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Higher curvature Bianchi identities, generalised geometry and L_∞ algebras

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Abstract: The Bianchi identities for bosonic fluxes in supergravity can receive higher derivative quantum and string corrections, the most well known being that of Heterotic theory $\mathrm{d}H = \frac{1}{4}\alpha'(\operatorname{tr} F^2 - \operatorname{tr} R^2)$. Less studied are the modifications at order R^4 that may arise, for example, in the Bianchi identity for the seven-form flux of M theory compactifications. We argue that such corrections appear to be incompatible with the exceptional generalised geometry description of the lower order supergravity, and seem to imply a gauge algebra for the bosonic potentials that cannot be written in terms of an (exceptional) Courant bracket. However, we show that this algebra retains the form of an L_∞ gauge field theory, which terminates at a level ten multibracket.

Contents

1 Generalised geometry and Bianchi identities

Generalised geometry was originally introduced in [1, 2] as a way of combining complex and symplectic geometry, by considering structures on the generalised tangent bundle E

$$T^* \to E \to T, \tag{1.1}$$

which is naturally equipped with a Dorfman or Courant bracket [3, 4]. $E \sim T \oplus T^*$ is in fact an example of an exact Courant algebroid [5] and it possesses a three-form H that is closed [6],

$$\mathrm{d}H = 0, \tag{1.2}$$

which can be thought of as the curvature of a "gerbe" B [7], i.e. $H = \mathrm{d}B$ locally, which specifies a spliting of the sequence (1.1). Physicists quickly realised that this formalism provides a way of geometrising the NSNS sector of type II supergravity [8–11], B being identified with the Kalb-Ramond field and H being its flux. The Dorfman bracket along a generalised vector $L_X, X \in E$ then generates the combined (infinitesimal) bosonic symmetries of the theory: diffeomorphisms $\mathcal{L}_x$ by taking the Lie derivative along a vector $x \in T$, and gauge B-shifts by $\mathrm{d}\lambda$, exact two-forms parametrised by one-forms $\lambda \in T^*$. Introducing also a metric, it is possible to unify all the NSNS fields into a single object, and rewrite all the supergravity equations as a generalised geometry equivalent of Einstein gravity [12] (see also [13] for an overview of the closely related subject of Double Field Theory that often implies many of these results).

1.1 Heterotic generalised geometry

In Heterotic theory, however, the field strength H is no longer closed. Supersymmetry and the Green-Schwarz anomaly cancelation mechanism [14] require that H satisfy a more complicated Bianchi identity. This can be handled in the generalised geometry formalism by enlarging the generalised tangent space. The resulting "Heterotic generalised geometry" [15–20] is given in terms of a bundle which is a transitive, but not exact, Courant algebroid E, that can be built as a result of two extensions

$$\begin{aligned} \mathfrak{g} \to \mathcal{A} \to T, \\ T^* \to E \to \mathcal{A}. \end{aligned} \tag{1.3}$$

The first sequence defines a Lie algebroid $\mathcal{A}$ known as the Atiyah algebroid for the quadratic Lie algebra $\mathfrak{g}$, which replaces the role of the tangent bundle T in the original generalised geometry. The bundle $E \sim T \oplus \mathfrak{g} \oplus T^*$ then encodes the information for local gauge fields: a two-form[1] B and a Yang-Mills one-form A taking values in $\mathfrak{g}$. These fields are not independent, they satisfy the global condition in terms of their respective field-strengths

$$\mathrm{d}H = \operatorname{tr} F^2. \tag{1.4}$$

By considering a Lie group G (with algebra $\mathfrak{g}$) composed of two factors, a "gravitational" Lorentz group, and the usual $SO(32)$ or $E_8 \times E_8$ (and choosing the correct normalisation of the metric in $\mathfrak{g}$), one obtains the Heterotic Bianchi identity:

$$\mathrm{d}H = \tfrac{1}{4}\alpha'(\operatorname{tr} F^2 - \operatorname{tr} R^2), \tag{1.5}$$

where R, the field strength of the Lorentz factor, is now identified with the gravitational curvature. Once more, the Courant bracket in E precisely reproduces the physical infinitesimal bosonic symmetries: diffeomorphims $\mathcal{L}_x$, B-shifts by $\mathrm{d}\lambda$, and now also non-Abelian gauge

[1]The heterotic B field we are considering here is not gauge invariant under YM transformations and is not a gerbe connection, it is a rather more complicated object [21].

transformations by some parameter $\Lambda \in \mathfrak{g}$. It is then possible to show that formulating the generalised equivalent of Einstein gravity in E precisely reproduces the known Heterotic supergravity to order α' [15]. The 'trick' of treating the gravitational term in (1.5) as if it were a Yang-Mills factor goes back to [22], though, as shown there, supersymmetry requires that the $\operatorname{tr} R^2$ be given by the curvature of a specific torsionful connection $\nabla - \frac{1}{2}H$. In [19] it was shown that this is entirely consistent with the generalised geometry set-up.

1.2 M theory and $E_{7(7)} \times \mathbb{R}^+$ generalised geometry

In M theory, the equation of motion for the four-form flux $\mathcal{F}$ in eleven-dimensional supergravity [23] is corrected by higher order terms, starting with eight derivatives [24, 25]

$$\mathrm{d} * \mathcal{F} = -\tfrac{1}{2}\mathcal{F}^2 + \kappa(\operatorname{tr} R^4 - \tfrac{1}{4}(\operatorname{tr} R^2)^2), \tag{1.6}$$

where κ is some constant which will be set to 1 as it will not influence the rest of our discussion, and with further terms which are functions of the flux expected to appear at the same order in derivatives but whose complete form is not yet known.

In order to find four-dimensional Minkowski backgrounds of M theory, one considers field ansätze that are compatible with the external global Lorentz symmetry. This means decomposing the eleven-dimensional manifold as a warped product $\mathcal{M}_{11} = \mathbb{R}^{3,1} \times_{\text{warped}} M$ where M is some seven-dimensional internal space, demanding that the fields depend only on internal coordinates, and keeping the components of the $\mathcal{F}$ flux which are external scalars, i.e. the purely internal four-form F and seven-form $\tilde{F}$. Their components are set in terms of the eleven-dimensional $\mathcal{F}$ simply by restricting

$$F = \mathcal{F}|_M, \quad \tilde{F} = (*\mathcal{F})|_M, \tag{1.7}$$

where $*\mathcal{F}$ is the eleven-dimensional Hodge dual. All other components of $\mathcal{F}$ are set to zero. The fact that $\mathcal{F}$ is closed in eleven dimensions together with the equation of motion (1.6) then imply the Bianchi identities for the internal fluxes

$$\mathrm{d}F = 0, \tag{1.8}$$

$$\mathrm{d}\tilde{F} = -\tfrac{1}{2}F^2 + \operatorname{tr} R^4 - \tfrac{1}{4}(\operatorname{tr} R^2)^2, \tag{1.9}$$

where the second equation should be taken as purely formal, since it vanishes identically in the seven-dimensional M. These induce internal local potentials, a three-form C and a six-form $\tilde{C}$, which together with a Riemannian metric for M and a warp factor make up the bosonic degrees of freedom of the theory.

Ignoring the higher-curvature terms, it was shown in [26] that this supergravity set-up (together with the fermionic sector) has a very natural interpretation as the analogue of Einstein gravity when formulated in $E_{7(7)} \times \mathbb{R}^+$ generalised geometry, also known as exceptional generalised geometry [27–29]. One introduces a generalised tangent bundle

$$E \sim T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^* \oplus T^* \otimes \Lambda^7 T^*, \tag{1.10}$$

which encodes the bosonic symmetries of the theory, namely diffeomorphism generated by vector-fields x and shifts by two-forms ω and five-forms σ of the gauge fields C and $\tilde{C}$ respectively. By construction, the generalised tangent bundle defines a global closed four-form F that can locally be expressed in terms of the potential

$$F = \mathrm{d}C, \tag{1.11}$$

and a seven-form such that

$$\tilde{F} = \mathrm{d}\tilde{C} - \tfrac{1}{2} C F. \tag{1.12}$$

The supergravity Bianchi identities inherited from eleven-dimensions are thus automatically satisfied. The gauge algebra is then given by the natural differential structure over E, the (exceptional) Dorfman bracket of two generalised vectors $X_1, X_2 \in E$, which takes the form[2]

$$\begin{aligned} L_{X_1} X_2 = & \mathcal{L}_{x_1} x_2 + \mathcal{L}_{x_1} \omega_2 - i_{x_2} \mathrm{d}\omega_1 + \mathcal{L}_{x_1} \sigma' - i_{x_2} \mathrm{d}\sigma_1 - \omega_2 \mathrm{d}\omega_1 \\ & + \mathcal{L}_{x_1} \tau_2 - j\sigma_2 \wedge \mathrm{d}\omega_1 - j\omega_2 \wedge \mathrm{d}\sigma_1, \end{aligned} \tag{1.13}$$

(this is also known as generalised Lie derivative, and its antisymmetrisation is known as the exceptional Courant bracket). The bundle E has a natural $E_{7(7)} \times \mathbb{R}^+$ structure and the bracket is compatible with this structure. The bosonic degrees of freedom turn out to simply be the components of a generalised metric for the generalised tangent space, reducing the structure group to its maximal compact subgroup $SU(8)/\mathbb{Z}_2$, and the corresponding generalised Ricci scalar precisely reproduces the supergravity bosonic action. That eleven-dimensional supergravity admited this larger symmetry had already been proven in [30]. This efficient rewriting has made it possible to tackle several physical problems in full generality (without needing to restrict to some subsector of the fluxes, for example), such as classifying supersymmetric backgrounds [31–39] or describing their moduli spaces and holographic duals [40, 41]. It would thus seem promising to apply the same techniques with the higher derivative corrections included [42].

1.3 M theory corrections

So now let us consider adding back the higher curvature terms originating in eleven dimensions (1.6). These are incompatible with the $E_{7(7)} \times \mathbb{R}^+$ generalised tangent bundle previously

[2] The j-notation corresponds to a projection to the $T^* \otimes \Lambda^7 T^*$ space, see [28, 29] for its precise definition, though it will not be needed for what follows here.

introduced, since by construction it forces $\tilde{F} = \mathrm{d}\tilde{C} - \frac{1}{2}CF$. On the contrary, the corrected Bianchi identity (1.9) implies the local form for the flux $\tilde{F}$

$$\tilde{F} = \mathrm{d}\tilde{C} - \tfrac{1}{2}CF + \omega_7(A) - \tfrac{1}{4}\omega_3(A)\,\mathrm{tr}\,R^2, \tag{1.14}$$

where A is the spin-connection for the Riemann curvature R and $\omega_n(A)$ denotes the Chern-Simons n-form for A such that $\mathrm{d}\omega_{2n-1}(A) = \mathrm{tr}\,R^n$, see appendix A for their explicit form.

Following the same trick as for the Heterotic case, we may treat at first the curvature R simply as the field strength for a generic Yang-Mills gauge field A taking values in some algebra $\mathfrak{g}$, though naturally it will eventually be necessary to identify $\mathfrak{g}$ with $spin(7)$ and express A in terms of gravitational degrees of freedom.[3] The Heterotic generalised geometric prescription would then lead us to consider structures over a generalised tangent space of the form

$$T \oplus \mathfrak{g} \oplus \Lambda^2T^* \oplus \Lambda^5T^* \oplus T^* \otimes \Lambda^7T^*, \tag{1.15}$$

in other words, replacing the tangent bundle component of (1.10) with the Atiyah algebroid.

In what follows, however, we will restrict ourselves to simpler versions of this problem, which will still suffice to show that the situation is more complex the one of Heterotic generalised geometry. In particular, we will find gauge algebras that are best described in terms of higher order L_∞-algebras.

In section 2 we will first look at a Bianchi identity

$$\mathrm{d}\tilde{F}_5 = \mathrm{tr}\,R^3, \tag{1.16}$$

where $\tilde{F}_5$ is a five-form which, even though it has no immediate physical motivation, is easier to handle and already displays the important features we wish to demonstrate. The corresponding generalised tangent space will be of the form

$$T \oplus \mathfrak{g} \oplus \Lambda^3T^*. \tag{1.17}$$

We will then move on in section 3 to the case

$$\mathrm{d}\tilde{F}_7 = \mathrm{tr}\,R^4, \tag{1.18}$$

where now $\tilde{F}_7$ is genuinely a seven-form, and so this corresponds to a special case of (1.14). The generalised tangent space is then

$$T \oplus \mathfrak{g} \oplus \Lambda^5T^*. \tag{1.19}$$

[3] Though an intriguing possibility is to consider a larger gauge group that could accommodate the flux degrees of freedom, such as taking $\mathfrak{g} = su(8)$ and relating A to the $SU(8)$ connections implied by supersymmetry. This could naturally give rise to a Bianchi identity which includes higher derivative flux terms.

In both cases we will find that the Bianchi identities imply a gauge algebra which cannot be expressed in terms of simply a Courant bracket. Instead it is of the type of the L_∞ field theory formalism of [43]. The analysis of the complete Bianchi identity implied by the corrected eleven-dimensional supergravity will be left for future work.

As an aside, we expect that similar conclusions would hold for $(2n-1)$-form fluxes $\tilde{F}_{(2n-1)}$ satisfying

$$\mathrm{d}\tilde{F}_{(2n-1)} = \operatorname{tr} R^n, \tag{1.20}$$

based on generalised tangent spaces

$$E \sim T \oplus \mathfrak{g} \oplus \Lambda^{(2n-3)} T^*, \tag{1.21}$$

though we will not attempt to prove this here. Note as well that in all these cases the Bianchi identities, when viewed in cohomology classes, correspond to obstructions to this construction, namely the requirement that n-th Chern character of the gauge vector bundle is trivial.

We also remark that the fact that the gauge algebras we are examining fit into the L_∞ setting is not surprising. It has already been shown that the "higher Courant algebroids" of the type $T \oplus \Lambda^p T^*$ have an associated L_∞ algebra [44], and the extra terms we are considering arise from adding an invariant polynomial to the Bianchi, which in the context of chiral anomalies lead to the well-known "descent equations" derived from the extended Cartan homotopy [45], with many of the terms in the brackets we present here being directly related to the extra homotopy operator.

1.4 L_∞ algebras and field theory

L_∞ algebras or strong homotopy Lie algebras, introduced in [46, 47] to the physics context, have found numerous applications in both mathematics and physics, see [48] for a recent review of the field. In particular, they can be found in the theories of Courant algebroids and generalised geometry. Courant algebroids were shown to have an L_3-algebra in [49]. In the case of Heterotic Courant algebroids, it has recently been proven that this algebra is directly connected to the physical problem of finding the moduli of finite deformations of the Strominger-Hull system [50]. Higher Courant algebroids over a space $T \oplus \Lambda^p T^*$ were proven to have L_∞ algebras for arbitrary p in [44] using a derived bracket construction, and in [51] a large class of "Leibniz algebroids", of which exceptional generalised geometries are examples, were likewise shown to admit L_∞ algebras. There has been much current work showing how such structures also appear in the related fields of Double/Exceptional Field Theory, for example in [52–64].

Recently, in [43] many of these ideas were systematised in a manner to be more immediately applicable to physics, by introducing the notion of "L_∞ gauge field theories". It is this approach that we will be following, and we start by quickly reviewing some of the concepts that will be relevant here.

There are a few alternative ways of defining an L_∞ algebra. Following the conventions of [43] we will be working with the "ℓ-picture" in terms of graded-antisymmetric multilinear brackets. Given a $\mathbb{Z}$-graded vector space

$$V = \oplus_{i\in\mathbb{Z}} V_i, \tag{1.22}$$

where the subscript denotes the degree, one defines an L_∞ algebra by endowing it with a series of multilinear products $\ell_n : \Lambda^n V \mapsto V$. These brackets are of degree $n-2$, i.e. for inputs $v_i \in V$, the total degree of $\ell_n(v_1,\dots,v_n)$ is

$$\deg \ell_n(v_1,\dots,v_n) = n - 2 + \sum_{i=1}^{n} \deg v_i. \tag{1.23}$$

They are also graded antisymmetric,

$$\ell_n(v_{\sigma(1)},\dots,v_{\sigma(n)}) = (-1)^{|\sigma|}\epsilon(\sigma)\ell_n(v_1,\dots,v_n), \tag{1.24}$$

for some permutation σ and where ϵ is the Koszul sign for the given permutation and grading of V. Crucially, for each n the brackets must also satisfy a Jacobi identity "up to higher homotopies", namely the generalised Jacobi identities

$$\sum_{i+j=n+1} (-1)^{i(j-1)} \sum_{\sigma} (-1)^{|\sigma|}\epsilon(\sigma)\ell_j(\ell_i(v_{\sigma(1)},\dots,v_{\sigma(i)}), v_{\sigma(i+1)},\dots,v_{\sigma(n)}) = 0, \tag{1.25}$$

or in abbreviated form

$$\sum_{i+j=n+1} (-1)^{(j-1)i}\ell_j\ell_i = 0. \tag{1.26}$$

Proceding with the proposal of [43] for a gauge field theory, one considers spaces of type[4]

$$V = \oplus_{i>0} V_i \oplus V_0 \oplus V_{-1}. \tag{1.27}$$

An important point here is that, since one allows a space with negative grading, there is a priori no guarantee that the L_∞ algebra will ever terminate even for a finite number of V_i. This is in contrast to L_n-algebras, defined such that the graded vector space is concentrated in degrees 0 to $n-1$ and therefore all brackets of degree higher than $n+1$ vanish trivially as a consequence of (1.23). However, we will see that the cases we consider in the next sections do indeed truncate and there is a finite number of brackets to consider.

[4]In [43] an extra subspace V_{-2} is also allowed, corresponding to the equations of motion, but we will not make use of it here.

In order to find the physical meaning of (1.27), one identifies elements $X \in V_0$ with gauge parameters and $\Psi \in V_{-1}$ are taken to be the gauge fields. Elements of $\oplus_{i>0} V_i$ are to be thought of as making up a tower of trivial gauge parameters. An L_∞ gauge field theory may then be defined with the symmetries given by

$$\delta_X \Psi = \Sigma_n \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}} \ell_{n+1}(X, \Psi^n), \tag{1.28}$$

satisfying a gauge algebra

$$[\delta_{X_1}, \delta_{X_2}]\Psi = \delta_{[X_1,X_2]}\Psi, \quad [X_1, X_2] = \Sigma_n \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}} \ell_{n+2}(X_1, X_2, \Psi^n). \tag{1.29}$$

Note in particular that in this formalism the gauge algebra of the parameters is permitted to depend explicitly on the fields. In what follows we will show how the higher curvature problem we are considering fits precisely into this picture.

2 $\mathrm{d}\tilde{F}_5 = \mathrm{tr}\, R^3$

We begin by considering a theory with a globally defined five-form flux $\tilde{F}$ and a Yang-Mills $\mathfrak{g}$-valued potential A with corresponding field strength R such that

$$\mathrm{d}_A R = 0, \tag{2.1}$$

$$\mathrm{d}\tilde{F}_5 = \mathrm{tr}\, R^3. \tag{2.2}$$

We can thus define a four-form potential $\tilde{C}$ for the flux by

$$\tilde{F}_5 = \mathrm{d}\tilde{C}_4 + \omega_5(A). \tag{2.3}$$

Much like the B field in Heterotic theory, we find that since $\tilde{F}_5$ is gauge invariant, $\tilde{C}_4$ must transform to compensate for a variation of the Chern-Simons five-form $\omega_5(A)$. That is, if $\Lambda \in \mathfrak{g}$ parametrises an infinitesimal gauge transformation, we must have that locally

$$\mathrm{d}\delta_\Lambda \tilde{C}_4 = -\delta_\Lambda \omega_5(A) = -\mathrm{d}\omega_4^1(\Lambda, A) = -\mathrm{d}\,\mathrm{tr}\,\mathrm{d}\Lambda\left(A\mathrm{d}A + \tfrac{1}{2}A^3\right), \tag{2.4}$$

from the properties of the Chern-Simons forms, see appendix A. It is also clear that $\tilde{F}$ remains invariant under shifts of $\tilde{C}_4$ by a closed four-form, locally parametrised by the exterior derivative of some three-form σ. Together with a diffeomorphism symmetry parametrised by some vector x, we have that the potentials obey the infinitesimal gauge transformations

$$\begin{aligned} \delta_X A &= \mathcal{L}_x A - \mathrm{d}\Lambda - [A, \Lambda], \\ \delta_X \tilde{C}_4 &= \mathcal{L}_x \tilde{C}_4 - \mathrm{d}\sigma - \mathrm{tr}\,\mathrm{d}\Lambda\left(A\mathrm{d}A + \tfrac{1}{2}A^3\right), \end{aligned} \tag{2.5}$$

where δ_X denotes the combined infinitesimal diffs, gauge and shifts in terms of parameters $X = x + \Lambda + \sigma$. This therefore suggests a generalised tangent space

$$E = T \oplus \mathfrak{g} \oplus \Lambda^3 T^*. \tag{2.6}$$

So far, this precisely matches the procedure for constructing the Heterotic generalised geometry, see for example [19]. However, let us look at how the algebra of transformations δ_X closes when acting on the fields. Taking two parameters $X_1, X_2 \in E$, we find that

$$\begin{aligned}
[\delta_{X_1}, \delta_{X_2}]A &= \mathcal{L}_{[x_1,x_2]}A - \mathrm{d}\left([\Lambda_1, \Lambda_2] + i_{x_1}\mathrm{d}\Lambda_2 - i_{x_2}\mathrm{d}\Lambda_1\right) \\
&\quad - [A, [\Lambda_1, \Lambda_2] + i_{x_1}\mathrm{d}\Lambda_2 - i_{x_2}\mathrm{d}\Lambda_1], \\
[\delta_{X_1}, \delta_{X_2}]\tilde{C}_4 &= \mathcal{L}_{[x_1,x_2]}\tilde{C}_4 - \mathrm{d}(i_{x_1}\mathrm{d}\sigma_2 - i_{x_2}\mathrm{d}\sigma_1 + \tfrac{1}{2}\mathrm{d}i_{x_1}\sigma_2 - \tfrac{1}{2}\mathrm{d}i_{x_2}\sigma_1) \\
&\quad - \mathrm{tr}\,\mathrm{d}\left([\Lambda_1, \Lambda_2] + i_{x_1}\mathrm{d}\Lambda_2 - i_{x_2}\mathrm{d}\Lambda_1\right)\left(A(\mathrm{d}A)^2 + \tfrac{1}{2}A^3\right) \\
&\quad + \mathrm{d}\,\mathrm{tr}\left(\Lambda_1\mathrm{d}\Lambda_2\mathrm{d}A - \Lambda_2\mathrm{d}\Lambda_1\mathrm{d}A\right),
\end{aligned} \tag{2.7}$$

so we have that indeed the algebra closes on a parameter X_3 given by

$$\begin{aligned}
&[\delta_{X_1}, \delta_{X_2}](A + \tilde{C}_4) = \delta_{X_3}(A + \tilde{C}_4), \\
&\qquad X_3 = [x_1, x_2] + [\Lambda_1, \Lambda_2] + i_{x_1}\mathrm{d}\Lambda_2 - i_{x_2}\mathrm{d}\Lambda_1 \\
&\qquad\qquad + i_{x_1}\mathrm{d}\sigma_2 - i_{x_2}\mathrm{d}\sigma_1 + \tfrac{1}{2}\mathrm{d}i_{x_1}\sigma_2 - \tfrac{1}{2}\mathrm{d}i_{x_2}\sigma_1 \\
&\qquad\qquad - \mathrm{tr}\left(\Lambda_1\mathrm{d}\Lambda_2\mathrm{d}A - \Lambda_2\mathrm{d}\Lambda_1\mathrm{d}A\right) \in E,
\end{aligned} \tag{2.8}$$

but note that this depends not just on X_1 and X_2 but also explicitly on the fields. Therefore, unlike the previous examples in generalised geometry, the gauge algebra does not define for us a bracket over just the space E. It does, nonetheless, fit into the L_∞ field theory setting.

2.1 An L_∞ gauge algebra for R^3

Let us then introduce the graded vector space:

$$V = V_3 \oplus V_2 \oplus V_1 \oplus V_0 \oplus V_{-1}, \tag{2.9}$$

where[5]

$$\begin{aligned}
&V_3 = C^\infty(M), \quad V_2 = T^*, \quad V_1 = \Lambda^2 T^*, \\
&V_0 = E = T \oplus \mathfrak{g} \oplus \Lambda^3 T^*, \quad V_{-1} = T^* \otimes \mathfrak{g} \oplus \Lambda^4 T^*,
\end{aligned} \tag{2.10}$$

[5] A more 'generalised' treatment in the sense of [29] would presumably involve introducing a space of "generalised frames" for E (that is a subspace of End(E) that preserves the defining generalised structures – $O(d,d)$ in NSNS generalised geometry, $E_{7(7)}$ in exceptional generalised geometry, etc.), and identifying its 'geometric subspace' with V_{-1} which is used to construct the physical brackets.

and we label elements in the subspaces as

$$f \in V_3, \quad \phi \in V_2, \quad \eta \in V_1, \quad X = x + \Lambda + \sigma \in V_0, \quad \Psi = A + \tilde{C} \in V_{-1}. \tag{2.11}$$

We will then endow V with a series of multilinear brackets to define an L_∞ algebra that will realise the gauge algebra (2.8). Comparing with (1.29), we can read off some of the multibrackets. Several more are necessary to complete the algebra, which can be obtained from the requirement that they satisfy the generalised Jacobi identities (1.25). Terms in the brackets involving only elements in $V_{i>0}$ or the vector + three-form part of V_0 will be necessarily the ones in [44], but we must introduce new products for terms involving the Lie algebra $\mathfrak{g}$.

Note that due to the grading and symmetry properties of the ℓ_n brackets (1.24), products involving multiple factors of X_i will always have to be antisymmetrised, and products involving products of Ψ_i will always have to be symmetrised. We denote this explicitly using the typical index notation of symmetrisers and antisymmetrisers.

We find that the (non-vanishing) L_∞ products are then:

at level one

$$\ell_1(f) = \mathrm{d}f, \quad \ell_1(\phi) = \mathrm{d}\phi, \quad \ell_1(\eta) = \mathrm{d}\eta, \quad \ell_1(X) = -\mathrm{d}\Lambda - \mathrm{d}\sigma, \quad \ell_1(\Psi) = 0, \tag{2.12}$$

at level two

$$\ell_2(X, \eta) = \tfrac{1}{2}\mathcal{L}_x \eta, \quad \ell_2(X, \phi) = \tfrac{1}{2}\mathcal{L}_x \phi, \quad \ell_2(X, f) = \tfrac{1}{2}\mathcal{L}_x f, \tag{2.13a}$$

$$\begin{aligned}\ell_2(X_1, X_2) &= [x_1, x_2] + [\Lambda_1, \Lambda_2] + \mathcal{L}_{x_1}\Lambda_2 - \mathcal{L}_{x_2}\Lambda_1 \\ &\quad + \mathcal{L}_{x_1}\sigma_2 - \mathcal{L}_{x_2}\sigma_1 - \tfrac{1}{2}\mathrm{d}i_{x_1}\sigma_2 + \tfrac{1}{2}\mathrm{d}i_{x_2}\sigma_1,\end{aligned} \tag{2.13b}$$

$$\ell_2(X, \Psi) = \mathcal{L}_x \Psi - [A, \Lambda], \tag{2.13c}$$

at level three

$$\ell_3(\phi, X_1, X_2) = -\tfrac{1}{6}(i_{x_{[1}}\mathcal{L}_{x_{2]}} + i_{[x_1, x_2]})\phi, \tag{2.14a}$$

$$\ell_3(\eta, X_1, X_2) = -\tfrac{1}{6}(i_{x_{[1}}\mathcal{L}_{x_{2]}} + i_{[x_1, x_2]})\eta, \tag{2.14b}$$

$$\ell_3(X_1, X_2, X_3) = -\tfrac{1}{2}\big(i_{x_{[1}}\mathcal{L}_{x_2} + i_{[x_{[1}, x_{2]}} + i_{x_{[1}} i_{x_2}\mathrm{d}\big)\sigma_{3]}, \tag{2.14c}$$

$$\ell_3(X_1, X_2, \Psi) = -2\,\mathrm{tr}\,\Lambda_{[1}\mathrm{d}\Lambda_{2]}\mathrm{d}A, \tag{2.14d}$$

$$\ell_3(X, \Psi_1, \Psi_2) = 2\,\mathrm{tr}\,\mathrm{d}\Lambda A_{(1}\mathrm{d}A_{2)}, \tag{2.14e}$$

at level four

$$\ell_4(X_1,X_2,X_3,X_4) = -12\,\mathrm{tr}\,\Lambda_{[1}\Lambda_2\Lambda_3\mathrm{d}\Lambda_{4]}, \tag{2.15a}$$

$$\ell_4(X_1,X_2,X_3,\Psi) = 3i_{x_{[1}}\,\mathrm{tr}\,\Lambda_2\mathrm{d}\Lambda_{3]}\mathrm{d}A + 6\,\mathrm{tr}\,\Lambda_{[1}\Lambda_2\mathrm{d}\Lambda_{3]}A, \tag{2.15b}$$

$$\ell_4(X,\Psi_1,\Psi_2,\Psi_3) = 3\,\mathrm{tr}\,\mathrm{d}\Lambda A_{(1}A_2A_{3)}, \tag{2.15c}$$

at level five

$$\begin{aligned}\ell_5(X_1,X_2,X_3,X_4,X_5) &= 30i_{x_{[1}}\,\mathrm{tr}\,\Lambda_2\Lambda_3\Lambda_4\mathrm{d}\Lambda_{5]} - 12\,\mathrm{tr}\,\Lambda_{[1}\Lambda_2\Lambda_3\Lambda_4\Lambda_{5]}\\ &\quad + \tfrac{1}{6}i_{x_{[1}}i_{x_2}\big(i_{x_3}\mathcal{L}_{x_4} + i_{[x_3,x_4]} + i_{x_3}i_{x_4}\mathrm{d}\big)\sigma_{5]},\end{aligned} \tag{2.16a}$$

$$\ell_5(X_1,X_2,X_3,X_4,\Psi) = -2i_{x_{[1}}i_{x_2}\,\mathrm{tr}\,\Lambda_3\mathrm{d}\Lambda_{4]}\mathrm{d}A - 12i_{x_{[1}}\,\mathrm{tr}\,\Lambda_2\Lambda_3\mathrm{d}\Lambda_{4]}A, \tag{2.16b}$$

and finally at level six

$$\ell_6(X_1,X_2,X_3,X_4,X_5,\Psi) = 10i_{x_{[1}}i_{x_2}\,\mathrm{tr}\,\Lambda_3\Lambda_4\mathrm{d}\Lambda_{5]}A. \tag{2.17}$$

All other brackets vanish.

We thus have that picking particular elements $A + \tilde{C}_4 = \Psi \in V_{-1}$ corresponds to specifying the data for the supergravity gauge fields as they satisfy the correct gauge algebra, that is

$$\delta_X\Psi = \ell_1(X) + \ell_2(X,\Psi) - \tfrac{1}{2}\ell_3(X,\Psi,\Psi) - \tfrac{1}{6}\ell_4(X,\Psi,\Psi,\Psi), \tag{2.18}$$

with

$$[\delta_{X_1},\delta_{X_2}]\Psi = \delta_{X_3}\Psi, \quad X_3 = \ell_2(X_1,X_2) + \ell_3(X_1,X_2,\Psi), \tag{2.19}$$

precisely matches (2.5) and (2.8).

We leave to appendix B the steps required to show that these multilinear products indeed satisfy the very non-trivial generalised Jacobi identities that characterise L_∞ algebras.

3 $\mathrm{d}\tilde{F}_7 = \mathrm{tr}\,R^4$

The analysis for a seven-form flux follows in much the same way as the five-form case we just considered, it is simply more computationally intensive. We again introduce a $\mathfrak{g}$-valued one-form potential A with field strength R, and a globally defined seven-form $\tilde{F}$ such that

$$\mathrm{d}\tilde{F}_7 = \mathrm{tr}\,R^4, \tag{3.1}$$

and so locally we define a six-form potential $\tilde{C}$ by

$$\tilde{F}_7 = \mathrm{d}\tilde{C}_6 + \omega_7(A). \tag{3.2}$$

As previously remarked, this is a toy example for the supergravity theory of section 1.2 when one truncates equation (1.14). Now, gauge invariance of $\tilde{F}_7$ once again implies that $\tilde{C}_6$ must vary as

$$\begin{aligned} \mathrm{d}\delta_\Lambda \tilde{C}_6 &= -\delta_\Lambda \omega_7(A) = -\mathrm{d}\omega_6^1(\Lambda, A) \\ &= -\mathrm{d}\,\mathrm{tr}\,\mathrm{d}\Lambda\big(A(\mathrm{d}A)^2 + \tfrac{2}{5}(A^3\mathrm{d}A + \mathrm{d}AA^3 + A^5) + \tfrac{1}{5}(A^2\mathrm{d}AA + A\mathrm{d}AA^2)\big), \end{aligned} \tag{3.3}$$

for some gauge parameter $\Lambda \in \mathfrak{g}$. We also have the usual diffeomorphism $\mathcal{L}_x$ and shift symmetries $\mathrm{d}\sigma$ generated by, respectively, a vector field $x \in T$ and a five-form $\sigma \in \Lambda^5 T^*$. We are thus led to consider a generalised tangent space

$$\begin{aligned} E &= T \oplus \mathfrak{g} \oplus \Lambda^5 T^*, \\ X &= x + \Lambda + \sigma \in E. \end{aligned} \tag{3.4}$$

This is a close cousin of the $SL(8,\mathbb{R}) \times \mathbb{R}^+$ "half-exceptional" generalised geometry obtained by truncating the $E_{7(7)} \times \mathbb{R}^+$ case, as was described in [65]. We can then group the infinitesimal symmetries as

$$\begin{aligned} \delta_X &= \text{ infinitesimal diffs, gauge and shifts} \\ \delta_X A &= \mathcal{L}_x A - \mathrm{d}\Lambda - [A, \Lambda] \\ \delta_X \tilde{C}_6 &= \mathcal{L}_x \tilde{C}_6 - \mathrm{d}\sigma \\ &\quad - \mathrm{tr}\,\mathrm{d}\Lambda\,\big(A(\mathrm{d}A)^2 + \tfrac{2}{5}(A^3\mathrm{d}A + \mathrm{d}AA^3 + A^5) + \tfrac{1}{5}(A^2\mathrm{d}AA + A\mathrm{d}AA^2)\big) \end{aligned} \tag{3.5}$$

As in the R^3 case, we find that the gauge algebra closes on terms that explicitly depend on the gauge fields. Taking two parameters $X_1, X_2 \in E$, we have

$$\begin{aligned} [\delta_{X_1}, \delta_{X_2}]A &= \mathcal{L}_{[x_1,x_2]}A - \mathrm{d}\left([\Lambda_1,\Lambda_2] + i_{x_1}\mathrm{d}\Lambda_2 - i_{x_2}\mathrm{d}\Lambda_1\right) \\ &\quad - [A, [\Lambda_1,\Lambda_2] + i_{x_1}\mathrm{d}\Lambda_2 - i_{x_2}\mathrm{d}\Lambda_1], \\ [\delta_{X_1}, \delta_{X_2}]\tilde{C}_6 &= \mathcal{L}_{[x_1,x_2]}\tilde{C}_6 - \mathrm{d}(i_{x_1}\mathrm{d}\sigma_2 - i_{x_2}\mathrm{d}\sigma_1 + \tfrac{1}{2}\mathrm{d}i_{x_1}\sigma_2 - \tfrac{1}{2}\mathrm{d}i_{x_2}\sigma_1) \\ &\quad - \mathrm{tr}\,\mathrm{d}\left([\Lambda_1,\Lambda_2] + i_{x_1}\mathrm{d}\Lambda_2 - i_{x_2}\mathrm{d}\Lambda_1\right)\big(A(\mathrm{d}A)^2 \\ &\qquad + \tfrac{2}{5}(A^3\mathrm{d}A + \mathrm{d}AA^3 + A^5) + \tfrac{1}{5}(A^2\mathrm{d}AA + A\mathrm{d}AA^2)\big) \\ &\quad + \mathrm{d}\,\mathrm{tr}\,\big(\Lambda_1\,\big(\mathrm{d}\Lambda_2\mathrm{d}A\mathrm{d}A + \tfrac{3}{5}\mathrm{d}\Lambda_2\mathrm{d}(A^3) + \tfrac{1}{5}\mathrm{d}(A^2\mathrm{d}\Lambda_2 A)\big) \\ &\qquad - \Lambda_2\,\big(\mathrm{d}\Lambda_1\mathrm{d}A\mathrm{d}A + \tfrac{3}{5}\mathrm{d}\Lambda_1\mathrm{d}(A^3) + \tfrac{1}{5}\mathrm{d}(A^2\mathrm{d}\Lambda_1 A)\big)\,\big), \end{aligned} \tag{3.6}$$

and therefore, the algebra of the gauge parameters is

$$\begin{aligned} [X_1, X_2] &= [x_1,x_2] + [\Lambda_1,\Lambda_2] + i_{x_1}\mathrm{d}\Lambda_2 - i_{x_2}\mathrm{d}\Lambda_1 \\ &\quad + i_{x_1}\mathrm{d}\sigma_2 - i_{x_2}\mathrm{d}\sigma_1 + \tfrac{1}{2}\mathrm{d}i_{x_1}\sigma_2 - \tfrac{1}{2}\mathrm{d}i_{x_2}\sigma_1 \\ &\quad - \mathrm{tr}\,\big(\Lambda_1\,\big(\mathrm{d}\Lambda_2\mathrm{d}A\mathrm{d}A + \tfrac{3}{5}\mathrm{d}\Lambda_2\mathrm{d}(A^3) + \tfrac{1}{5}\mathrm{d}(A^2)\mathrm{d}\Lambda_2 A - \tfrac{1}{5}A^2\mathrm{d}\Lambda_2\mathrm{d}A\big) \\ &\qquad - \Lambda_2\,\big(\mathrm{d}\Lambda_1\mathrm{d}A\mathrm{d}A + \tfrac{3}{5}\mathrm{d}\Lambda_1\mathrm{d}(A^3) + \tfrac{1}{5}\mathrm{d}(A^2)\mathrm{d}\Lambda_1 A - \tfrac{1}{5}A^2\mathrm{d}\Lambda_1\mathrm{d}A\big)\,\big) \in E. \end{aligned} \tag{3.7}$$

Let us then see how this fits with the L_∞ formalism.

3.1 An L_∞ gauge algebra for R^4

We start by building a seven term graded vector space

$$V = V_5 \oplus V_4 \oplus V_3 \oplus V_2 \oplus V_1 \oplus V_0 \oplus V_{-1}, \tag{3.8}$$

where

$$\begin{aligned} &V_5 = C^\infty(M), \quad V_4 = T^*, \quad V_3 = \Lambda^2 T^*, \quad V_2 = \Lambda^3 T^*, \quad V_1 = \Lambda^4 T^*, \\ &V_0 = E = T \oplus \mathfrak{g} \oplus \Lambda^5 T^*, \quad V_{-1} = T^* \otimes \mathfrak{g} \oplus \Lambda^6 T^*, \end{aligned} \tag{3.9}$$

whose elements we will generically label as

$$\xi \in V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5, \quad X = x + \Lambda + \sigma \in V_0, \quad \Psi = A + \tilde{C} \in V_{-1}. \tag{3.10}$$

We now construct the L_∞ products as before. We read off some of the brackets by comparing (3.5) and (3.7) with (1.28) and (1.29) respectively, and use the generalised Jacobi conditions to complete the algebra. The terms in the products which are independent of V_{-1} or the $\mathfrak{g}$ part of V_0 must reproduce the results of [44]. The full list of multilinear brackets is then as follows:

at level one

$$\ell_1(\xi) = \mathrm{d}\xi, \quad \ell_1(X) = -\mathrm{d}\Lambda - \mathrm{d}\sigma, \quad \ell_1(\Psi) = 0, \tag{3.11}$$

at level two

$$\ell_2(X, \xi) = \tfrac{1}{2}\mathcal{L}_x \xi, \tag{3.12a}$$

$$\begin{aligned} \ell_2(X_1, X_2) = {}& [x_1, x_2] + [\Lambda_1, \Lambda_2] + \mathcal{L}_{x_1}\Lambda_2 - \mathcal{L}_{x_2}\Lambda_1 \\ & + \mathcal{L}_{x_1}\sigma_2 - \mathcal{L}_{x_2}\sigma_1 - \tfrac{1}{2}\mathrm{d}i_{x_1}\sigma_2 + \tfrac{1}{2}\mathrm{d}i_{x_2}\sigma_1, \end{aligned} \tag{3.12b}$$

$$\ell_2(X, \Psi) = \mathcal{L}_x \Psi - [A, \Lambda], \tag{3.12c}$$

at level three

$$\ell_3(\xi, X_1, X_2) = -\tfrac{1}{6}(i_{x_{[1}}\mathcal{L}_{x_{2]}} + i_{[x_1, x_2]})\xi, \tag{3.13a}$$

$$\ell_3(X_1, X_2, X_3) = -\tfrac{1}{2}\big(i_{x_{[1}}\mathcal{L}_{x_2} + i_{[x_{[1}, x_2]} + i_{x_{[1}} i_{x_2}\mathrm{d}\big)\sigma_{3]}, \tag{3.13b}$$

at level four

$$\ell_4(X_1, X_2, \Psi_1, \Psi_2) = 4\,\mathrm{tr}\,\Lambda_{[1}\mathrm{d}\Lambda_{2]}\mathrm{d}A_{(1}\mathrm{d}A_{2)}, \tag{3.14a}$$

$$\ell_4(X, \Psi_1, \Psi_2, \Psi_3) = 6\,\mathrm{tr}\,\mathrm{d}\Lambda A_{(1}\mathrm{d}A_2\mathrm{d}A_{3)}, \tag{3.14b}$$

at level five

$$\ell_5(\xi, X_1, X_2, X_3, X_4) = \tfrac{1}{30} i_{x_{[1}} i_{x_2} (i_{x_3} \mathcal{L}_{x_{4]}} + i_{[x_3, x_{4]}]})\xi, \tag{3.15a}$$

$$\ell_5(X_1, X_2, X_3, X_4, X_5) = \tfrac{1}{6} i_{x_{[1}} i_{x_2} (i_{x_3} \mathcal{L}_{x_4} + i_{[x_3, x_4]} + i_{x_3} i_{x_4} \mathrm{d})\sigma_{5]}, \tag{3.15b}$$

$$\ell_5(X_1, X_2, X_3, X_4, \Psi) = -\tfrac{24}{5} \operatorname{tr}\big(2\Lambda_{[1}\Lambda_2\Lambda_3 \mathrm{d}\Lambda_{4]} - \Lambda_{[1}\Lambda_2 \mathrm{d}\Lambda_3 \Lambda_{4]}\big)\mathrm{d}A, \tag{3.15c}$$

$$\begin{aligned}\ell_5(X_1, X_2, X_3, \Psi_1, \Psi_2) &= -6 i_{x_{[1}} \operatorname{tr} \Lambda_2 \mathrm{d}\Lambda_{3]} \mathrm{d}A_{(1} \mathrm{d}A_{2)} \\ &\quad - \tfrac{12}{5} \operatorname{tr}\big(2\Lambda_{[1}\Lambda_2 \mathrm{d}\Lambda_{3]} \mathrm{d}A_{(1} A_{2)} + 3\Lambda_{[1}\Lambda_2 \mathrm{d}\Lambda_{3]} A_{(1} \mathrm{d}A_{2)} - \Lambda_{[1} \mathrm{d}\Lambda_2 \Lambda_{3]} \mathrm{d}(A_{(1}A_{2)}) \\ &\quad - \Lambda_{[1} \mathrm{d}\Lambda_2 \mathrm{d}A_{(1} \Lambda_{3]} A_{2)} + \Lambda_{[1} \mathrm{d}\Lambda_2 A_{(1} \Lambda_{3]} \mathrm{d}A_{2)}\big),\end{aligned} \tag{3.15d}$$

$$\ell_5(X_1, X_2, \Psi_1, \Psi_2, \Psi_3) = \tfrac{12}{5} \operatorname{tr}\big(3\Lambda_{[1} \mathrm{d}\Lambda_{2]} \mathrm{d}(A_{(1}A_2A_{3)}) + \Lambda_{[1} \mathrm{d}(A_{(1}A_2 \mathrm{d}\Lambda_{2]} A_{3)})\big), \tag{3.15e}$$

$$\begin{aligned}\ell_5(X, \Psi_1, \Psi_2, \Psi_3, \Psi_4) &= -\tfrac{24}{5} \operatorname{tr} \mathrm{d}\Lambda \big(2A_{(1}A_2A_3 \mathrm{d}A_{4)} + A_{(1}A_2 \mathrm{d}A_3 A_{4)} \\ &\quad + A_{(1} \mathrm{d}A_2 A_3 A_{4)} + 2\mathrm{d}A_{(1}A_2A_3A_{4)}\big),\end{aligned} \tag{3.15f}$$

at level six

$$\ell_6(X_1, X_2, X_3, X_4, X_5, X_6) = -144 \operatorname{tr} \Lambda_{[1}\Lambda_2\Lambda_3\Lambda_4\Lambda_5 \mathrm{d}\Lambda_{6]}, \tag{3.16a}$$

$$\begin{aligned}\ell_6(X_1, X_2, X_3, X_4, X_5, \Psi) &= 12 i_{x_{[1}} \operatorname{tr}\big(2\Lambda_2\Lambda_3\Lambda_4 \mathrm{d}\Lambda_{5]} - \Lambda_2\Lambda_3 \mathrm{d}\Lambda_4\Lambda_{5]}\big)\mathrm{d}A \\ &\quad + 24 \operatorname{tr}\big(2\Lambda_{[1}\Lambda_2\Lambda_3\Lambda_4 \mathrm{d}\Lambda_{5]} + \Lambda_{[1}\Lambda_2 \mathrm{d}\Lambda_3\Lambda_4\Lambda_{5]}\big)A,\end{aligned} \tag{3.16b}$$

$$\begin{aligned}\ell_6(X_1, X_2, X_3, X_4, \Psi_1, \Psi_2) &= 4 i_{x_{[1}} i_{x_2} \operatorname{tr} \Lambda_3 \mathrm{d}\Lambda_{4]} \mathrm{d}A_{(1} \mathrm{d}A_{2)} \\ &\quad + \tfrac{24}{5} i_{x_{[1}} \operatorname{tr}\big(2\Lambda_2\Lambda_3 \mathrm{d}\Lambda_{4]} \mathrm{d}A_{(1}A_{2)} + 3\Lambda_2\Lambda_3 \mathrm{d}\Lambda_{4]} A_{(1} \mathrm{d}A_{2)} \\ &\quad - \Lambda_2 \mathrm{d}\Lambda_3\Lambda_{4]} \mathrm{d}(A_{(1}A_{2)}) - \Lambda_2 \mathrm{d}\Lambda_3 \mathrm{d}A_{(1}\Lambda_{4]} A_{2)} + \Lambda_2 \mathrm{d}\Lambda_3 A_{(1}\Lambda_{4]} \mathrm{d}A_{2)}\big) \\ &\quad - \tfrac{48}{5} \operatorname{tr}\big(\Lambda_{[1} \mathrm{d}(\Lambda_2\Lambda_3)\Lambda_{4]} A_{(1}A_{2)} + \Lambda_{[1}\Lambda_2 \mathrm{d}\Lambda_3 A_{(1}\Lambda_{4]} A_{2)} + \Lambda_{[1} \mathrm{d}\Lambda_2 A_{(1}\Lambda_3\Lambda_{4]} A_{2)}\big),\end{aligned} \tag{3.16c}$$

$$\begin{aligned}\ell_6(X_1, X_2, X_3, \Psi_1, \Psi_2, \Psi_3) &= -\tfrac{18}{5} i_{x_{[1}} \operatorname{tr}\big(3\Lambda_2 \mathrm{d}\Lambda_{3]} \mathrm{d}(A_{(1}A_2A_{3)}) \\ &\quad + \Lambda_2 \mathrm{d}(A_{(1}A_2 \mathrm{d}\Lambda_{3]} A_{3)})\big) - \tfrac{36}{5} \operatorname{tr}\big(3\Lambda_{[1}\Lambda_2 \mathrm{d}\Lambda_{3]} A_{(1}A_2A_{3)} \\ &\quad - \Lambda_{[1}\Lambda_2 A_{(1}A_2 \mathrm{d}\Lambda_{3]} A_{3)}\big),\end{aligned} \tag{3.16d}$$

$$\ell_6(X, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = -48 \operatorname{tr} \mathrm{d}\Lambda A_{(1}A_2A_3A_4A_{5)}, \tag{3.16e}$$

at level seven

$$\ell_7(X_1,X_2,X_3,X_4,X_5,X_6,X_7) = -\tfrac{1}{6} i_{x_{[1}} i_{x_2} i_{x_3} i_{x_4} (i_{x_5}\mathcal{L}_{x_6} + i_{[x_5,x_6]} + i_{x_5} i_{x_6}\mathrm{d})\sigma_{7]} \\ + 504 i_{x_{[1}} \operatorname{tr} \Lambda_2\Lambda_3\Lambda_4\Lambda_5\Lambda_6\mathrm{d}\Lambda_{7]} - 144 \operatorname{tr} \Lambda_{[1}\Lambda_2\Lambda_3\Lambda_4\Lambda_5\Lambda_6\Lambda_{7]}, \tag{3.17a}$$

$$\ell_7(X_1,X_2,X_3,X_4,X_5,X_6,\Psi) = -12 i_{x_{[1}} i_{x_2} \operatorname{tr}\big(2\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_{6]} - \Lambda_3\Lambda_4\mathrm{d}\Lambda_5\Lambda_{6]}\big)\mathrm{d}A \\ - 72 i_{x_{[1}} \operatorname{tr}\big(2\Lambda_2\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_{6]} + \Lambda_2\Lambda_3\mathrm{d}\Lambda_4\Lambda_5\Lambda_{6]}\big)A, \tag{3.17b}$$

$$\ell_7(X_1,X_2,X_3,X_4,X_5,\Psi_1,\Psi_2) = -4 i_{x_{[1}} i_{x_2} \operatorname{tr}\big(2\Lambda_3\Lambda_4\mathrm{d}\Lambda_{5]}\mathrm{d}A_{(1}A_{2)} \\ + 3\Lambda_3\Lambda_4\mathrm{d}\Lambda_{5]}A_{(1}\mathrm{d}A_{2)} - \Lambda_3\mathrm{d}\Lambda_4\Lambda_{5]}\mathrm{d}(A_{(1}A_{2)}) - \Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A_{(1}\Lambda_{5]}A_{2)} \\ + \Lambda_3\mathrm{d}\Lambda_4 A_{(1}\Lambda_{5]}\mathrm{d}A_{2)}\big) + 24 i_{x_{[1}} \operatorname{tr}\big(\Lambda_2\mathrm{d}(\Lambda_3\Lambda_4)\Lambda_{5]}A_{(1}A_{2)} + \Lambda_2\Lambda_3\mathrm{d}\Lambda_4 A_{(1}\Lambda_{5]}A_{2)} \\ + \Lambda_2\mathrm{d}\Lambda_3 A_{(1}\Lambda_4\Lambda_{5]}A_{2)}\big), \tag{3.17c}$$

$$\ell_7(X_1,X_2,X_3,X_4,\Psi_1,\Psi_2,\Psi_3) = \tfrac{12}{5} i_{x_{[1}} i_{x_2} \operatorname{tr}\big(3\Lambda_3\mathrm{d}\Lambda_{4]}\mathrm{d}(A_{(1}A_2A_{3)}) \\ + \Lambda_3\mathrm{d}(A_{(1}A_2\mathrm{d}\Lambda_{4]}A_{3)})\big) + \tfrac{72}{5} i_{x_{[1}} \operatorname{tr}\big(3\Lambda_2\Lambda_3\mathrm{d}\Lambda_{4]}A_{(1}A_2A_{3)} \\ - \Lambda_2\Lambda_3 A_{(1}A_2\mathrm{d}\Lambda_{4]}A_{3)}\big), \tag{3.17d}$$

at level eight

$$\ell_8(X_1,X_2,X_3,X_4,X_5,X_6,X_7,\Psi) = 84 i_{x_{[1}} i_{x_2} \operatorname{tr}\big(2\Lambda_3\Lambda_4\Lambda_5\Lambda_6\mathrm{d}\Lambda_{7]} \\ + \Lambda_3\Lambda_4\mathrm{d}\Lambda_5\Lambda_6\Lambda_{7]}\big)A, \tag{3.18a}$$

$$\ell_8(X_1,X_2,X_3,X_4,X_5,X_6,\Psi_1,\Psi_2) = -2 i_{x_{[1}} i_{x_2} i_{x_3} i_{x_4} \operatorname{tr}\Lambda_5\mathrm{d}\Lambda_{6]}\mathrm{d}A_{(1}\mathrm{d}A_{2)} \\ - 24 i_{x_{[1}} i_{x_2} \operatorname{tr}\big(\Lambda_3\mathrm{d}(\Lambda_4\Lambda_5)\Lambda_{6]}A_{(1}A_{2)} + \Lambda_3\Lambda_4\mathrm{d}\Lambda_5 A_{(1}\Lambda_{6]}A_{2)} \\ + \Lambda_3\mathrm{d}\Lambda_4 A_{(1}\Lambda_5\Lambda_{6]}A_{2)}\big), \tag{3.18b}$$

$$\ell_8(X_1,X_2,X_3,X_4,X_5,\Psi_1,\Psi_2,\Psi_3) = -12 i_{x_{[1}} i_{x_2} \operatorname{tr}\big(3\Lambda_3\Lambda_4\mathrm{d}\Lambda_{5]}A_{(1}A_2A_{3)} \\ - \Lambda_3\Lambda_4 A_{(1}A_2\mathrm{d}\Lambda_{5]}A_{3)}\big), \tag{3.18c}$$

at level nine

$$\ell_9(X_1,X_2,X_3,X_4,X_5,X_6,X_7,\Psi_1,\Psi_2) = \tfrac{14}{5} i_{x_{[1}} i_{x_2} i_{x_3} i_{x_4} \operatorname{tr}\big(3\Lambda_5\Lambda_6\mathrm{d}\Lambda_{7]}\mathrm{d}A_{(1}A_{2)}, \\ + 2\Lambda_5\Lambda_6\mathrm{d}\Lambda_{7]}A_{(1}\mathrm{d}A_{2)} - \Lambda_5\mathrm{d}\Lambda_6\Lambda_{7]}\mathrm{d}(A_{(1}A_{2)}) - \Lambda_5\mathrm{d}\Lambda_6\mathrm{d}A_{(1}\Lambda_{7]}A_{2)} \\ + \Lambda_5\mathrm{d}\Lambda_6 A_{(1}\Lambda_{7]}\mathrm{d}A_{2)}\big), \tag{3.19a}$$

$$\ell_9(X_1,X_2,X_3,X_4,X_5,X_6,\Psi_1,\Psi_2,\Psi_3) \\ = -\tfrac{6}{5} i_{x_{[1}} i_{x_2} i_{x_3} i_{x_4} \operatorname{tr}\big(3\Lambda_5\mathrm{d}\Lambda_{6]}\mathrm{d}(A_{(1}A_2A_{3)}) + \Lambda_5\mathrm{d}(A_{(1}A_2\mathrm{d}\Lambda_{6]}A_{3)})\big), \tag{3.19b}$$

and finally at level ten

$$\begin{aligned}\ell_{10}&(X_1,X_2,X_3,X_4,X_5,X_6,X_7,\Psi_1,\Psi_2,\Psi_3)\\ &=\tfrac{42}{5}i_{x_{[1}}i_{x_2}i_{x_3}i_{x_4}\,\mathrm{tr}\left(3\Lambda_5\Lambda_6\mathrm{d}\Lambda_{7]}A_{(1}A_2A_{3)}-\Lambda_5\Lambda_6A_{(1}A_2\mathrm{d}\Lambda_{7]}A_{3)}\right).\end{aligned}\tag{3.20}$$

All other brackets vanish. The proof that they satisfy the generalised Jacobi identities is given in appendix C.

Picking a specific point Ψ in the space V_{-1} will correspond to specifying the supergravity data, since the gauge algebra obeyed by Ψ

$$\begin{aligned}\delta_X\Psi=\ell_1(X)+\ell_2(X,\Psi)-\tfrac{1}{2}\ell_3(X,\Psi,\Psi)-\tfrac{1}{6}\ell_4(X,\Psi,\Psi,\Psi)\\ +\tfrac{1}{24}\ell_5(X,\Psi,\Psi,\Psi,\Psi)+\tfrac{1}{120}\ell_6(X,\Psi,\Psi,\Psi,\Psi,\Psi),\end{aligned}\tag{3.21}$$

and

$$\begin{aligned}[][X_1,X_2]=\ell_2(X_1,X_2)+\ell_3(X_1,X_2,\Psi)-\tfrac{1}{2}\ell_4(X_1,X_2,\Psi,\Psi)\\ -\tfrac{1}{6}\ell_5(X_1,X_2,\Psi,\Psi,\Psi),\end{aligned}\tag{3.22}$$

is such that its components $\Psi=A+\tilde{C}_6$ match (3.5) and (3.7) by construction.

Despite no longer being able to describe these higher order gauge algebras in terms of just a Leibniz bracket on the generalised tangent space, we thus have that the extra structure of E is still enough to ensure that we can find an L_∞ algebra, and that this algebra has a finite number of brackets. And while there should not be much difficulty in adding the extra geometrical data that make up the physical degrees of freedom such as the Riemannian metric, it will require further study to see whether this weaker differential structure will be enough to give a natural geometric description of the dynamics of higher-derivative-corrected supergravity.

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A General definitions

We mostly follow the conventions of [19], though we generally omit the wedge symbol for the product of differential forms. We also use the (anti)symmetrisation of indices notation with weight one, so for example

$$\Lambda_{[1}\Lambda_2\Lambda_{3]}=\tfrac{1}{3!}(\Lambda_1\Lambda_2\Lambda_3-\Lambda_1\Lambda_3\Lambda_2+\Lambda_2\Lambda_3\Lambda_1-\Lambda_2\Lambda_1\Lambda_3+\Lambda_3\Lambda_1\Lambda_2-\Lambda_3\Lambda_2\Lambda_1).\tag{A.1}$$

Given a Lie algebra-valued one-form A we define its curvature by

$$R(A) = \mathrm{d}A + A^2, \tag{A.2}$$

which satisfies

$$\mathrm{d}_A R = \mathrm{d}R + [A, R] = 0, \tag{A.3}$$

and is invariant under the infinitesimal gauge transformations of the potential

$$\delta_\Lambda A = -\mathrm{d}_A \Lambda = -\mathrm{d}\Lambda - [A, \Lambda]. \tag{A.4}$$

As is well known from the study of anomalies [45, 66–68], taking the trace of powers of the curvature one can define invariant polynomials

$$\mathrm{d}\,\mathrm{tr}\,R^n = \delta\,\mathrm{tr}\,R^n = 0, \tag{A.5}$$

from the n-th Chern character $\mathrm{tr}\,R^n$ of the gauge vector bundle. Poincaré's lemma then implies that one can locally define the Chern-Simons forms $\omega_{(2n-1)}$

$$\mathrm{d}\omega_{(2n-1)}(A) = \mathrm{tr}\,R^n, \tag{A.6}$$

and applying the lemma once again, now for δ, gives

$$\mathrm{d}\omega^1_{(2n-2)}(\Lambda, A) = \delta_\Lambda \omega_{(2n-1)}(A), \tag{A.7}$$

where the superscript denotes the powers of the gauge parameter, since in principle one can continue "descending" along this chain.

We can list some of the Chern-Simons forms that will be important for us explicitly (up to exact terms, as is clear from their defining relations)

$$\omega_7(A) = \mathrm{tr}\left(A(\mathrm{d}A)^3 + \tfrac{8}{5}A^3(\mathrm{d}A)^2 + \tfrac{4}{5}A\mathrm{d}AA^2\mathrm{d}A + 2A^5\mathrm{d}A + \tfrac{4}{7}A^7\right), \tag{A.8}$$

$$\begin{aligned}\omega^1_6(\Lambda, A) = \mathrm{tr}\,\mathrm{d}\Lambda\big(&A(\mathrm{d}A)^2 + \tfrac{2}{5}(A^3\mathrm{d}A + \mathrm{d}AA^3 + A^5) \\ &+ \tfrac{1}{5}(A^2\mathrm{d}AA + A\mathrm{d}AA^2)\big),\end{aligned} \tag{A.9}$$

$$\omega_5(A) = \mathrm{tr}\left(A(\mathrm{d}A)^2 + \tfrac{3}{2}A^3\mathrm{d}A + \tfrac{3}{5}A^5\right), \tag{A.10}$$

$$\omega^1_4(\Lambda, A) = \mathrm{tr}\,\mathrm{d}\Lambda\left(A\mathrm{d}A + \tfrac{1}{2}A^3\right), \tag{A.11}$$

$$\omega_3(A) = \mathrm{tr}\left(A\mathrm{d}A + \tfrac{2}{3}A^3\right), \tag{A.12}$$

$$\omega^1_2(\Lambda, A) = \mathrm{tr}\,\mathrm{d}\Lambda A. \tag{A.13}$$

The last two are not used in this work, but are the ones that are featured in Heterotic generalised geometry. These agree with the usual ones in the literature [67] up to exact terms corresponding to our convention choice of having the differential acting on the parameter Λ.

A.1 Helpful identities

Since substantial use of them was made in proving some of the generalised Jacobi identities in the next appendices, here we list some helpful identities that follow from the Lie-Cartan algebra on differential forms

$$\begin{aligned} &\mathcal{L}_x \mathrm{d} = \mathrm{d}\mathcal{L}_x, \quad \mathcal{L}_{x_1} i_{x_2} - i_{x_2}\mathcal{L}_{x_1} = i_{[x_1,x_2]}, \\ &\mathrm{d}i_x + i_x \mathrm{d} = \mathcal{L}_x, \quad \mathcal{L}_{x_{[1}}\mathcal{L}_{x_{2]}} = \tfrac{1}{2}\mathcal{L}_{[x_1,x_2]}. \end{aligned} \tag{A.14}$$

The following tend to be useful in expressions involving two vector fields (these operators maintain the degree of the form they are acting on)

$$\mathrm{d}(i_{x_{[1}}\mathcal{L}_{x_{2]}} + i_{[x_1,x_2]}) + (i_{x_{[1}}\mathcal{L}_{x_{2]}} + i_{[x_1,x_2]})\mathrm{d} = \tfrac{3}{2}\mathcal{L}_{[x_1,x_2]}, \tag{A.15}$$

$$\mathrm{d}(i_{x_{[1}}\mathcal{L}_{x_{2]}} + i_{[x_1,x_2]} + i_{x_1} i_{x_2}\mathrm{d}) = \tfrac{3}{2}\mathcal{L}_{[x_1,x_2]} - 3 i_{x_{[1}}\mathcal{L}_{x_{2]}}\mathrm{d}, \tag{A.16}$$

three vector fields (these lower the degree by one)

$$\mathcal{L}_{[x_{[1},x_2]} i_{x_{3]}} - i_{x_{[1}}\mathcal{L}_{[x_2,x_{3]}]} = i_{[[x_{[1},x_2],x_{3]}]} = 0, \tag{A.17}$$

and four vector fields (these lower by two and three)

$$i_{[x_{[1},x_2]} i_{[x_3,x_{4]}]} = 0, \tag{A.18}$$

$$\begin{aligned} &\mathrm{d}i_{x_{[1}} i_{x_2}(i_{[x_3,x_{4]}]} + i_{x_3}\mathcal{L}_{x_{4]}}) = -i_{x_{[1}} i_{x_2}(i_{[x_3,x_{4]}]} + i_{x_3}\mathcal{L}_{x_{4]}})\mathrm{d} \\ &\qquad + 5(\mathcal{L}_{x_{[1}}\mathcal{L}_{x_2} i_{x_3} i_{x_{4]}} + \mathcal{L}_{x_{[1}} i_{x_2} i_{[x_3,x_{4]}]}), \end{aligned} \tag{A.19}$$

$$i_{x_{[1}} i_{x_2}(i_{x_3}\mathcal{L}_{x_{4]}} + i_{[x_3,x_{4]}]} + i_{x_3} i_{x_{4]}}\mathrm{d}) = 5(i_{x_{[1}}\mathcal{L}_{x_2} + i_{[x_{[1},x_2]}) i_{x_3} i_{x_{4]}} + \mathrm{d}i_{x_{[1}} i_{x_2} i_{x_3} i_{x_{4]}}. \tag{A.20}$$

B Computation of generalised Jacobi identities from section 2

We verify the generalised Jacobi relations by exhaustively going term by term through the list of identities (1.25) for each level n and for each possible set of inputs for the brackets in section 2. Fortunately, because of both the grading of the vector space V and the subdivisions inside V_0 and V_{-1}, many of these vanish trivially. For example, all brackets ℓ_n of level $n > 2$ whose image is in V_{-1} actually only map to the four-form subspace, i.e. they are $\tilde{C}$-type objects. On the other hand, the brackets of level $n > 2$ that take an object in V_{-1} as input are all independent of $\tilde{C}$. So chaining together those sets of brackets is trivial.

The first generalised Jacobi identity is easy to verify from (2.12)

$$(\ell_1)^2 = 0 \quad \square$$

and shows that the graded vector space V of an L_∞ algebra forms a chain complex with the operator ℓ_1.

B.1 $\ell_1\ell_2 - \ell_2\ell_1 = 0$ relations

The next set of relations establishes ℓ_1 as a derivation on ℓ_2, given by the brackets (2.13).

First check on inputs in $V_1 \otimes V_0$

$$\begin{aligned}
&\ell_1(\ell_2(X,\eta)) = \tfrac{1}{2}\mathrm{d}\mathcal{L}_x\eta, \\
&\ell_2(X,\ell_1(\eta)) = \mathcal{L}_x\mathrm{d}\eta - \tfrac{1}{2}\mathrm{d}i_x\mathrm{d}\eta = \tfrac{1}{2}\mathrm{d}\mathcal{L}_x\eta \\
&\Rightarrow \ell_1(\ell_2(X,\eta)) = \ell_2(\ell_1(X),\eta) + \ell_2(X,\ell_1(\eta)) \quad \square
\end{aligned}$$

and similarly replacing η with $f \in V_3$ or $\phi \in V_2$.

And on $V_0 \otimes V_0$

$$\begin{aligned}
&\ell_1(\ell_2(X_1,X_2)) = -\mathrm{d}[\Lambda_1,\Lambda_2] - \mathrm{d}i_{x_1}\mathrm{d}\Lambda_2 + \mathrm{d}i_{x_2}\mathrm{d}\Lambda_1 - \mathrm{d}i_{x_1}\mathrm{d}\sigma_2 + \mathrm{d}i_{x_2}\mathrm{d}\sigma_1 \\
&\quad = -\mathrm{d}[\Lambda_1,\Lambda_2] - \mathcal{L}_{x_1}\mathrm{d}\Lambda_2 + \mathcal{L}_{x_2}\mathrm{d}\Lambda_1 - \mathcal{L}_{x_1}\mathrm{d}\sigma_2 + \mathcal{L}_{x_2}\mathrm{d}\sigma_1 \\
&\ell_2(X_1,\ell_1(X_2)) = \mathcal{L}_{x_1}(-\mathrm{d}\Lambda_2 - \mathrm{d}\sigma_2) - [-\mathrm{d}\Lambda_2,\Lambda_1] \\
&\Rightarrow \ell_1(\ell_2(X_1,X_2)) = \ell_2(\ell_1(X_1),\eta) + \ell_2(X_1,\ell_1(X_2)) \quad \square
\end{aligned}$$

For inputs in $(X,\Psi) \in V_0 \otimes V_{-1}$, the check is trivial.

B.2 $\ell_1\ell_3 + \ell_3\ell_1 + \ell_2\ell_2 = 0$ relations

Next is the check of what would be the 'traditional' Jacobi identity, where the L_∞ algebra starts to clearly diverge from the normal graded Lie algebras. They involve brackets (2.14) plus the previous ones.

First for objects in $V_1 \otimes V_0 \otimes V_0$

$$\begin{aligned}
&\ell_1(\ell_3(X_1,X_2,\eta)) = -\tfrac{1}{6}\mathrm{d}(i_{x_{[1}}\mathcal{L}_{x_{2]}} + i_{[x_1,x_2]})\eta, \\
&\ell_3(\ell_1(X_1),X_2,\eta) = 0, \\
&\ell_3(\ell_1(\eta),X_1,X_2) = -\tfrac{1}{6}\big(i_{x_{[1}}\mathcal{L}_{x_{2]}} + i_{[x_1,x_2]}\big)\mathrm{d}\eta = \tfrac{1}{6}\big(\mathrm{d}i_{x_{[1}}\mathcal{L}_{x_{2]}} - \mathcal{L}_{x_{[1}}\mathcal{L}_{x_{2]}} + \mathrm{d}i_{[x_1,x_2]} - \mathcal{L}_{[x_1,x_2]}\big)\eta \\
&\quad = \tfrac{1}{6}\mathrm{d}\big(i_{x_{[1}}\mathcal{L}_{x_{2]}} + i_{[x_1,x_2]}\big)\eta - \tfrac{1}{4}\mathcal{L}_{[x_1,x_2]}\eta, \\
&\ell_2(\ell_2(X_1,X_2),\eta) = \tfrac{1}{2}\mathcal{L}_{[x_1,x_2]}\eta, \\
&\ell_2(X_1,\ell_2(X_2,\eta)) = \tfrac{1}{4}\mathcal{L}_{x_1}(\mathcal{L}_{x_2}\eta), \\
&\Rightarrow \ell_1(\ell_3(X_1,X_2,\Psi)) + \ell_3(\ell_1(X_1),X_2,\eta) + \ell_3(X_1,\ell_1(X_2),\eta) + \ell_3(X_1,X_2,\ell_1(\eta)) \\
&+ \ell_2(\ell_2(X_1,X_2),\eta) - \ell_2(X_1,\ell_2(X_2,\eta)) + \ell_2(X_2,\ell_2(X_1,\eta)) = 0 \quad \square
\end{aligned}$$

and similarly for $f \in V_3$ and $\phi \in V_2$ instead of η.

Next for $V_0 \otimes V_0 \otimes V_0$

$$
\begin{aligned}
&\ell_1(\ell_3(X_1, X_2, X_3)) = -\tfrac{1}{2}\mathrm{d}\big(i_{x_{[1}}\mathcal{L}_{x_2} + i_{[x_{[1}, x_{2]}} + i_{x_{[1}} i_{x_2}\mathrm{d}\big)\sigma_{3]} = -\tfrac{3}{2}\mathrm{d}\big(i_{x_{[1}}\mathrm{d} i_{x_2} + i_{x_{[1}} i_{x_2}\mathrm{d}\big)\sigma_{3]},\\
&\ell_3(\ell_1(X_1), X_2, X_3) = 0,\\
&\ell_2(\ell_2(X_{[1}, X_2), X_{3]}) = \tfrac{1}{2}\mathrm{d}(i_{x_{[1}}\mathrm{d} i_{x_2} + i_{x_{[1}} i_{x_2}\mathrm{d})\sigma_{3]},\\
&\Rightarrow \ell_1(\ell_3(X_1, X_2, X_3)) + \ell_3(\ell_1(X_1), X_2, X_3) + \ell_3(X_1, \ell_1(X_2), X_3) + \ell_3(X_1, X_2, \ell_1(X_3))\\
&+ \ell_2(\ell_2(X_1, X_2), X_3) + \ell_2(\ell_2(X_2, X_3), X_1) + \ell_2(\ell_2(X_3, X_1), X_2) = 0 \quad \square
\end{aligned}
$$

Finally for $V_0 \otimes V_0 \otimes V_{-1}$

$$
\begin{aligned}
&\ell_1(\ell_3(X_1, X_2, \Psi)) = \operatorname{tr}\mathrm{d}\Lambda_1\mathrm{d}\Lambda_2\mathrm{d}A - \operatorname{tr}\mathrm{d}\Lambda_2\mathrm{d}\Lambda_1\mathrm{d}A,\\
&\ell_3(\ell_1(X_1), X_2, \Psi) = \operatorname{tr}\mathrm{d}\Lambda_2\mathrm{d}\Lambda_1\mathrm{d}A,\\
&\ell_2(\ell_2(X_1, X_2), \Psi) = \mathcal{L}_{[x_1, x_2]}\Psi - \big[A, [\Lambda_1, \Lambda_2] + \mathcal{L}_{x_1}\Lambda_2 - \mathcal{L}_{x_2}\Lambda_1\big],\\
&\ell_2(X_1, \ell_2(X_2, \Psi)) = \mathcal{L}_{x_1}\big(\mathcal{L}_{x_2}\Psi - [A, \Lambda_2]\big) - \big[\mathcal{L}_{x_2}A - [A, \Lambda_2], \Lambda_1\big],\\
&\Rightarrow \ell_1(\ell_3(X_1, X_2, \Psi)) + \ell_3(\ell_1(X_1), X_2, \Psi) + \ell_3(X_1, \ell_1(X_2), \Psi) + \ell_3(X_1, X_2, \ell_1(\Psi))\\
&\quad + \ell_2(\ell_2(X_1, X_2), \Psi) - \ell_2(X_1, \ell_2(X_2, \Psi)) + \ell_2(X_2, \ell_2(X_1, \Psi)) = 0 \quad \square
\end{aligned}
$$

B.3 $\ell_1\ell_4 - \ell_4\ell_1 - \ell_2\ell_3 + \ell_3\ell_2 = 0$ relations

The next identities introduce brackets (2.15) together with the previous.

We check with inputs in $V_1 \otimes (V_0)^3$

$$
\begin{aligned}
&\ell_3(\ell_2(X_3, \eta), X_1, X_2) = -\tfrac{1}{12}(i_{x_{[1}}\mathcal{L}_{x_{2]}} + i_{[x_1, x_2]})\mathcal{L}_{x_3}\eta,\\
&\ell_3(\eta, \ell_2(X_1, X_2), X_3) = -\tfrac{1}{6}(\tfrac{1}{2}i_{[x_1, x_2]}\mathcal{L}_{x_3} - \tfrac{1}{2}i_{x_3}\mathcal{L}_{[x_1, x_2]} + i_{[[x_1, x_2], x_3]})\eta = -\tfrac{1}{12}(i_{[x_1, x_2]}\mathcal{L}_{x_3} - i_{x_3}\mathcal{L}_{[x_1, x_2]})\eta,\\
&\ell_2(\ell_3(\eta, X_1, X_2), X_3) = \tfrac{1}{12}\mathcal{L}_{x_3}(i_{x_{[1}}\mathcal{L}_{x_{2]}} + i_{[x_1, x_2]})\eta = \tfrac{1}{12}\mathcal{L}_{x_3}(-\mathcal{L}_{x_{[1}} i_{x_{2]}} + 2i_{[x_1, x_2]})\eta,\\
&\Rightarrow \ell_3(\ell_2(X_1, X_2), X_3, \eta) + \ell_3(\ell_2(X_3, X_1), X_2, \eta) + \ell_3(\ell_2(X_2, X_3), X_1, \eta)\\
&+ \ell_3(\ell_2(X_1, \eta), X_2, X_3) + \ell_3(\ell_2(X_3, \eta), X_1, X_2) + \ell_3(\ell_2(X_2, \eta), X_3, X_1)\\
&+ \ell_2(\ell_3(\eta, X_1, X_2), X_3) + \ell_2(\ell_3(\eta, X_2, X_3), X_1) + \ell_2(\ell_3(\eta, X_3, X_1), X_2) - \ell_2(\ell_3(X_1, X_2, X_3), \eta)\\
&\quad = 0 \quad \square
\end{aligned}
$$

and similarly with $\phi \in V_2$ instead of η.

Next for $(V_0)^4$

$$
\begin{aligned}
&\ell_1(\ell_4(X_1,X_2,X_3,X_4)) = -12\mathrm{d}\big(\operatorname{tr}\Lambda_{[1}\Lambda_2\Lambda_3\mathrm{d}\Lambda_{4]}\big) = 24\operatorname{tr}\Lambda_{[1}\Lambda_2\mathrm{d}\Lambda_3\mathrm{d}\Lambda_{4]},\\
&\ell_4(\ell_1(X_4),X_1,X_2,X_3) = 6\operatorname{tr}\Lambda_{[1}\Lambda_2\mathrm{d}\Lambda_{3]}\mathrm{d}\Lambda_4,\\
&\ell_3(\ell_2(X_3,X_4),X_1,X_2) = -\tfrac{1}{6}\big(i_{x_{[1}}\mathcal{L}_{x_{2]}} + i_{[x_1,x_2]} + i_{x_1}i_{x_2}\mathrm{d}\big)(2\mathcal{L}_{x_{[3}}\sigma_{4]} - \mathrm{d}i_{x_{[3}}\sigma_{4]})\\
&-\tfrac{1}{3}\big(\tfrac{1}{2}i_{x_{[2}}\mathcal{L}_{[x_3,x_4]} - \tfrac{1}{2}i_{[x_3,x_4]}\mathcal{L}_{x_{[2}} - i_{[x_3,x_4]}i_{x_{[2}}\mathrm{d}\big)\sigma_{1]},\\
&\ell_2(\ell_3(X_1,X_2,X_3),X_4) = \tfrac{1}{4}\mathcal{L}_{x_4}\big(i_{x_{[1}}\mathcal{L}_{x_2} + i_{[x_{[1},x_2]} + i_{x_{[1}}i_{x_2}\mathrm{d}\big)\sigma_{3]},\\
&\Rightarrow \ell_1(\ell_4(X_1,X_2,X_3,X_4)) - 4\ell_4(\ell_1(X_{[1}),X_2,X_3,X_{4]})\\
&+6\ell_3(\ell_2(X_{[1},X_2),X_3,X_{4]}) - 4\ell_2(\ell_3(X_{[1},X_2,X_3),X_{4]})\\
&\quad = \big(-i_{x_{[1}}\mathcal{L}_{[x_2,x_3]} - 2\mathcal{L}_{x_{[1}}i_{[x_2,x_3]} + 4i_{[x_{[1},x_2]}i_{x_3}\mathrm{d} - 2\mathcal{L}_{x_{[1}}i_{x_2}i_{x_3}\mathrm{d}\big)\sigma_{4]}\\
&\quad + \big(\tfrac{1}{2}i_{x_{[1}}\mathcal{L}_{[x_2,x_3]} + \mathcal{L}_{x_{[1}}i_{[x_2,x_3]} - 2i_{[x_{[1},x_2]}i_{x_3}\mathrm{d} + \mathcal{L}_{x_{[1}}i_{x_2}i_{x_3}\mathrm{d}\big)\sigma_{4]}\\
&\quad + \big(i_{x_{[1}}\mathcal{L}_{[x_2,x_3]} - \mathcal{L}_{x_{[1}}i_{[x_2,x_3]} - 2i_{[x_{[1},x_2]}i_{x_3}\mathrm{d}\big)\sigma_{4]}\\
&\quad + \big(-\tfrac{1}{2}i_{x_{[1}}\mathcal{L}_{[x_2,x_3]} + 2\mathcal{L}_{x_{[1}}i_{[x_2,x_3]} + \mathcal{L}_{x_{[1}}i_{x_2}i_{x_3}\mathrm{d}\big)\sigma_{4]} = 0 \quad \square
\end{aligned}
$$

For $(V_0)^3 \otimes V_{-1}$ we have

$$
\begin{aligned}
&\ell_3(\ell_2(X_{[1},X_2),X_{3]},\Psi) = -2\operatorname{tr}\mathcal{L}_{x_{[1}}(\Lambda_2\mathrm{d}\Lambda_3)\mathrm{d}A + 2\operatorname{tr}\Lambda_{[1}\mathrm{d}\Lambda_2\Lambda_{3]}\mathrm{d}A,\\
&\ell_3(X_{[1},X_2,\ell_2(X_{3]},\Psi)) = -2\operatorname{tr}\Lambda_{[1}\mathrm{d}\Lambda_2\big(\mathcal{L}_{x_{3]}}\mathrm{d}A - \mathrm{d}[A,\Lambda_{3]}]\big),\\
&\ell_2(X_{[1},\ell_3(X_2,X_{3]},\Psi)) = -2\mathcal{L}_{x_{[1}}\operatorname{tr}\Lambda_2\mathrm{d}\Lambda_{3]}\mathrm{d}A + \mathrm{d}i_{x_{[1}}\operatorname{tr}\Lambda_2\mathrm{d}\Lambda_{3]}\mathrm{d}A,\\
&\ell_1(\ell_4(X_1,X_2,X_3,\Psi)) = 3\mathrm{d}i_{x_{[1}}\operatorname{tr}\Lambda_2\mathrm{d}\Lambda_{3]}\mathrm{d}A + 6\mathrm{d}\operatorname{tr}\Lambda_{[1}\Lambda_2\mathrm{d}\Lambda_{3]}A\\
&\Rightarrow \ell_1(\ell_4(X_1,X_2,X_3,\Psi)) - 3\ell_4(\ell_1(X_{[1}),X_2,X_{3]},\Psi) - 3\ell_2(X_{[1},\ell_3(X_2,X_{3]},\Psi))\\
&+3\ell_3(X_{[1},X_2,\ell_2(X_{3]},\Psi)) + 3\ell_3(\ell_2(X_{[1},X_2),X_{[3},\Psi) = 0 \quad \square
\end{aligned}
$$

And for $(V_0)^2 \otimes (V_{-1})^2$

$$\begin{aligned}
&\ell_3(\ell_2(X_1,X_2),\Psi_1,\Psi_2) = \mathrm{tr}(\mathrm{d}[\Lambda_1,\Lambda_2] + \mathcal{L}_{x_1}\mathrm{d}\Lambda_2 - \mathcal{L}_{x_2}\mathrm{d}\Lambda_1)(A_1\mathrm{d}A_2 + A_2\mathrm{d}A_1),\\
&\ell_3(X_1,\ell_2(X_2,\Psi_1),\Psi_2) = \mathrm{tr}\,\mathrm{d}\Lambda_1(\mathcal{L}_{x_2}A_1 - [A_1,\Lambda_2])\mathrm{d}A_2 + \mathrm{tr}\,\mathrm{d}\Lambda_1 A_2(\mathcal{L}_{x_2}\mathrm{d}A_1 - \mathrm{d}[A_1,\Lambda_2]),\\
&\ell_2(X_1,\ell_3(X_2,\Psi_1,\Psi_2)) = \mathrm{tr}\,\mathcal{L}_{x_1}(\mathrm{d}\Lambda_2 A_1\mathrm{d}A_2 + \mathrm{d}\Lambda_2 A_2\mathrm{d}A_1),\\
&\ell_2(\ell_3(X_1,X_2,\Psi_1),\Psi_2) = 0,\\
&\ell_4(X_1,\ell_1(X_2),\Psi_1,\Psi_2) = -\tfrac{1}{2}\,\mathrm{tr}\,\mathrm{d}\Lambda_1(\mathrm{d}\Lambda_2 A_1 A_2 + A_1\mathrm{d}\Lambda_2 A_2 + A_1 A_2\mathrm{d}\Lambda_2\\
&\quad + \mathrm{d}\Lambda_2 A_2 A_1 + A_2\mathrm{d}\Lambda_2 A_1 + A_2 A_1\mathrm{d}\Lambda_2),\\
&\Rightarrow \ell_1(\ell_4(X_1,X_2,\Psi_1,\Psi_2)) - 2\ell_4(\ell_1(X_{[1}),X_{2]},\Psi_1,\Psi_2) - 2\ell_4(X_1,X_2,\ell_1(\Psi_{(1}),\Psi_{2)})\\
&- 2\ell_2(\ell_3(X_1,X_2,\Psi_{(1}),\Psi_{2)}) - 2\ell_2(X_{[1},\ell_3(X_{2]},\Psi_1,\Psi_2)) - 4\ell_3(X_{[1},\ell_2(X_{2]},\Psi_{(1}),\Psi_{2)})\\
&+ \ell_3(\ell_2(X_1,X_2),\Psi_1,\Psi_2) + \ell_3(X_1,X_2,\ell_2(\Psi_1,\Psi_2)) = 0 \quad \square
\end{aligned}$$

B.4 $\ell_1\ell_5 + \ell_5\ell_1 + \ell_2\ell_4 + \ell_4\ell_2 + \ell_3\ell_3 = 0$ relations

To simplify notation in what follows, products containing multiple elements $X_i = x_i + \Lambda_i + \sigma_i \in V_0$ are always implicitly antisymmetrised, so one should read $\ell_3(\ell_3(\eta,X_1,X_2),X_3,X_4) = \ell_3(\ell_3(\eta,X_{[1},X_2),X_3,X_{4]})$, and products containing multiple elements $\Psi_i = A_i + \tilde{C}_i \in V_{-1}$ are always implicitly symmetrised so for example $\ell_3(X_1,\ell_2(X_2,\Psi_1),\Psi_2) = \ell_3(X_{[1},\ell_2(X_{2]},\Psi_{(1}),\Psi_{2)})$. We will also omit terms that vanish trivially.

These identities start involving brackets (2.16) together with the previous. We begin with elements in $V_1 \otimes (V_0)^4$

$$\begin{aligned}
&\ell_5(\ell_1(\eta),X_1,X_2,X_3,X_4) = \tfrac{1}{5}\tfrac{1}{6}\, i_{x_1} i_{x_2}\big(i_{x_3}\mathcal{L}_{x_{4]}} + i_{[x_3,x_{4]}]}\big)\mathrm{d}\eta,\\
&\ell_3(\ell_3(X_3,X_4,\eta),X_1,X_2) = \tfrac{1}{36}(i_{x_{[1}}\mathcal{L}_{x_{2]}} + i_{[x_1,x_{2]}]})(i_{x_{[3}}\mathcal{L}_{x_{4]}} + i_{[x_3,x_{4]}]})\eta,\\
&\Rightarrow \ell_5(\ell_1(\eta),X_1,X_2,X_3,X_4) + 6\ell_3(\ell_3(\eta,X_1,X_2),X_3,X_4)\\
&\quad = \tfrac{1}{5}\tfrac{1}{6}\, i_{x_{[1}} i_{x_2}\big(i_{x_3}\mathcal{L}_{x_{4]}} + i_{[x_3,x_{4]}]}\big)\mathrm{d}\eta + \tfrac{1}{6}(i_{x_{[1}}\mathcal{L}_{x_2} + i_{[x_{[1},x_2]})(i_{x_3}\mathcal{L}_{x_{4]}} + i_{[x_3,x_{4]}]})\eta\\
&\quad = \tfrac{1}{5}\tfrac{1}{6}\big(5\mathcal{L}_{x_1}\mathcal{L}_{x_2} i_{x_3} i_{x_4} + 5\mathcal{L}_{x_1} i_{x_2} i_{[x_3,x_4]}\big)\eta + \tfrac{1}{5}\tfrac{1}{6}\mathrm{d}(\mathcal{L}_{x_1} i_{x_2} i_{x_3} i_{x_4} - 4 i_{x_1} i_{x_2} i_{[x_3,x_4]})\eta\\
&\quad - \tfrac{1}{6}(\mathcal{L}_{x_1}\mathcal{L}_{x_2} i_{x_3} i_{x_4} + \mathcal{L}_{x_1} i_{x_2} i_{[x_3,x_4]})\eta \quad = 0 \quad \square
\end{aligned}$$

Next for $(V_0)^5$

$$\ell_1(\ell_5(X_1,X_2,X_3,X_4,X_5)) = -60\,\mathrm{tr}\,\Lambda_1\Lambda_2\Lambda_3\Lambda_4\mathrm{d}\Lambda_5 + \tfrac{1}{6}\mathrm{d}i_{x_1}i_{x_2}\big(i_{x_3}\mathcal{L}_{x_4} + i_{[x_3,x_4]} + i_{x_3}i_{x_4}\mathrm{d}\big)\sigma_5$$
$$+ 30\mathrm{d}i_{x_1}\,\mathrm{tr}\,\Lambda_2\Lambda_3\Lambda_4\mathrm{d}\Lambda_5,$$

$$\ell_5(\ell_1(X_1),X_2,X_3,X_4,X_5) = 12 i_{x_1}\,\mathrm{tr}\,\Lambda_2\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}\Lambda_5 = 6 i_{x_1}\mathrm{d}\,\mathrm{tr}\,\Lambda_2\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}\Lambda_5$$

$$\ell_3(\ell_3(X_1,X_2,X_3),X_4,X_5) = \tfrac{1}{2}\tfrac{1}{6}(i_{x_1}\mathcal{L}_{x_2} + i_{[x_1,x_2]})\big(i_{x_3}\mathcal{L}_{x_4} + i_{[x_3,x_4]} + i_{x_3}i_{x_4}\mathrm{d}\big)\sigma_5,$$

$$\ell_2(\ell_4(X_1,X_2,X_3,X_4),X_5) = 6\mathcal{L}_{x_1}(\mathrm{tr}\,\Lambda_2\Lambda_3\Lambda_4\mathrm{d}\Lambda_5),$$

$$\ell_4(\ell_2(X_1,X_2),X_3,X_4,X_5) = 6\,\mathrm{tr}\,\Lambda_1\Lambda_2\Lambda_3\Lambda_4\mathrm{d}\Lambda_5 - 6\mathcal{L}_{x_1}\,\mathrm{tr}\,\Lambda_2\Lambda_3\Lambda_4\mathrm{d}\Lambda_5,$$

$$\Rightarrow \ell_1(\ell_5(X_1,X_2,X_3,X_4,X_5)) + 5\ell_5(\ell_1(X_1),X_2,X_3,X_4,X_5)$$
$$+ 10\ell_3(\ell_3(X_1,X_2,X_3),X_4,X_5) + 5\ell_2(\ell_4(X_1,X_2,X_3,X_4),X_5)$$
$$+ 10\ell_4(\ell_2(X_1,X_2),X_3,X_4,X_5)$$
$$= \tfrac{5}{6}(i_{x_1}\mathcal{L}_{x_2} + i_{[x_1,x_2]})\big(i_{x_3}\mathcal{L}_{x_4} + i_{[x_3,x_4]} + i_{x_3}i_{x_4}\mathrm{d}\big)\sigma_5 + \tfrac{1}{6}\mathrm{d}i_{x_1}i_{x_2}\big(i_{x_3}\mathcal{L}_{x_4} + i_{[x_3,x_4]} + i_{x_3}i_{x_4}\mathrm{d}\big)\sigma_5$$
$$= \tfrac{5}{6}\big(i_{x_1}i_{[x_2,x_3]}\mathcal{L}_{x_4} - i_{x_1}i_{x_2}\mathcal{L}_{x_3}\mathcal{L}_{x_4} - i_{x_1}i_{x_2}i_{[x_3,x_4]}\mathrm{d} + i_{x_1}i_{x_2}i_{x_3}\mathcal{L}_{x_4}\mathrm{d}\big)\sigma_5$$
$$- \tfrac{5}{6}\big(i_{x_1}i_{[x_2,x_3]}\mathcal{L}_{x_4} - i_{x_1}i_{x_2}\mathcal{L}_{x_3}\mathcal{L}_{x_4} - i_{x_1}i_{x_2}i_{[x_3,x_4]}\mathrm{d} + i_{x_1}i_{x_2}i_{x_3}\mathcal{L}_{x_4}\mathrm{d}\big)\sigma_5 = 0 \quad \Box$$

For $(V_0)^4 \otimes V_{-1}$

$$\ell_1(\ell_5(X_1,X_2,X_3,X_4,\Psi)) = -12\mathrm{d}i_{x_1}\,\mathrm{tr}\,\Lambda_2\Lambda_3\mathrm{d}\Lambda_4 A - 2\mathrm{d}i_{x_1}i_{x_2}\,\mathrm{tr}\,\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A$$
$$= -12\mathcal{L}_{x_1}\,\mathrm{tr}\,\Lambda_2\Lambda_3\mathrm{d}\Lambda_4 A + 12 i_{x_1}\mathrm{d}(\mathrm{tr}\,\Lambda_2\Lambda_3\mathrm{d}\Lambda_4 A) - 2\mathrm{d}i_{x_1}i_{x_2}\,\mathrm{tr}\,\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A,$$

$$\ell_3(X_1,X_2,\ell_3(X_3,X_4,\Psi)) = \tfrac{1}{3}\big(i_{x_1}\mathcal{L}_{x_2} + i_{[x_1,x_2]} + i_{x_1}i_{x_2}\mathrm{d}\big)(\mathrm{tr}\,\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A),$$

$$\ell_2(X_1,\ell_4(X_2,X_3,X_4,\Psi)) = \tfrac{3}{2}\mathcal{L}_{x_1}(i_{x_2}\,\mathrm{tr}\,\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A + 2\,\mathrm{tr}\,\Lambda_2\Lambda_3\mathrm{d}\Lambda_4 A),$$

$$\ell_4(X_1,X_2,X_3,\ell_2(X_4,\Psi)) = 3 i_{x_1}\,\mathrm{tr}\,\Lambda_2\mathrm{d}\Lambda_3(\mathcal{L}_{x_4}\mathrm{d}A - \mathrm{d}[A,\Lambda_4]) + 6\,\mathrm{tr}\,\Lambda_1\Lambda_2\mathrm{d}\Lambda_3(\mathcal{L}_{x_4}A - [A,\Lambda_4]),$$

$$\ell_4(\ell_2(X_1,X_2),X_3,X_4,\Psi) = i_{[x_1,x_2]}\,\mathrm{tr}\,\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A$$
$$- 2 i_{x_1}\,\mathrm{tr}(\Lambda_2\Lambda_3 + \mathcal{L}_{x_2}\Lambda_3)\mathrm{d}\Lambda_4\mathrm{d}A + 2 i_{x_1}\,\mathrm{tr}\,\Lambda_2(\mathrm{d}\Lambda_3\Lambda_4 + \Lambda_3\mathrm{d}\Lambda_4 + \mathcal{L}_{x_3}\mathrm{d}\Lambda_4)\mathrm{d}A$$
$$+ 2\,\mathrm{tr}(\Lambda_1\Lambda_2 + \mathcal{L}_{x_{[1}}\Lambda_2)\Lambda_3\mathrm{d}\Lambda_4 A - 2\,\mathrm{tr}\,\Lambda_1(\Lambda_2\Lambda_3 + \mathcal{L}_{x_2}\Lambda_3)\mathrm{d}\Lambda_4 A$$
$$+ 2\,\mathrm{tr}\,\Lambda_1\Lambda_2(\mathrm{d}\Lambda_3\Lambda_{4]} + \Lambda_3\mathrm{d}\Lambda_4 + \mathcal{L}_{x_3}\mathrm{d}\Lambda_4)A,$$

$$\Rightarrow \ell_1(\ell_5(X_1,X_2,X_3,X_4,\Psi)) - 4\ell_2(X_1,\ell_4(X_2,X_3,X_4,\Psi)) - 4\ell_4(X_1,X_2,X_3,\ell_2(X_4,\Psi))$$
$$+ 6\ell_4(\ell_2(X_1,X_2),X_3,X_4,\Psi) + 6\ell_3(X_1,X_2,\ell_3(X_3,X_4,\Psi))$$
$$= -6 i_{x_1}\mathcal{L}_{x_2}\,\mathrm{tr}\,\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A + 2\big(i_{x_1}\mathcal{L}_{x_2} + i_{[x_1,x_2]} + i_{x_1}i_{x_2}\mathrm{d}\big)\,\mathrm{tr}\,\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A$$
$$- 2\mathrm{d}i_{x_1}i_{x_2}\,\mathrm{tr}\,\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A = 0 \quad \Box$$

And finally for $(V_0)^2 \otimes (V_{-1})^3$

$$
\begin{aligned}
&\ell_2(X_1, \ell_4(X_2, \Psi_1, \Psi_2, \Psi_3)) = 3\mathcal{L}_{x_1} \operatorname{tr} \mathrm{d}\Lambda_2 A_1 A_2 A_3, \\
&\ell_4(\ell_2(X_1, X_2), \Psi_1, \Psi_2, \Psi_3) = 3 \operatorname{tr}(2\mathrm{d}(\Lambda_1\Lambda_2) + 2\mathcal{L}_{x_1}\mathrm{d}\Lambda_2) A_1 A_2 A_3, \\
&\ell_4(X_1, \ell_2(X_2, \Psi_1), \Psi_2, \Psi_3) = \operatorname{tr} \mathrm{d}\Lambda_1 \big((\mathcal{L}_{x_2} A_1 - [A_1, \Lambda_2]) A_2 A_3 + A_1 (\mathcal{L}_{x_2} A_2 - [A_2, \Lambda_2]) A_3 \\
&\quad + A_1 A_2 (\mathcal{L}_{x_2} A_3 - [A_3, \Lambda_2]) \big), \\
&\Rightarrow \ell_4(\ell_2(X_1, X_2), \Psi_1, \Psi_2, \Psi_3) - 6\ell_4(X_1, \ell_2(X_2, \Psi_1), \Psi_2, \Psi_3) - 2\ell_2(X_1, \ell_4(X_2, \Psi_1, \Psi_2, \Psi_3)) \\
&\quad = 0 \quad \square
\end{aligned}
$$

B.5 $\ell_1\ell_6 - \ell_6\ell_1 - \ell_2\ell_5 + \ell_5\ell_2 + \ell_3\ell_4 - \ell_4\ell_3 = 0$ relations

These identities involve brackets (2.17) together with all the previous. The first we check has inputs in $(V_0)^6$

$$
\begin{aligned}
&\ell_2(\ell_5(X_1, X_2, X_3, X_4, X_5), X_6) = -6\mathcal{L}_{x_1} \operatorname{tr} \Lambda_2\Lambda_3\Lambda_4\Lambda_5\Lambda_6 - 15\mathcal{L}_{x_1} i_{x_2} \operatorname{tr} \Lambda_3\Lambda_4\Lambda_5 \mathrm{d}\Lambda_6 \\
&\quad + \tfrac{1}{6}\big(5\mathcal{L}_{x_1} i_{x_2} i_{x_3} i_{[x_4,x_5]} - \tfrac{5}{4}\mathcal{L}_{[x_1,x_2]} i_{x_3} i_{x_4} i_{x_5}\big)\sigma_6, \\
&\ell_5(\ell_2(X_1, X_2), X_3, X_4, X_5, X_6) = \tfrac{1}{5}\big(30 i_{[x_1,x_2]} \operatorname{tr} \Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6 + 60 i_{x_1}\mathcal{L}_{x_2} \operatorname{tr} \Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6 \\
&\quad + 60 i_{x_1} \operatorname{tr} \Lambda_2\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6 - 24\mathcal{L}_{x_1} \operatorname{tr} \Lambda_2\Lambda_3\Lambda_4\Lambda_5\Lambda_6\big) \\
&\quad + \tfrac{1}{6}\tfrac{1}{5}\big(2 i_{[x_1,x_2]} i_{x_3} (i_{x_4}\mathcal{L}_{x_5} + + i_{x_4} i_{x_5}\mathrm{d}) + i_{x_1} i_{x_2} (i_{[x_3,x_4]}\mathcal{L}_{x_5} + i_{[x_3,x_4]} i_{x_5}\mathrm{d}) \\
&\quad - i_{x_1} i_{x_2} (i_{x_3}\mathcal{L}_{[x_4,x_5]} + i_{x_3} i_{[x_4,x_5]}\mathrm{d}) + i_{x_1} i_{x_2} (i_{x_3}\mathcal{L}_{x_4} + i_{[x_3,x_4]} + i_{x_3} i_{x_4}\mathrm{d})(2\mathcal{L}_{x_5} - \mathrm{d} i_{x_5})\big)\sigma_6 \\
&\quad = \tfrac{1}{5}\big(30 i_{[x_1,x_2]} \operatorname{tr} \Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6 + 60 i_{x_1}\mathcal{L}_{x_2} \operatorname{tr} \Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6 - 12\mathcal{L}_{x_1} \operatorname{tr} \Lambda_2\Lambda_3\Lambda_4\Lambda_5\Lambda_6\big) \\
&\quad + \tfrac{1}{30}\big(15\mathcal{L}_{x_1} i_{x_2} i_{x_3} i_{[x_4,x_5]} - 5\mathcal{L}_{[x_1,x_2]} i_{x_3} i_{x_4} i_{x_5} - 5\mathcal{L}_{x_1} i_{x_2} i_{x_3} i_{[x_4,x_5]} + \tfrac{5}{2}\mathcal{L}_{[x_1,x_2]} i_{x_3} i_{x_4} i_{x_5}\big)\sigma_6, \\
&\ell_3(\ell_4(X_1, X_2, X_3, X_4), X_5, X_6) = 2(i_{x_1}\mathcal{L}_{x_2} + i_{[x_1,x_2]}) \operatorname{tr} \Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6, \\
&\ell_6(\ell_1(X_1), X_2, X_3, X_4, X_5, X_6) = -10 i_{x_1} i_{x_2} \operatorname{tr} \Lambda_3\Lambda_4\mathrm{d}\Lambda_5\mathrm{d}\Lambda_6, \\
&\Rightarrow 6\ell_2(\ell_5(X_1, X_2, X_3, X_4, X_5), X_6) - 15\ell_5(\ell_2(X_1, X_2), X_3, X_4, X_5, X_6) \\
&- 15\ell_3(\ell_4(X_1, X_2, X_3, X_4), X_5, X_6) + 6\ell_6(\ell_1(X_1), X_2, X_3, X_4, X_5, X_6) \\
&\quad = -30(\mathcal{L}_{x_1} i_{x_2} - i_{x_1}\mathcal{L}_{x_2}) \operatorname{tr} \Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_{6]} - 60 i_{x_{[1}} i_{x_2} \operatorname{tr} \Lambda_3\Lambda_4\mathrm{d}\Lambda_5\mathrm{d}\Lambda_6 = 0 \quad \square
\end{aligned}
$$

And we also need $(V_0)^5 \otimes V_{-1}$

$$
\begin{aligned}
&\ell_1(\ell_6(X_1,X_2,X_3,X_4,X_5,\Psi)) = 10\mathrm{d}i_{x_1}i_{x_2}\operatorname{tr}\Lambda_3\Lambda_4\mathrm{d}\Lambda_5 A,\\
&\ell_2(\ell_5(X_1,X_2,X_3,X_4,\Psi),X_5) = \tfrac{1}{2}\mathcal{L}_{x_1}(12i_{x_2}\operatorname{tr}\Lambda_3\Lambda_4\mathrm{d}\Lambda_{5]}A + 2i_{x_2}i_{x_3}\operatorname{tr}\Lambda_4\mathrm{d}\Lambda_5\mathrm{d}A),\\
&\ell_5(\ell_2(X_1,X_2),X_3,X_4,X_5,\Psi) = \tfrac{1}{4}\big(-12i_{[x_1,x_2]}\operatorname{tr}\Lambda_3\Lambda_4\mathrm{d}\Lambda_5 A - 2i_{[x_1,x_2]}i_{x_3}\operatorname{tr}\Lambda_4\mathrm{d}\Lambda_5\mathrm{d}A\\
&\quad + 12i_{x_3}\operatorname{tr}\Lambda_4\Lambda_5\mathrm{d}(2\mathcal{L}_{x_1}\Lambda_2 + 2\Lambda_1\Lambda_2)A + 2i_{x_3}i_{x_4}\operatorname{tr}\Lambda_5\mathrm{d}(2\mathcal{L}_{x_1}\Lambda_2 + 2\Lambda_1\Lambda_2)\mathrm{d}A\\
&\quad - 12i_{x_4}\operatorname{tr}\Lambda_5(2\mathcal{L}_{x_1}\Lambda_2 + 2\Lambda_1\Lambda_2)\mathrm{d}\Lambda_3 A - 2i_{x_4}i_{x_5}\operatorname{tr}(2\mathcal{L}_{x_1}\Lambda_2 + 2\Lambda_1\Lambda_2)\mathrm{d}\Lambda_3\mathrm{d}A\\
&\quad + 12i_{x_5}\operatorname{tr}(2\mathcal{L}_{x_1}\Lambda_2 + 2\Lambda_1\Lambda_2)\Lambda_3\mathrm{d}\Lambda_4 A + 2i_{x_5}i_{[x_1,x_2]}\operatorname{tr}\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A\big)\\
&\quad = -3i_{[x_1,x_2]}\operatorname{tr}\Lambda_3\Lambda_4\mathrm{d}\Lambda_5 A - i_{[x_1,x_2]}i_{x_3}\operatorname{tr}\Lambda_4\mathrm{d}\Lambda_5\mathrm{d}A + i_{x_1}i_{x_2}\operatorname{tr}\Lambda_3\mathrm{d}\Lambda_4\Lambda_5\mathrm{d}A\\
&\quad - i_{x_1}i_{x_2}\operatorname{tr}\mathcal{L}_{x_3}(\Lambda_4\mathrm{d}\Lambda_5)\mathrm{d}A + 6i_{x_1}\operatorname{tr}\mathcal{L}_{x_2}(\Lambda_3\Lambda_4\mathrm{d}\Lambda_5)A + 6i_{x_1}\operatorname{tr}\Lambda_2\Lambda_3\mathrm{d}(\Lambda_4\Lambda_5)A,\\
&\ell_5(\ell_2(X_1,\Psi),X_2,X_3,X_4,X_5) = -12i_{x_2}\operatorname{tr}\Lambda_3\Lambda_4\mathrm{d}\Lambda_5(\mathcal{L}_{x_1}A - [A,\Lambda_1])\\
&\quad - 2i_{x_2}i_{x_3}\operatorname{tr}\Lambda_4\mathrm{d}\Lambda_5(\mathcal{L}_{x_1}\mathrm{d}A - \mathrm{d}[A,\Lambda_1])\\
&\quad = -12i_{x_2}\operatorname{tr}\Lambda_3\Lambda_4\mathrm{d}\Lambda_5\mathcal{L}_{x_1}A - 12i_{x_1}\operatorname{tr}\Lambda_2\Lambda_3\Lambda_4\mathrm{d}\Lambda_5 A - 12i_{x_1}\operatorname{tr}\Lambda_2\Lambda_3\mathrm{d}\Lambda_4\Lambda_5 A\\
&\quad - 2i_{x_2}i_{x_3}\operatorname{tr}\Lambda_4\mathrm{d}\Lambda_5\mathcal{L}_{x_1}\mathrm{d}A - 2i_{x_2}i_{x_3}\operatorname{tr}\Lambda_1\mathrm{d}\Lambda_4\Lambda_5\mathrm{d}A - 2i_{x_1}i_{x_2}\mathrm{d}\operatorname{tr}\Lambda_3\Lambda_4\mathrm{d}\Lambda_5 A,\\
&\ell_3(\ell_4(X_1,X_2,X_3,\Psi),X_4,X_5) = -\tfrac{1}{2}(i_{x_1}\mathcal{L}_{x_2} + i_{[x_1,x_2]})(i_{x_3}\operatorname{tr}\Lambda_4\mathrm{d}\Lambda_5\mathrm{d}A + 2\operatorname{tr}\Lambda_3\Lambda_4\mathrm{d}\Lambda_5 A),\\
&\Rightarrow \ell_1(\ell_6(X_1,X_2,X_3,X_4,X_5,\Psi)) + 5\ell_2(\ell_5(X_1,X_2,X_3,X_4,\Psi),X_5)\\
&+ 10\ell_3(\ell_4(X_1,X_2,X_3,\Psi),X_4,X_5) + 5\ell_5(\ell_2(X_1,\Psi),X_2,X_3,X_4,X_5)\\
&+ 10\ell_5(\ell_2(X_1,X_2),X_3,X_4,X_5,\Psi)\\
&\quad = -10\mathrm{d}i_{x_1}i_{x_2}\operatorname{tr}\Lambda_3\Lambda_4\mathrm{d}\Lambda_5 A + 5(6\mathcal{L}_{x_1}i_{x_2}\operatorname{tr}\Lambda_3\Lambda_4\mathrm{d}\Lambda_5 A + \mathcal{L}_{x_1}i_{x_2}i_{x_3}\operatorname{tr}\Lambda_4\mathrm{d}\Lambda_5\mathrm{d}A)\\
&\quad - 5\big((2i_{x_1}\mathcal{L}_{x_2} + \mathcal{L}_{x_1}i_{x_2})i_{x_3}\operatorname{tr}\Lambda_4\mathrm{d}\Lambda_5\mathrm{d}A + (4i_{x_1}\mathcal{L}_{x_2} + 2\mathcal{L}_{x_1}i_{x_2})\operatorname{tr}\Lambda_3\Lambda_4\mathrm{d}\Lambda_5 A\big)\\
&\quad - 5(6\mathcal{L}_{x_1}i_{x_2} - 6i_{x_1}\mathcal{L}_{x_2} + 2i_{x_1}i_{x_2}\mathrm{d})\operatorname{tr}\Lambda_3\Lambda_4\mathrm{d}\Lambda_5 A\\
&\quad - 10(\mathcal{L}_{x_1}i_{x_2}i_{x_3} + i_{x_1}\mathcal{L}_{x_2}i_{x_3} + i_{x_1}i_{x_2}\mathcal{L}_{x_3})\operatorname{tr}\Lambda_4\mathrm{d}\Lambda_5\mathrm{d}A\\
&\quad = 10(\mathrm{d}i_{x_1}i_{x_2} - i_{x_1}i_{x_2}\mathrm{d} - \mathcal{L}_{x_1}i_{x_2} + i_{x_1}\mathcal{L}_{x_2})\operatorname{tr}\Lambda_3\Lambda_4\mathrm{d}\Lambda_5 A\\
&\quad - 10(\mathcal{L}_{x_1}i_{x_2}i_{x_3} + 2i_{x_1}\mathcal{L}_{x_2}i_{x_3} + i_{x_1}i_{x_2}\mathcal{L}_{x_3})\operatorname{tr}\Lambda_4\mathrm{d}\Lambda_5\mathrm{d}A\\
&\quad = -10(i_{[x_1,x_2]}i_{x_3} + i_{x_1}i_{[x_2,x_3]})\operatorname{tr}\Lambda_4\mathrm{d}\Lambda_5\mathrm{d}A = 0 \quad \square
\end{aligned}
$$

B.6 $\ell_4\ell_4 + \ell_5\ell_3 + \ell_3\ell_5 + \ell_2\ell_6 + \ell_6\ell_2 = 0$ relations

These do not involve any new brackets, and the only non-trivial check that must be made is for inputs in $(V_0)^6 \otimes V_{-1}$

$$\ell_5(\ell_3(X_1, X_2, \Psi), X_3, X_4, X_5, X_6) = -\tfrac{1}{3}\tfrac{1}{5} i_{x_3} i_{x_4} (i_{x_5}\mathcal{L}_{x_6} + i_{[x_5,x_6]} + i_{x_5} i_{x_6}\mathrm{d}) \operatorname{tr}\Lambda_1 \mathrm{d}\Lambda_2 \mathrm{d}A,$$

$$\ell_3(\ell_5(X_1, X_2, X_3, X_4, \Psi), X_5, X_6) = \tfrac{1}{3}(i_{x_5}\mathcal{L}_{x_6} + i_{[x_5,x_6]})(6 i_{x_1} \operatorname{tr}\Lambda_2\Lambda_3\mathrm{d}\Lambda_4 A + i_{x_1} i_{x_2} \operatorname{tr}\Lambda_3 \mathrm{d}\Lambda_4 \mathrm{d}A),$$

$$\ell_2(\ell_6(X_1, X_2, X_3, X_4, X_5, \Psi), X_6) = -5\mathcal{L}_{x_6} i_{x_1} i_{x_2} \operatorname{tr}\Lambda_3\Lambda_4\mathrm{d}\Lambda_5 A,$$

$$\ell_6(\ell_2(X_1, \Psi), X_2, X_3, X_4, X_5, X_6) = -10 i_{x_2} i_{x_3} \operatorname{tr}\Lambda_4\Lambda_5\mathrm{d}\Lambda_6(\mathcal{L}_{x_1}A - [A, \Lambda_1]),$$

$$\ell_6(\ell_2(X_1, X_2), X_3, X_4, X_5, X_6, \Psi) = 4 i_{[x_1,x_2]} i_{x_3} \operatorname{tr}\Lambda_4\Lambda_5\mathrm{d}\Lambda_6 A + 4 i_{x_5} i_{x_6} \operatorname{tr}\mathcal{L}_{x_1}(\Lambda_2\Lambda_3\mathrm{d}\Lambda_4)A$$
$$+ 4 i_{x_3} i_{x_4} \operatorname{tr}\Lambda_5\Lambda_6\Lambda_1\mathrm{d}\Lambda_2 A + 4 i_{x_3} i_{x_4} \operatorname{tr}\Lambda_5\Lambda_6\mathrm{d}\Lambda_1\Lambda_2 A,$$

$$\Rightarrow 15\ell_5(\ell_3(X_1, X_2, \Psi), X_3, X_4, X_5, X_6) + 15\ell_3(\ell_5(X_1, X_2, X_3, X_4, \Psi), X_5, X_6)$$
$$- 6\ell_2(\ell_6(X_1, X_2, X_3, X_4, X_5, \Psi), X_6) + 15\ell_6(\ell_2(X_1, X_2), X_3, X_4, X_5, X_6, \Psi)$$
$$- 6\ell_6(\ell_2(X_1, \Psi), X_2, X_3, X_4, X_5, X_6) = 0 \quad \square$$

B.7 Higher relations

The identities

$$\ell_3\ell_6 - \ell_6\ell_3 - \ell_4\ell_5 + \ell_5\ell_4 = 0,$$
$$\ell_5\ell_5 + \ell_4\ell_6 + \ell_6\ell_4 = 0,$$
$$\ell_5\ell_6 - \ell_6\ell_5 = 0,$$
$$\ell_6\ell_6 = 0,$$

are all trivially satisfied.

C Computation of generalised Jacobi identities from section 3

As in the previous section, we verify the generalised Jacobi relations by exhaustively going through every term of the identities (1.25), for each level n and for each possible set of inputs for the brackets in section 3. And as was the case in the previous section, the extra sub-structure of the vector spaces V_0 and V_{-1} means that many of those terms vanish trivially. We omit those from the following. We also omit the explicit use of index symmetrisers for the sake of readability. As such, products containing multiple elements $X_i = x_i + \Lambda_i + \sigma_i \in V_0$ are always implicitly antisymmetrised, so $\ell_2(X_1, \ell_2(X_2, X_3)) = \ell_2(X_{[1}, \ell_2(X_2, X_{3]}))$, and products containing multiple elements $\Psi_i = A_i + \tilde{C}_i \in V_{-1}$ are always implicitly symmetrised so for example $\ell_3(X_1, \ell_2(X_2, \Psi_1), \Psi_2) = \ell_3(X_{[1}, \ell_2(X_{2]}, \Psi_{(1)}), \Psi_{2)})$.

As always, the first generalised Jacobi identity is simply, from equations (3.11),

$$(\ell_1)^2 = 0 \quad \square \tag{C.1}$$

The $\ell_1\ell_2 - \ell_2\ell_1 = 0$ relations with brackets (3.12) are the same as in the R^3 case in B.1, and the $\ell_1\ell_3 + \ell_3\ell_1 + \ell_2\ell_2 = 0$ relations with brackets (3.13) are either the same as in B.2 or trivial.

C.1 $\ell_1\ell_4 - \ell_4\ell_1 - \ell_2\ell_3 + \ell_3\ell_2 = 0$ relations

The first non-trivial identities involving brackets (3.14) are the same as in B.3, except for a set of inputs in $V_0 \otimes V_0 \otimes V_{-1} \otimes V_{-1}$

$$\begin{aligned}
&\ell_1(\ell_4(X_1, X_2, \Psi_1, \Psi_2)) = -4 \operatorname{tr} \mathrm{d}\Lambda_1 \mathrm{d}\Lambda_2 \mathrm{d}A_1 \mathrm{d}A_2, \\
&\ell_4(X_1, \ell_1(X_2), \Psi_1, \Psi_2) = -2 \operatorname{tr} \mathrm{d}\Lambda_1 \mathrm{d}\Lambda_2 \mathrm{d}A_1 \mathrm{d}A_2, \\
&\Rightarrow \ell_1(\ell_4(X_1, \Psi_1, \Psi_2, \Psi_3)) - 2\ell_4(X_1, \ell_1(X_2), \Psi_1, \Psi_2) = 0 \quad \square
\end{aligned}$$

C.2 $\ell_1\ell_5 + \ell_5\ell_1 + \ell_2\ell_4 + \ell_4\ell_2 + \ell_3\ell_3 = 0$ relations

For the next identities we need brackets (3.15).

First we consider the action on elements of $(V_0)^2 \otimes (V_{-1})^3$

$$\begin{aligned}
&\ell_5(X_1, \ell_1(X_2), \Psi_1, \Psi_2, \Psi_3) = \tfrac{6}{5} \operatorname{tr} \mathrm{d}\Lambda_1 \big(2\mathrm{d}A_1 \mathrm{d}\Lambda_2 A_2 A_3 + 2\mathrm{d}A_1 A_2 \mathrm{d}\Lambda_2 A_3 + 2\mathrm{d}A_1 A_2 A_3 \mathrm{d}\Lambda_2 \\
&\quad + 2A_1 \mathrm{d}\Lambda_2 A_2 \mathrm{d}A_3 + 2A_1 A_2 \mathrm{d}\Lambda_2 \mathrm{d}A_3 + 2\mathrm{d}\Lambda_2 A_1 A_2 \mathrm{d}A_3 + \mathrm{d}\Lambda_2 \mathrm{d}A_1 A_2 A_3 + A_1 \mathrm{d}A_2 \mathrm{d}\Lambda_2 A_3 \\
&\quad + A_1 \mathrm{d}A_2 A_3 \mathrm{d}\Lambda_2 + \mathrm{d}\Lambda_2 A_1 \mathrm{d}A_2 A_3 + A_1 \mathrm{d}\Lambda_2 \mathrm{d}A_2 A_3 + A_1 A_2 \mathrm{d}A_3 \mathrm{d}\Lambda_2\big) \\
&\quad = \tfrac{6}{5} \operatorname{tr} \mathrm{d}\Lambda_1 \mathrm{d}\big(A_1 A_2 \mathrm{d}\Lambda_2 A_3 - 3\mathrm{d}\Lambda_2 A_1 A_2 A_3\big) + 6 \operatorname{tr} \mathrm{d}\Lambda_1 \big(\mathrm{d}\Lambda_2 A_1 \mathrm{d}A_2 A_3 + A_1 A_2 \mathrm{d}\Lambda_2 \mathrm{d}A_3\big), \\
&\ell_2(X_1, \ell_4(X_2, \Psi_1, \Psi_2, \Psi_3)) = 6\mathcal{L}_{x_1} \operatorname{tr} \mathrm{d}\Lambda_2 A_1 \mathrm{d}A_2 \mathrm{d}A_3, \\
&\ell_4(\ell_2(X_1, X_2), \Psi_1, \Psi_2, \Psi_3) = 12 \operatorname{tr}(\mathcal{L}_{x_1}\Lambda_2) A_1 \mathrm{d}A_2 \mathrm{d}A_3 + 12 \operatorname{tr} \mathrm{d}(\Lambda_1\Lambda_2) A_1 \mathrm{d}A_2 \mathrm{d}A_3, \\
&\ell_4(X_1, \ell_2(X_2, \Psi_1), \Psi_2, \Psi_3) = 2 \operatorname{tr} \mathrm{d}\Lambda_1(\mathcal{L}_{x_2} A_1 \mathrm{d}A_2 \mathrm{d}A_3) - 2 \operatorname{tr} \mathrm{d}\Lambda_1 [A_1, \Lambda_2] \mathrm{d}A_2 \mathrm{d}A_3 \\
&\quad - 2 \operatorname{tr} \mathrm{d}\Lambda_1 A_1 \mathrm{d}[A_2, \Lambda_2] \mathrm{d}A_3 - 2 \operatorname{tr} \mathrm{d}\Lambda_1 A_1 \mathrm{d}A_2 \mathrm{d}[A_3, \Lambda_2] \\
&\quad = -2 \operatorname{tr} \mathrm{d}\Lambda_2(\mathcal{L}_{x_1} A_1 \mathrm{d}A_2 \mathrm{d}A_3) + 2 \operatorname{tr} \mathrm{d}(\Lambda_1\Lambda_2) A_1 \mathrm{d}A_2 \mathrm{d}A_3 \\
&\quad + 2 \operatorname{tr} \mathrm{d}\Lambda_1 \big(\mathrm{d}\Lambda_2 A_1 \mathrm{d}A_2 A_3 + A_1 A_2 \mathrm{d}\Lambda_2 \mathrm{d}A_3\big), \\
&\ell_1(\ell_5(X_1, X_2, \Psi_1, \Psi_2, \Psi_3)) = \tfrac{12}{5} \operatorname{tr} \mathrm{d}\Lambda_1 \mathrm{d}\big(3\mathrm{d}\Lambda_2 A_1 A_2 A_3 - A_1 A_2 \mathrm{d}\Lambda_2 A_3\big), \\
&\Rightarrow \ell_1(\ell_5(X_1, X_2, \Psi_1, \Psi_2, \Psi_3)) + 2\ell_5(X_1, \ell_1(X_2), \Psi_1, \Psi_2, \Psi_3) - 2\ell_2(X_1, \ell_4(X_2, \Psi_1, \Psi_2, \Psi_3)) \\
&+ \ell_4(\ell_2(X_1, X_2), \Psi_1, \Psi_2, \Psi_3) - 6\ell_4(X_1, \ell_2(X_2, \Psi_1), \Psi_2, \Psi_3) = 0 \quad \square
\end{aligned}$$

Next on $(V_0)^3 \otimes (V_{-1})^2$

$$\ell_5(X_1,X_2,\ell_1(X_3),\Psi_1,\Psi_2) = \tfrac{4}{5}\operatorname{tr}\Lambda_1\mathrm{d}\big(3\mathrm{d}\Lambda_2\mathrm{d}\Lambda_3A_1A_2 + 3\mathrm{d}\Lambda_2A_1\mathrm{d}\Lambda_3A_2 + 3\mathrm{d}\Lambda_2A_1A_2\mathrm{d}\Lambda_3$$
$$+ \mathrm{d}\Lambda_2A_1\mathrm{d}\Lambda_3A_2 + A_1\mathrm{d}\Lambda_2\mathrm{d}\Lambda_3A_2 - A_1A_2\mathrm{d}\Lambda_2\mathrm{d}\Lambda_3\big),$$

$$\ell_2(X_1,\ell_4(X_2,X_3,\Psi_1,\Psi_2)) = 4(\mathcal{L}_{x_1} - \tfrac{1}{2}\mathrm{d}i_{x_1})\operatorname{tr}\Lambda_2\mathrm{d}\Lambda_3\mathrm{d}A_1\mathrm{d}A_2,$$

$$\ell_4(\ell_2(X_1,X_2),X_3,\Psi_1,\Psi_2) = 4\operatorname{tr}(\mathcal{L}_{x_1}\Lambda_2\mathrm{d}\Lambda_3)\mathrm{d}A_1\mathrm{d}A_2 + 4\operatorname{tr}\Lambda_1\Lambda_2\mathrm{d}\Lambda_3\mathrm{d}A_1\mathrm{d}A_2$$
$$- 4\operatorname{tr}\Lambda_1\mathrm{d}(\Lambda_2\Lambda_3)\mathrm{d}A_1\mathrm{d}A_2$$
$$= -4\operatorname{tr}\Lambda_1\mathrm{d}\Lambda_2\Lambda_3\mathrm{d}A_1\mathrm{d}A_2 + 4\operatorname{tr}(\mathcal{L}_{x_1}\Lambda_2\mathrm{d}\Lambda_3)\mathrm{d}A_1\mathrm{d}A_2,$$

$$\ell_4(X_1,X_2,\ell_2(X_3,\Psi_1),\Psi_2) = 2\operatorname{tr}\Lambda_1\mathrm{d}\Lambda_2(\mathcal{L}_{x_3}\mathrm{d}A_1\mathrm{d}A_2)$$
$$- 2\operatorname{tr}\Lambda_1\mathrm{d}\Lambda_2\mathrm{d}[A_1,\Lambda_3]\mathrm{d}A_2 - 2\operatorname{tr}\Lambda_1\mathrm{d}\Lambda_2\mathrm{d}A_1\mathrm{d}[A_2,\Lambda_3]$$
$$= 2\operatorname{tr}\Lambda_2\mathrm{d}\Lambda_3(\mathcal{L}_{x_1}\mathrm{d}A_1\mathrm{d}A_2) - 2\operatorname{tr}\Lambda_1\mathrm{d}\Lambda_2\big(-A_1\mathrm{d}\Lambda_3\mathrm{d}A_2 - \mathrm{d}\Lambda_3A_1\mathrm{d}A_2$$
$$- \Lambda_3\mathrm{d}A_1\mathrm{d}A_2 + \mathrm{d}A_1\mathrm{d}A_2\Lambda_3 - \mathrm{d}A_1A_2\mathrm{d}\Lambda_3 - \mathrm{d}A_1\mathrm{d}\Lambda_3A_2\big),$$

$$\ell_1(\ell_5(X_1,X_2,X_3,\Psi_1,\Psi_2)) = -6\mathrm{d}i_{x_1}\operatorname{tr}\Lambda_2\mathrm{d}\Lambda_3\mathrm{d}A_1\mathrm{d}A_2 - \tfrac{12}{5}\mathrm{d}\operatorname{tr}\Lambda_1\big(3\Lambda_2\mathrm{d}\Lambda_3\mathrm{d}A_1A_2$$
$$+ 2\Lambda_2\mathrm{d}\Lambda_3A_1\mathrm{d}A_2 - \mathrm{d}\Lambda_2\Lambda_3\mathrm{d}(A_1A_2) - \Lambda_2\mathrm{d}\Lambda_3\mathrm{d}(A_1A_2)\big) + \tfrac{12}{5}\mathrm{d}\operatorname{tr}\Lambda_1\mathrm{d}\Lambda_2(\mathrm{d}A_1\Lambda_3A_2 - A_1\Lambda_3\mathrm{d}A_2),$$

$$\Rightarrow \ell_1(\ell_5(X_1,X_2,X_3,\Psi_1,\Psi_2)) + 3\ell_5(X_1,X_2,\ell_1(X_3),\Psi_1,\Psi_2) - 3\ell_2(X_1,\ell_4(X_2,X_3,\Psi_1,\Psi_2))$$
$$+ 3\ell_4(\ell_2(X_1,X_2),X_3,\Psi_1,\Psi_2) + 6\ell_4(X_1,X_2,\ell_2(X_3,\Psi_1),\Psi_2) = 0 \quad \square$$

On $(V_0)^4 \otimes V_{-1}$

$$\ell_5(X_1,X_2,X_3,\ell_1(X_4),\Psi) = \tfrac{6}{5}\mathrm{d}\operatorname{tr}\big(2\Lambda_1\Lambda_2\Lambda_3\mathrm{d}\Lambda_4 - \Lambda_1\Lambda_2\mathrm{d}\Lambda_3\Lambda_4\big)\mathrm{d}A,$$
$$\ell_1(\ell_5(X_1,X_2,X_3,X_4,\Psi)) = -\tfrac{24}{5}\mathrm{d}\operatorname{tr}\big(2\Lambda_1\Lambda_2\Lambda_3\mathrm{d}\Lambda_4 - \Lambda_1\Lambda_2\mathrm{d}\Lambda_3\Lambda_4\big)\mathrm{d}A,$$
$$\Rightarrow 4\ell_5(X_1,X_2,X_3,\ell_1(X_4),\Psi) + \ell_1(\ell_5(X_1,X_2,X_3,X_4,\Psi)) = 0 \quad \square$$

On $(V_0)^5$

$$\ell_1(\ell_5(X_1,X_2,X_3,X_4,X_5)) = \tfrac{1}{6}\mathrm{d}i_{x_1}i_{x_2}\big(i_{x_3}\mathcal{L}_{x_4} + i_{[x_3,x_4]} + i_{x_3}i_{x_4}\mathrm{d}\big)\sigma_5$$
$$= -\tfrac{5}{6}\big(i_{x_1}i_{[x_2,x_3]}\mathcal{L}_{x_4} - i_{x_1}i_{x_2}\mathcal{L}_{x_3}\mathcal{L}_{x_4} - i_{x_1}i_{x_2}i_{[x_3,x_4]}\mathrm{d} + i_{x_1}i_{x_2}i_{x_3}\mathcal{L}_{x_4}\mathrm{d}\big)\sigma_5,$$
$$\ell_3(\ell_3(X_3,X_4,X_5),X_1,X_2) = \tfrac{1}{2}\tfrac{1}{6}(i_{x_1}\mathcal{L}_{x_2} + i_{[x_1,x_2]})\big(i_{x_3}\mathcal{L}_{x_4} + i_{[x_3,x_4]} + i_{x_3}i_{x_4}\mathrm{d}\big)\sigma_5,$$
$$\Rightarrow \ell_1(\ell_5(X_1,X_2,X_3,X_4,X_5)) + 10\ell_3(\ell_3(X_1,X_2,X_3),X_4,X_5) = 0 \quad \square$$

And finally on $(\oplus_{i>0}V_i) \otimes (V_0)^4$

$$\ell_1(\ell_5(\xi,X_1,X_2,X_3,X_4)) = \tfrac{1}{6}\tfrac{1}{5}\mathrm{d}i_{x_1}i_{x_2}(i_{x_3}\mathcal{L}_{x_4} + i_{[x_3,x_4]})\xi,$$
$$\ell_5(\ell_1(\xi),X_1,X_2,X_3,X_4) = \tfrac{1}{6}\tfrac{1}{5}i_{x_1}i_{x_2}(i_{x_3}\mathcal{L}_{x_4} + i_{[x_3,x_4]})\mathrm{d}\xi,$$
$$\ell_3(\ell_3(\xi,X_1,X_2),X_3,X_4) = \tfrac{1}{36}(i_{x_1}\mathcal{L}_{x_2} + i_{[x_1,x_2]})(i_{x_3}\mathcal{L}_{x_4} + i_{[x_3,x_4]})\xi,$$
$$\Rightarrow \ell_1(\ell_5(\xi,X_1,X_2,X_3,X_4)) + \ell_5(\ell_1(\xi),X_1,X_2,X_3,X_4) + 6\ell_3(\ell_3(\xi,X_1,X_2),X_3,X_4) = 0 \quad \square$$

C.3 $\ell_1\ell_6 - \ell_6\ell_1 - \ell_2\ell_5 + \ell_5\ell_2 + \ell_3\ell_4 - \ell_4\ell_3 = 0$ relations

Now we start using brackets (3.16). We first check the identity for objects in $(V_0)^2 \otimes (V_{-1})^4$

$$\begin{aligned}
&\ell_6(X_1, \ell_1(X_2), \Psi_1, \Psi_2, \Psi_3, \Psi_4) = \tfrac{48}{5} \operatorname{tr} \mathrm{d}\Lambda_1\big(2\mathrm{d}\Lambda_2 A_1 A_2 A_3 A_4 + A_1 A_2 \mathrm{d}\Lambda_2 A_3 A_4\big),\\
&\ell_5(\ell_2(X_1, X_2), \Psi_1, \Psi_2, \Psi_3, \Psi_4) - 2\ell_5(X_1, \ell_2(X_2, \Psi_1), \Psi_2, \Psi_3, \Psi_4)\\
&\quad = \tfrac{48}{5} \operatorname{tr} \mathrm{d}\Lambda_1\big(2\mathrm{d}\Lambda_2 A_1 A_2 A_3 A_4 + A_1 A_2 \mathrm{d}\Lambda_2 A_3 A_4\big) - \tfrac{48}{5}\mathcal{L}_{x_1} \operatorname{tr} \mathrm{d}\Lambda_2\big(2A_1 A_2 A_3 \mathrm{d}A_4\\
&\quad + A_1 A_2 \mathrm{d}A_3 A_4 + A_1 \mathrm{d}A_2 A_3 A_4 + 2\mathrm{d}A_1 A_2 A_3 A_4\big),\\
&\ell_2(X_1, \ell_5(X_2, \Psi_1, \Psi_2, \Psi_3, \Psi_4)) = -\tfrac{24}{5}\mathcal{L}_{x_1} \operatorname{tr} \mathrm{d}\Lambda_2\big(2A_1 A_2 A_3 \mathrm{d}A_4 + A_1 A_2 \mathrm{d}A_3 A_4\\
&\quad + A_1 \mathrm{d}A_2 A_3 A_4 + 2\mathrm{d}A_1 A_2 A_3 A_4\big)\\
&\Rightarrow -\ell_6(X_1, \ell_1(X_2), \Psi_1, \Psi_2, \Psi_3, \Psi_4) - 2\ell_2(X_1, \ell_5(X_2, \Psi_1, \Psi_2, \Psi_3, \Psi_4))\\
&+ \ell_5(\ell_2(X_1, X_2), \Psi_1, \Psi_2, \Psi_3, \Psi_4) - 2\ell_5(X_1, \ell_2(X_2, \Psi_1), \Psi_2, \Psi_3, \Psi_4) = 0 \quad \square
\end{aligned}$$

Then for $(V_0)^3 \otimes (V_{-1})^3$

$$\begin{aligned}
&\ell_1(\ell_6(X_1, X_2, X_3, \Psi_1, \Psi_2, \Psi_3)) = \tfrac{18}{5}\mathrm{d}i_{x_1} \operatorname{tr} \Lambda_2 \mathrm{d}(3\mathrm{d}\Lambda_3 A_1 A_2 A_3 - A_1 A_2 \mathrm{d}\Lambda_3 A_3)\\
&\quad - \tfrac{36}{5}\mathrm{d} \operatorname{tr}\big(3\Lambda_1\Lambda_2 \mathrm{d}\Lambda_3 A_1 A_2 A_3 - \Lambda_1\Lambda_2 A_1 A_2 \mathrm{d}\Lambda_3 A_3\big),\\
&\ell_2(X_1, \ell_5(X_2, X_3, \Psi_1, \Psi_2, \Psi_3)) = -\tfrac{12}{5}(\mathcal{L}_{x_1} - \tfrac{1}{2}\mathrm{d}i_{x_1}) \operatorname{tr} \Lambda_2 \mathrm{d}\big(3\mathrm{d}\Lambda_3 A_1 A_2 A_3 - A_1 A_2 \mathrm{d}\Lambda_3 A_3\big),\\
&\ell_5(\ell_2(X_1, X_2), X_3, \Psi_1, \Psi_2, \Psi_3) + 3\ell_5(X_1, X_2, \ell_2(X_3, \Psi_1), \Psi_2, \Psi_3)\\
&\quad = -\tfrac{12}{5}\mathcal{L}_{x_1} \operatorname{tr} \Lambda_2 \mathrm{d}\big(3\mathrm{d}\Lambda_3 A_1 A_2 A_3 - A_1 A_2 \mathrm{d}\Lambda_3 A_3\big) + \tfrac{12}{5} \operatorname{tr} \Lambda_1\Lambda_2\big(3\mathrm{d}\Lambda_3 \mathrm{d}(A_1 A_2 A_3)\\
&\quad + \mathrm{d}(A_1 A_2)\mathrm{d}\Lambda_3 A_3 - A_1 A_2 \mathrm{d}\Lambda_3 \mathrm{d}A_3\big) - \tfrac{12}{5} \operatorname{tr} \Lambda_1\big(3\mathrm{d}(\Lambda_2\Lambda_3)\mathrm{d}(A_1 A_2 A_3)\\
&\quad + \mathrm{d}(A_1 A_2)\mathrm{d}(\Lambda_2\Lambda_3)A_3 - A_1 A_2 \mathrm{d}(\Lambda_2\Lambda_3)\mathrm{d}A_3\big)\\
&\quad - \tfrac{12}{5} \operatorname{tr} \Lambda_1\big(3\mathrm{d}\Lambda_2 \mathrm{d}(A_1 A_2 A_3 \Lambda_3 - \Lambda_3 A_1 A_2 A_3) + \mathrm{d}(\Lambda_2 A_1 A_2 - A_1 A_2 \Lambda_2)\mathrm{d}\Lambda_3 A_3\\
&\quad + \mathrm{d}(A_1 A_2)\mathrm{d}\Lambda_2[A_3, \Lambda_3] + (A_1 A_2 \Lambda_2 - \Lambda_2 A_1 A_2)\mathrm{d}\Lambda_3 \mathrm{d}A_3 - A_1 A_2 \mathrm{d}\Lambda_2 \mathrm{d}[A_3, \Lambda_3]\big)\\
&\quad = -\tfrac{12}{5}\mathcal{L}_{x_1} \operatorname{tr} \Lambda_2 \mathrm{d}\big(3\mathrm{d}\Lambda_3 A_1 A_2 A_3 - A_1 A_2 \mathrm{d}\Lambda_3 A_3\big)\\
&\quad + \tfrac{12}{5}\mathrm{d} \operatorname{tr}\big(3\Lambda_1\Lambda_2 \mathrm{d}\Lambda_3 A_1 A_2 A_3 - \Lambda_1\Lambda_2 A_1 A_2 \mathrm{d}\Lambda_3 A_3\big),\\
&\Rightarrow \ell_1(\ell_6(X_1, X_2, X_3, \Psi_1, \Psi_2, \Psi_3)) - 3\ell_2(X_1, \ell_5(X_2, X_3, \Psi_1, \Psi_2, \Psi_3))\\
&+ 3\ell_5(\ell_2(X_1, X_2), X_3, \Psi_1, \Psi_2, \Psi_3) + 9\ell_5(X_1, X_2, \ell_2(X_3, \Psi_1), \Psi_2, \Psi_3) = 0 \quad \square
\end{aligned}$$

For $(V_0)^4 \otimes (V_{-1})^2$

$$\begin{aligned}
&\ell_1(\ell_6(X_1,X_2,X_3,X_4,\Psi_1,\Psi_2)) = 4\mathrm{d}i_{x_1}i_{x_2}\,\mathrm{tr}\,\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A_1\mathrm{d}A_2 + \tfrac{24}{5}\mathrm{d}i_{x_1}\,\mathrm{tr}\,\Lambda_2\big(2\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A_1A_2\\
&\quad + 3\Lambda_3\mathrm{d}\Lambda_4A_1\mathrm{d}A_2 - \mathrm{d}\Lambda_3\Lambda_4\mathrm{d}(A_1A_2) - \mathrm{d}\Lambda_3(\mathrm{d}A_1\Lambda_4A_2 - A_1\Lambda_4\mathrm{d}A_2)\big)\\
&\quad - \tfrac{48}{5}\mathrm{d}\,\mathrm{tr}\,\big(\Lambda_1\mathrm{d}(\Lambda_2\Lambda_3)\Lambda_4A_1A_2 + \Lambda_1\Lambda_2\mathrm{d}\Lambda_3A_1\Lambda_4A_2 + \Lambda_1\mathrm{d}\Lambda_2A_1\Lambda_3\Lambda_4A_2\big),\\
&\ell_6(X_1,X_2,X_3,\ell_1(X_4),\Psi_1,\Psi_2) = -\tfrac{6}{5}i_{x_1}\,\mathrm{tr}\,\big((3\Lambda_2\mathrm{d}\Lambda_3\mathrm{d}\Lambda_4 - 3\mathrm{d}\Lambda_2\Lambda_3\mathrm{d}\Lambda_4 - \mathrm{d}\Lambda_2\mathrm{d}\Lambda_3\Lambda_4)\mathrm{d}(A_1A_2)\\
&\quad - 4\Lambda_2\mathrm{d}\Lambda_3\mathrm{d}A_1\mathrm{d}\Lambda_4A_2 - 4\Lambda_2\mathrm{d}\Lambda_3A_1\mathrm{d}\Lambda_4\mathrm{d}A_2 + \mathrm{d}\Lambda_2\mathrm{d}\Lambda_3A_1\Lambda_4\mathrm{d}A_2 - \mathrm{d}\Lambda_2\mathrm{d}\Lambda_3\mathrm{d}A_1\Lambda_4A_2\big)\\
&\quad + \tfrac{12}{5}\,\mathrm{tr}\,\big(3\Lambda_1\Lambda_2\mathrm{d}\Lambda_3\mathrm{d}\Lambda_4A_1A_2 + 3\mathrm{d}\Lambda_1\Lambda_2\Lambda_3\mathrm{d}\Lambda_4A_1A_2 - \mathrm{d}\Lambda_1\mathrm{d}\Lambda_2\Lambda_3\Lambda_4A_1A_2\\
&\quad + 4\Lambda_1\Lambda_2\mathrm{d}\Lambda_3A_1\mathrm{d}\Lambda_4A_2 + \Lambda_1\Lambda_2A_1\mathrm{d}\Lambda_3\mathrm{d}\Lambda_4A_2\big),\\
&3\ell_5(\ell_2(X_1,X_2),X_3,X_4,\Psi_1,\Psi_2) - 4\ell_5(X_1,X_2,X_3,\ell_2(X_4,\Psi_1),\Psi_2) = -6i_{[x_1,x_2]}\,\mathrm{tr}\,\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A_1\mathrm{d}A_2\\
&\quad + 12i_{x_1}\mathcal{L}_{x_2}\,\mathrm{tr}\,\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A_1\mathrm{d}A_2 - 12i_{x_1}\,\mathrm{tr}\,\big(\Lambda_2\mathrm{d}\Lambda_3\Lambda_4\mathrm{d}A_1\mathrm{d}A_2 - \Lambda_2\mathrm{d}\Lambda_3\mathrm{d}(A_1\mathrm{d}\Lambda_4A_2)\\
&\quad - \Lambda_2\mathrm{d}\Lambda_3\mathrm{d}\Lambda_4A_1\mathrm{d}A_2 - \mathrm{d}\Lambda_2\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A_1A_2 - \Lambda_2\mathrm{d}(\Lambda_3\Lambda_4)\mathrm{d}A_1\mathrm{d}A_2\big)\\
&\quad - \tfrac{24}{5}\mathcal{L}_{x_1}\,\mathrm{tr}\,\big(3\Lambda_2\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A_1A_2 + 2\Lambda_2\Lambda_3\mathrm{d}\Lambda_4A_1\mathrm{d}A_2 - \Lambda_2\mathrm{d}(\Lambda_3\Lambda_4)\mathrm{d}(A_1A_2)\\
&\quad - \Lambda_2\mathrm{d}\Lambda_3(\mathrm{d}A_1\Lambda_4A_2 - A_1\Lambda_4\mathrm{d}A_2) - \Lambda_2\mathrm{d}\Lambda_3(\mathrm{d}A_1\Lambda_4A_2 - A_1\Lambda_4\mathrm{d}A_2)\big)\\
&\quad + \tfrac{24}{5}\,\mathrm{tr}\,\big(4\Lambda_1\Lambda_2\mathrm{d}\Lambda_3A_1\mathrm{d}\Lambda_4A_2 + \mathrm{d}(\Lambda_1\Lambda_2\mathrm{d}\Lambda_3A_1\Lambda_4A_2) - \Lambda_1\mathrm{d}\Lambda_2\mathrm{d}(A_1\Lambda_3\Lambda_4A_2)\\
&\quad + (2\Lambda_1\Lambda_2\mathrm{d}\Lambda_3\mathrm{d}\Lambda_4 + 3\mathrm{d}\Lambda_1\Lambda_2\Lambda_3\mathrm{d}\Lambda_4 - \Lambda_1\mathrm{d}\Lambda_2\Lambda_3\mathrm{d}\Lambda_4 + \mathrm{d}\Lambda_1\Lambda_2\mathrm{d}\Lambda_3\Lambda_4)A_1A_2\\
&\quad - \Lambda_1\mathrm{d}(\Lambda_2\Lambda_3)\Lambda_4\mathrm{d}(A_1A_2)\big),\\
&\ell_2(X_1,\ell_5(X_2,X_3,X_4,\Psi_1,\Psi_2)) = -2\mathcal{L}_{x_1}i_{x_2}\,\mathrm{tr}\,\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A_1\mathrm{d}A_2 - \tfrac{6}{5}\mathcal{L}_{x_1}\,\mathrm{tr}\,\Lambda_2\big(2\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A_1A_2\\
&\quad + 3\Lambda_3\mathrm{d}\Lambda_4A_1\mathrm{d}A_2 - \mathrm{d}\Lambda_3\Lambda_4\mathrm{d}(A_1A_2) - \mathrm{d}\Lambda_3(\mathrm{d}A_1\Lambda_4A_2 - A_1\Lambda_4\mathrm{d}A_2)\big),\\
&\ell_3(\ell_4(X_1,X_2,\Psi_1,\Psi_2),X_3,X_4) = -\tfrac{2}{3}\big(i_{x_1}\mathcal{L}_{x_2} + i_{[x_1,x_2]} + i_{x_1}i_{x_2}\mathrm{d}\big)\,\mathrm{tr}\,\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A_1\mathrm{d}A_2,\\
&\Rightarrow \ell_1(\ell_6(X_1,X_2,X_3,X_4,\Psi_1,\Psi_2)) - 4\ell_6(X_1,X_2,X_3,\ell_1(X_4),\Psi_1,\Psi_2)\\
&+ 6\ell_5(\ell_2(X_1,X_2),X_3,X_4,\Psi_1,\Psi_2) - 8\ell_5(X_1,X_2,X_3,\ell_2(X_4,\Psi_1),\Psi_2)\\
&- 4\ell_2(X_1,\ell_5(X_2,X_3,X_4,\Psi_1,\Psi_2)) + 6\ell_3(\ell_4(X_1,X_2,\Psi_1,\Psi_2),X_3,X_4) = 0 \quad \square
\end{aligned}$$

For $(V_0)^5 \otimes V_{-1}$

$$
\begin{aligned}
&\ell_1(\ell_6(X_1,X_2,X_3,X_4,X_5,\Psi)) = 12\mathrm{d}i_{x_5}\,\mathrm{tr}\,\big(2\Lambda_1\Lambda_2\Lambda_3\mathrm{d}\Lambda_4 - \Lambda_1\Lambda_2\mathrm{d}\Lambda_3\Lambda_4\big)\mathrm{d}A\\
&\quad + 24\mathrm{d}\,\mathrm{tr}\,\big(2\Lambda_1\Lambda_2\Lambda_3\Lambda_4\mathrm{d}\Lambda_5 + \Lambda_1\Lambda_2\mathrm{d}\Lambda_3\Lambda_4\Lambda_5\big)A,\\
&\ell_6(X_1,X_2,X_3,X_4,\ell_1(X_5),\Psi) = \tfrac{12}{5}i_{x_4}\,\mathrm{tr}\,\Lambda_1\big(3\Lambda_2\mathrm{d}\Lambda_3\mathrm{d}A\mathrm{d}\Lambda_5 + 2\Lambda_2\mathrm{d}\Lambda_3\mathrm{d}\Lambda_5\mathrm{d}A\\
&\quad - \mathrm{d}(\Lambda_2\Lambda_3)(\mathrm{d}A\mathrm{d}\Lambda_5 - \mathrm{d}\Lambda_5\mathrm{d}A) - \mathrm{d}\Lambda_2(\mathrm{d}A\Lambda_3\mathrm{d}\Lambda_5 - \mathrm{d}\Lambda_5\Lambda_3\mathrm{d}A)\big)\\
&\quad + \tfrac{24}{5}\,\mathrm{tr}\,\big(\Lambda_1\mathrm{d}(\Lambda_2\Lambda_3)\Lambda_4(\mathrm{d}\Lambda_5A + A\mathrm{d}\Lambda_5) + \Lambda_1\Lambda_2\mathrm{d}\Lambda_3\mathrm{d}\Lambda_5\Lambda_4A\\
&\quad + \Lambda_1\Lambda_2\mathrm{d}\Lambda_3A\Lambda_4\mathrm{d}\Lambda_5 + \Lambda_1\mathrm{d}\Lambda_2\mathrm{d}\Lambda_5\Lambda_3\Lambda_4A + \Lambda_1\mathrm{d}\Lambda_2A\Lambda_3\Lambda_4\mathrm{d}\Lambda_5\big)\\
&\quad = -\tfrac{12}{5}i_{x_5}\mathrm{d}\,\mathrm{tr}\,\big(2\Lambda_1\Lambda_2\Lambda_3\mathrm{d}\Lambda_4 - \Lambda_1\Lambda_2\mathrm{d}\Lambda_3\Lambda_4\big)\mathrm{d}A\\
&\quad + \tfrac{24}{5}\,\mathrm{tr}\,\big(2\Lambda_1\mathrm{d}\Lambda_2\Lambda_3\Lambda_4\mathrm{d}\Lambda_5 + 2\Lambda_1\Lambda_2\mathrm{d}\Lambda_3\Lambda_4\mathrm{d}\Lambda_5 + \mathrm{d}\Lambda_1\Lambda_2\Lambda_3\mathrm{d}\Lambda_4\Lambda_5\\
&\quad + \Lambda_1\mathrm{d}\Lambda_2\mathrm{d}\Lambda_3\Lambda_4\Lambda_5 + \mathrm{d}\Lambda_1\Lambda_2\mathrm{d}\Lambda_3\Lambda_4\Lambda_5 - \Lambda_1\Lambda_2\mathrm{d}\Lambda_3\mathrm{d}\Lambda_4\Lambda_5\big)A,\\
&\ell_2(\ell_5(X_1,X_2,X_3,X_4,\Psi),X_5) = \tfrac{12}{5}\mathcal{L}_{x_5}\big(\,\mathrm{tr}\,\big(2\Lambda_1\Lambda_2\Lambda_3\mathrm{d}\Lambda_4 - \Lambda_1\Lambda_2\mathrm{d}\Lambda_3\Lambda_4\big)\mathrm{d}A\big),\\
&\ell_5(\ell_2(X_1,X_2),X_3,X_4,X_5,\Psi) = -\tfrac{12}{5}\,\mathrm{tr}\,\big(\mathcal{L}_{x_5}(2\Lambda_1\Lambda_2\Lambda_3\mathrm{d}\Lambda_4 - \Lambda_1\Lambda_2\mathrm{d}\Lambda_3\Lambda_4) - 3\Lambda_1\Lambda_2\Lambda_3\mathrm{d}\Lambda_4\Lambda_5\big)\mathrm{d}A,\\
&\ell_5(X_1,X_2,X_3,X_4,\ell_2(X_5,\Psi)) = -\tfrac{24}{5}\,\mathrm{tr}\,\big(2\Lambda_1\Lambda_2\Lambda_3\mathrm{d}\Lambda_4 - \Lambda_1\Lambda_2\mathrm{d}\Lambda_3\Lambda_4\big)\mathcal{L}_{x_5}\mathrm{d}A\\
&\quad + \tfrac{24}{5}\,\mathrm{tr}\,\big((2\Lambda_1\Lambda_2\Lambda_3\Lambda_4\mathrm{d}\Lambda_5 - 3\Lambda_1\Lambda_2\Lambda_3\mathrm{d}\Lambda_4\Lambda_5 + \Lambda_1\Lambda_2\mathrm{d}\Lambda_3\Lambda_4\Lambda_5)\mathrm{d}A\\
&\quad + (2\mathrm{d}\Lambda_1\Lambda_2\Lambda_3\Lambda_4\mathrm{d}\Lambda_5 - \mathrm{d}\Lambda_1\Lambda_2\Lambda_3\mathrm{d}\Lambda_4\Lambda_5 + 2\Lambda_1\Lambda_2\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}\Lambda_5 - \Lambda_1\Lambda_2\mathrm{d}\Lambda_3\Lambda_4\mathrm{d}\Lambda_5)A\big),\\
&\Rightarrow \ell_1(\ell_6(X_1,X_2,X_3,X_4,X_5,\Psi)) - 5\ell_6(X_1,X_2,X_3,X_4,\ell_1(X_5),\Psi)\\
&+ 10\ell_5(\ell_2(X_1,X_2),X_3,X_4,X_5,\Psi) + 5\ell_5(X_1,X_2,X_3,X_4,\ell_2(X_5,\Psi))\\
&+ 5\ell_2(\ell_5(X_1,X_2,X_3,X_4,\Psi),X_5) = 0 \quad \square
\end{aligned}
$$

And finally for inputs in $(V_0)^6$

$$
\begin{aligned}
&\ell_1(\ell_6(X_1,X_2,X_3,X_4,X_5,X_6)) = -144\mathrm{d}\,\mathrm{tr}\,\Lambda_1\Lambda_2\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6,\\
&\ell_6(X_1,X_2,X_3,X_4,X_5,\ell_1(X_6)) = -24\,\mathrm{tr}\,\big(2\Lambda_1\Lambda_2\Lambda_3\Lambda_4\mathrm{d}\Lambda_5 + \Lambda_1\Lambda_2\mathrm{d}\Lambda_3\Lambda_4\Lambda_5\big)\mathrm{d}\Lambda_6,\\
&\ell_2(\ell_5(X_1,X_2,X_3,X_4,X_5),X_6) = \tfrac{1}{2}\tfrac{1}{6}\mathcal{L}_{x_1}i_{x_2}i_{x_3}(i_{x_4}\mathcal{L}_{x_5} + i_{[x_4,x_5]} + i_{x_4}i_{x_5}\mathrm{d})\sigma_6,\\
&\ell_5(\ell_2(X_1,X_2),X_3,X_4,X_5,X_6) = \tfrac{1}{5}\tfrac{1}{6}\big(2i_{[x_1,x_2]}i_{x_3}(i_{x_4}\mathcal{L}_{x_5} + i_{[x_4,x_5]} + i_{x_4}i_{x_5}\mathrm{d})\sigma_6\\
&\quad + i_{x_1}i_{x_2}(i_{[x_3,x_4]}\mathcal{L}_{x_5} + i_{[x_3,x_4]}i_{x_5}\mathrm{d})\sigma_6 - i_{x_1}i_{x_2}(i_{x_3}\mathcal{L}_{[x_4,x_5]} - i_{[x_3,x_4]}i_{x_5}\mathrm{d})\sigma_6\\
&\quad + i_{x_1}i_{x_2}(i_{x_3}\mathcal{L}_{x_4} + i_{[x_3,x_4]} + i_{x_3}i_{x_4}\mathrm{d})(2\mathcal{L}_{x_5}\sigma_6 - \mathrm{d}i_{x_5}\sigma_6)\big),\\
&\Rightarrow \ell_1(\ell_6(X_1,X_2,X_3,X_4,X_5,X_6)) - 6\ell_6(X_1,X_2,X_3,X_4,X_5,\ell_1(X_6))\\
&+ 15\ell_5(\ell_2(X_1,X_2),X_3,X_4,X_5,X_6) - 6\ell_2(\ell_5(X_1,X_2,X_3,X_4,X_5),X_6) = 0 \quad \square
\end{aligned}
$$

C.4 $\ell_1\ell_7 + \ell_7\ell_1 + \ell_4\ell_4 + \ell_5\ell_3 + \ell_3\ell_5 + \ell_2\ell_6 + \ell_6\ell_2 = 0$ relations

We begin using brackets (3.17). The first check is for inputs in $(V_0)^2 \otimes (V_{-1})^5$

$$
\begin{aligned}
&\ell_6(\ell_2(X_1, X_2), \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = -48\,\mathrm{tr}(2\mathrm{d}\Lambda_1\Lambda_2 + 2\Lambda_1\mathrm{d}\Lambda_2 + 2\mathcal{L}_{x_1}\mathrm{d}\Lambda_2)A_1A_2A_3A_4A_5,\\
&\ell_6(X_1, \ell_2(X_2, \Psi_1), \Psi_2, \Psi_3, \Psi_4, \Psi_5) = -\tfrac{48}{5}\,\mathrm{tr}\,\mathrm{d}\Lambda_1\mathcal{L}_{x_2}(A_1A_2A_3A_4A_5)\\
&\quad + \tfrac{48}{5}\,\mathrm{tr}\,\mathrm{d}\Lambda_1(A_1A_2A_3A_4A_5\Lambda_2 - \Lambda_2A_1A_2A_3A_4A_5),\\
&\ell_2(X_1, \ell_6(X_2, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5)) = -48\mathcal{L}_{x_1}\,\mathrm{tr}\,\mathrm{d}\Lambda_2A_1A_2A_3A_4A_5,\\
&\Rightarrow \ell_6(\ell_2(X_1, X_2), \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) - 10\ell_6(X_1, \ell_2(X_2, \Psi_1), \Psi_2, \Psi_3, \Psi_4, \Psi_5)\\
&- 2\ell_2(X_1, \ell_6(X_2, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5)) = 0 \quad \square
\end{aligned}
$$

Next non-trivial is for $(V_0)^4 \otimes (V_{-1})^3$

$$
\begin{aligned}
&\ell_1(\ell_7(X_1, X_2, X_3, X_4, \Psi_1, \Psi_2, \Psi_3)) = \tfrac{12}{5}\mathrm{d}i_{x_1}i_{x_2}\,\mathrm{tr}\,\Lambda_3\big(3\mathrm{d}\Lambda_4\mathrm{d}(A_1A_2A_3)\\
&\quad + \mathrm{d}(A_1A_2)\mathrm{d}\Lambda_4A_3 - A_1A_2\mathrm{d}\Lambda_4\mathrm{d}A_3\big) + \tfrac{72}{5}\mathrm{d}i_{x_1}\,\mathrm{tr}\,\big(3\Lambda_2\Lambda_3\mathrm{d}\Lambda_4A_1A_2A_3 - \Lambda_2\Lambda_3A_1A_2\mathrm{d}\Lambda_4A_3\big),\\
&\ell_6(\ell_2(X_1, X_2), X_3, X_4, \Psi_1, \Psi_2, \Psi_3) - 2\ell_6(X_1, X_2, X_3, \ell_2(X_4, \Psi_1), \Psi_2, \Psi_3)\\
&\quad = -\tfrac{6}{5}i_{[x_1,x_2]}\,\mathrm{tr}\,\Lambda_3\big(3\mathrm{d}\Lambda_4\mathrm{d}(A_1A_2A_3) + \mathrm{d}(A_1A_2)\mathrm{d}\Lambda_4A_3 - A_1A_2\mathrm{d}\Lambda_4\mathrm{d}A_3\big)\\
&\quad + \tfrac{12}{5}i_{x_1}\,\mathrm{tr}\,\big(\mathrm{d}(3\Lambda_2\Lambda_3\mathrm{d}\Lambda_4A_1A_2A_3) - \mathrm{d}(\Lambda_2\Lambda_3A_1A_2\mathrm{d}\Lambda_4A_3)\big)\\
&\quad + \tfrac{12}{5}i_{x_1}\mathcal{L}_{x_2}\,\mathrm{tr}\,\big(3\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}(A_1A_2A_3) + \Lambda_3\mathrm{d}(A_1A_2)\mathrm{d}\Lambda_4A_3 - \Lambda_3A_1A_2\mathrm{d}\Lambda_4\mathrm{d}A_3\big)\\
&\quad - \tfrac{24}{5}\mathcal{L}_{x_1}\,\mathrm{tr}\,\big(3\Lambda_2\Lambda_3\mathrm{d}\Lambda_4A_1A_2A_3 - \Lambda_2\Lambda_3A_1A_2\mathrm{d}\Lambda_4A_3\big),\\
&\ell_2(X_1, \ell_6(X_2, X_3, X_4, \Psi_1, \Psi_2, \Psi_3)) = -\tfrac{9}{5}\mathcal{L}_{x_1}i_{x_2}\,\mathrm{tr}\,\Lambda_3\big(3\mathrm{d}\Lambda_4\mathrm{d}(A_1A_2A_3) + \mathrm{d}(A_1A_2)\mathrm{d}\Lambda_4A_3\\
&\quad - A_1A_2\mathrm{d}\Lambda_4\mathrm{d}A_3\big) - \tfrac{18}{5}\mathcal{L}_{x_1}\,\mathrm{tr}\,\big(3\Lambda_2\Lambda_3\mathrm{d}\Lambda_4A_1A_2A_3 - \Lambda_2\Lambda_3A_1A_2\mathrm{d}\Lambda_4A_3\big),\\
&\ell_3(\ell_5(X_1, X_2, \Psi_1, \Psi_2, \Psi_3), X_3, X_4) = -\tfrac{2}{5}\big(i_{x_3}\mathcal{L}_{x_4} + i_{[x_3,x_4]} + i_{x_3}i_{x_4}\mathrm{d}\big)\,\mathrm{tr}\,\Lambda_1\big(3\mathrm{d}\Lambda_2\mathrm{d}(A_1A_2A_3)\\
&\quad + \mathrm{d}(A_1A_2)\mathrm{d}\Lambda_2A_3 - A_1A_2\mathrm{d}\Lambda_2\mathrm{d}A_3\big),\\
&\Rightarrow \ell_1(\ell_7(X_1, X_2, X_3, X_4, \Psi_1, \Psi_2, \Psi_3) + 6\ell_6(\ell_2(X_1, X_2), X_3, X_4, \Psi_1, \Psi_2, \Psi_3)\\
&- 12\ell_6(X_1, X_2, X_3, \ell_2(X_4, \Psi_1), \Psi_2, \Psi_3) - 4\ell_2(X_1, \ell_6(X_2, X_3, X_4, \Psi_1, \Psi_2, \Psi_3))\\
&+ 6\ell_3(\ell_5(X_1, X_2, \Psi_1, \Psi_2, \Psi_3), X_3, X_4) = 0 \quad \square
\end{aligned}
$$

And for $(V_0)^5 \otimes (V_{-1})^2$

$$
\begin{aligned}
&\ell_1(\ell_7(X_1,X_2,X_3,X_4,X_5,\Psi_1,\Psi_2)) = -4\mathrm{d}i_{x_4}i_{x_5}\,\mathrm{tr}\,\big(3\Lambda_1\Lambda_2\mathrm{d}\Lambda_3\mathrm{d}A_1A_2\\
&\quad + 2\Lambda_1\Lambda_2\mathrm{d}\Lambda_3A_1\mathrm{d}A_2 - \Lambda_1\mathrm{d}(\Lambda_2\Lambda_3)\mathrm{d}(A_1A_2) - \Lambda_1\mathrm{d}\Lambda_2\mathrm{d}A_1\Lambda_3A_2 + \Lambda_1\mathrm{d}\Lambda_2A_1\Lambda_3\mathrm{d}A_2\big)\\
&\quad + 24\mathrm{d}i_{x_5}\,\mathrm{tr}\,\big(\Lambda_1\mathrm{d}(\Lambda_2\Lambda_3)\Lambda_4A_1A_2 + \Lambda_1\Lambda_2\mathrm{d}\Lambda_3A_1\Lambda_4A_2 + \Lambda_1\mathrm{d}\Lambda_2A_1\Lambda_3\Lambda_4A_2\big),\\
&\ell_7(X_1,X_2,X_3,X_4,\ell_1(X_5),\Psi_1,\Psi_2) = -\tfrac{4}{5}i_{x_1}i_{x_2}\,\mathrm{tr}\,\big(3\mathrm{d}\Lambda_3\Lambda_4\mathrm{d}\Lambda_5\mathrm{d}(A_1A_2)\\
&\quad - 3\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}\Lambda_5\mathrm{d}(A_1A_2) + \mathrm{d}\Lambda_3\mathrm{d}\Lambda_4\Lambda_5\mathrm{d}(A_1A_2) + 4\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A_1\mathrm{d}\Lambda_5A_2 + 4\Lambda_3\mathrm{d}\Lambda_4A_1\mathrm{d}\Lambda_5\mathrm{d}A_2\\
&\quad - \Lambda_3\mathrm{d}A_1\mathrm{d}\Lambda_4\mathrm{d}\Lambda_5A_2 + \Lambda_3A_1\mathrm{d}\Lambda_4\mathrm{d}\Lambda_5\mathrm{d}A_2\big)\\
&\quad - \tfrac{24}{5}i_{x_1}\,\mathrm{tr}\,\big(3\mathrm{d}\Lambda_2\Lambda_3\Lambda_4\mathrm{d}\Lambda_5A_1A_2 + 3\Lambda_2\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}\Lambda_5A_1A_2 - \mathrm{d}\Lambda_2\mathrm{d}\Lambda_3\Lambda_4\Lambda_5A_1A_2\\
&\quad + 4\Lambda_2\Lambda_3\mathrm{d}\Lambda_4A_1\mathrm{d}\Lambda_5A_2 + \Lambda_2\Lambda_3A_1\mathrm{d}\Lambda_4\mathrm{d}\Lambda_5A_2\big),\\
&\ell_2(X_5,\ell_6(X_1,X_2,X_3,X_4,\Psi_1,\Psi_2)) = 2\mathcal{L}_{x_5}i_{x_1}i_{x_2}\,\mathrm{tr}\,\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A_1\mathrm{d}A_2\\
&\quad - \tfrac{12}{5}\mathcal{L}_{x_5}i_{x_4}\,\mathrm{tr}\,\big(3\Lambda_1\Lambda_2\mathrm{d}\Lambda_3\mathrm{d}A_1A_2 + 2\Lambda_1\Lambda_2\mathrm{d}\Lambda_3A_1\mathrm{d}A_2 - \Lambda_1\mathrm{d}(\Lambda_2\Lambda_3)\mathrm{d}(A_1A_2)\\
&\quad - \Lambda_1\mathrm{d}\Lambda_2(\mathrm{d}A_1\Lambda_3A_2 - A_1\Lambda_3\mathrm{d}A_2)\big)\\
&\quad - \tfrac{24}{5}\mathcal{L}_{x_5}\,\mathrm{tr}\,\big(\Lambda_1\mathrm{d}(\Lambda_2\Lambda_3)\Lambda_4A_1A_2 + \Lambda_1\Lambda_2\mathrm{d}\Lambda_3A_1\Lambda_4A_2 + \Lambda_1\mathrm{d}\Lambda_2A_1\Lambda_3\Lambda_4A_2\big),\\
&\ell_3(\ell_5(X_1,X_2,X_3,\Psi_1,\Psi_2),X_4,X_5) = (i_{x_4}\mathcal{L}_{x_5} + i_{[x_4,x_5]})i_{x_1}\,\mathrm{tr}\,\Lambda_2\mathrm{d}\Lambda_3\mathrm{d}A_1\mathrm{d}A_2\\
&\quad + \tfrac{2}{5}(i_{x_4}\mathcal{L}_{x_5} + i_{[x_4,x_5]})\,\mathrm{tr}\,\big(3\Lambda_1\Lambda_2\mathrm{d}\Lambda_3\mathrm{d}A_1A_2 + 2\Lambda_1\Lambda_2\mathrm{d}\Lambda_3A_1\mathrm{d}A_2 - \Lambda_1\mathrm{d}(\Lambda_2\Lambda_3)\mathrm{d}(A_1A_2)\\
&\quad - \Lambda_1\mathrm{d}\Lambda_2\mathrm{d}A_1\Lambda_3A_2 + \Lambda_1\mathrm{d}\Lambda_2A_1\Lambda_3\mathrm{d}A_2\big),\\
&\ell_6(X_1,X_2,X_3,X_4,\ell_2(X_5,\Psi_1),\Psi_2) + \ell_6(\ell_2(X_1,X_2),X_3,X_4,X_5,\Psi_1,\Psi_2)\\
&\quad = 2i_{[x_1,x_2]}i_{x_3}\,\mathrm{tr}\,\Lambda_4\mathrm{d}\Lambda_5\mathrm{d}A_1\mathrm{d}A_2 + 2i_{x_1}i_{x_2}\mathcal{L}_{x_3}\,\mathrm{tr}\,\Lambda_4\mathrm{d}\Lambda_5\mathrm{d}A_1\mathrm{d}A_2\\
&\quad + \tfrac{6}{5}(i_{[x_4,x_5]} - 2i_{x_4}\mathcal{L}_{x_5})\,\mathrm{tr}\,\big(3\Lambda_1\Lambda_2\mathrm{d}\Lambda_3\mathrm{d}A_1A_2 + 2\Lambda_1\Lambda_2\mathrm{d}\Lambda_3A_1\mathrm{d}A_2 - \Lambda_1\mathrm{d}(\Lambda_2\Lambda_3)\mathrm{d}(A_1A_2)\\
&\quad - \Lambda_1\mathrm{d}\Lambda_2(\mathrm{d}A_1\Lambda_3A_2 - A_1\Lambda_3\mathrm{d}A_2)\big)\\
&\quad + 2i_{x_1}i_{x_2}\,\mathrm{tr}\,\big(\mathrm{d}\Lambda_3\Lambda_4\mathrm{d}\Lambda_5\mathrm{d}A_1A_2 - \Lambda_3\Lambda_4\mathrm{d}\Lambda_5\mathrm{d}A_1\mathrm{d}A_2 + \Lambda_3\mathrm{d}\Lambda_4\mathrm{d}\Lambda_5A_1\mathrm{d}A_2\\
&\quad + \Lambda_3\mathrm{d}\Lambda_4A_1\mathrm{d}\Lambda_4\mathrm{d}A_2 + \Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A_1\mathrm{d}\Lambda_5A_2\big)\\
&\quad - \tfrac{24}{5}\mathcal{L}_{x_5}\,\mathrm{tr}\,\big(\Lambda_1\mathrm{d}(\Lambda_2\Lambda_3)\Lambda_4A_1A_2 + \Lambda_1\Lambda_2\mathrm{d}\Lambda_3A_1\Lambda_4A_2 + \Lambda_1\mathrm{d}\Lambda_2A_1\Lambda_3\Lambda_4A_2\big)\\
&\quad + \tfrac{12}{5}i_{x_5}\mathrm{d}\,\mathrm{tr}\,\big(\Lambda_1\mathrm{d}(\Lambda_2\Lambda_3)\Lambda_4A_1A_2 + \Lambda_1\Lambda_2\mathrm{d}\Lambda_3A_1\Lambda_4A_2 + \Lambda_1\mathrm{d}\Lambda_2A_1\Lambda_3\Lambda_4A_2\big)\\
&\quad + \tfrac{12}{5}i_{x_1}\,\mathrm{tr}\,\big(3\mathrm{d}\Lambda_2\Lambda_3\Lambda_4\mathrm{d}\Lambda_5A_1A_2 + 3\Lambda_2\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}\Lambda_5A_1A_2 - \mathrm{d}\Lambda_2\mathrm{d}\Lambda_3\Lambda_4\Lambda_5A_1A_2\\
&\quad + 4\Lambda_2\Lambda_3\mathrm{d}\Lambda_4A_1\mathrm{d}\Lambda_5A_2 + \Lambda_2\Lambda_3A_1\mathrm{d}\Lambda_4\mathrm{d}\Lambda_5A_2\big),\\
&\Rightarrow \ell_1(\ell_7(X_1,X_2,X_3,X_4,X_5,\Psi_1,\Psi_2)) + 5\ell_7(X_1,X_2,X_3,X_4,\ell_1(X_5),\Psi_1,\Psi_2)\\
&+ 10\ell_3(\ell_5(X_1,X_2,X_3,\Psi_1,\Psi_2),X_4,X_5) + 10\ell_6(X_1,X_2,X_3,X_4,\ell_2(X_5,\Psi_1),\Psi_2)\\
&+ 10\ell_6(\ell_2(X_1,X_2),X_3,X_4,X_5,\Psi_1,\Psi_2) - 5\ell_2(X_5,\ell_6(X_1,X_2,X_3,X_4,\Psi_1,\Psi_2))\\
&\quad = 0 \quad \square
\end{aligned}
$$

For $(V_0)^6 \otimes V_{-1}$

$$\ell_1(\ell_7(X_1,X_2,X_3,X_4,X_5,X_6,\Psi)) = -12\mathrm{d}i_{x_1}i_{x_2}\,\mathrm{tr}\left(2\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6 - \Lambda_3\Lambda_4\mathrm{d}\Lambda_5\Lambda_6\right)\mathrm{d}A$$
$$- 72\mathrm{d}i_{x_1}\,\mathrm{tr}\left(2\Lambda_2\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6 + \Lambda_2\Lambda_3\mathrm{d}\Lambda_4\Lambda_5\Lambda_6\right)A,$$

$$\ell_7(X_1,X_2,X_3,X_4,X_5,\ell_1(X_6),\Psi) = 2i_{x_1}i_{x_2}\mathrm{d}\,\mathrm{tr}\left(2\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6 - \Lambda_3\Lambda_4\mathrm{d}\Lambda_5\Lambda_6\right)\mathrm{d}A$$
$$- 12i_{x_1}\,\mathrm{tr}\left(2\Lambda_2\mathrm{d}(\Lambda_3\Lambda_4)\Lambda_5\mathrm{d}\Lambda_6 + \mathrm{d}\Lambda_2\Lambda_3\mathrm{d}(\Lambda_4\Lambda_5)\Lambda_6 + \Lambda_2\mathrm{d}\Lambda_3\mathrm{d}\Lambda_4\Lambda_5\Lambda_6\right.$$
$$\left. - \Lambda_2\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}\Lambda_5\Lambda_6\right)A,$$

$$\ell_2(\ell_6(X_1,X_2,X_3,X_4,X_5,\Psi),X_6) = 6\mathcal{L}_{x_1}i_{x_2}\,\mathrm{tr}\left(2\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6 - \Lambda_3\Lambda_4\mathrm{d}\Lambda_5\Lambda_6\right)\mathrm{d}A$$
$$+ 12\mathcal{L}_{x_1}\,\mathrm{tr}\left(2\Lambda_2\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6 + \Lambda_2\Lambda_3\mathrm{d}\Lambda_4\Lambda_5\Lambda_6\right)A,$$

$$\ell_3(\ell_5(X_1,X_2,X_3,X_4,\Psi),X_5,X_6) = \tfrac{4}{5}(i_{[x_1,x_2]} + i_{x_1}\mathcal{L}_{x_2})\,\mathrm{tr}\left(2\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6 - \Lambda_3\Lambda_4\mathrm{d}\Lambda_5\Lambda_6\right)\mathrm{d}A,$$

$$15\ell_6(\ell_2(X_1,X_2),X_3,X_4,X_5,X_6,\Psi) - 6\ell_6(X_1,X_2,X_3,X_4,X_5,\ell_2(X_6,\Psi))$$
$$= 36(i_{[x_1,x_2]} - 2i_{x_1}\mathcal{L}_{x_2})\,\mathrm{tr}\left(2\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6 - \Lambda_3\Lambda_4\mathrm{d}\Lambda_5\Lambda_6\right)\mathrm{d}A$$
$$+ 144\mathcal{L}_{x_1}\,\mathrm{tr}\left(2\Lambda_2\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6 + \Lambda_2\Lambda_3\mathrm{d}\Lambda_4\Lambda_5\Lambda_6\right)A$$
$$+ 72i_{x_1}\,\mathrm{tr}\left(2\Lambda_2\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6 + \Lambda_2\Lambda_3\mathrm{d}\Lambda_4\Lambda_5\Lambda_6\right)\mathrm{d}A + 72i_{x_1}\,\mathrm{tr}\left(- 2\mathrm{d}\Lambda_2\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6\right.$$
$$\left. - 2\Lambda_2\Lambda_3\Lambda_4\mathrm{d}\Lambda_5\mathrm{d}\Lambda_6 + \mathrm{d}\Lambda_2\Lambda_3\Lambda_4\mathrm{d}\Lambda_5\Lambda_6 + \Lambda_2\Lambda_3\mathrm{d}\Lambda_3\Lambda_5\mathrm{d}\Lambda_6\right)A,$$

$$\Rightarrow \ell_1(\ell_7(X_1,X_2,X_3,X_4,X_5,X_6,\Psi)) + 6\ell_7(X_1,X_2,X_3,X_4,X_5,\ell_1(X_6),\Psi)$$
$$- 6\ell_2(\ell_6(X_1,X_2,X_3,X_4,X_5,\Psi),X_6) + 15\ell_3(\ell_5(X_1,X_2,X_3,X_4,\Psi),X_5,X_6)$$
$$+ 15\ell_6(\ell_(X_1,X_2),X_3,X_4,X_5,X_6,\Psi) - 6\ell_6(X_1,X_2,X_3,X_4,X_5,\ell_(X_6,\Psi)) = 0 \quad \square$$

And finally for $(V_0)^7$

$$\ell_1(\ell_7(X_1,X_2,X_3,X_4,X_5,X_6,X_7)) = 504\mathrm{d}i_{x_1}\,\mathrm{tr}\,\Lambda_2\Lambda_3\Lambda_4\Lambda_5\Lambda_6\mathrm{d}\Lambda_7 - 144\mathrm{d}\,\mathrm{tr}\,\Lambda_1\Lambda_2\Lambda_3\Lambda_4\Lambda_5\Lambda_6\Lambda_7$$
$$- \tfrac{1}{6}\mathrm{d}i_{x_1}i_{x_2}i_{x_3}i_{x_4}(i_{x_5}\mathcal{L}_{x_6} + i_{[x_5,x_6]} + i_{x_5}i_{x_6}\mathrm{d})\sigma_7,$$

$$\ell_7(X_1,X_2,X_3,X_4,X_5,X_6,\ell_1(X_7)) = 72i_{x_1}\,\mathrm{tr}\left(2\Lambda_2\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6\mathrm{d}\Lambda_7 + \Lambda_2\Lambda_3\mathrm{d}\Lambda_4\Lambda_5\Lambda_6\mathrm{d}\Lambda_7\right)$$
$$= 72i_{x_1}\mathrm{d}\,\mathrm{tr}\,\Lambda_2\Lambda_3\Lambda_4\Lambda_5\Lambda_6\mathrm{d}\Lambda_7,$$

$$\ell_2(\ell_6(X_1,X_2,X_3,X_4,X_5,X_6),X_7) = 72\mathcal{L}_{x_7}\,\mathrm{tr}\,\Lambda_1\Lambda_2\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6,$$

$$\ell_6(\ell_2(X_1,X_2),X_3,X_4,X_5,X_6,X_7) = -48\mathcal{L}_{x_1}\,\mathrm{tr}\,\Lambda_2\Lambda_3\Lambda_4\Lambda_5\Lambda_6\mathrm{d}\Lambda_7 + 48\,\mathrm{tr}\,\Lambda_1\Lambda_2\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6\Lambda_7,$$

$$\ell_3(\ell_5(X_1,X_2,X_3,X_4,X_5),X_6,X_7) = -\tfrac{1}{12}\tfrac{1}{3}(i_{x_1}\mathcal{L}_{x_2} + i_{[x_1,x_2]})i_{x_3}i_{x_4}(i_{x_5}\mathcal{L}_{x_6} + i_{[x_5,x_6]} + i_{x_5}i_{x_6}\mathrm{d})\sigma_7,$$

$$\ell_5(\ell_3(X_1,X_2,X_3),X_4,X_5,X_6,X_7) = -\tfrac{1}{12}\tfrac{1}{5}i_{x_1}i_{x_2}(i_{x_3}\mathcal{L}_{x_4} + i_{[x_3,x_4]})(i_{x_5}\mathcal{L}_{x_6} + i_{[x_5,x_6]} + i_{x_5}i_{x_6}\mathrm{d})\sigma_7,$$

$$\ell_1(\ell_7(X_1,X_2,X_3,X_4,X_5,X_6,X_7)) + 7\ell_7(X_1,X_2,X_3,X_4,X_5,X_6,\ell_1(X_7))$$
$$+ 7\ell_2(\ell_6(X_1,X_2,X_3,X_4,X_5,X_6),X_7) + 21\ell_6(\ell_2(X_1,X_2),X_3,X_4,X_5,X_6,X_7)$$
$$+ 21\ell_3(\ell_5(X_1,X_2,X_3,X_4,X_5),X_6,X_7) + 35\ell_5(\ell_3(X_1,X_2,X_3),X_4,X_5,X_6,X_7)$$
$$= 0 \quad \square$$

C.5 $\ell_1\ell_8 - \ell_8\ell_1 + \ell_7\ell_2 - \ell_2\ell_7 + \ell_3\ell_6 - \ell_6\ell_3 - \ell_4\ell_5 + \ell_5\ell_4 = 0$ relations

Now we start checking brackets (3.18), with the first non-trivial relation involving elements in $(V_0)^5 \otimes (V_{-1})^3$

$$
\begin{aligned}
&\ell_1(\ell_8(X_1,X_2,X_3,X_4,X_5,\Psi_1,\Psi_2,\Psi_3)) = -12\mathrm{d}i_{x_1}i_{x_2}\,\mathrm{tr}\,\big(3\Lambda_3\Lambda_4\mathrm{d}\Lambda_5A_1A_2A_3\\
&\quad - \Lambda_3\Lambda_4A_1A_2\mathrm{d}\Lambda_5A_3\big),\\
&\ell_3(\ell_6(X_1,X_2,X_3,\Psi_1,\Psi_2,\Psi_3),X_4,X_5) = \tfrac{3}{5}(i_{x_4}\mathcal{L}_{x_5} + i_{[x_4,x_5]})i_{x_1}\,\mathrm{tr}\,\Lambda_2\big(3\mathrm{d}\Lambda_3\mathrm{d}(A_1A_2A_3)\\
&\quad + \mathrm{d}(A_1A_2\mathrm{d}\Lambda_3A_3)\big) + \tfrac{6}{5}(i_{x_4}\mathcal{L}_{x_5} + i_{[x_4,x_5]})\,\mathrm{tr}\,\big(3\Lambda_1\Lambda_2\mathrm{d}\Lambda_3A_1A_2A_3 - \Lambda_1\Lambda_2A_1A_2\mathrm{d}\Lambda_3A_3\big),\\
&\ell_2(\ell_7(X_1,X_2,X_3,X_4,\Psi_1,\Psi_2,\Psi_3),X_5) = -\tfrac{6}{5}\mathcal{L}_{x_1}i_{x_2}i_{x_3}\,\mathrm{tr}\,\Lambda_4\big(3\mathrm{d}\Lambda_5\mathrm{d}(A_1A_2A_3)\\
&\quad + \mathrm{d}(A_1A_2)\mathrm{d}\Lambda_5A_3 - A_1A_2\mathrm{d}\Lambda_5\mathrm{d}A_3\big) - \tfrac{36}{5}\mathcal{L}_{x_1}i_{x_2}\,\mathrm{tr}\,\big(3\Lambda_3\Lambda_4\mathrm{d}\Lambda_5A_1A_2A_3\\
&\quad - \Lambda_3\Lambda_4A_1A_2\mathrm{d}\Lambda_5A_3\big),\\
&10\ell_7(\ell_2(X_1,X_2),X_3,X_4,X_5,\Psi_1,\Psi_2,\Psi_3) + 15\ell_7(X_1,X_2,X_3,X_4,\ell_2(X_5,\Psi_1),\Psi_2,\Psi_3)\\
&\quad = 12(i_{x_1}i_{x_2}\mathcal{L}_{x_3} + i_{[x_1,x_2]}i_{x_3})\,\mathrm{tr}\,\Lambda_4\big(3\mathrm{d}\Lambda_5\mathrm{d}(A_1A_2A_3) + \mathrm{d}(A_1A_2)\mathrm{d}\Lambda_5A_3 - A_1A_2\mathrm{d}\Lambda_5\mathrm{d}A_3\big)\\
&\quad - 12(6i_{x_1}\mathcal{L}_{x_2} - 3i_{[x_1,x_2]} - 2i_{x_1}i_{x_2}\mathrm{d})\,\mathrm{tr}\,\big(3\Lambda_3\Lambda_4\mathrm{d}\Lambda_5A_1A_2A_3 - \Lambda_3\Lambda_4A_1A_2\mathrm{d}\Lambda_5A_3\big),\\
&\Rightarrow \ell_1(\ell_8(X_1,X_2,X_3,X_4,X_5,\Psi_1,\Psi_2,\Psi_3)) + 10\ell_3(\ell_6(X_1,X_2,X_3,\Psi_1,\Psi_2,\Psi_3),X_4,X_5)\\
&+ 10\ell_7(\ell_2(X_1,X_2),X_3,X_4,X_5,\Psi_1,\Psi_2,\Psi_3) + 15\ell_7(X_1,X_2,X_3,X_4,\ell_2(X_5,\Psi_1),\Psi_2,\Psi_3)\\
&+ 5\ell_2(\ell_7(X_1,X_2,X_3,X_4,\Psi_1,\Psi_2,\Psi_3),X_5) = 0 \quad \square
\end{aligned}
$$

Next we check $(V_0)^6 \otimes (V_{-1})^2$

$$\ell_1(\ell_8(X_1, X_2, X_3, X_4, X_5, X_6, \Psi_1, \Psi_2) = -2\mathrm{d}i_{x_1}i_{x_2}i_{x_3}i_{x_4}\,\mathrm{tr}\,\Lambda_5\mathrm{d}\Lambda_6\mathrm{d}A_1\mathrm{d}A_2$$
$$-\,24\mathrm{d}i_{x_1}i_{x_2}\,\mathrm{tr}\,\big(\Lambda_3\mathrm{d}(\Lambda_4\Lambda_5)\Lambda_6A_1A_2 + \Lambda_3\Lambda_4\mathrm{d}\Lambda_5A_1\Lambda_6A_2 + \Lambda_3\mathrm{d}\Lambda_4A_1\Lambda_5\Lambda_6A_2\big),$$

$$\ell_8(X_1, X_2, X_3, X_4, X_5, \ell_1(X_6), \Psi_1, \Psi_2) = 4i_{x_1}i_{x_2}\,\mathrm{tr}\,\big(4\Lambda_3\Lambda_4\mathrm{d}\Lambda_5A_1\mathrm{d}\Lambda_6A_2$$
$$+\,(3\Lambda_3\Lambda_4\mathrm{d}\Lambda_5\mathrm{d}\Lambda_6 + 3\mathrm{d}\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6 - \mathrm{d}\Lambda_3\mathrm{d}\Lambda_4\Lambda_3\Lambda_4)A_1A_2 - \mathrm{d}\Lambda_3\mathrm{d}\Lambda_4A_1\Lambda_5\Lambda_6A_2\big),$$

$$\ell_5(\ell_4(X_1, X_2, \Psi_1, \Psi_2), X_3, X_4, X_5, X_6) = \tfrac{2}{3}\tfrac{1}{5}i_{x_3}i_{x_4}(i_{x_5}\mathcal{L}_{x_6} + i_{[x_5,x_6]} + i_{x_5}i_{x_6}\mathrm{d})\,\mathrm{tr}\,\Lambda_1\mathrm{d}\Lambda_2\mathrm{d}A_1\mathrm{d}A_2,$$

$$\ell_3(\ell_6(X_1, X_2, X_3, X_4, \Psi_1, \Psi_2), X_5, X_6) = -\tfrac{2}{3}(i_{x_5}\mathcal{L}_{x_6} + i_{[x_5,x_6]})i_{x_1}i_{x_2}\,\mathrm{tr}\,\Lambda_3\mathrm{d}\Lambda_4\mathrm{d}A_1\mathrm{d}A_2$$
$$-\,\tfrac{4}{5}(i_{x_5}\mathcal{L}_{x_6} + i_{[x_5,x_6]})i_{x_1}\,\mathrm{tr}\,\Lambda_2\big(2\Lambda_3\mathrm{d}\Lambda_{4]}\mathrm{d}A_1A_2 + 3\Lambda_3\mathrm{d}\Lambda_4A_1\mathrm{d}A_2 - \mathrm{d}\Lambda_3\Lambda_4\mathrm{d}(A_1A_2)$$
$$-\,\mathrm{d}\Lambda_3(\mathrm{d}A_1\Lambda_4A_2 - A_1\Lambda_4\mathrm{d}A_2)\big) + \tfrac{8}{5}(i_{x_5}\mathcal{L}_{x_6} + i_{[x_5,x_6]})\,\mathrm{tr}\,\big(\Lambda_1\mathrm{d}(\Lambda_2\Lambda_3)\Lambda_4A_1A_2$$
$$+\,\Lambda_1\Lambda_2\mathrm{d}\Lambda_3A_1\Lambda_4A_2 + \Lambda_1\mathrm{d}\Lambda_2A_1\Lambda_3\Lambda_4A_2\big),$$

$$\ell_2(\ell_7(X_1, X_2, X_3, X_4, X_5, \Psi_1, \Psi_2), X_6) = -2\mathcal{L}_{x_1}i_{x_2}i_{x_3}\,\mathrm{tr}\,\big(2\Lambda_4\Lambda_5\mathrm{d}\Lambda_6\mathrm{d}A_1A_2$$
$$+\,3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6A_1\mathrm{d}A_2 - \Lambda_4\mathrm{d}\Lambda_5\Lambda_6\mathrm{d}(A_1A_2) - \Lambda_4\mathrm{d}\Lambda_5\mathrm{d}A_1\Lambda_6A_2 + \Lambda_4\mathrm{d}\Lambda_5A_1\Lambda_6\mathrm{d}A_2\big)$$
$$+\,12\mathcal{L}_{x_1}i_{x_2}\,\mathrm{tr}\,\big(\Lambda_3\mathrm{d}(\Lambda_4\Lambda_5)\Lambda_6A_1A_2 + \Lambda_3\Lambda_4\mathrm{d}\Lambda_5A_1\Lambda_6A_2 + \Lambda_3\mathrm{d}\Lambda_4A_1\Lambda_5\Lambda_6A_2\big),$$

$$15\ell_7(\ell_2(X_1, X_2), X_3, X_4, X_5, X_6, \Psi_1, \Psi_2) - 12\ell_7(X_1, X_2, X_3, X_4, X_5, \ell_2(X_6, \Psi_1), \Psi_2) =$$
$$-\,24(i_{[x_1,x_2]}i_{x_3} + i_{x_1}i_{x_2}\mathcal{L}_{x_3})\,\mathrm{tr}\,\Lambda_4\big(2\Lambda_5\mathrm{d}\Lambda_6\mathrm{d}A_1A_2 + 3\Lambda_5\mathrm{d}\Lambda_6A_1\mathrm{d}A_2 - \mathrm{d}\Lambda_5\Lambda_6\mathrm{d}(A_1A_2)$$
$$-\,\mathrm{d}\Lambda_5\mathrm{d}A_1\Lambda_6A_2 + \mathrm{d}\Lambda_5A_1\Lambda_6\mathrm{d}A_2\big) + 72(i_{[x_1,x_2]} - 2i_{x_1}\mathcal{L}_{x_2})\,\mathrm{tr}\,\big(\Lambda_3\mathrm{d}(\Lambda_4\Lambda_5)\Lambda_6A_1A_2$$
$$+\,\Lambda_3\Lambda_4\mathrm{d}\Lambda_5A_1\Lambda_6A_2 + \Lambda_3\mathrm{d}\Lambda_4A_1\Lambda_5\Lambda_6A_2\big) - 24i_{x_1}i_{x_2}\,\mathrm{tr}\,\big(\Lambda_3\mathrm{d}(\Lambda_4\Lambda_5)\Lambda_6\mathrm{d}(A_1A_2)$$
$$+\,(\Lambda_3\mathrm{d}\Lambda_4\Lambda_5\mathrm{d}\Lambda_6 - 2\Lambda_3\Lambda_4\mathrm{d}\Lambda_5\mathrm{d}\Lambda_6 - \mathrm{d}\Lambda_3\Lambda_4\mathrm{d}\Lambda_5\Lambda_6 - 3\mathrm{d}\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6)A_1A_2$$
$$-\,4\Lambda_3\Lambda_4\mathrm{d}\Lambda_5A_1\mathrm{d}\Lambda_6A_2 - \mathrm{d}(\Lambda_3\Lambda_3\mathrm{d}\Lambda_5A_1\Lambda_6A_2) + \Lambda_3\mathrm{d}\Lambda_4\mathrm{d}(A_1\Lambda_5\Lambda_6A_2)\big),$$

$$\Rightarrow \ell_1(\ell_8(X_1, X_2, X_3, X_4, X_5, X_6, \Psi_1, \Psi_2) - 6\ell_8(X_1, X_2, X_3, X_4, X_5, \ell_1(X_6), \Psi_1, \Psi_2)$$
$$+\,15\ell_3(\ell_6(X_1, X_2, X_3, X_4, \Psi_1, \Psi_2), X_5, X_6) - 6\ell_2(\ell_7(X_1, X_2, X_3, X_4, X_5, \Psi_1, \Psi_2), X_6)$$
$$+\,15\ell_7(\ell_2(X_1, X_2), X_3, X_4, X_5, X_6, \Psi_1, \Psi_2) - 12\ell_7(X_1, X_2, X_3, X_4, X_5, \ell_2(X_6, \Psi_1), \Psi_2)$$
$$= 0 \quad \square$$

For $(V_0)^7 \otimes V_{-1}$ we have

$$\ell_1(\ell_8(X_1,X_2,X_3,X_4,X_5,X_6,X_7,\Psi)) = 84\mathrm{d}i_{x_1}i_{x_2}\,\mathrm{tr}\,\big(2\Lambda_3\Lambda_4\Lambda_5\Lambda_6\mathrm{d}\Lambda_7 + \Lambda_3\Lambda_4\mathrm{d}\Lambda_5\Lambda_6\Lambda_7\big)A,$$

$$\ell_8(X_1,X_2,X_3,X_4,X_5,X_6,\ell_1(X_7),\Psi) = 12i_{x_1}i_{x_2}\,\mathrm{tr}\,\big(\mathrm{d}\Lambda_3\Lambda_4\mathrm{d}\Lambda_5\Lambda_6\Lambda_7 + \mathrm{d}\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6\Lambda_7$$
$$- \Lambda_3\Lambda_4\mathrm{d}\Lambda_5\mathrm{d}\Lambda_6\Lambda_7 + 2\Lambda_3\mathrm{d}\Lambda_4\Lambda_5\Lambda_6\mathrm{d}\Lambda_7 + \Lambda_3\mathrm{d}\Lambda_4\mathrm{d}\Lambda_5\Lambda_6\Lambda_7 + 2\Lambda_3\Lambda_4\mathrm{d}\Lambda_5\Lambda_6\mathrm{d}\Lambda_7\big)A,$$

$$\ell_3(\ell_6(X_1,X_2,X_3,X_4,X_5,\Psi),X_6,X_7) = -2(i_{x_1}\mathcal{L}_{x_2}i_{x_3} + i_{[x_1,x_2]}i_{x_3})\,\mathrm{tr}\,\big(2\Lambda_4\Lambda_5\Lambda_6\mathrm{d}\Lambda_7$$
$$- \Lambda_4\Lambda_5\mathrm{d}\Lambda_6\Lambda_7\big)\mathrm{d}A - 4(i_{x_1}\mathcal{L}_{x_2} + i_{[x_1,x_2]})\,\mathrm{tr}\,\big(2\Lambda_3\Lambda_4\Lambda_5\Lambda_6\mathrm{d}\Lambda_7 + \Lambda_3\Lambda_4\mathrm{d}\Lambda_5\Lambda_6\Lambda_7\big)A,$$

$$\ell_2(\ell_7(X_1,X_2,X_3,X_4,X_5,X_6,\Psi),X_7) = 6\mathcal{L}_{x_1}i_{x_2}i_{x_3}\,\mathrm{tr}\,\big(2\Lambda_4\Lambda_5\Lambda_6\mathrm{d}\Lambda_7 - \Lambda_4\Lambda_5\mathrm{d}\Lambda_6\Lambda_7\big)\mathrm{d}A$$
$$+ 36\mathcal{L}_{x_1}i_{x_2}\,\mathrm{tr}\,\big(2\Lambda_3\Lambda_4\Lambda_5\Lambda_6\mathrm{d}\Lambda_7 + \Lambda_3\Lambda_4\mathrm{d}\Lambda_5\Lambda_6\Lambda_7\big)A,$$

$$3\ell_7(\ell_2(X_1,X_2),X_3,X_4,X_5,X_6,X_7,\Psi) + \ell_7(X_1,X_2,X_3,X_4,X_5,X_6,\ell_2(X_7,\Psi))$$
$$= -12(i_{[x_1,x_2]}i_{x_3} + i_{x_1}i_{x_2}\mathcal{L}_{x_3})\,\mathrm{tr}\,\big(2\Lambda_4\Lambda_5\Lambda_6\mathrm{d}\Lambda_7 - \Lambda_4\Lambda_5\mathrm{d}\Lambda_6\Lambda_7\big)\mathrm{d}A$$
$$- 36(i_{[x_1,x_2]} - 2i_{x_1}\mathcal{L}_{x_2})\,\mathrm{tr}\,\big(2\Lambda_3\Lambda_4\Lambda_5\Lambda_6\mathrm{d}\Lambda_7 + \Lambda_3\Lambda_4\mathrm{d}\Lambda_5\Lambda_6\Lambda_7\big)A$$
$$+ 12i_{x_1}i_{x_2}\,\mathrm{tr}\,\Big(\big(2\Lambda_3\Lambda_4\Lambda_5\Lambda_6\mathrm{d}\Lambda_7 + \Lambda_3\Lambda_4\mathrm{d}\Lambda_5\Lambda_6\Lambda_7\big)\mathrm{d}A + \big(\Lambda_3\Lambda_4\mathrm{d}\Lambda_3\Lambda_5\Lambda_6\mathrm{d}\Lambda_7$$
$$+ \mathrm{d}\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6\Lambda_7 - 2\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6\mathrm{d}\Lambda_7 - 2\mathrm{d}\Lambda_3\Lambda_4\Lambda_5\Lambda_6\mathrm{d}\Lambda_7\big)A\Big),$$

$$\Rightarrow \ell_1(\ell_8(X_1,X_2,X_3,X_4,X_5,X_6,X_7,\Psi)) - 7\ell_8(X_1,X_2,X_3,X_4,X_5,X_6,\ell_1(X_7),\Psi)$$
$$+ 21\ell_3(\ell_6(X_1,X_2,X_3,X_4,X_5,\Psi),X_6,X_7) + 7\ell_2(\ell_7(X_1,X_2,X_3,X_4,X_5,X_6,\Psi),X_7)$$
$$+ 21\ell_7(\ell_2(X_1,X_2),X_3,X_4,X_5,X_6,X_7,\Psi) + 7\ell_7(X_1,X_2,X_3,X_4,X_5,X_6,\ell_2(X_7,\Psi)$$
$$= 0 \quad \square$$

And finally with $(V_0)^8$

$$\ell_8(X_1,X_2,X_3,X_4,X_5,X_6,X_7,\ell_1(X_8)) = -84i_{x_1}i_{x_2}\mathrm{d}\,\mathrm{tr}\,\Lambda_3\Lambda_4\Lambda_5\Lambda_6\Lambda_7\mathrm{d}\Lambda_8$$
$$= -84i_{x_1}i_{x_2}\,\mathrm{tr}(2\Lambda_3\Lambda_4\Lambda_5\Lambda_6\mathrm{d}\Lambda_7 + \Lambda_3\Lambda_4\mathrm{d}\Lambda_5\Lambda_6\Lambda_7)\mathrm{d}\Lambda_8,$$

$$\ell_3(\ell_6(X_1,X_2,X_3,X_4,X_5,X_6),X_7,X_8) = 24(i_{x_7}\mathcal{L}_{x_8} + i_{[x_7,x_8]})\,\mathrm{tr}\,\Lambda_{[1}\Lambda_2\Lambda_3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6,$$

$$\ell_2(\ell_7(X_1,X_2,X_3,X_4,X_5,X_6,X_7),X_8) = \tfrac{1}{12}\mathcal{L}_{x_8}i_{x_1}i_{x_2}i_{x_3}i_{x_4}(i_{x_5}\mathcal{L}_{x_6} + i_{[x_5,x_6]} + i_{x_5}i_{x_6}\mathrm{d})\sigma_7$$
$$+ -252\mathcal{L}_{x_8}i_{x_1}\,\mathrm{tr}\,\Lambda_2\Lambda_3\Lambda_4\Lambda_5\Lambda_6\mathrm{d}\Lambda_7 + 72\mathcal{L}_{x_8}\,\mathrm{tr}\,\Lambda_1\Lambda_2\Lambda_3\Lambda_4\Lambda_5\Lambda_6\Lambda_7,$$

$$\ell_7(\ell_2(X_1,X_2),X_3,X_4,X_5,X_6,X_7,X_8) = -\tfrac{1}{6}\tfrac{1}{7}i_{x_1}i_{x_2}i_{x_3}i_{x_4}\big(i_{x_5}i_{x_6}\mathcal{L}_{x_7}\mathrm{d} - \tfrac{1}{2}i_{x_5}\mathcal{L}_{[x_6,x_7]}$$
$$+ 6i_{[x_5,x_6]}\mathcal{L}_{x_7} + 6i_{[x_5,x_6]}i_{x_7}\mathrm{d}\big)\sigma_8 - \tfrac{288}{7}\mathcal{L}_{x_1}\,\mathrm{tr}\,\Lambda_2\Lambda_3\Lambda_4\Lambda_5\Lambda_6\Lambda_7\Lambda_8$$
$$+ 72\big(i_{[x_1,x_2]}\,\mathrm{tr}\,\Lambda_3\Lambda_4\Lambda_5\Lambda_6\Lambda_7\mathrm{d}\Lambda_8 - 2i_{x_1}\mathcal{L}_{x_2}\,\mathrm{tr}\,\Lambda_3\Lambda_4\Lambda_5\Lambda_6\Lambda_7\mathrm{d}\Lambda_8$$
$$+ \tfrac{2}{7}i_{x_1}\mathrm{d}\,\mathrm{tr}\,\Lambda_2\Lambda_3\Lambda_4\Lambda_5\Lambda_6\Lambda_7\Lambda_8\big),$$

$$\Rightarrow 28\ell_3(\ell_6(X_1,X_2,X_3,X_4,X_5,X_6),X_7,X_8) - 8\ell_2(\ell_7(X_1,X_2,X_3,X_4,X_5,X_6,X_7),X_8)$$
$$+ 28\ell_7(\ell_2(X_1,X_2),X_3,X_4,X_5,X_6,X_7,X_8) - 8\ell_8(X_1,X_2,X_3,X_4,X_5,X_6,X_7,\ell_1(X_8))$$
$$= 0 \quad \square$$

C.6 $\ell_1\ell_9 + \ell_9\ell_1 + \ell_8\ell_2 + \ell_2\ell_8 + \ell_3\ell_7 + \ell_7\ell_3 + \ell_4\ell_6 + \ell_6\ell_4 + \ell_5\ell_5 = 0$ relations

These identities require brackets (3.19). We must compute for inputs in $(V_0)^6 \otimes (V_{-1})^3$

$$\ell_1(\ell_9(X_1, X_2, X_3, X_4, X_5, X_6, \Psi_1, \Psi_2, \Psi_3)) = -\tfrac{6}{5}\mathrm{d}i_{x_1}i_{x_2}i_{x_3}i_{x_4}\,\mathrm{tr}\,\Lambda_5\big(3\mathrm{d}\Lambda_6\mathrm{d}(A_1A_2A_3) + \mathrm{d}(A_1A_2)\mathrm{d}\Lambda_6A_3 - A_1A_2\mathrm{d}\Lambda_6\mathrm{d}A_3\big),$$

$$\ell_5(\ell_5(X_1, X_2, \Psi_1, \Psi_2, \Psi_3), X_3, X_4, X_5, X_6) = \tfrac{2}{5}\tfrac{1}{5}i_{x_1}i_{x_2}(i_{x_3}\mathcal{L}_{x_4} + i_{[x_3,x_4]} + i_{x_3}i_{x_4}\mathrm{d})\,\mathrm{tr}\,\Lambda_5\big(3\mathrm{d}\Lambda_6\mathrm{d}(A_1A_2A_3) + \mathrm{d}(A_1A_2)\mathrm{d}\Lambda_6A_3 - A_1A_2\mathrm{d}\Lambda_6\mathrm{d}A_3\big),$$

$$\ell_3(\ell_7(X_1, X_2, X_3, X_4, \Psi_1, \Psi_2, \Psi_3), X_5, X_6) = -\tfrac{2}{5}(i_{x_1}\mathcal{L}_{x_2} + i_{[x_1,x_2]})i_{x_3}i_{x_4}\,\mathrm{tr}\,\Lambda_5\big(3\mathrm{d}\Lambda_6\mathrm{d}(A_1A_2A_3) + \mathrm{d}(A_1A_2)\mathrm{d}\Lambda_6A_3 - A_1A_2\mathrm{d}\Lambda_6\mathrm{d}A_3\big) - \tfrac{12}{5}(i_{x_1}\mathcal{L}_{x_2} + i_{[x_1,x_2]})i_{x_3}\,\mathrm{tr}\,\big(3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6A_1A_2A_3 - \Lambda_4\Lambda_5A_1A_2\mathrm{d}\Lambda_6A_3\big),$$

$$\ell_2(\ell_8(X_1, X_2, X_3, X_4, X_5, \Psi_1, \Psi_2, \Psi_3), X_6) = 6\mathcal{L}_{x_6}i_{x_1}i_{x_2}\,\mathrm{tr}\,\big(3\Lambda_3\Lambda_4\mathrm{d}\Lambda_5A_1A_2A_3 - \Lambda_3\Lambda_4A_1A_2\mathrm{d}\Lambda_5A_3\big),$$

$$15\ell_8(\ell_2(X_1, X_2), X_3, X_4, X_5, X_6, \Psi_1, \Psi_2, \Psi_3) - 18\ell_8(X_1, X_2, X_3, X_4, X_5, \ell_2(X_6, \Psi_1), \Psi_2, \Psi_3) = -72(i_{[x_1,x_2]}i_{x_3} + i_{x_1}i_{x_2}\mathcal{L}_{x_3})\,\mathrm{tr}\,\big(3\Lambda_4\Lambda_5\mathrm{d}\Lambda_6A_1A_2A_3 - \Lambda_4\Lambda_5A_1A_2\mathrm{d}\Lambda_6A_3\big),$$

$$\Rightarrow \ell_1(\ell_9(X_1, X_2, X_3, X_4, X_5, X_6, \Psi_1, \Psi_2, \Psi_3)) + 15\ell_5(\ell_5(X_1, X_2, \Psi_1, \Psi_2, \Psi_3), X_3, X_4, X_5, X_6) + 15\ell_3(\ell_7(X_1, X_2, X_3, X_4, \Psi_1, \Psi_2, \Psi_3), X_5, X_6) - 6\ell_2(\ell_8(X_1, X_2, X_3, X_4, X_5, \Psi_1, \Psi_2, \Psi_3), X_6) + 15\ell_8(\ell_2(X_1, X_2), X_3, X_4, X_5, X_6, \Psi_1, \Psi_2, \Psi_3) - 18\ell_8(X_1, X_2, X_3, X_4, X_5, \ell_2(X_6, \Psi_1), \Psi_2, \Psi_3) = 0 \quad \square$$

For $(V_0)^7 \otimes (V_{-1})^2$

$$\ell_1(\ell_9(X_1, X_2, X_3, X_4, X_5, X_6, X_7, \Psi_1, \Psi_2)) = \tfrac{14}{5}\mathrm{d}i_{x_1}i_{x_2}i_{x_3}i_{x_4}\,\mathrm{tr}\,\big(2\Lambda_5\Lambda_6\mathrm{d}\Lambda_7\mathrm{d}A_1A_2$$
$$+ 3\Lambda_5\Lambda_6\mathrm{d}\Lambda_7A_1\mathrm{d}A_2 - \Lambda_5\mathrm{d}\Lambda_6\Lambda_7\mathrm{d}(A_1A_2) - \Lambda_5\mathrm{d}\Lambda_6\mathrm{d}A_1\Lambda_7A_2 + \Lambda_5\mathrm{d}\Lambda_6A_1\Lambda_7\mathrm{d}A_2\big),$$

$$\ell_9(X_1, X_2, X_3, X_4, X_5, X_6, \ell_1(X_7), \Psi_1, \Psi_2) = \tfrac{2}{5}i_{x_1}i_{x_2}i_{x_3}i_{x_4}\,\mathrm{tr}\,\big(3\mathrm{d}\Lambda_5\Lambda_6\mathrm{d}\Lambda_7\mathrm{d}(A_1A_2)$$
$$- 3\Lambda_5\mathrm{d}\Lambda_6\mathrm{d}\Lambda_7\mathrm{d}(A_1A_2) + \mathrm{d}\Lambda_5\mathrm{d}\Lambda_6\Lambda_7\mathrm{d}(A_1A_2) + 4\Lambda_5\mathrm{d}\Lambda_6\mathrm{d}A_1\mathrm{d}\Lambda_7A_2 + 4\Lambda_5\mathrm{d}\Lambda_6A_1\mathrm{d}\Lambda_7\mathrm{d}A_2$$
$$- \Lambda_5\mathrm{d}A_1\mathrm{d}\Lambda_6\mathrm{d}\Lambda_7A_2 + \Lambda_5A_1\mathrm{d}\Lambda_6\mathrm{d}\Lambda_7\mathrm{d}A_2\big),$$

$$\ell_5(\ell_5(X_1, X_2, X_3, \Psi_1, \Psi_2), X_4, X_5, X_6, X_7) = -\tfrac{1}{5}i_{x_4}i_{x_5}(i_{x_6}\mathcal{L}_{x_7} + i_{[x_6,x_7]})i_{x_1}\,\mathrm{tr}\,\Lambda_2\mathrm{d}\Lambda_3\mathrm{d}A_1\mathrm{d}A_2$$
$$- \tfrac{2}{25}i_{x_4}i_{x_5}(i_{x_6}\mathcal{L}_{x_7} + i_{[x_6,x_7]})\,\mathrm{tr}\,\Lambda_1\big(2\Lambda_2\mathrm{d}\Lambda_3\mathrm{d}A_1A_2 + 3\Lambda_2\mathrm{d}\Lambda_3A_1\mathrm{d}A_2 - \mathrm{d}\Lambda_2\Lambda_3\mathrm{d}(A_1A_2)$$
$$- \mathrm{d}\Lambda_2(\mathrm{d}A_1\Lambda_3A_2 - A_1\Lambda_3\mathrm{d}A_2)\big),$$

$$\ell_3(\ell_7(X_1, X_2, X_3, X_4, X_5, \Psi_1, \Psi_2), X_6, X_7) = \tfrac{2}{3}(i_{x_1}\mathcal{L}_{x_2} + i_{[x_1,x_2]})i_{x_3}i_{x_4}\,\mathrm{tr}\,\big(2\Lambda_5\Lambda_6\mathrm{d}\Lambda_7\mathrm{d}A_1A_2$$
$$+ 3\Lambda_5\Lambda_6\mathrm{d}\Lambda_7A_1\mathrm{d}A_2 - \Lambda_5\mathrm{d}\Lambda_6\Lambda_7\mathrm{d}(A_1A_2) - \Lambda_5\mathrm{d}\Lambda_6\mathrm{d}A_1\Lambda_7A_2 + \Lambda_5\mathrm{d}\Lambda_6A_1\Lambda_7\mathrm{d}A_2\big)$$
$$- 4(i_{x_1}\mathcal{L}_{x_2}i_{x_3} + i_{[x_1,x_2]}i_{x_3})\,\mathrm{tr}\,\big(\Lambda_4\mathrm{d}(\Lambda_5\Lambda_6)\Lambda_7A_1A_2 + \Lambda_4\Lambda_5\mathrm{d}\Lambda_6A_1\Lambda_7A_2 + \Lambda_4\mathrm{d}\Lambda_5A_1\Lambda_6\Lambda_7A_2\big),$$

$$\ell_2(\ell_8(X_1, X_2, X_3, X_4, X_5, X_6, \Psi_1, \Psi_2), X_7) = \mathcal{L}_{x_7}i_{x_1}i_{x_2}i_{x_3}i_{x_4}\,\mathrm{tr}\,\Lambda_5\mathrm{d}\Lambda_6\mathrm{d}A_1\mathrm{d}A_2$$
$$+ 12\mathcal{L}_{x_7}i_{x_1}i_{x_2}\,\mathrm{tr}\,\big(\Lambda_3\mathrm{d}(\Lambda_4\Lambda_5)\Lambda_6A_1A_2 + \Lambda_3\Lambda_4\mathrm{d}\Lambda_5A_1\Lambda_6A_2 + \Lambda_3\mathrm{d}\Lambda_4A_1\Lambda_5\Lambda_6A_2\big),$$

$$21\ell_8(\ell_2(X_1, X_2), X_3, X_4, X_5, X_6, X_7, \Psi_1, \Psi_2) + 14\ell_8(X_1, X_2, X_3, X_4, X_5, X_6, \ell_2(X_7, \Psi_1), \Psi_2)$$
$$= -14(2i_{[x_1,x_2]}i_{x_3}i_{x_4}i_{x_5} + i_{x_1}i_{x_2}i_{x_3}i_{x_4}\mathcal{L}_{x_5})\,\mathrm{tr}\,\Lambda_6\mathrm{d}\Lambda_7\mathrm{d}A_1\mathrm{d}A_2$$
$$- 168(i_{[x_1,x_2]}i_{x_3} + i_{x_1}i_{x_2}\mathcal{L}_{x_3})\,\mathrm{tr}\,\big(\Lambda_3\mathrm{d}(\Lambda_4\Lambda_5)\Lambda_6A_1A_2 + \Lambda_3\Lambda_4\mathrm{d}\Lambda_5A_1\Lambda_6A_2 + \Lambda_3\mathrm{d}\Lambda_4A_1\Lambda_5\Lambda_6A_2\big)$$
$$- 14i_{x_1}i_{x_2}i_{x_3}i_{x_4}\,\mathrm{tr}\,\big(\mathrm{d}\Lambda_5\Lambda_6\mathrm{d}\Lambda_7\mathrm{d}A_1A_2 - \Lambda_5\Lambda_6\mathrm{d}\Lambda_7\mathrm{d}A_1\mathrm{d}A_2 + \Lambda_5\mathrm{d}\Lambda_6\mathrm{d}\Lambda_7A_1\mathrm{d}A_2$$
$$+ \Lambda_5\mathrm{d}\Lambda_6A_1\mathrm{d}\Lambda_7\mathrm{d}A_2 + \Lambda_5\mathrm{d}\Lambda_6\mathrm{d}A_1\mathrm{d}\Lambda_7A_2\big),$$

$$\Rightarrow \ell_1(\ell_9(X_1, X_2, X_3, X_4, X_5, X_6, X_7, \Psi_1, \Psi_2)) + 7\ell_9(X_1, X_2, X_3, X_4, X_5, X_6, \ell_1(X_7), \Psi_1, \Psi_2)$$
$$+ 35\ell_5(\ell_5(X_1, X_2, X_3, \Psi_1, \Psi_2), X_4, X_5, X_6, X_7) + 7\ell_2(\ell_8(X_1, X_2, X_3, X_4, X_5, X_6, \Psi_1, \Psi_2), X_7)$$
$$+ 21\ell_8(\ell_2(X_1, X_2), X_3, X_4, X_5, X_6, X_7, \Psi_1, \Psi_2) + 14\ell_8(X_1, X_2, X_3, X_4, X_5, X_6, \ell_2(X_7, \Psi_1), \Psi_2)$$
$$= 0 \quad \square$$

And for $(V_0)^8 \otimes V_{-1}$

$$\begin{aligned}
&\ell_9(X_1,X_2,X_3,X_4,X_5,X_6,X_7,\ell_1(X_8),\Psi) = -\tfrac{7}{5} i_{x_1} i_{x_2} i_{x_3} i_{x_4} \mathrm{d}\,\mathrm{tr}\,\big(2\Lambda_5\Lambda_6\Lambda_7\mathrm{d}\Lambda_8 \\
&\quad - \Lambda_5\Lambda_6\mathrm{d}\Lambda_7\Lambda_8\big)\mathrm{d}A, \\
&\ell_5(\ell_5(X_1,X_2,X_3,X_4,\Psi),X_5,X_6,X_7,X_8) = -\tfrac{4}{5}\tfrac{1}{5} i_{x_1} i_{x_2} (i_{x_3}\mathcal{L}_{x_4} + i_{[x_3,x_4]})\,\mathrm{tr}\,\big(2\Lambda_5\Lambda_6\Lambda_7\mathrm{d}\Lambda_8 \\
&\quad - \Lambda_5\Lambda_6\mathrm{d}\Lambda_7\Lambda_8\big)\mathrm{d}A, \\
&\ell_3(\ell_7(X_1,X_2,X_3,X_4,X_5,X_6,\Psi),X_7,X_8) = 2(i_{x_1}\mathcal{L}_{x_2} + i_{[x_1,x_2]}) i_{x_3} i_{x_4}\,\mathrm{tr}\,\big(2\Lambda_5\Lambda_6\Lambda_7\mathrm{d}\Lambda_8 \\
&\quad - \Lambda_5\Lambda_6\mathrm{d}\Lambda_7\Lambda_8\big)\mathrm{d}A + 12(i_{x_1}\mathcal{L}_{x_2} + i_{[x_1,x_2]}) i_{x_3}\,\mathrm{tr}\,\big(2\Lambda_4\Lambda_5\Lambda_6\Lambda_7\mathrm{d}\Lambda_8 + \Lambda_4\Lambda_5\mathrm{d}\Lambda_6\Lambda_7\Lambda_8\big)A, \\
&\ell_2(\ell_3(X_1,X_2,X_3,X_4,X_5,X_6,X_7,\Psi),X_8) = 42\mathcal{L}_{x_1} i_{x_2} i_{x_3}\,\mathrm{tr}\,\big(2\Lambda_4\Lambda_5\Lambda_6\Lambda_7\mathrm{d}\Lambda_8 \\
&\quad + \Lambda_4\Lambda_5\mathrm{d}\Lambda_6\Lambda_7\Lambda_8\big)A, \\
&28\ell_8(\ell_2(X_1,X_2),X_3,X_4,X_5,X_6,X_7,X_8,\Psi) - 8\ell_8(X_1,X_2,X_3,X_4,X_5,X_6,X_7,\ell_2(X_8,\Psi)) \\
&\quad = 672(i_{x_1} i_{x_2}\mathcal{L}_{x_3} + i_{[x_1,x_2]} i_{x_3})\,\mathrm{tr}\,\big(2\Lambda_4\Lambda_5\Lambda_6\Lambda_7\mathrm{d}\Lambda_8 + \Lambda_4\Lambda_5\mathrm{d}\Lambda_6\Lambda_7\Lambda_8\big)A, \\
&\Rightarrow 8\ell_9(X_1,X_2,X_3,X_4,X_5,X_6,X_7,\ell_1(X_8),\Psi) + 70\ell_5(\ell_5(X_1,X_2,X_3,X_4,\Psi),X_5,X_6,X_7,X_8) \\
&+ 28\ell_3(\ell_7(X_1,X_2,X_3,X_4,X_5,X_6,\Psi),X_7,X_8) - 8\ell_2(\ell_3(X_1,X_2,X_3,X_4,X_5,X_6,X_7,\Psi),X_8) \\
&+ 28\ell_8(\ell_2(X_1,X_2),X_3,X_4,X_5,X_6,X_7,X_8,\Psi) - 8\ell_8(X_1,X_2,X_3,X_4,X_5,X_6,X_7,\ell_2(X_8,\Psi)) \\
&\quad = 0 \quad \square
\end{aligned}$$

C.7 $\ell_1\ell_{10} - \ell_{10}\ell_1 - \ell_2\ell_9 + \ell_9\ell_2 - \ell_8\ell_3 + \ell_3\ell_8 - \ell_4\ell_7 + \ell_7\ell_4 + \ell_5\ell_6 - \ell_6\ell_5 = 0$ relations

These relations require the last level of brackets (3.20). We have non-trivial identities for $(V_0)^7 \otimes (V_{-1})^3$

$$\begin{aligned}
&\ell_1(\ell_{10}(X_1, X_2, X_3, X_4, X_5, X_6, X_7, \Psi_1, \Psi_2, \Psi_3)) = \tfrac{42}{5}\mathrm{d}i_{x_1}i_{x_2}i_{x_3}i_{x_4}\,\mathrm{tr}\,\big(3\Lambda_5\Lambda_6\mathrm{d}\Lambda_7A_1A_2A_3 \\
&\quad - \Lambda_5\Lambda_6A_1A_2\mathrm{d}\Lambda_7A_3\big), \\
&\ell_3(\ell_8(X_1, X_2, X_3, X_4, X_5, \Psi_1, \Psi_2, \Psi_3), X_6, X_7) \\
&\quad = 2(i_{x_1}\mathcal{L}_{x_2} + i_{[x_1,x_2]})i_{x_3}i_{x_4}\,\mathrm{tr}\,\big(3\Lambda_5\Lambda_6\mathrm{d}\Lambda_7A_1A_2A_3 - \Lambda_5\Lambda_6A_1A_2\mathrm{d}\Lambda_7A_3\big), \\
&\ell_5(\ell_6(X_1, X_2, X_3, \Psi_1, \Psi_2, \Psi_3), X_4, X_5, X_6, X_7) \\
&\quad = -\tfrac{3}{25}i_{x_1}i_{x_2}(i_{x_3}\mathcal{L}_{x_4} + i_{[x_3,x_4]})i_{x_5}\,\mathrm{tr}\,\Lambda_6\big(3\mathrm{d}\Lambda_7\mathrm{d}(A_1A_2A_3) + \mathrm{d}(A_1A_2)\mathrm{d}\Lambda_7A_3 - A_1A_2\mathrm{d}\Lambda_7\mathrm{d}A_3\big) \\
&\quad - \tfrac{6}{25}i_{x_1}i_{x_2}(i_{x_3}\mathcal{L}_{x_4} + i_{[x_3,x_4]})\,\mathrm{tr}\,\big(3\Lambda_5\Lambda_6\mathrm{d}\Lambda_7A_1A_2A_3 - \Lambda_5\Lambda_6A_1A_2\mathrm{d}\Lambda_7A_3\big), \\
&\ell_2(\ell_9(X_1, X_2, X_3, X_4, X_5, X_6, \Psi_1, \Psi_2, \Psi_3), X_7) = \tfrac{3}{5}\mathcal{L}_{x_1}i_{x_2}i_{x_3}i_{x_4}i_{x_5}\,\mathrm{tr}\,\Lambda_6\big(3\mathrm{d}\Lambda_7\mathrm{d}(A_1A_2A_3) \\
&\quad + \mathrm{d}(A_1A_2)\mathrm{d}\Lambda_7A_3 - A_1A_2\mathrm{d}\Lambda_7\mathrm{d}A_3\big), \\
&\ell_9(\ell_2(X_1, X_2), X_3, X_4, X_5, X_6, X_7, \Psi_1, \Psi_2, \Psi_3) + \ell_9(X_1, X_2, X_3, X_4, X_5, X_6, \ell_2(X_7, \Psi_1), \Psi_2, \Psi_3) \\
&\quad = -\tfrac{2}{5}(i_{x_1}i_{x_2}i_{x_3}i_{x_4}\mathcal{L}_{x_5} + 2i_{[x_1,x_2]}i_{x_3}i_{x_4}i_{x_5})\,\mathrm{tr}\,\Lambda_6\big(3\mathrm{d}\Lambda_7\mathrm{d}(A_1A_2A_3) + \mathrm{d}(A_1A_2)\mathrm{d}\Lambda_7A_3 \\
&\quad - A_1A_2\mathrm{d}\Lambda_7\mathrm{d}A_3\big) - \tfrac{12}{5}i_{x_1}i_{x_2}i_{x_3}i_{x_4}\mathrm{d}\,\mathrm{tr}\,\big(3\Lambda_5\Lambda_6\mathrm{d}\Lambda_7A_1A_2A_3 - \Lambda_5\Lambda_6A_1A_2\mathrm{d}\Lambda_7A_3\big), \\
&\Rightarrow \ell_1(\ell_{10}(X_1, X_2, X_3, X_4, X_5, X_6, X_7, \Psi_1, \Psi_2, \Psi_3)) \\
&+ 21\ell_3(\ell_8(X_1, X_2, X_3, X_4, X_5, \Psi_1, \Psi_2, \Psi_3), X_6, X_7) \\
&+ 35\ell_5(\ell_6(X_1, X_2, X_3, \Psi_1, \Psi_2, \Psi_3), X_4, X_5, X_6, X_7) \\
&+ 7\ell_2(\ell_9(X_1, X_2, X_3, X_4, X_5, X_6, \Psi_1, \Psi_2, \Psi_3), X_7) \\
&+ 21\ell_9(\ell_2(X_1, X_2), X_3, X_4, X_5, X_6, X_7, \Psi_1, \Psi_2, \Psi_3) \\
&+ 21\ell_9(X_1, X_2, X_3, X_4, X_5, X_6, \ell_2(X_7, \Psi_1), \Psi_2, \Psi_3) = 0 \quad \square
\end{aligned}$$

And for $(V_0)^8 \otimes (V_{-1})^2$

$$\ell_{10}(X_1, X_2, X_3, X_4, X_5, X_6, X_7, \ell_1(X_8), \Psi_1, \Psi_2) = -\tfrac{14}{5} i_{x_1} i_{x_2} i_{x_3} i_{x_4} \operatorname{tr}\big((3\Lambda_5\Lambda_6 \mathrm{d}\Lambda_7 \mathrm{d}\Lambda_8 + 3\mathrm{d}\Lambda_5\Lambda_6\Lambda_7\mathrm{d}\Lambda_8 - \mathrm{d}\Lambda_5\mathrm{d}\Lambda_6\Lambda_7\Lambda_8)A_1A_2 - \mathrm{d}\Lambda_5\mathrm{d}\Lambda_6 A_1\Lambda_7\Lambda_8A_2 + 4\Lambda_5\Lambda_6\mathrm{d}\Lambda_7A_1\mathrm{d}\Lambda_8A_2\big),$$

$$\ell_3(\ell_8(X_1, X_2, X_3, X_4, X_5, X_6, \Psi_1, \Psi_2), X_7, X_8) = \tfrac{1}{3}(i_{x_1}\mathcal{L}_{x_2} + i_{[x_1,x_2]}) i_{x_3} i_{x_4} i_{x_5} i_{x_6} \operatorname{tr} \Lambda_7 \mathrm{d}\Lambda_8 \mathrm{d}A_1 \mathrm{d}A_2 + 4(i_{x_1}\mathcal{L}_{x_2} + i_{[x_1,x_2]}) i_{x_3} i_{x_4} \operatorname{tr}\big(\Lambda_5\mathrm{d}(\Lambda_6\Lambda_7)\Lambda_8A_1A_2 + \Lambda_5\Lambda_6\mathrm{d}\Lambda_7A_1\Lambda_8A_2 + \Lambda_5\mathrm{d}\Lambda_6A_1\Lambda_7\Lambda_8A_2\big),$$

$$\begin{aligned}\ell_7(\ell_4(X_1, X_2, \Psi_1, \Psi_2), X_3, X_4, X_5, X_6, X_7, X_8) &= -\tfrac{2}{3}\tfrac{1}{7} i_{x_1} i_{x_2} i_{x_3} i_{x_4}(i_{x_5}\mathcal{L}_{x_6} + i_{[x_5,x_6]} + i_{x_5}i_{x_6}\mathrm{d}) \operatorname{tr} \Lambda_7 \mathrm{d}\Lambda_8 \mathrm{d}A_1 \mathrm{d}A_2 \\ &= -\tfrac{2}{3}\tfrac{1}{7}(7 i_{x_1}\mathcal{L}_{x_2} + 14 i_{[x_1,x_2]}) i_{x_3} i_{x_4} i_{x_5} i_{x_6} \operatorname{tr} \Lambda_7 \mathrm{d}\Lambda_8 \mathrm{d}A_1 \mathrm{d}A_2,\end{aligned}$$

$$\ell_5(\ell_6(X_1, X_2, X_3, X_4, \Psi_1, \Psi_2), X_5, X_6, X_7, X_8) = \tfrac{2}{3}\tfrac{1}{5} i_{x_1} i_{x_2}(i_{x_3}\mathcal{L}_{x_4} + i_{[x_3,x_4]}) i_{x_5} i_{x_6} \operatorname{tr} \Lambda_7 \mathrm{d}\Lambda_8 \mathrm{d}A_1 \mathrm{d}A_2 + \tfrac{4}{5}\tfrac{1}{5} i_{x_1} i_{x_2}(i_{x_3}\mathcal{L}_{x_4} + i_{[x_3,x_4]}) i_{x_5} \operatorname{tr}\big(3\Lambda_5\Lambda_6\mathrm{d}\Lambda_7\mathrm{d}A_1A_2 + 2\Lambda_5\Lambda_6\mathrm{d}\Lambda_7A_1\mathrm{d}A_2 - \Lambda_5\mathrm{d}(\Lambda_6\Lambda_7)\mathrm{d}(A_1A_2) - \Lambda_5\mathrm{d}\Lambda_6\mathrm{d}A_1\Lambda_7A_2 + \Lambda_5\mathrm{d}\Lambda_6A_1\Lambda_7\mathrm{d}A_2\big) - \tfrac{8}{5}\tfrac{1}{5} i_{x_1} i_{x_2}(i_{x_3}\mathcal{L}_{x_4} + i_{[x_3,x_4]}) \operatorname{tr}\big(\Lambda_5\mathrm{d}(\Lambda_6\Lambda_7)\Lambda_8A_1A_2 + \Lambda_5\Lambda_6\mathrm{d}\Lambda_7A_1\Lambda_8A_2 + \Lambda_5\mathrm{d}\Lambda_6A_1\Lambda_7\Lambda_8A_2\big),$$

$$\ell_2(\ell_9(X_1, X_2, X_3, X_4, X_5, X_6, X_7, \Psi_1, \Psi_2), X_8) = \tfrac{7}{5}\mathcal{L}_{x_1} i_{x_2} i_{x_3} i_{x_4} i_{x_5} \operatorname{tr}\big(3\Lambda_5\Lambda_6\mathrm{d}\Lambda_7\mathrm{d}A_1A_2 + 2\Lambda_5\Lambda_6\mathrm{d}\Lambda_7A_1\mathrm{d}A_2 - \Lambda_5\mathrm{d}(\Lambda_6\Lambda_7)\mathrm{d}(A_1A_2) - \Lambda_5\mathrm{d}\Lambda_6\mathrm{d}A_1\Lambda_7A_2 + \Lambda_5\mathrm{d}\Lambda_6A_1\Lambda_7\mathrm{d}A_2\big),$$

$$\begin{aligned}&28\ell_9(\ell_2(X_1, X_2), X_3, X_4, X_5, X_6, X_7, X_8, \Psi_1, \Psi_2) - 16\ell_9(X_1, X_2, X_3, X_4, X_5, X_6, X_7, \ell_2(X_8, \Psi_1), \Psi_2) \\ &= \tfrac{56}{5}(4 i_{[x_1,x_2]} i_{x_3} i_{x_4} i_{x_5} + 2 i_{x_1} i_{x_2} i_{x_3} i_{x_4}\mathcal{L}_{x_5}) \operatorname{tr}\big(3\Lambda_5\Lambda_6\mathrm{d}\Lambda_7\mathrm{d}A_1A_2 \\ &+ 2\Lambda_5\Lambda_6\mathrm{d}\Lambda_7A_1\mathrm{d}A_2 - \Lambda_5\mathrm{d}(\Lambda_6\Lambda_7)\mathrm{d}(A_1A_2) - \Lambda_5\mathrm{d}\Lambda_6\mathrm{d}A_1\Lambda_7A_2 + \Lambda_5\mathrm{d}\Lambda_6A_1\Lambda_7\mathrm{d}A_2\big) \\ &+ \tfrac{112}{5} i_{x_1} i_{x_2} i_{x_3} i_{x_4} \operatorname{tr}\big(\Lambda_5\mathrm{d}(\Lambda_6\Lambda_7)\Lambda_8\mathrm{d}(A_1A_2) + (\Lambda_5\mathrm{d}\Lambda_6\Lambda_7\mathrm{d}\Lambda_8 - 2\Lambda_5\Lambda_6\mathrm{d}\Lambda_7\mathrm{d}\Lambda_8 \\ &- \mathrm{d}\Lambda_5\Lambda_6\mathrm{d}\Lambda_7\Lambda_8 - 3\mathrm{d}\Lambda_5\Lambda_6\Lambda_7\mathrm{d}\Lambda_8)A_1A_2 - 4\Lambda_5\Lambda_6\mathrm{d}\Lambda_7A_1\mathrm{d}\Lambda_8A_2 \\ &- \mathrm{d}(\Lambda_5\Lambda_6\mathrm{d}\Lambda_7A_1\Lambda_8A_2) + \Lambda_5\mathrm{d}\Lambda_6\mathrm{d}(A_1\Lambda_7\Lambda_8A_2)\big),\end{aligned}$$

$$\begin{aligned}\Rightarrow\ &-8\ell_{10}(X_1, X_2, X_3, X_4, X_5, X_6, X_7, \ell_1(X_8), \Psi_1, \Psi_2) \\ &- 8\ell_2(\ell_9(X_1, X_2, X_3, X_4, X_5, X_6, X_7, \Psi_1, \Psi_2), X_8) \\ &+ 70\ell_5(\ell_6(X_1, X_2, X_3, X_4, \Psi_1, \Psi_2), X_5, X_6, X_7, X_8) \\ &+ 28\ell_7(\ell_4(X_1, X_2, \Psi_1, \Psi_2), X_3, X_4, X_5, X_6, X_7, X_8) \\ &+ 28\ell_9(\ell_2(X_1, X_2), X_3, X_4, X_5, X_6, X_7, X_8, \Psi_1, \Psi_2) \\ &- 16\ell_9(X_1, X_2, X_3, X_4, X_5, X_6, X_7, \ell_2(X_8, \Psi_1), \Psi_2) = 0 \quad \square\end{aligned}$$

C.8 $\ell_2\ell_{10}+\ell_{10}\ell_2+\ell_3\ell_9+\ell_9\ell_3+\ell_8\ell_4+\ell_4\ell_8+\ell_5\ell_7+\ell_7\ell_5+\ell_6\ell_6=0$ relations

No new brackets are needed to satisfy the remaining identities, but there are still non-trivial checks for $(V_0)^8\otimes(V_{-1})^3$

$$
\begin{aligned}
&\ell_3(\ell_9(X_1,X_2,X_3,X_4,X_5,X_6,\Psi_1,\Psi_2,\Psi_3),X_7,X_8)\\
&\quad=\tfrac{1}{5}(i_{x_1}\mathcal{L}_{x_2}+i_{[x_1,x_2]})i_{x_3}i_{x_4}i_{x_5}i_{x_6}\,\mathrm{tr}\,\Lambda_7\big(3\mathrm{d}\Lambda_8\mathrm{d}(A_1A_2A_3)+\mathrm{d}(A_1A_2\mathrm{d}\Lambda_8A_3)\big),\\
&\ell_5(\ell_7(X_1,X_2,X_3,X_4,\Psi_1,\Psi_2,\Psi_3),X_5,X_6,X_7,X_8)\\
&\quad=\tfrac{2}{25}i_{x_1}i_{x_2}(i_{x_3}\mathcal{L}_{x_4}+i_{[x_3,x_4]})i_{x_5}i_{x_6}\,\mathrm{tr}\,\Lambda_7\big(3\mathrm{d}\Lambda_8\mathrm{d}(A_1A_2A_3)+\mathrm{d}(A_1A_2\mathrm{d}\Lambda_8A_3)\big)\\
&\quad+\tfrac{12}{25}i_{x_1}i_{x_2}(i_{x_3}\mathcal{L}_{x_4}+i_{[x_3,x_4]})i_{x_5}\,\mathrm{tr}\,\big(3\Lambda_6\Lambda_7\mathrm{d}\Lambda_8A_1A_2A_3-\Lambda_6\Lambda_7A_1A_2\mathrm{d}\Lambda_8A_3\big),\\
&\ell_7(\ell_5(X_1,X_2,\Psi_1,\Psi_2,\Psi_3),X_3,X_4,X_5,X_6,X_7,X_8)=-\tfrac{1}{7}\tfrac{2}{5}i_{x_1}i_{x_2}i_{x_3}i_{x_4}(i_{x_5}\mathcal{L}_{x_6}+i_{[x_5,x_6]}\\
&\quad+i_{x_5}i_{x_6}\mathrm{d})\,\mathrm{tr}\,\Lambda_7\big(3\mathrm{d}\Lambda_8\mathrm{d}(A_1A_2A_3)+\mathrm{d}(A_1A_2)\mathrm{d}\Lambda_8A_3-A_1A_2\mathrm{d}\Lambda_8\mathrm{d}A_3\big),\\
&\ell_2(\ell_{10}(X_1,X_2,X_3,X_4,X_5,X_6,X_7,\Psi_1,\Psi_2,\Psi_3),X_8)\\
&\quad=\tfrac{21}{5}\mathcal{L}_{x_1}i_{x_2}i_{x_3}i_{x_4}i_{x_5}\,\mathrm{tr}\,\big(3\Lambda_6\Lambda_7\mathrm{d}\Lambda_8A_1A_2A_3-\Lambda_6\Lambda_7A_1A_2\mathrm{d}\Lambda_8A_3\big),\\
&28\ell_{10}(\ell_2(X_1,X_2),X_3,X_4,X_5,X_6,X_7,X_8,\Psi_1,\Psi_2,\Psi_3)\\
&-24\ell_{10}(X_1,X_2,X_3,X_4,X_5,X_6,X_7,\ell_2(X_8,\Psi_1),\Psi_2,\Psi_3)=\tfrac{6}{5}(4i_{[x_1,x_2]}i_{x_3}i_{x_4}i_{x_5}\\
&\quad+2i_{x_1}i_{x_2}i_{x_3}i_{x_4}\mathcal{L}_{x_5})\,\mathrm{tr}\,\big(3\Lambda_6\Lambda_7\mathrm{d}\Lambda_8A_1A_2A_3-\Lambda_6\Lambda_7A_1A_2\mathrm{d}\Lambda_8A_3\big),\\
&\Rightarrow-8\ell_2(\ell_{10}(X_1,X_2,X_3,X_4,X_5,X_6,X_7,\Psi_1,\Psi_2,\Psi_3),X_8)\\
&+28\ell_7(\ell_5(X_1,X_2,\Psi_1,\Psi_2,\Psi_3),X_3,X_4,X_5,X_6,X_7,X_8)\\
&+28\ell_{10}(\ell_2(X_1,X_2),X_3,X_4,X_5,X_6,X_7,X_8,\Psi_1,\Psi_2,\Psi_3)\\
&-24\ell_{10}(X_1,X_2,X_3,X_4,X_5,X_6,X_7,\ell_2(X_8,\Psi_1),\Psi_2,\Psi_3)\\
&+28\ell_3(\ell_9(X_1,X_2,X_3,X_4,X_5,X_6,\Psi_1,\Psi_2,\Psi_3),X_7,X_8)\\
&+70\ell_5(\ell_7(X_1,X_2,X_3,X_4,\Psi_1,\Psi_2,\Psi_3),X_5,X_6,X_7,X_8)=0\quad\square
\end{aligned}
$$

C.9 Higher relations

The relations

$$\ell_3\ell_{10} - \ell_{10}\ell_3 - \ell_4\ell_9 + \ell_9\ell_4 - \ell_8\ell_5 + \ell_5\ell_8 - \ell_6\ell_7 + \ell_7\ell_6 = 0, \tag{C.2}$$

$$\ell_4\ell_{10} + \ell_{10}\ell_4 + \ell_5\ell_9 + \ell_9\ell_5 + \ell_8\ell_6 + \ell_6\ell_8 + \ell_7\ell_7 = 0, \tag{C.3}$$

$$\ell_5\ell_{10} - \ell_{10}\ell_5 - \ell_6\ell_9 + \ell_9\ell_6 + \ell_7\ell_8 - \ell_8\ell_7 = 0, \tag{C.4}$$

$$\ell_6\ell_{10} + \ell_{10}\ell_6 + \ell_7\ell_9 + \ell_9\ell_7 + \ell_8\ell_8 = 0, \tag{C.5}$$

$$\ell_7\ell_{10} - \ell_{10}\ell_7 + \ell_9\ell_8 - \ell_8\ell_9 = 0, \tag{C.6}$$

$$\ell_8\ell_{10} + \ell_{10}\ell_8 + \ell_9\ell_9 = 0, \tag{C.7}$$

$$\ell_9\ell_{10} - \ell_{10}\ell_9 = 0, \tag{C.8}$$

$$\ell_{10}\ell_{10} = 0, \tag{C.9}$$

are all trivial to verify.

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