}^E(x, o)$. Since $\angle_{L_j}^E(x, o) + \angle_{R_k}^E(x, o) + \angle_o^E(L_j, R_k) = 2\pi - \angle_x^E(L_j, R_k) > \pi$. Hence $\angle_v(v_0, L_j) + \angle_v(L_j, R_k) + \angle_v(R_k, v_n) > \pi$. By Lemma 5.6 (2), Corollary 5.14 and Lemma 5.15, we have $\text{length}_\angle(\omega) \geq \angle_v(v_0, L_j) + \angle_v(L_j, R_k) + \angle_v(R_k, v_n) > \pi$.

Case 2: Suppose case (1) is not true and $\omega \cap \{L_2, \dots, L_{n-1}\} \neq \emptyset$. We suppose in addition that after the last vertex of ω in $\{L_2, \dots, L_{n-1}\}$ (say L_i), ω still contains at least one vertex from $\{R_2, \dots, R_{n-1}\}$.

Then L_i is followed by a sub-path ω' of ω with $\omega' \subset \partial\Pi$, and then a vertex R_k . Suppose the first and the last vertices of ω' are v_m and $v_{m'}$ respectively. Since L_i and v_m are adjacent, we have $m \leq i$ by Lemma 5.12 (1). Similarly, $m' \geq n - j$. By Lemmas 5.6 (2) and 5.15, $\text{length}_\angle(\omega) \geq \angle_v(v_0, L_i) + \angle_v(L_i, v_m) + \text{length}_\angle(\omega') + \angle_v(v_{m'}, R_j) + \angle_v(R_j, v_n)$. By Lemma 5.12,

$$\angle_v(v_0, L_i) = \angle_v(L_i, v_m) = \frac{i}{4n}2\pi; \quad \angle_v(R_j, v_n) = \angle_v(v_{m'}, R_j) = \frac{j}{4n}2\pi.$$

We are done if $i + j \geq n$. Now we assume $i + j < n$. Then $m \leq i < n - j \leq m'$ and $\text{length}_\angle(\omega') \geq \frac{n-j-i}{2n}2\pi$. Hence we still have $\text{length}_\angle(\omega) \geq \pi$.

Case 3: Suppose case (1) is not true and $\omega \cap \{L_2, \dots, L_{n-1}\} \neq \emptyset$. We suppose in addition that after the last vertex of ω in $\{L_2, \dots, L_{n-1}\}$ (say L_i), ω does not contain any vertex from $\{R_2, \dots, R_{n-1}\}$.

Then L_i is followed by a sub-path ω' of ω traveling from v_m to v_n . It follows that $\text{length}_\angle(\omega) \geq \angle_v(v_0, L_i) + \angle_v(L_i, v_m) + \text{length}_\angle(\omega')$. By Lemma 5.12, $m \leq i$ and $\angle_v(v_0, L_i) = \angle_v(L_i, v_m) = \frac{i}{4n}2\pi$. It follows that $\text{length}_\angle(\omega') \geq \frac{n-m}{2n}2\pi \geq \frac{n-i}{2n}2\pi$, and hence $\text{length}_\angle(\omega) \geq \pi$.

Case 4: Suppose $\omega \cap \{R_2, \dots, R_{n-1}\} \neq \emptyset$ and $\omega \cap \{L_2, \dots, L_{n-1}\} = \emptyset$. This is similar to the previous case.

Case 5: The remaining case is that ω does not contain any interior vertices. Then it is clear that $\text{length}_\angle(\omega) \geq \pi$. □

Proof of Proposition 5.1 (for interior vertices) In view of Corollary 5.14, it suffices to prove any 2-full simplex n -cycle in Λ_v with $n \geq 4$ has angular length $\geq 2\pi$. But this follows from Lemmas 5.16 and 5.17. □

The following is an analog of Lemma 5.11 in the case of interior vertex. It will be crucial for applications in [34].

Lemma 5.18 *Suppose v is interior and ω is an edge path in Λ_v from v_0 to v_n of angular length π . Then either $\omega \subset \Lambda_v^+$ or $\omega \subset \Lambda_v^-$. If $\omega \subset \Lambda_v^+$, then the following are the only possibilities for ω :*

- (1) ω does not contain interior vertices, i.e. $\omega = v_0 \rightarrow v_1 \rightarrow \dots \rightarrow v_n$;
- (2) $\omega = v_0 \rightarrow v_1 \rightarrow \dots \rightarrow v_{n-i_1} \rightarrow R_{i_1} \rightarrow \dots \rightarrow R_{i_m} \rightarrow v_n$, where $i_1 > \dots > i_m \geq 2$;
- (3) $\omega = v_0 \rightarrow L_{i_1} \rightarrow \dots \rightarrow L_{i_m} \rightarrow v_{i_m} \rightarrow v_{i_m+1} \rightarrow \dots \rightarrow v_n$, where $2 \leq i_1 < \dots < i_m$;
- (4) $\omega = v_0 \rightarrow L_{i_1} \rightarrow \dots \rightarrow L_{i_m} \rightarrow v_{i_m} \rightarrow v_{i_m+1} \rightarrow \dots \rightarrow v_{n-i'_1} \rightarrow R_{i'_1} \rightarrow \dots \rightarrow R_{i'_{m'}}$ $\rightarrow v_n$, where $2 \leq i_1 < \dots < i_m, i'_1 > \dots > i'_{m'} \geq 2$, and $i_m \leq n - i'_1$.

A similar statement holds when $\omega \subset \Lambda_v^-$.

Proof We argue as Lemma 5.11 to show that ω is embedded, and that $\omega \subset \Lambda_v^+$ or $\omega \subset \Lambda_v^-$. Now we assume $\omega \subset \Lambda_v^+$.

A *left interior component* of ω is a maximal connected sub-path of ω such that each of its vertices is one of the L_i . We define a *right interior component* in a similar way. By case 1 of Lemma 5.17, there is at least one real vertex between a left interior component and a right interior component.

We show there is at most one left interior component. Suppose the contrary is true. Let L_i be the first vertex of the last left interior component. The vertex in ω preceding L_i is a real vertex, which we denote by v_{i_0} . Since ω is embedded, $i_0 > 0$. Let ω' be the edge path consisting of the edge $\overline{v_0 L_i}$ together with the part of ω from L_i to v_n . By Lemma 5.17, $\text{length}_\angle(\omega') \geq \pi$. Since $\angle_v(v_0, L_i) = \angle_v(v_{i_0}, L_i)$ and $v_{i_0} \neq v_0$, $\text{length}_\angle(\omega') < \text{length}_\angle(\omega) = \pi$, which leads to a contradiction.

The same argument also shows that if $L_i \in \omega$ then the vertex of ω preceding L_i can not be some $v_{i'}$ with $i' \neq 0$. Thus, if there were a left interior component, then the vertex of ω following v_0 would be contained in such component.

Suppose there is a left interior component. Let L_i be the last vertex in this component and let $v_{i'}$ be any real vertex in the sub-path of ω from L_i to v_n . Then $i' \geq i$. To see this, we suppose the contrary $i' < i$ is true. Let ω' be the edge path consisting of $\overline{v_0 L_{i'}}$, $\overline{L_{i'} v_{i'}}$, and the part of ω from $v_{i'}$ to v_n . By Lemmas 5.6 (2) and 5.15, the angular length of the sub-path of ω from v_0 to L_i (from L_i to $v_{i'}$) is $\geq \angle_v(v_0, L_i)$ (resp. $\angle_v(L_i, v_{i'})$). By Lemma 5.12 (3), $\angle_v(v_0, L_i) > \angle_v(v_0, L_{i'})$ and $\angle_v(L_i, v_{i'}) > \angle_v(L_{i'}, v_{i'})$. Thus $\text{length}_\angle(\omega') < \text{length}_\angle(\omega) = \pi$, which is contradictory to Lemma 5.17.

We claim that if there are two consecutive vertices L_i and L_j in ω such that ω reaches L_i first, then $i < j$. To see this, note that by the proof of Corollary 5.14 (we can think the center of the circle in Fig. 10 left is v_0), $\angle_v(v_0, L_{i'}) + \angle_v(L_{i'}, L_{j'}) = \angle_v(v_0, L_{j'})$ for $i' < j'$. Thus if $i > j$, then the concatenation of $\overline{v_0 L_j}$ and the sub-path of ω from L_j to v_n has angular length $< \text{length}_\angle(\omega) = \pi$, which contradicts Lemma 5.17.

We can repeat the above discussion to obtain analogous statements for right interior components. Then the lemma follows. □

6 The complexes for 2-dimensional Artin groups

In this section we finalize the proof of one of the main results of the article, namely Theorem 1.2 from Introduction. More precisely, for any two-dimensional Artin group A_Γ we construct a metric simplicial complex X_Γ , by gluing together complexes X_n for

2-generated subgroups of A_Γ . In Lemma 6.4 we prove that X_Γ is simply connected, and in Lemma 6.6 we show that links of vertices in X_Γ are 2π -large. As an immediate consequence we obtain the main result of this section:

Theorem 6.1 X_Γ is metrically systolic. Consequently, each 2-dimensional Artin group is metrically systolic.

Let A_Γ be an Artin group with defining graph Γ . Let $\Gamma' \subset \Gamma$ be a full subgraph with induced edge labeling and let $A_{\Gamma'}$ be the Artin group with defining graph Γ' . Then there is a natural homomorphism $A_{\Gamma'} \rightarrow A_\Gamma$. By [48], this homomorphism is injective. Subgroups of A_Γ of the form $A_{\Gamma'}$ are called *standard subgroups*.

Let P_Γ be the standard presentation complex of A_Γ , and let X_Γ^* be the universal cover of P_Γ . We orient each edge in P_Γ and label each edge in P_Γ by a generator of A_Γ . Thus edges of X_Γ^* have induced orientation and labeling. There is a natural embedding $P_{\Gamma'} \hookrightarrow P_\Gamma$. Since $A_{\Gamma'} \rightarrow A_\Gamma$ is injective, $P_{\Gamma'} \hookrightarrow P_\Gamma$ lifts to various embeddings $X_{\Gamma'}^* \rightarrow X_\Gamma^*$. Subcomplexes of X_Γ^* arising in such way are called *standard subcomplexes*.

A *block* of X_Γ^* is a standard subcomplex which comes from an edge in Γ . This edge is called the *defining edge* of the block. Two blocks with the same defining edge are either disjoint, or identical.

We define precells of X_Γ^* as in 4.1, and subdivide each precell as in Fig. 7 to obtain a simplicial complex X_Γ^Δ . Interior vertices and real vertices of X_Γ^Δ are defined in a similar way. We record the following simple observations.

- Lemma 6.2** (1) Each element of A_Γ maps one block of X_Γ^Δ to another block with the same defining edge;
- (2) if $g \in A_\Gamma$ such that g maps an interior vertex of a block of X_Γ^Δ to another interior vertex of the same block, then g stabilizes this block;
- (3) the stabilizer of each block of X_Γ^Δ is a conjugate of a standard subgroup of A_Γ .

Within each block of X_Γ^Δ , we add edges between interior vertices as in Sect. 4.2. Then we take the flag completion to obtain X_Γ . By Lemma 6.2, the newly added edges are compatible with the action of deck transformations $A_\Gamma \curvearrowright X_\Gamma^\Delta$. Thus the action $A_\Gamma \curvearrowright X_\Gamma^\Delta$ extends to a simplicial action $A_\Gamma \curvearrowright X_\Gamma$, which is proper and cocompact. A *block* in X_Γ is defined to be the full subcomplex spanned by vertices in a block of X_Γ^Δ . Two blocks of X_Γ that have the same defining edge are either disjoint or identical.

Lemma 6.3 Any isomorphism between a block in X_Γ^* and the space X_n^* (cf. Sect. 4.1) that preserves the labeling and orientation of edges extends to an isomorphism between a block in X_Γ and the space X_n (cf. Sect. 4.2).

Proof By our construction, it suffices to show that if two vertices v_1 and v_2 in a block $B \subset X_\Gamma^*$ are not adjacent in this block, then they are not adjacent in X_Γ^* . However, this follows from a result of Charney and Paris ([21]) that $B^{(1)}$ is convex with respect to the path metric on the 1-skeleton of X_Γ^* . \square

Lemma 6.4 X_Γ is simply connected.

Proof Let f be an edge of X_Γ not in X_Γ^Δ . Then there are two cells Π_1 and Π_2 such that f connects the interior vertices $o_1 \in \Pi_1$ and $o_2 \in \Pi_2$. By construction, Π_1 and Π_2 are in the same block. Thus f and a vertex of $\Pi_1 \cap \Pi_2$ span a triangle. By flagness of X_Γ , f is homotopic rel its end points to the concatenation of other two sides of this triangle, which is inside X_Γ^Δ .

Now we show that each loop in X_Γ is null-homotopic. Up to homotopy, we assume this loop is a concatenation of edges of X_Γ . If some edges of this loop are not in X_Γ^Δ , then we can homotop these edges rel their end points to paths in X_Γ^Δ by the previous paragraph. Thus this loop is homotopic to a loop in X_Γ^Δ , which must be null-homotopic since X_Γ^Δ is simply connected. \square

Next, we assign lengths to edges of X_Γ in an A_Γ -invariant way.

Let $B \subset X_\Gamma$ be a block with its defining edge labeled by n . By Lemma 6.3, there is a simplicial isomorphism $i : B \rightarrow X_n$ that is label and orientation preserving. Note that all the edges between real vertices of X_n has the same length, which we denote p . We define the length of an edge $e \subset B$ to be $\text{length}(i(e))/p$. Then, for each vertex $b \in B$, the isomorphism $\text{lk}(b, B^{(1)}) \rightarrow \text{lk}(i(b), X_n^{(1)})$ induced by i preserves the angular lengths of edges. In particular, Proposition 5.1 holds for B .

We repeat this process for each block of X_Γ . Each edge of X_Γ belongs to at least one block, so it has been assigned at least one value of length. If an edge belongs to two different blocks, then both endpoints of this edge are real vertices, hence all values of lengths assigned to this edge equal to 1 by the previous paragraph. In summary, each edge of X_Γ has a well-defined length. Moreover, such assignment of lengths is A_Γ -invariant by Lemma 6.2.

Lemma 6.5 *Each simplex of X_Γ is contained in a block.*

Proof Suppose there is an interior vertex v of the simplex Δ . Let Π be the cell containing v and B be the unique block containing v . Then any real vertex of X_Γ adjacent to v is contained in Π and any interior vertex of X_Γ adjacent to v is contained in B . Since B is a full subcomplex, we have $\Delta \subset B$. If Δ does not contain any interior vertices, then Δ is a point, or an edge, and the lemma is clear. \square

In particular, each triangle of X_Γ is contained in a block, its side lengths satisfy the strict triangle inequality by Lemma 4.7. We define the angular metric on the link of each vertex of X_Γ as before.

Lemma 6.6 *Let $v \in X_\Gamma^{(1)}$ be a vertex and let $\Lambda_v = \text{lk}(v, X_\Gamma^{(1)})$.*

- (1) *The angular lengths of the three sides of each triangle in Λ_v satisfy the triangle inequality.*
- (2) *Λ_v is 2π -large.*

Proof The 3-simplex spanned by v and a triangle in Λ_v is inside a block by Lemma 6.5. Then (1) follows from Corollaries 5.4 and 5.14.

Now we prove (2). If v is an interior vertex, then there is a unique block $B \subset X_\Gamma$ containing this vertex, and any other vertex in $X_\Gamma^{(1)}$ adjacent to v is contained in this

block. Since B is a full subcomplex of X_Γ , $\text{lk}(v, X_\Gamma^{(1)}) = \text{lk}(v, B^{(1)})$, which is 2π -large by Proposition 5.1.

We assume v is a real vertex. Let ω be a 2-full simple n -cycle in $\text{lk}(x, X_\Gamma^{(1)})$ for $n \geq 4$. If ω is contained in a block, then we know that $\text{length}_\angle(\omega) \geq 2\pi$ by Proposition 5.1. The case when ω is not contained in a block follows from Lemma 6.7. \square

Lemma 6.7 *Let $v \in X_\Gamma$ be a real vertex and let $\Lambda_v = \text{lk}(v, X_\Gamma^{(1)})$. Let ω be a simple cycle with angular length $\leq 2\pi$ in the link of v . Then exactly one of the following four situations happens:*

- (1) ω is contained in one block;
- (2) ω travels through two different blocks B_1 and B_2 such that their defining edges intersect in a vertex a , and ω has angular length π inside each block; moreover, there are exactly two vertices in $\omega \cap B_1 \cap B_2$ and they correspond to an incoming a -edge and an outgoing a -edge based at v ;
- (3) ω travels through three blocks B_1, B_2, B_3 such that the defining edges of these blocks form a triangle $\Delta(abc) \subset \Gamma$ and $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1$ where n_1, n_2 and n_3 are labels of the edges of this triangle; moreover, ω is a 6-cycle with its vertices alternating between real and fake such that the three real vertices in ω correspond to an a -edge, a b -edge and a c -edge based at v ;
- (4) ω travels through four blocks such that the defining edges of these blocks form a full 4-cycle in Γ ; moreover, ω is a 4-cycle with one edge of angular length $\pi/2$ in each block.

Note that in cases (2), (3) and (4), ω actually has angular length 2π .

Proof Note that each interior vertex of Λ_v is contained in a unique block. Since each edge of Λ_v contains at least one interior vertex (otherwise we would have a triangle in X_Γ with all its vertices being real, which is impossible), each edge of Λ_v is contained in a unique block. Thus, there is a decomposition $\omega = \{\omega_i\}_{i \in \mathbb{Z}/n\mathbb{Z}}$ such that

- (1) each ω_i is a maximal sub-path of ω that is contained in a block (we denote this block by B_i);
- (2) $\omega_i \cap \omega_{i+1}$ is made of one or two real vertices.

Let $\{v_i\}_{i \in \mathbb{Z}/n\mathbb{Z}}$ be real vertices in ω such that the endpoints of ω_i are v_i and v_{i+1} . It follows from Lemma 5.10 that $\text{length}_\angle(\omega_i) \geq \pi/2$. Thus $n \leq 4$. The case $n = 4$ leads to case (4) in the lemma. It remains to consider the case $n = 3$ and $n = 2$.

Each v_i arises from an edge between x and v_i . This edge is inside X_Γ^* , hence it is labeled by a generator of A_Γ , corresponding to a vertex $z_i \in \Gamma$. Since v_i corresponds to either an incoming, or an outgoing edge labeled by z_i , we will also write $v_i = z_i^i$ or $v_i = z_i^o$. Let e_i be the defining edge of B_i . Then

$$z_{i+1} \in e_i \cap e_{i+1}. \tag{5}$$

If $n = 2$, then (5) implies that $z_0 = z_1$. Thus Lemma 5.10 (2) implies that $\text{length}_\angle(\omega_i) \geq \pi$ for $i = 0, 1$. Thus case (2) in the lemma follows.

Suppose $n = 3$. Recall that two blocks of X_Γ with the same defining edge are either disjoint or identical. Thus $e_i \neq e_{i+1}$ (otherwise both ω_i and ω_{i+1} are contained

in B_i). By (5), either $z_0 = z_1 = z_2$, or e_0, e_1 and e_2 span a triangle Δ in Γ . The former case is not possible because of the parity. Let n_i be the label of e_i . Note that $z_i \neq z_{i+1}$ for each i . Hence $\text{length}_{\angle}(\omega_i) \geq \frac{n_i-1}{2n_i}2\pi$ by Lemma 5.10 (1). Thus $\text{length}_{\angle}(\omega) \geq (3 - \frac{1}{n_0} - \frac{1}{n_1} - \frac{1}{n_2})\pi \geq 2\pi$, where the last inequality follows from the fact that A_{Γ} is 2-dimensional. Case (3) in the lemma follows. \square

7 Ending remarks and open questions

7.1 Open questions

The class of metrically systolic complexes contains the class of all 2-dimensional $CAT(0)$ piecewise Euclidean simplicial complexes and the class of systolic complexes. This motivates the following natural questions.

Question 7.1 *Let G be a metrically systolic group. Is every abelian subgroup of G quasi-isometrically embedded? Are solvable subgroups of G virtually abelian?*

At least we know the answer is positive for 2-dimensional Artin groups. A proof is given in Sect. 7.2. See [6, Chapter II.7] and [41] for results in the $CAT(0)$ and systolic settings.

Question 7.2 *Let G be a metrically systolic group. Is the centralizer of any infinite order element in G abstractly commensurable with $F_k \times \mathbb{Z}$? Here F_k is the free group with k generators and F_0 denotes the trivial group.*

The answer is affirmative for systolic groups [41].

Question 7.3 *Are metrically systolic groups semihyperbolic? Biautomatic?*

Biautomaticity for systolic groups has been established by Januszkiewicz-Świątkowski [35,46].

Question 7.4 *Does every finite group acting on a metrically systolic complex fix a point?*

A fixed point theorem for $CAT(0)$ spaces follows from convexity of the distance function [6, Chapter II.2]. A fixed point theorem for systolic complexes has been proven in [19].

Question 7.5 *Let X be a metrically systolic complex. Is X contractible? Does X satisfy $S^k FRC$ in the sense of [36] for all $k \geq 2$?*

It is proved in Sect. 3.3 that X has trivial second homotopy group and X is $S^2 FRC$. It is proved in [17,36] that the answer to Question 7.5 is affirmative for systolic complexes.

Question 7.6 *Is there a notion of boundary for metrically systolic complexes which generalizes both the $CAT(0)$ case and the systolic case?*

See [6, Chapter II.8] for the definition of $CAT(0)$ boundaries and [40] for the definition of boundaries of systolic complexes.

7.2 Abelian and solvable subgroups

For each Artin group A_Γ , Charney and Davis [13] defined an associated *modified Deligne complex* D_Γ . Now we recall their construction in the 2-dimensional case. Vertices of D_Γ are in one to one correspondence with left cosets of the form $gA_{\Gamma'}$, where $g \in A_\Gamma$ and Γ' is either the empty-subgraph of Γ (in which case $A_{\Gamma'}$ is the identity subgroup), or a vertex of Γ , or an edge of Γ . The *rank* of a vertex $gA_{\Gamma'}$ is the number of vertices in Γ' . Note that the set V of the vertices has a partial order induced by inclusion of sets. A collection $\{v_i\}_{i=1}^k \subset V$ spans a $(k - 1)$ -dimensional simplex if $\{v_i\}_{i=1}^k$ form a chain with respect to the partial order. It is clear that D_Γ is a 2-dimensional simplicial complex, and A_Γ acts on D_Γ without inversions, i.e. if an element of A_Γ fixes a simplex of D_Γ , then it fixes the simplex pointwise.

We endow D_Γ with a piecewise Euclidean metric such that each triangle $\Delta(g_1A_s, g_2A_s, g_3A_{\overline{st}})$ is a Euclidean triangle with angle $\pi/2$ at g_2A_s and angle $\frac{\pi}{2n}$ at $g_3A_{\overline{st}}$ with n being the label of the edge \overline{st} of Γ . By [13, Proposition 4.4.5], D_Γ is $CAT(0)$ with such metric. As being observed in [22, Lemma 6], the action $A_\Gamma \curvearrowright D_\Gamma$ is semisimple.

Theorem 7.7 *Let A_Γ be a 2-dimensional Artin group. Then every abelian subgroup of A_Γ is quasi-isometrically embedded.*

Proof Let $A \subset A_\Gamma$ be an abelian subgroup. By [13, Theorem B] and [13, Corollary 1.4.2], the presentation complex of A_Γ is a $K(A_\Gamma, 1)$ space. Thus A is a free abelian with rank ≤ 2 . First we assume $A \cong \mathbb{Z}$. Since $A_\Gamma \curvearrowright D_\Gamma$ is semisimple, by [6, Chapter II.6], either A acts by translation on a $CAT(0)$ geodesic line $\ell \subset D_\Gamma$, or A fixes a point $x \in D_\Gamma$. In the former case, we conclude A is quasi-isometrically embedded by noting that any orbit map from A_Γ to D_Γ is coarsely Lipschitz. In the later case, since A_Γ acts on D_Γ without inversions, A fixes a vertex in D_Γ . Thus, up to conjugation, we may assume that A is contained in a standard subgroup $A_{\Gamma'}$ with Γ' being a vertex or an edge. Any dihedral Artin group is $CAT(0)$ [7], so A is quasi-isometrically embedded in $A_{\Gamma'}$ (alternatively, dihedral Artin groups are $C(4)$ -T(4) [43], hence they are biautomatic [28]). By [21, Theorem 1.2], $A_{\Gamma'}$ is quasi-isometrically embedded in A_Γ . Hence A is quasi-isometrically embedded in A_Γ .

Now we assume $A \cong \mathbb{Z} \oplus \mathbb{Z}$. By [6, Theorem II.7.20], either there is an A -invariant flat plane $P \subset X_\Gamma$ upon which A acts geometrically, or there is an A -invariant $CAT(0)$ geodesic line $\ell \subset X_\Gamma$ upon which A acts cocompactly, or A fixes a point. The first and the third case can be handled in a similar way. Now we assume the second case. There is a group homomorphism $h : A \rightarrow \mathbb{R}$ by considering translation length of elements of A along ℓ . Since A acts on D_Γ by cellular isometries, there exists $\epsilon > 0$ such that any element of A with nonzero translation length has translation length $> \epsilon$. Hence $h(A) \cong \mathbb{Z}$. Thus by passing to a finite index subgroup of A , we assume $A = \langle a_1, a_2 \rangle$ such that a_1 fixes a point in ℓ and a_2 has nonzero translation length.

Let $p : A_\Gamma \rightarrow \ell$ be the composition of an orbit map from A_Γ to D_Γ and the $CAT(0)$ nearest point projection from D_Γ to ℓ . Then there exists $L > 0$ such that p is L -Lipschitz. Suppose the translation length of a_2 is L' . For $i = 1, 2$, suppose $\langle a_i \rangle \rightarrow A_\Gamma$ is an L_i -bi-Lipschitz embedding. For $k = n_1a_1 + n_2a_2 \in A$, let $\|k\|_\infty =$

$\max\{|n_1|, |n_2|\}$. Let $*$ be the identity element in A_Γ and let d_{A_Γ} denote the word metric on A_Γ with respect to the standard generating set. If $|n_1| \geq 2L_1L_2|n_2|$, then

$$d_{A_\Gamma}(k, *) \geq \frac{|n_1|}{L_1} - L_2n_2 \geq \frac{|n_1|}{2L_1} \geq \frac{1}{2L_1} \|k\|_\infty.$$

If $|n_1| < 2L_1L_2|n_2|$, then

$$\begin{aligned} d_{A_\Gamma}(k, *) &\geq L^{-1}d_\ell(p(n_1a_1 + n_2a_2), p(*)) = L^{-1}d_\ell(p(n_2a_2), p(*)) \\ &\geq L^{-1}L'|n_2| > \frac{L'}{2LL_1L_2} \|k\|_\infty. \end{aligned}$$

Now we conclude from the above two inequalities that $A \rightarrow A_\Gamma$ is a quasi-isometric embedding. \square

Corollary 7.8 *Nontrivial solvable subgroups of 2-dimensional Artin groups are either \mathbb{Z} or virtually \mathbb{Z}^2 .*

Proof Observe that, by Theorem 7.7 A_Γ is translation discrete, and by [20, Theorem 3.4] solvable subgroups of finite cohomological dimension in translation discrete groups are virtually \mathbb{Z}^n . Since, by [13, Theorem B] and [13, Corollary 1.4.2] the cohomological dimension is in our case at most 2 we have the assertion. \square

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