Exponential maps of a polynomial ring in two variables

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Keywords: exponential map, polynomial ring, locally finite iterative higher derivation, Rentschler-Miyanishi Theorem, Jung-van der Kulk Theorem

MSC 2010: Primary: 14R10, Secondary: 13A50, 14R20

Abstract

Let k be a field. We show that the ring of invariants of every exponential map on $k[x, y]$ is generated by the image of x or y under a composition of triangular automorphisms. From this we obtain two well-known theorems on $k[x, y]$ with no restriction on the characteristic of k: the Rentschler-Miyanishi Theorem and the Jung-van der Kulk Theorem.

1 Introduction

Throughout this paper k is a field, $p \geq 0$ is the characteristic of k, and $k^* = k \setminus 0$.

Let $R = k[x, y]$ be a polynomial ring in two variables x and y. The 1942 theorem of Jung [8] states that, when the characteristic of k is zero, the automorphism group of R is generated by linear automorphisms and triangular automorphisms of R. In other words, all automorphisms of R are tame. Many proofs of Jung's Theorem have appeared over the decades. (See [6].) Jung's result was first extended to allow the prime characteristic case in 1953 by van der Kulk [9]. Apart from the proof of van der Kulk, another approach to the prime characteristic case was given by Makar-Limanov in [10]. (See [1] and [5] as well.) A well-known paper of Rentschler [15] used

*Supported by the Fundação de Amparo a Pesquisa do Estado de São Paulo, Brazil processo no. 2015/25280-1.

locally nilpotent derivations on R to provide a short proof of Jung's Theorem. The main result, now known as Rentschler's Theorem, states that, assuming k has characteristic zero, every locally nilpotent derivation of R is equivalent up to the conjugation by an automorphism to $f(x)\frac{\partial}{\partial y}$ for some polynomial $f(x)$. A prime characteristic version of Rentschler's Theorem was given by Miyanishi in [13]. Locally nilpotent derivations are less potent when the characteristic of k is prime, as they no longer correspond to algebraic actions of the additive group k^+ on $\operatorname{Spec} R$. In the prime characteristic case, Miyanishi instead considered locally finite iterative higher derivations.

In this paper, we use exponential maps which are equivalent to both k^+-actions and locally finite iterative higher derivations. (Both notions are defined below.) This approach leads to an algebraic proof of the Rentschler-Miyanishi Theorem that is independent of characteristic. From there we obtain the Jung-van der Kulk Theorem. The only other characteristic free proof of these theorems which is known to us is based on the techniques developed in [10].

2 Exponential maps

Let A be a ring. Let $\delta_t : A \to A[t]$ be a homomorphism. We say $\delta = \delta_t$ is an *exponential map on A* if

(i) $\varepsilon_0 \delta_t$ is the identity on A, where $\varepsilon_0 : A[t] \to A$ is evaluation at $t = 0$, and

(ii) $\delta_s \delta_t = \delta_{s+t}$, where $\delta_s : A \to A[s]$ is extended to a homomorphism $A[t] \to A[s,t]$ by $\delta_s(t) = t$.

Define the *ring of δ-invariants* to be the subalgebra

$$A^\delta = \{a \in A \mid \delta(a) = a\}.$$

Note that the standard inclusion of A in $A[t]$ is an exponential map on A. We call an exponential map *nontrivial* if it is not the standard inclusion.

If A is a k-algebra, we additionally require an exponential map to be a k-algebra homomorphism. Exponential maps on a k-algebra A are equivalent to algebraic actions of the additive group k^+ on $\operatorname{Spec}(A)$. See [12] for details. If δ is an exponential map on A, the ring of δ-invariants coincides with the ring of invariants of the corresponding k^+-action.

If $\delta : A \to A[t]$ is an exponential map on A, then for each $a \in A$ we can write

$$\delta(a) = \sum_{i=0}^{\infty} \delta^{(i)}(a) t^i.$$

The sequence of maps $\{\delta^{(i)}\}_{i=0}^\infty$ is a *locally finite iterative higher derivation on A*. By definition, this is a sequence of linear maps on A with all of the following properties:

(i) For each $a \in A$, the sequence $\{\delta^{(i)}(a)\}_{i=0}^\infty$ has finitely many nonzero terms.

(ii) $\delta^{(0)}$ is the identity map on A.

(iii) (*Leibniz rule*) For all integers $n \geq 0$ and for all $a, b \in A$,

$$\delta^{(n)}(ab) = \sum_{i+j=n} \delta^{(i)}(a) \delta^{(j)}(b).$$

(iv) (*iterative property*) For all nonnegative integers i and j,

$$\delta^{(i)}\delta^{(j)} = \binom{i+j}{i}\delta^{(i+j)}.$$

Note that $\delta^{(1)}$ is a *locally nilpotent* derivation on A, meaning that for each $a \in A$ there exists sufficiently large n for which $(\delta^{(1)})^n(a) = 0$. When the characteristic of k is zero, we have $\delta^{(i)} = \frac{1}{i!}(\delta^{(1)})^i$ for each i, so that $\delta = \exp(t\delta^{(1)})$ and $A^\delta = \ker \delta^{(1)}$.

Locally nilpotent derivations gained prominence with Rentschler's 1968 paper [15] and have been an essential tool since. To work over a prime characteristic field, it is beneficial to use exponential maps and locally finite iterative higher derivations in place of locally nilpotent derivations.

Suppose $\varphi : A \to B$ is an isomorphism of either rings or k-algebras and $\delta : A \to A[t]$ is an exponential map on A. Extend φ to an isomorphism $A[t] \to B[t]$ by $\varphi(t) = t$. Then $\varphi\delta\varphi^{-1}$ defines an exponential map on B with $B^{\varphi\delta\varphi^{-1}} = \varphi(A^\delta)$. Note also that $(\varphi\delta\varphi^{-1})^{(i)} = \varphi\delta^{(i)}\varphi^{-1}$ for all integers $i \geq 0$.

Given an exponential map $\delta : A \to A[t]$ on A, we can define the δ-*degree* of an element $a \in A$ by

$$\deg_\delta a = \deg_t \delta(a)$$

where $\deg_t 0 = -\infty$. Note that

$$A^\delta = \{a \in A \mid \deg_\delta a \leq 0\}.$$

In this paper we focus entirely on integral domains. If A is an integral domain, the function $\deg_\delta$ is a degree function on A, meaning for all $a, b \in A$ it satisfies

(i) $\deg_\delta(ab) = \deg_\delta a + \deg_\delta b$, and

(ii) $\deg_\delta(a + b) \leq \max\{\deg_\delta a, \deg_\delta b\}$.

3 Exponential maps on integral domains

First we collect some useful information regarding exponential maps on integral domains. The first two lemmas are commonly used and their proofs are omitted. Proofs are given in [3], for example, where the degree function of an exponential map is a key ingredient. However, the results can be found in earlier papers such as [14].

Lemma 3.1. *If δ is a nontrivial exponential map on an integral domain A then*

(a) A^δ is both factorially closed and algebraically closed in A.

(b) If $a \in A$ and $i \in \mathbf{Z}$ with $i \geq 0$, then $\deg_\delta(\delta^{(i)}(a)) \leq \deg_\delta a - i$. In particular, if $a \neq 0$ then $\delta^{(\deg_\delta a)}(a) \in A^\delta$.

Lemma 3.2. *If δ is a nontrivial exponential map on an integral domain A with characteristic $p \geq 0$ and $x \in A$ has the minimal positive δ-degree then*

(a) For each $i \in \mathbf{Z}^+$ we have $\delta^{(i)}(x) \in A^\delta$. Moreover, $\delta^{(i)}(x) = 0$ whenever $i \geq 2$ is not a power of p. In particular, if $p > 0$ then $\deg_\delta x$ is a power of p.

(b) Let $c = \delta^{(\deg_\delta x)}(x)$. Then $A \subseteq A^\delta[c^{-1}, x] \subset F$ where F is the field of fractions of A.

The next two lemmas provide useful extensions of Lemma 3.1(a).

Lemma 3.3. *Let δ be an exponential map on an integral domain A with characteristic $p \geq 0$. If $c_1 f^a + c_2 g^b \in A^\delta \setminus 0$ where $a > 1,\ b > 1 \in \mathbf{Z}$ and neither a nor b is a power of p, and $c_1, c_2 \in A^\delta \setminus 0$ then $f, g \in A^\delta$.*

Proof. We assume δ is nontrivial since otherwise there is nothing to show. If $p > 0$, write $a = p^j a_1$ and $b = p^l b_1$ where $j, l, a_1, b_1 \in \mathbf{Z}^+$ with $a_1 > 1$, $b_1 > 1$, and $p \nmid a_1$, $p \nmid b_1$. Then $c_1 f^a + c_2 g^b = c_1 (f^{p^j})^{a_1} + c_2 (g^{p^l})^{b_1}$. If we show that $f^{p^j}, g^{p^l} \in A^\delta \setminus 0$, then the result follows since A^δ is factorially closed by Lemma 3.1(a). Therefore we can assume neither a nor b is divisible by p.

It suffices to show $f \in A^\delta$, again since A^δ is factorially closed. Suppose $f \notin A^\delta$. Let

$$m = \min\{i \in \mathbf{Z}^+ \mid \delta^{(i)}(f) \neq 0 \text{ or } \delta^{(i)}(g) \neq 0\}.$$

Since $\delta^{(i)}(f) = \delta^{(i)}(g) = 0$ for $1 \leq i < m$, according to the Liebniz rule we have

$$0 = \delta^{(m)}(c_1 f^a + c_2 g^b) = a c_1 f^{a-1} \delta^{(m)}(f) + b c_2 g^{b-1} \delta^{(m)}(g). \tag{1}$$

Since $p \nmid a$ and $p \nmid b$, it follows that both $\delta^{(m)}(f) \neq 0$ and $\delta^{(m)}(g) \neq 0$.

Let $x \in A$ such that x has minimal positive δ-degree. Then $A \subseteq \mathrm{Frac}(A^\delta)[x]$ by Lemma 3.2(b) and we can view f and g as polynomials in x. Since $c_1 f^a + c_2 g^b \in \mathrm{Frac}(A^\delta) \setminus 0$, we see that f and g are relatively prime in $\mathrm{Frac}(A^\delta)[x]$. So from Equation (1), since $a > 1$ and $b > 1$, we have $f \mid \delta^{(m)}(g)$ and $g \mid \delta^{(m)}(f)$ in $\mathrm{Frac}(A^\delta)[x]$. Therefore $\deg_\delta f \leq \deg_\delta(\delta^{(m)}(g))$ and $\deg_\delta g \leq \deg_\delta(\delta^{(m)}(f))$. So, using Lemma 3.1(b), we have

$$\begin{aligned}
\deg_\delta f &\leq \deg_\delta(\delta^{(m)}(g)) \\
&\leq \deg_\delta g - m \\
&\leq \deg_\delta(\delta^{(m)}(f)) - m \\
&\leq \deg_\delta f - 2m.
\end{aligned}$$

This is only possible if $f = 0$, a contradiction. □

Lemma 3.4. *Let δ be an exponential map on an integral domain A with characteristic $p \geq 0$. If $c_1 f^a + c_2 g^{p^j} \in A^\delta \setminus 0$ where $a > 1,\ j > 0 \in \mathbf{Z},\ p \nmid a,\ c_1, c_2 \in A^\delta \setminus 0$ and f is prime in A then $f, g \in A^\delta$.*

Proof. We assume δ is nontrivial and $p > 0$ since otherwise there is nothing to show. If $j > 1$ and $g^{p^{j-1}} \in A^\delta \setminus 0$, then $g \in A^\delta$ since A^δ is factorially closed by Lemma 3.1(a). Therefore we can assume $j = 1$. It suffices to show $f \in A^\delta$, again since A^δ is factorially closed. Suppose $f \notin A^\delta$. Let $m = \min\{i \in \mathbf{Z}^+ \mid \delta^{(i)}(f) \neq 0\}$. We now examine how $\delta^{((a-1)m)}$ acts on f^a. By the Leibniz rule

$$\delta^{((a-1)m)}(f^a) = \sum_{i_1 + i_2 + \cdots + i_a = (a-1)m} \delta^{(i_1)}(f)\delta^{(i_2)}(f)\cdots\delta^{(i_a)}(f).$$

Since $\delta^{(0)}(f) = f$ and $\delta^{(i)}(f) = 0$ for $0 < i < m$, we have

$$\delta^{((a-1)m)}(f^a) = a f \delta^{(m)}(f)^{a-1} + f^2 h$$

for some $h \in A$. Since $c_1 f^a + c_2 g^p \in A^\delta \setminus 0$, we also have

$$c_1 \delta^{((a-1)m)}(f^a) = -c_2 \delta^{((a-1)m)}(g^p)$$

Now, $\delta^{(b)}(g^p) = (\delta^{(\frac{b}{p})}(g))^p$ as the characteristic of A is p. (Of course $\delta^{(b)}(g^p) = 0$ if $\frac{b}{p} \notin \mathbf{Z}$). Hence

$$c_1(af\delta^{(m)}(f)^{a-1} + f^2 h) = -c_2(\delta^{((a-1)m/p)}(g))^p.$$

Therefore f divides $\delta^{((a-1)m/p)}(g)$ in A since f is prime. So f^2 divides $\delta^{((a-1)m/p)}(g)^p$ in A since $p \geq 2$. Therefore f^2 divides $f\delta^{(m)}(f)^{a-1}$ in A, and so f divides $\delta^{(m)}(f)$ in A since f is prime. But this implies $\deg_\delta f \leq \deg_\delta(\delta^{(m)}(f))$ which is only possible if $f \in A^\delta$. This is a contradiction, completing the proof. □

4 Exponential maps and automorphisms of $k[x, y]$

We now turn our attention to the polynomial ring $R = k[x, y]$. We will use gradings and homogenization techniques to further investigate exponential maps on R. Let $a, b \in \mathbf{Z}^+$ be relatively prime. Define weights $w(x) = a$ and $w(y) = b$ and extend this to a degree function on R. This degree function determines a grading on R. If $r \in R$, let $\operatorname{gr} r$ denote the homogeneous leading form of r.

Let $\delta : R \to R[t]$ be a nontrivial exponential map on R. Define

$$w(t) = \min\left\{ \frac{w(r) - w(\delta^{(i)}(r))}{i} \;\middle|\; r \in \{x, y\}, i \in \mathbf{Z}^+ \right\} \tag{2}$$

Then we can extend w to a degree function on $R[t]$ and we have $w(\delta^{(n)}(r)t^n) \leq w(r)$ for all $r \in R$ and all integers $n \geq 0$. For each $r \in R$ define

$$S(r) = \{n \mid w(\delta^{(n)}(r)t^n) = w(r)\}$$

and define $\operatorname{gr}\delta$ on homogeneous elements by

$$(\operatorname{gr}\delta)(\operatorname{gr} r) = \sum_{n \in S(r)} \operatorname{gr}(\delta^{(n)}(r))t^n.$$

We can extend this linearly to obtain a map $\operatorname{gr}\delta : R \to R[t]$. In fact $\operatorname{gr}\delta$ is a nontrivial exponential map on R which we call the homogenization of δ. Moreover, we have

$$\operatorname{gr}(R^\delta) \subseteq R^{\operatorname{gr}\delta}$$

where

$$\operatorname{gr}(R^\delta) = \{\operatorname{gr} r \in R \mid r \in R^\delta\}.$$

See Proposition 2.2 in [4] for a proof of these facts.

Lemma 4.1. *Suppose $R = k[x, y]$ is graded by $w(x) = a$ and $w(y) = b$ for some relatively prime integers $a, b \in \mathbf{Z}^+$. Let δ be a nontrivial exponential map on R, and let $\operatorname{gr}\delta$ denote the homogenization of δ. Let $f \in R^\delta$ such that $\operatorname{gr} f \notin k$. Then either $\operatorname{gr} f = \lambda(x - \mu y^a)^m$ or $\operatorname{gr} f = \lambda(y - \mu x^b)^m$ for some $\lambda, \mu \in k$ with $\lambda \neq 0$ and some $m \in \mathbf{Z}^+$.*

Proof. Write $\operatorname{gr} f = x^i y^j g(x,y)$ for some integers $i, j \geq 0$ and some $g(x,y) \in R$ which is homogeneous with respect to the grading on R and for which $x \nmid g(x,y)$ and $y \nmid g(x,y)$. If $g(x,y)$ is not constant, then

$$g(x,y) = \lambda \prod_l (x^b - \mu_l y^a)$$

where $\lambda, \mu_l \in \overline{k}^*$, an algebraic closure of k. Note that $\operatorname{gr} \delta$ extends to a nontrivial exponential map D on $S = \overline{k}[x,y]$ with $D(x) = (\operatorname{gr} \delta)(x)$ and $D(y) = (\operatorname{gr} \delta)(y)$, and $\operatorname{gr} f \in S^D$. Furthermore, all factors of $\operatorname{gr} f$ must belong to S^D by Lemma 3.1(a). If g contains two distinct factors then $x,\ y \in S^D$ and D is trivial. Hence $g(x,y) = \lambda(x^b - \mu y^a)^m$ for some $\mu \in \overline{k}^*, m \in \mathbf{Z}^+$. Similarly D is trivial if any two of the integers $i,\ j,\ m$ are not equal to zero. Therefore either $\operatorname{gr} f = \lambda x^i$, or $\operatorname{gr} f = \lambda y^j$, or $\operatorname{gr} f = \lambda(x^b - \mu y^a)^m$.

Since $\operatorname{gr} f \in R$ it is clear that $\lambda \in k$. If $p \nmid m$ then μ is also in k. To show that $\mu \in k$ if $m \neq 0$ observe that $D(x^b - \mu y^a) = x^b - \mu y^a$. Hence $D^{(j)}(x^b - \mu y^a) = 0$ for all $j > 0$. Since $y \notin S^D$, there exists some $i \in \mathbf{Z}^+$ such that $D^{(i)}(y^a) \neq 0$. So

$$\mu = \frac{D^{(i)}(x^b)}{D^{(i)}(y^a)} = \frac{(\operatorname{gr} \delta)^{(i)}(x^b)}{(\operatorname{gr} \delta)^{(i)}(y^a)} \in k(x,y) \cap \overline{k}^* = k^*.$$

Therefore $\operatorname{gr} f = \lambda(x^b - \mu y^a)^m$ where $\lambda,\ \mu \in k$ with $\lambda \neq 0$.

Finally, if both $a > 1$ and $b > 1$, then from Lemmas 3.3 and 3.4 we must have $x, y \in R^{\operatorname{gr} \delta}$ which again is not possible. Therefore either $a = 1$ or $b = 1$, completing the proof. $\square$

Theorem 4.2. *Let k be a field, and let $R = k[x,y]$. Let δ be a nontrivial exponential map on R. Let $f \in R^\delta \setminus k$. Then there exists a finite sequence of automorphisms $\Delta_1, \ldots, \Delta_n \in \operatorname{Aut}_k R$ such that*

1. *For each $i = 1, \ldots, n$, the automorphism Δ_i is given by either*

$$\Delta_i(x) = x, \quad \Delta_i(y) = y + \mu_i x^{c_i}$$

   or

$$\Delta_i(x) = x + \mu_i y^{c_i}, \quad \Delta_i(y) = y$$

   for some $\mu_i \in k$ and some $c_i \in \mathbf{Z}^+$; and

2. *$\alpha(f) \in k[x]$ or $\alpha(f) \in k[y]$, where $\alpha = \Delta_n \cdots \Delta_1$.*

Furthermore, if $\alpha(f) \in k[x]$ (resp. $k[y]$), then $R^{\alpha\delta\alpha^{-1}} = k[x]$ (resp. $k[y]$) and $R^\delta = k[\alpha^{-1}(x)]$ (resp. $k[\alpha^{-1}(y)]$). Therefore $R^\delta = k[h]$ where h is the image of x or y under a composition of triangular automorphisms.

Proof. If $f \in k[x]$ or $f \in k[y]$, then we are done. Now suppose $f \notin k[x]$ and $f \notin k[y]$. Then $d_x = \deg_x f \neq 0$ and $d_y = \deg_y f \neq 0$ where $\deg_x f$ is the degree of f relative to x and $\deg_y f$ is the degree of f relative to y. Choose relatively prime integers $a, b \in \mathbf{Z}^+$ for which $ad_x = bd_y$ and define weights $w(x) = a$ and $w(y) = b$ to obtain a grading on R. If $w(f) > ad_x = bd_y$ then $\operatorname{gr} f$ must have both x and y as factors contrary to Lemma 4.1. Hence $w(f) = ad_x = bd_y$ and $\operatorname{gr} f$ contains monomials x^{d_x} and y^{d_y} with nonzero coefficients. So according to Lemma 4.1 we have $\operatorname{gr} f = \lambda(x - \mu y^a)^m$ or $\operatorname{gr} f = \lambda(y - \mu x^b)^m$ for some $\lambda, \mu \in k^*$ and $m \in \mathbf{Z}^+$. In the first case, if

$\operatorname{gr} f = \lambda(x - \mu y^a)^m$, consider the automorphism $\beta \in \operatorname{Aut}_k R$ given by $\beta(x) = x + \mu y^a$ and $\beta(y) = y$. Then $\beta(f) \in R^{\beta\delta\beta^{-1}}$ and $\operatorname{gr}(\beta(f)) = \lambda x^m$. Moreover, we have $\deg_y \beta(f) < d_y$ while $\deg_x \beta(f) = d_x$. Similarly in the second case, if $\operatorname{gr} f = \lambda(y - \mu x^b)^m$, consider the automorphism $\gamma \in \operatorname{Aut}_k R$ given by $\gamma(x) = x$ and $\gamma(y) = y + \mu x^b$. Then $\gamma(f) \in R^{\gamma\delta\gamma^{-1}}$ and $\operatorname{gr}(\gamma(f)) = \lambda y^m$. Moreover in this case, $\deg_x \gamma(f) < d_x$ while $\deg_y \gamma(f) = d_y$.

In either case, taking $\Delta_1 = \beta$ or $\Delta_1 = \gamma$, we replace δ by $\Delta_1\delta\Delta_1^{-1}$ and f by $\Delta_1(f) \in R^{\Delta_1\delta\Delta_1^{-1}} \setminus k$ with smaller total degree. If $\Delta_1(f)$ does not belong to $k[x]$ or $k[y]$, then we can repeat the process. Since the total degree will decrease at each step, this process can be repeated only finitely many times until we obtain an automorphic image of f which belongs to $k[x]$ or $k[y]$. This proves items (1) and (2).

For the remaining statement, suppose $\alpha(f) \in k[x]$. (The case when $\alpha(f) \in k[y]$ is handled analogously.) Since $\alpha(f) \in R^{\alpha\delta\alpha^{-1}} \setminus k$ as well, we conclude $x \in R^{\alpha\delta\alpha^{-1}}$ by Lemma 3.1(a). If there exists $g \in R^{\alpha\delta\alpha^{-1}} \setminus k[x]$, then again using Lemma 3.1(a) we must have $y \in R^{\alpha\delta\alpha^{-1}}$. But this is not possible since $\alpha\delta\alpha^{-1}$ is nontrivial. Therefore $R^{\alpha\delta\alpha^{-1}} = k[x]$. Finally, since $R^{\alpha\delta\alpha^{-1}} = \alpha(R^\delta)$, we have $R^\delta = \alpha^{-1}(k[x]) = k[\alpha^{-1}(x)]$. □

The following well-known description of the exponential maps of $k[x, y]$ is due to Rentschler in characteristic zero [15]. The prime characteristic case was established by Miyanishi in [13].

Theorem 4.3 (Rentschler, Miyanishi)**.** *Let k be a field with characteristic $p \geq 0$, and let $R = k[x, y]$. Let Γ be the subgroup of $\operatorname{Aut}_k R$ generated by triangular automorphisms of the form*

$$\Delta_{\mu,c}(x) = x, \quad \Delta_{\mu,c}(y) = y + \mu x^c$$

with $c \in \mathbf{Z}$, $c \geq 0$, and $\mu \in k$, and by the transposition $\tau(x) = y$, $\tau(y) = x$. Let $\delta : R \to R[t]$ be a nontrivial exponential map on R. Then there exists $\gamma \in \Gamma$ such that $\gamma\delta\gamma^{-1}(x) = x$ and y is an element of minimal positive $\gamma\delta\gamma^{-1}$-degree, so that

$$\gamma\delta\gamma^{-1}(y) = y + f_0(x)t + \sum_{i=1}^{n} f_i(x)t^{p^i}$$

for some $n \in \mathbf{Z}$ with $n \geq 0$ and some $f_0(x), \ldots, f_n(x) \in k[x]$, with $n = 0$ if $p = 0$.

Proof. By Theorem 4.2 there exists $\gamma \in \Gamma$ such that $R^{\gamma\delta\gamma^{-1}} = k[x]$ or $R^{\gamma\delta\gamma^{-1}} = k[y]$. Composing γ with τ if necessary, we can assume $R^{\gamma\delta\gamma^{-1}} = k[x]$. Therefore $\gamma\delta\gamma^{-1}(x) = x$. Noting that $\gamma\delta\gamma^{-1}$ extends to an exponential map on $k(x)[y]$ with ring of invariants $k(x)$, necessarily y must be an element of minimal positive $\gamma\delta\gamma^{-1}$-degree, and the result follows from Lemma 3.2(a). □

The following well-known description of the automorphism group of $k[x, y]$ is due to Jung in characteristic zero [8] and van der Kulk in prime characteristic [9].

Theorem 4.4 (Jung, van der Kulk)**.** *Let k be a field, and let $R = k[x, y]$. Let Γ be the subgroup of $\operatorname{Aut}_k R$ defined in Theorem 4.3. Then $Aut_k R$ is generated by Γ along with the linear automorphisms of the form $x \to \lambda_1 x$, $y \to \lambda_2 y$, where $\lambda_1, \lambda_2 \in k^*$. Therefore every automorphism of $k[x, y]$ is tame.*

Proof. Let $R = k[x, y]$. Let $\alpha \in \operatorname{Aut}_k R$. Let $f = \alpha(x)$ and $g = \alpha(y)$. Define an exponential map $\delta : R \to R[t]$ by $\delta(f) = f$ and $\delta(g) = g + t$. Since $R = k[f, g]$, we have $R^\delta = k[f]$ using Lemma 3.1(a). By Theorem 4.3 there exists an automorphism $\gamma \in \Gamma$ such that $R^{\gamma\delta\gamma^{-1}} = k[x]$

and y is an element of minimal positive $\gamma\delta\gamma^{-1}$-degree. Furthermore, as in Theorem 4.2 we have $R^\delta = k[\gamma^{-1}(x)]$. Thus $f = \mu\gamma^{-1}(x) + \lambda$ for some $\mu \in k^*$ and $\lambda \in k$. So $f = \gamma^{-1}(\mu x + \lambda)$.

Turning to $\gamma(g)$, note that for each integer $i \geq 0$ we have

$$(\gamma\delta\gamma^{-1})^{(i)}(\gamma(g)) = (\gamma\delta^{(i)}\gamma^{-1})(\gamma(g)) = \gamma\delta^{(i)}(g).$$

As $\delta^{(1)}(g) = 1$ and $\delta^{(i)}(g) = 0$ for $i > 1$, we therefore have

$$(\gamma\delta\gamma^{-1})(\gamma(g)) = \gamma(g) + t.$$

But y is an element of minimal positive $\gamma\delta\gamma^{-1}$-degree, and so $\deg_{\gamma\delta\gamma^{-1}}(y) = 1$ and

$$(\gamma\delta\gamma^{-1})(y) = y + h(x)t$$

for some $h(x) \in k[x]$. Since $(\gamma\delta\gamma^{-1})^{(1)}(y) = h(x)$ and $(\gamma\delta\gamma^{-1})^{(1)}(x) = 0$, we have

$$(\gamma\delta\gamma^{-1})^{(1)}(R) \subseteq h(x)R.$$

But

$$1 = (\gamma\delta\gamma^{-1})^{(1)}(\gamma(g)) \in h(x)R.$$

Therefore $h(x)$ must be a unit in R. So we have $h(x) = \xi \in k^*$ and

$$(\gamma\delta\gamma^{-1})(y) = y + \xi t.$$

Moreover, we have $\gamma(g) - \xi^{-1}y \in R^{\gamma\delta\gamma^{-1}} = k[x]$. So $\gamma(g) - \xi^{-1}y = q(x)$ for some $q(x) \in k[x]$. Therefore $g = \gamma^{-1}(\xi^{-1}y + q(x))$.

Let $\beta \in \operatorname{Aut}_k R$ be the triangular automorphism given by $\beta(x) = \mu x + \lambda$ and $\beta(y) = \xi^{-1}y + q(x)$. We have now shown that $f = \gamma^{-1}\beta(x)$ and $g = \gamma^{-1}\beta(y)$. Thus $\alpha = \gamma^{-1}\beta$, proving that α is tame.

Note that β can be written as a composition $\beta = \gamma_3\gamma_2\beta_1$ where β_1 is the linear automorphism given by $\beta_1(x) = \mu x$, $\beta_1(y) = \xi^{-1}y$, and $\gamma_2, \gamma_3 \in \Gamma$ are given by $\gamma_2(x) = x + \mu^{-1}\lambda$, $\gamma_2(y) = y$ and $\gamma_3(x) = x$, $\gamma_3(y) = y + \xi^{-1}q(x)$. This proves the claim. $\square$

5 A remark on the AK invariant

If A is a ring, we define the *AK invariant of* A to be the intersection of the rings of invariants over all exponential maps on A. The AK invariant can sometimes be used to distinguish a k-algebra from a polynomial ring over k, as any isomorphism of k-algebras restricts to an isomorphism of their AK invariants. The first example of this was given in [11] to show that the Koras-Russell hypersurface $x + x^2y + z^2 + t^3 = 0$ in $\mathbf{C}^4$ is not algebraically isomorphic to $\mathbf{C}^3$. This can be extended to prime characteristic, as shown in [2]. More recently, Gupta used similar techniques in [7] to establish the first counterexample to the Zariski Cancellation Problem. She showed that, when k has prime characteristic p, the Asanuma hypersurface $x^my + z^{p^e} + t + t^{sp} = 0$ (where $m, e, s \in \mathbf{Z}^+$ such that $p^e \nmid sp$ and $sp \nmid p^e$) is not algebraically isomorphic to k^3. In both results, the idea is to use homogenization techniques to tease out information on the rings of invariants of exponential maps on the coordinate ring. We note here that for each of these results, it is possible to streamline a section of the proofs using Lemmas 3.3 and 3.4. Hopefully these lemmas will be useful in other applications.

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