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Groups, Geometry, and Dynamics

Dynamical correspondences of L^2-Betti numbers

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Abstract. We investigate dynamical analogues of the L^2-Betti numbers for modules over integral group ring of a discrete sofic group. In particular, we show that the L^2-Betti numbers exactly measure the failure of addition formula for dynamical invariants.

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1. Introduction

There are a couple of connections established among invariants in dynamical systems, group rings, and L^2-invariants. These connections are obtained via a type of dynamical system called algebraic actions. Given a discrete group Γ, each $\mathbb{Z}\Gamma$-module $\mathcal{M}$ can be treated as an action of Γ on the discrete abelian group $\mathcal{M}$ by group automorphisms. The Pontryagin dual $\widehat{\mathcal{M}}$ of $\mathcal{M}$ naturally inherits an action of Γ by continuous automorphisms from the module structure of $\mathcal{M}$. Conversely, by Pontryagin duality, each action of Γ on a compact Hausdorff abelian group arise this way and thus we call such a dynamical system an *algebraic action* [37].

A surprising fact is that one can recover certain algebraic information about $\mathcal{M}$ by taking advantage of purely dynamical information about $\Gamma \curvearrowright \widehat{\mathcal{M}}$. However, the dynamical information itself does not use the algebraic structure of $\widehat{\mathcal{M}}$. For example, Li and Thom showed that, in the setting of amenable group actions, the entropy of $\Gamma \curvearrowright \widehat{\mathcal{M}}$ coincides with the L^2-torsion of $\mathcal{M}$ (see [30]). One ingredient of establishing this connection is Peters' algebraic characterization of entropy [34]. This correspondence has interesting applications to the vanishing results on L^2-torsion and Euler characteristic [30, 7]. In the same spirit, Li and the author showed that the mean topological dimension of $\Gamma \curvearrowright \widehat{\mathcal{M}}$ coincides with the von Neumann–Lück rank of $\mathcal{M}$ (see [27]). Establishing this correspondence relies on the study of the mean rank as an algebraic invariant of $\mathbb{Z}\Gamma$-modules and Lück's result on dimension-flatness for amenable groups [33, Theorem 6.73].

Based on this connection, the mean dimension of algebraic actions for amenable groups is well understood [27].

Mean topological dimension is a newly-introduced dynamical invariant by Gromov [14], systematically studied by Lindenstrauss and Weiss [31], and remains to be further explored [8]. As a dynamical analogue of the covering dimension, it is closely related to the topological entropy, and takes a crucial role in embedding problem of dynamical systems [15, 16, 17, 18, 19, 20, 31].

On the other hand, using Lück's extended von Neumann dimension for any module over the group von Neumann algebra $\mathcal{L}\Gamma$ of a discrete group Γ (see [33, Chapter 6]), for any $\mathbb{Z}\Gamma$-module $\mathcal{M}$ we call the von Neumann–Lück dimension for $\mathcal{L}\Gamma \otimes_{\mathbb{Z}\Gamma} \mathcal{M}$ as the *von Neumann–Lück rank* $\mathrm{vrk}(\mathcal{M})$ of $\mathcal{M}$. Von Neumann-Lück dimension is a length function on $\mathcal{L}\Gamma$-modules [33, Theorem 6.7] and von Neumann–Lück rank is a length function on $\mathbb{Z}\Gamma$-modules when Γ is amenable [28, Definition 2.1], [27, Section 5.2], and [29, Theorem 3.3.4].

Mean rank is also a length function on $\mathbb{Z}\Gamma$-modules of an amenable group Γ (see [27, Section 3]). As a dynamical analogue of the rank of abelian groups, it serves as a bridge connecting mean dimension and von Neumann–Lück rank [27, Theorem 1.1].

Towards more general groups, Bowen and Kerr and Li developed an entropy theory based on the idea of approximating the dynamical data by external finite models when the acting group can be approximated by finite groups [2, 25]. The groups admitting this approximation are the so-called *sofic groups*, which include residually finite groups and amenable groups [13, 40]. The extended notion of entropy extends the classic notion but no longer decreases when passing to a factor system. Similarly mean dimension has been extended to the case of sofic group actions [26]. To deal with this nonamenable phenomenon, Li and the author introduced the relative sofic invariants, established an alternative addition formula, and used them to relate mean dimension with von Neumann–Lück rank for sofic groups [28]. Similar approaches also independently appear in the works of other experts. Kerr reformulated the definition of sofic measure entropy via restricting the approximate models of a finite partition to another coarser partition [24]. Hayes gave a formula for this invariant in terms of a given compact model in [22]. A similar notion for Rokhlin entropy, called outer Rokhlin entropy, was developed by Seward in [38]. Using the microstate technique, Hayes proved that von Neumann–Lück rank of a finitely presented $\mathbb{Z}\Gamma$-module $\mathcal{M}$ coincides with sofic mean dimension of $\Gamma \curvearrowright \widehat{\mathcal{M}}$ under certain conditions [21].

Via a projective resolution of any $\mathbb{Z}\Gamma$-module $\mathcal{M}$, we can treat von Neumann–Lück rank of $\mathcal{M}$ as the 0-th L^2-Betti number $\beta_0^{(2)}(\mathcal{M})$ of $\mathcal{M}$ (see Proposition 3.5). From [28, Theorem 1.3], we know the sofic mean dimension $\mathrm{mdim}_\Sigma(\widehat{\mathcal{M}})$ of $\Gamma \curvearrowright \widehat{\mathcal{M}}$ correspondences to $\beta_0^{(2)}(\mathcal{M})$ when Γ is a countable sofic group and $\mathcal{M}$ is countable. Here Σ is a fixed sofic approximation sequence for Γ. For the higher L^2-Betti numbers of $\mathcal{M}$, Hanfeng Li asked the following question.

Question 1.1. If Γ is sofic, what dynamical invariants of $\Gamma \curvearrowright \widehat{\mathcal{M}}$ correspond to the j-th L^2-Betti numbers $\beta_j^{(2)}(\mathcal{M})$ of $\mathcal{M}$ for $j \geq 1$?

In this paper, motivated by the above question, we mainly study dynamical analogues of the L^2-Betti numbers $\beta_j^{(2)}(\mathcal{C}_*)$ of a chain complex $\mathcal{C}_*$ of $\mathbb{Z}\Gamma$-modules:

$$\cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0(= C_{-1}).$$

In the spirit of Elek [11] (also for the notational convenience), we introduce the j-th mean rank $\mathrm{mrk}_j(\mathcal{C}_*)$ of $\mathcal{C}_*$ and the j-th mean dimension $\mathrm{mdim}_j(\mathcal{C}_*)$ of $\widehat{\mathcal{C}_*} := \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}_*, \mathbb{R}/\mathbb{Z})$ for any sofic group Γ (see Definition 3.1). These definitions use the relative sofic invariants as opposed to Elek's approach where he considered the case that Γ is amenable and therefore there is no nonamenable phenomenon appeared.

Let $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright Y$ be two algebraic actions, X and Y be metrizable spaces, and $\pi\colon X \to Y$ be a Γ-equivariant continuous homomorphism. We say π satisfies *Juzvinskiĭ formula for mean dimension* if $\mathrm{mdim}_\Sigma(X) = \mathrm{mdim}_\Sigma(\ker \pi) + \mathrm{mdim}_\Sigma(\mathrm{im}\, \pi)$. The main result of this paper is as follows.

Theorem 1.2. *Suppose that* $\mathrm{vrk}(C_j) < \infty$ *for some* $j \geq 0$. *Then*

(1) $\beta_j^{(2)}(\mathcal{C}_*) = \mathrm{vrk}(\mathrm{coker}\, \partial_{j+1}) - \mathrm{vrk}(\mathrm{im}\, \partial_j | C_{j-1})$.

(2) *If* Γ *is sofic, we have* $\beta_j^{(2)}(\mathcal{C}_*) = \mathrm{mrk}_j(\mathcal{C}_*)$. *If furthermore* C_j *and* C_{j-1} *are countable, we have* $\mathrm{mrk}_j(\mathcal{C}_*) = \mathrm{mdim}_j(\widehat{\mathcal{C}_*})$.

(3) *If* $\mathrm{im}\, \partial_{j+1} = \ker \partial_j$ *and* $\mathrm{vrk}(C_{j-1}) < \infty$, *we have that* $\beta_j^{(2)}(\mathcal{C}_*) = 0$ *if and only if*

$$\mathrm{vrk}(C_{j-1}) = \mathrm{vrk}(\mathrm{coker}\, \partial_j) + \mathrm{vrk}(\mathrm{im}\, \partial_j).$$

If furthermore Γ *is sofic,* C_j *and* C_{j-1} *are countable, we have that* $\beta_j^{(2)}(\mathcal{C}_*) = 0$ *if and only if* $\widehat{\partial_j}$ *satisfies Juzvinskiĭ formula for mean dimension.*

From Theorem 1.2, the notion of j-th mean rank provides an equivalent algebraic definition of L^2-Betti numbers from module theory. Secondly, the L^2-Betti numbers exactly measure the failure of the additivity of dynamical invariants. In [11], Elek introduced an analogue of the L^2-Betti numbers for amenable linear subshifts. It was shown that Juzvinskiĭ formula for entropy can fail when the group Γ has nonzero Euler characteristic [10]. Hayes proved that Juzvinskiĭ formula for entropy fails when Γ has nonzero L^2-torsion [23]. Gaboriau and Seward established some inequalities relating Juzvinskiĭ formula for entropy with L^2-Betti numbers [12]. Bowen and Gutman established Juzvinskiĭ formula for the f-invariant of finitely generated free group actions in some cases [3].

To respond to Question 1.1, we introduce the j-th mean dimension $\mathrm{mdim}_j(\widehat{\mathcal{M}})$ of $\Gamma \curvearrowright \widehat{\mathcal{M}}$ and j-th mean rank $\mathrm{mrk}_j(\mathcal{M})$ of $\mathcal{M}$ (Definition 3.3 and Definition 3.1). As the first application, the following corollary may shed some light on Question 1.1.

Corollary 1.3. *When* $\mathrm{mrk}_j(\mathcal{M})$ *is defined, we have* $\mathrm{mrk}_j(\mathcal{M}) = \beta_j^{(2)}(\mathcal{M})$. *If furthermore* $\mathcal{M}$ *is countable, we have* $\mathrm{mdim}_j(\widehat{\mathcal{M}}) = \beta_j^{(2)}(\mathcal{M})$.

For the second application, we give a dynamical characterization of Lück's dimension-flatness. We say Γ satisfies *Lück's dimension-flatness over* $\mathbb{Z}$ if $\beta_j^{(2)}(\mathcal{M})$ vanishes for any $j \geq 1$ and $\mathbb{Z}\Gamma$-module $\mathcal{M}$. It was proven that amenable groups satisfy Lück's dimension-flatness [33, Theorem 6.37]. We say Γ satisfies *Juzvinskiĭ formula for von Neumann–Lück rank* if $\mathrm{vrk}(\mathcal{M}) = \mathrm{vrk}(\ker\varphi) + \mathrm{vrk}(\mathrm{im}\,\varphi)$ for any $\mathbb{Z}\Gamma$-module homomorphism $\varphi\colon \mathcal{M} \to \mathcal{N}$ of $\mathbb{Z}\Gamma$-modules $\mathcal{M}$ and $\mathcal{N}$. It is similarly defined when we talk about whether Γ satisfies Juzvinskiĭ formula for mean rank.

Corollary 1.4. Γ *satisfies Lück's dimension-flatness over* $\mathbb{Z}$ *if and only if* Γ *satisfies Juzvinskiĭ formula for von Neumann–Lück rank. If* Γ *is sofic, then* Γ *satisfies Lück's dimension-flatness over* $\mathbb{Z}$ *if and only if* Γ *satisfies Juzvinskiĭ formula for mean rank and mean dimension.*

We remark that the first statement of the above corollary can also be proved using standard properties of Tor functor and additivity of von Neumann–Lück dimension. In the light of results on the failure of Juzvinskiĭ formula [21, Proposition 7.2], [23, Corollary 6.24], and [12, Theorem 6.3], we show that taking subgroups respects the property of Lück's dimension-flatness in Proposition 4.4. As a consequence, if $\beta_j^{(2)}(H) > 0$ for some subgroup H of Γ and some $j \geq 1$, then Γ violoates Juzvinskiĭ formula for mean dimension.

Lück conjectured that a group is amenable if and only if it satisfies *Lück's dimension-flatness* [33, Conjecture 6.48]. Bartholdi and Kielak implicitly proved this conjecture using a new characterization of amenability [1, Theorem 1.1]. It follows that

Corollary 1.5. *A countable group is amenable if and only if it satisfies Juzvinskiĭ formula for von Neumann–Lück rank.*

This paper is organized as follows. We recall some background knowledge in Section 2. In Section 3 we introduce the j-th mean rank, j-th mean dimension, and establish some basic properties. We prove the main results and show some applications in Section 4.

Throughout this paper, Γ will be a countable discrete group. For any set S, we denote by $\mathcal{F}(S)$ the set of all nonempty finite subsets of S. All modules are assumed to be left modules unless specified. For any $d \in \mathbb{N}$, we write $[d]$ for the set $\{1, \cdots, d\}$ and $\mathrm{Sym}(d)$ for the permutation group of $[d]$.

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2. Preliminaries

2.1. Group rings. The *integral group ring of* Γ, denoted by $\mathbb{Z}\Gamma$, consists of all finitely supported functions $f\colon \Gamma \to \mathbb{Z}$. We shall write f as $\sum_{s\in\Gamma} f_s s$, where $f_s \in \mathbb{Z}$ for all $s \in \Gamma$ and $f_s = 0$ for all except finitely many $s \in \Gamma$. The algebraic operations on $\mathbb{Z}\Gamma$ are defined by

$$\sum_{s\in\Gamma} f_s s + \sum_{s\in\Gamma} g_s s = \sum_{s\in\Gamma} (f_s + g_s)s, \quad \text{and} \quad \Big(\sum_{s\in\Gamma} f_s s\Big)\Big(\sum_{t\in\Gamma} g_t t\Big) = \sum_{s,t\in\Gamma} f_s g_t (st).$$

We similarly have the product if one of f and g sits in $\mathbb{C}^\Gamma$.

For any countable $\mathbb{Z}\Gamma$-module $\mathcal{M}$, treated as a discrete abelian group, its Pontryagin dual $\widehat{\mathcal{M}}$ consisting of all continuous group homomorphisms $\mathcal{M} \to \mathbb{R}/\mathbb{Z}$, coincides with $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}, \mathbb{R}/\mathbb{Z})$. By Pontryagin duality, $\widehat{\mathcal{M}}$ is a compact metrizable space under compact-open topology. Furthermore, the $\mathbb{Z}\Gamma$-module structure of $\mathcal{M}$ naturally induces an adjoint action $\Gamma \curvearrowright \widehat{\mathcal{M}}$ by continuous automorphisms. To be precise,

$$\langle s\chi, u\rangle := \langle \chi, s^{-1}u\rangle$$

for all $\chi \in \widehat{\mathcal{M}}, u \in \mathcal{M}$, and $s \in \Gamma$.

2.2. Relative von Neuman-Lück rank . Let $\ell^2(\Gamma)$ be the Hilbert space of square summable functions $f\colon \Gamma \to \mathbb{C}$, i.e. $\sum_{s\in\Gamma} |f_s|^2 < +\infty$. Then Γ has two canonical commuting unitary representations on $\ell^2(\Gamma)$, namely the *left regular representation* λ and the *right regular representation* ρ defined by

$$\lambda(s)(x) = sx \quad \text{and} \quad \rho(s)(x) = xs^{-1}$$

for all $x \in \ell^2(\Gamma)$ and $s \in \Gamma$. Here we treat Γ as a subset of $\mathbb{C}\Gamma$. The (*left*) *group von Neumann algebra* of Γ, denoted by $\mathcal{L}\Gamma$, consists of all bounded linear operators $\ell^2(\Gamma) \to \ell^2(\Gamma)$ commuting with $\rho(s)$ for all $s \in \Gamma$.

Denote by δ_{e_Γ} the unit vector of $\ell^2(\Gamma)$ being 1 at the identity element e_Γ of Γ, and 0 everywhere else. The canonical *trace* on $\mathcal{L}\Gamma$ is the linear functional $\mathrm{tr}_{\mathcal{L}\Gamma}\colon \mathcal{L}\Gamma \to \mathbb{C}$ given by $\mathrm{tr}_{\mathcal{L}\Gamma}(T) = \langle T\delta_{e_\Gamma}, \delta_{e_\Gamma}\rangle$. For each $n \in \mathbb{N}$, the extension of $\mathrm{tr}_{\mathcal{L}\Gamma}$ to $M_n(\mathcal{L}\Gamma)$ sending $(T_{j,k})_{1\le j,k\le n}$ to $\sum_{j=1}^n \mathrm{tr}_{\mathcal{L}\Gamma}(T_{j,j})$ will still be denoted by $\mathrm{tr}_{\mathcal{L}\Gamma}$.

For any finitely generated projective $\mathcal{L}\Gamma$-module $\mathbb{P}$, one has $\mathbb{P} \cong (\mathcal{L}\Gamma)^{1\times n}P$ for some $n \in \mathbb{N}$ and some $P \in M_n(\mathcal{L}\Gamma)$ with $P^2 = P$. The *von Neumann dimension* of $\mathbb{P}$ is defined as

$$\dim'_{\mathcal{L}\Gamma}(\mathbb{P}) := \mathrm{tr}_{\mathcal{L}\Gamma}(P) \in [0, n],$$

which does not depend on the choice of n and P. For an arbitrary $\mathcal{L}\Gamma$-module $\mathbb{M}$, its *von Neumann–Lück dimension* [33, Definition 6.6] is defined as

$$\dim_{\mathcal{L}\Gamma}(\mathbb{M}) := \sup_{\mathbb{P}} \dim'_{\mathcal{L}\Gamma}(\mathbb{P}),$$

for $\mathbb{P}$ ranging over all finitely generated projective $\mathcal{L}\Gamma$-submodules of $\mathbb{M}$.

The following theorem collects the fundamental properties of the von Neumann–Lück dimension [33, Theorem 6.7]. Given a unital ring R, a length function on left $R\Gamma$-modules is a function on left $R\Gamma$-modules satisfying certain conditions ([28, Definition 2.1]).

Theorem 2.1. $\dim_{\mathcal{L}\Gamma}$ *extends* $\dim'_{\mathcal{L}\Gamma}$ *and is a length function on* $\mathcal{L}\Gamma$*-modules with* $\dim_{\mathcal{L}\Gamma}(\mathcal{L}\Gamma) = 1$.

Definition 2.2. For any $\mathbb{Z}\Gamma$-modules $\mathcal{M}_1 \subseteq \mathcal{M}_2$, the *von Neumann–Lück rank of* $\mathcal{M}_1$ *relative to* $\mathcal{M}_2$ is defined as

$$\mathrm{vrk}(\mathcal{M}_1|\mathcal{M}_2) := \dim_{\mathcal{L}\Gamma}(\mathrm{im}\, 1 \otimes i),$$

where $1 \otimes i$ is the natural map $\mathcal{L}\Gamma \otimes \mathcal{M}_1 \to \mathcal{L}\Gamma \otimes \mathcal{M}_2$.

Note that when $\mathcal{M}_1 = \mathcal{M}_2$, we have $\mathrm{vrk}(\mathcal{M}_1) = \mathrm{vrk}(\mathcal{M}_1|\mathcal{M}_2)$.

2.3. Amenable and sofic groups . The group Γ is called *amenable* if for any $K \in \mathcal{F}(\Gamma)$ and any $\delta > 0$ there is a $F \in \mathcal{F}(\Gamma)$ with $|KF \setminus F| < \delta|F|$.

A sequence of maps $\Sigma = \{\sigma_i : \Gamma \to \mathrm{Sym}(d_i)\}_{i\in\mathbb{N}}$ is called a *sofic approximation* for Γ if it satisfies:

(1) $\lim_{i\to\infty} |\{v \in [d_i] : \sigma_{i,s}\sigma_{i,t}(v) = \sigma_{i,st}(v)\}|/d_i = 1$ for all $s, t \in \Gamma$,
(2) $\lim_{i\to\infty} |\{v \in [d_i] : \sigma_{i,s}(v) \neq \sigma_{i,t}(v)\}|/d_i = 1$ for all distinct $s, t \in \Gamma$,
(3) $\lim_{i\to\infty} d_i = +\infty$.

The group Γ is called a *sofic group* if it admits a sofic approximation.

Any amenable group is sofic since one can use a sequence of asymptotically-invariant subsets of the amenable group, i.e. *Følner sequence*, to construct a sofic approximation. Residually finite groups are also sofic since a sequence of exhausting finite-index subgroups naturally induces a sofic approximation in which each approximating map is actually a group homomorphism. We refer the reader to [5, 6] for more information on sofic groups.

Throughout the rest of this paper, Γ will be a countable sofic group, and $\Sigma = \{\sigma_i : \Gamma \to \mathrm{Sym}(d_i)\}_{i\in\mathbb{N}}$ will be a sofic approximation for Γ.

2.4. Relative mean dimension and relative mean rank. We first recall the notion of the covering dimension. For any finite open cover $\mathcal{U}$ of a compact metrizable space Z, denote the overlapping number of $\mathcal{U}$ by $\mathrm{ord}(\mathcal{U})$, i.e. $\mathrm{ord}(\mathcal{U}) = \max_{x\in X}\sum_{U\in\mathcal{U}} 1_U(x) - 1$. Set

$$\mathcal{D}(\mathcal{U}) = \inf_{\mathcal{V}} \mathrm{ord}(\mathcal{V})$$

for $\mathcal{V}$ ranging over all finite open covers of Z finer than $\mathcal{U}$, i.e. each element of $\mathcal{V}$ is contained in some element of $\mathcal{U}$. Then the *covering dimension* of Z is defined as $\sup_{\mathcal{U}} \mathcal{D}(\mathcal{U})$ for $\mathcal{U}$ ranging over all finite open covers of Z.

Let Γ act continuously on a compact metrizable space X.

Definition 2.3. Let ρ be a compatible metric on X. For any $d \in \mathbb{N}$, there is a compatible metric on X^d defined by

$$\rho_2(\varphi,\psi) = \Big(\frac{1}{d}\sum_{v\in[d]} \rho(\varphi_v,\psi_v)^2\Big)^{1/2}.$$

Let σ be a map from Γ to $\mathrm{Sym}(d)$, $F \in \mathcal{F}(\Gamma)$, and $\delta > 0$. The set of approximately equivariant maps $\mathrm{Map}(\rho, F, \delta, \sigma)$ is defined to be the set of all maps $\varphi\colon [d] \to X$ such that $\rho_2(s\varphi, \varphi\circ\sigma(s)) \le \delta$ for all $s \in F$.

Now let Γ act on another compact metrizable space Y and $\pi\colon X \to Y$ be a surjective Γ-equivariant continuous map. Denote by $\mathrm{Map}(\pi,\rho,F,\delta,\sigma)$ the set of all $\pi\circ\varphi$ for φ ranging in $\mathrm{Map}(\rho,F,\delta,\sigma)$. Note that $\mathrm{Map}(\pi,\rho,F,\delta,\sigma)$ is a closed subset of Y^d. For any finite open cover $\mathcal{U}$ of Y, denote by $\mathcal{U}^d$ the open cover of Y^d consisting of $\Pi_{v\in[d]}U_v$, where each U_v sits in $\mathcal{U}$. Restricting $\mathcal{U}^d$ to $\mathrm{Map}(\pi,\rho,F,\delta,\sigma)$, we obtain a finite open cover $\mathcal{U}^d|_{\mathrm{Map}(\pi,\rho,F,\delta,\sigma)} := \{U \cap \mathrm{Map}(\pi,\rho,F,\delta,\sigma)\}_{U\in\mathcal{U}^d}$ of $\mathrm{Map}(\pi,\rho,F,\delta,\sigma)$.

Definition 2.4. For any finite open cover $\mathcal{U}$ of Y we define

$$\mathrm{mdim}_\Sigma(\pi,\mathcal{U},\rho,F,\delta) = \varlimsup_{i\to\infty} \frac{\mathcal{D}(\mathcal{U}^d|_{\mathrm{Map}(\pi,\rho,F,\delta,\sigma_i)})}{d_i}.$$

If $\mathrm{Map}(\rho,F,\delta,\sigma_i)$ is empty for all sufficiently large i, we set

$$\mathrm{mdim}_\Sigma(\pi,\mathcal{U},\rho,F,\delta) = -\infty.$$

We define the *mean topological dimension of* $\Gamma \curvearrowright Y$ *relative to the extension* $\Gamma \curvearrowright X$ as

$$\mathrm{mdim}_\Sigma(Y|X) := \sup_{\mathcal{U}} \inf_{F\in\mathcal{F}(\Gamma)} \inf_{\delta>0} \mathrm{mdim}_\Sigma(\pi,\mathcal{U},\rho,F,\delta),$$

where $\mathcal{U}$ ranges over finite open covers of Y. By a similar argument as in [26, Lemma 2.9], we know $\mathrm{mdim}_\Sigma(Y|X)$ does not depend on the choice of ρ. The *sofc mean topological dimension of* $\Gamma \curvearrowright X$ is defined as

$$\mathrm{mdim}_\Sigma(X) := \mathrm{mdim}_\Sigma(X|X)$$

for $\pi: X \to X$ being the identity map.

Example 2.5. Let $\mathcal{M}_1 \subseteq \mathcal{M}_2$ be countable $\mathbb{Z}\Gamma$-modules. Then the induced map $\widehat{\mathcal{M}_2} \to \widehat{\mathcal{M}_1}$ is a surjective Γ-equivariant continuous map of compact metrizable spaces. Thus $\mathrm{mdim}_\Sigma(\widehat{\mathcal{M}_1}|\widehat{\mathcal{M}_2})$ is well-defined.

Now we recall the notion of the relative mean rank. For any $\mathbb{Z}\Gamma$-module $\mathcal{M}$, denote by $\mathscr{F}(\mathcal{M})$ the set of finitely generated abelian subgroups of $\mathcal{M}$. Let $\mathscr{A}, \mathscr{B} \in \mathscr{F}(\mathcal{M})$, $F \in \mathcal{F}(\Gamma)$, and σ be a map from Γ to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$. Denote by $\mathscr{M}(\mathscr{A}, \mathscr{B}, F, \sigma)$ the image of $\mathscr{A}^d$ in $\mathcal{M}^d/\mathscr{M}(\mathscr{B}, F, \sigma)$ under the quotient map $\mathcal{M}^d \to \mathcal{M}^d/\mathscr{M}(\mathscr{B}, F, \sigma)$. Here $\mathscr{M}(\mathscr{B}, F, \sigma)$ denotes the abelian subgroup of $\mathcal{M}^d \cong \mathbb{Z}^d \otimes_{\mathbb{Z}} \mathcal{M}$ generated by the elements $\delta_v \otimes b - \delta_{sv} \otimes sb$ for all $v \in [d], b \in \mathscr{B}$, and $s \in F$.

Definition 2.6. Let $\mathcal{M}_1 \subseteq \mathcal{M}_2$ be $\mathbb{Z}\Gamma$-modules. For any $\mathscr{A} \in \mathscr{F}(\mathcal{M}_1), \mathscr{B} \in \mathscr{F}(\mathcal{M}_2)$, and $F \in \mathcal{F}(\Gamma)$, set

$$\mathrm{mrk}_\Sigma(\mathscr{A}|\mathscr{B}, F) = \overline{\lim_{i\to\infty}} \frac{\mathrm{rk}(\mathscr{M}(\mathscr{A}, \mathscr{B}, F, \sigma_i))}{d_i}.$$

We define the *mean rank of* $\mathcal{M}_1$ *relative to* $\mathcal{M}_2$ as

$$\mathrm{mrk}_\Sigma(\mathcal{M}_1|\mathcal{M}_2) = \sup_{\mathscr{A}\in\mathscr{F}(\mathcal{M}_1)} \inf_{F\in\mathcal{F}(\Gamma)} \inf_{\mathscr{B}\in\mathscr{F}(\mathcal{M}_2)} \mathrm{mrk}_\Sigma(\mathscr{A}|\mathscr{B}, F).$$

The *sofic mean rank* of $\mathcal{M}_1$ is then defined as

$$\mathrm{mrk}_\Sigma(\mathcal{M}_1) := \mathrm{mrk}_\Sigma(\mathcal{M}_1|\mathcal{M}_1).$$

Applying [28, Theorem 1.1],[28, Theorem 7.2], [28, Theorem 10.1], and running a similar argument as in the proof of [28, Proposition 8.5] for relative mean rank, we have:

Theorem 2.7. *For any* $\mathbb{Z}\Gamma$*-modules* $\mathcal{M}_1 \subseteq \mathcal{M}_2$*, we have* $\mathrm{mrk}_\Sigma(\mathcal{M}_1|\mathcal{M}_2) = \mathrm{vrk}(\mathcal{M}_1|\mathcal{M}_2)$ *and*

$$\mathrm{mrk}_\Sigma(\mathcal{M}_2) = \mathrm{mrk}_\Sigma(\mathcal{M}_1|\mathcal{M}_2) + \mathrm{mrk}_\Sigma(\mathcal{M}_2/\mathcal{M}_1).$$

If furthermore $\mathcal{M}_2$ *is countable, we have* $\mathrm{mdim}_\Sigma(\widehat{\mathcal{M}_1}|\widehat{\mathcal{M}_2}) = \mathrm{mrk}_\Sigma(\mathcal{M}_1|\mathcal{M}_2)$.

The following proposition collects basic properties of the sofic mean rank [28, Section 3].

Proposition 2.8. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be $\mathbb{Z}\Gamma$-modules. The following are true.*

(1) $\mathrm{mrk}_\Sigma(\mathbb{Z}\Gamma) = 1$.

(2) *We have*

$$\mathrm{mrk}_\Sigma(\mathcal{M}_1|\mathcal{M}_1 \oplus \mathcal{M}_2) = \mathrm{mrk}_\Sigma(\mathcal{M}_1)$$

and

$$\mathrm{mrk}_\Sigma(\mathcal{M}_1 \oplus \mathcal{M}_2) = \mathrm{mrk}_\Sigma(\mathcal{M}_1) + \mathrm{mrk}_\Sigma(\mathcal{M}_2);$$

(3) *if $\mathcal{M}_1 \subseteq \mathcal{M}_2$ and $\mathcal{M}_1$ is the union of an increasing net of $\mathbb{Z}\Gamma$-submodules $\{\mathcal{M}'_j\}_{j\in\mathcal{J}}$, then*

$$\mathrm{mrk}_\Sigma(\mathcal{M}'_j|\mathcal{M}_2) \nearrow \mathrm{mrk}_\Sigma(\mathcal{M}_1|\mathcal{M}_2).$$

If furthermore $\mathrm{mrk}_\Sigma(\mathcal{M}_2) < \infty$, then

$$\mathrm{mrk}_\Sigma(\mathcal{M}_2/\mathcal{M}'_j) \searrow \mathrm{mrk}_\Sigma(\mathcal{M}_2/\mathcal{M}_1).$$

(4) *Assume that $\mathcal{M}_1 \subseteq \mathcal{M}_2$, $\mathcal{M}_1$ is finitely generated, and $\mathcal{M}_2$ is the union of an increasing net of $\mathbb{Z}\Gamma$-submodules $\{\mathcal{M}'_j\}_{j\in\mathcal{J}}$ of $\mathcal{M}_2$ containing $\mathcal{M}_1$. Then*

$$\mathrm{mrk}_\Sigma(\mathcal{M}_1|\mathcal{M}'_j) \searrow \mathrm{mrk}_\Sigma(\mathcal{M}_1|\mathcal{M}_2).$$

3. L^2-Betti number, j-th mean rank, and j-th mean dimension

Let $\mathcal{C}_*$ be a chain complex of $\mathbb{Z}\Gamma$-modules:

$$\cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0 (= C_{-1}).$$

Applying the covariant tensor functor $\mathcal{L}\Gamma \otimes_{\mathbb{Z}\Gamma} \cdot$, we get a chain complex $\mathcal{L}\Gamma \otimes_{\mathbb{Z}\Gamma} \mathcal{C}_*$ of $\mathcal{L}\Gamma$-modules:

$$\cdots \xrightarrow{1\otimes\partial_2} \mathcal{L}\Gamma \otimes C_1 \xrightarrow{1\otimes\partial_1} \mathcal{L}\Gamma \otimes C_0 \to 0;$$

applying the contravariant Pontryagin dual functor $\mathrm{Hom}_{\mathbb{Z}}(\cdot, \mathbb{R}/\mathbb{Z}) := \widehat{\cdot}$, we get a chain complex $\widehat{\mathcal{C}_*}$ of algebraic actions such that the maps $\{\widehat{\partial_j}\}_j$ are Γ-equivariant:

$$\cdots \xleftarrow{\widehat{\partial_2}} \widehat{C_1} \xleftarrow{\widehat{\partial_1}} \widehat{C_0} \longleftarrow 0.$$

Since $\mathbb{R}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, when $\mathrm{im}\,\partial_{j+1} = \ker \partial_j$, we have $\ker \widehat{\partial_{j+1}} = \mathrm{im}\,\widehat{\partial_j}$.

Definition 3.1. For each $j \geq 0$, the *j-th L^2-Betti number of* $\mathcal{C}_*$ is defined as

$$\beta_j^{(2)}(\mathcal{C}_*) = \dim_{\mathcal{L}\Gamma} H_j(\mathcal{L}\Gamma \otimes_{\mathbb{Z}\Gamma} \mathcal{C}_*).$$

If $\mathrm{vrk}(C_j) < \infty$ for all $j \geq 1$ and $C_j = 0$ as j is large enough, we define the *Euler characteristic* of $\mathcal{C}_*$ as

$$\chi(\mathcal{C}_*) := \sum_{j \geq 0} (-1)^j \,\mathrm{vrk}(C_j).$$

When $\mathrm{vrk}(C_j) < \infty$ for some $j \geq 0$ and Γ is sofic, we define the *j-th mean rank of* $\mathcal{C}_*$ as

$$\mathrm{mrk}_j(\mathcal{C}_*) = \mathrm{mrk}_\Sigma(\mathrm{coker}\,\partial_{j+1}) - \mathrm{mrk}_\Sigma(\mathrm{im}\,\partial_j | C_{j-1}).$$

If furthermore C_j and C_{j-1} are countable, we define the *j-th mean topological dimension of* $\mathcal{C}_*$ as

$$\mathrm{mdim}_j(\widehat{\mathcal{C}_*}) := \mathrm{mdim}_\Sigma(\ker \widehat{\partial_{j+1}}) - \mathrm{mdim}_\Sigma(\mathrm{im}\,\widehat{\partial_j} | \widehat{C_{j-1}}).$$

Remark 3.2. (1) When $\mathcal{C}_*$ is a chain complex of $\mathbb{C}\Gamma$-modules, since $\mathbb{C}\Gamma$ is flat as a $\mathbb{Z}\Gamma$-module, we have $\beta_j^{(2)}(\mathcal{C}_*) = \dim_{\mathcal{L}\Gamma} H_j(\mathcal{L}\Gamma \otimes_{\mathbb{C}\Gamma} \mathcal{C}_*)$, which extends the definition of L^2-Betti numbers for chain complexes of $\mathbb{C}\Gamma$-modules [33, Definition1.16, Theorem 6.24].

(2) By Theorem 2.7, we know that the j-th mean rank and j-th mean dimension are well defined.

(3) Wall gave some criteria when a chain complex of $\mathbb{Z}\Gamma$-modules can be realized as the chain complex of a Γ-CW complex [39, Theorem 2].

Let $\mathcal{M}$ be a $\mathbb{Z}\Gamma$-module. A projective resolution of $\mathcal{M}$ is an exact sequence of $\mathbb{Z}\Gamma$-modules

$$\cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} \mathcal{M} \longrightarrow 0$$

in which each C_j is a projective $\mathbb{Z}\Gamma$-module. Denote by $\mathcal{C}_*$ its deleted projective resolution

$$\cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0,$$

which is a chain complex of $\mathbb{Z}\Gamma$-modules. We similarly have the notion of free resolution. Apply the notion of free module, we know that any $\mathbb{Z}\Gamma$-module admits a free resolution [36, Proposition 10.32].

Definition 3.3. For each $j \geq 0$, we define the *j-th L^2-Betti number of* $\mathcal{M}$ as

$$\beta_j^{(2)}(\mathcal{M}) := \beta_j^{(2)}(\mathcal{C}_*).$$

The *j-th L^2-Betti number* $\beta_j^{(2)}(\Gamma)$ *of* Γ is defined as the j-th L^2-Betti number of the trivial $\mathbb{Z}\Gamma$-module $\mathbb{Z}$.

If $\mathrm{vrk}(C_j) < \infty$ for all $j \geq 1$ and $C_j = 0$ as j is large enough, we define the *Euler characteristic* of $\mathcal{M}$ as

$$\chi(\mathcal{M}) := \chi(\mathcal{C}_*).$$

When $\mathrm{vrk}(C_j) < \infty$ for some $j \geq 0$ and Γ is sofic, we define the *j-th mean rank of* $\mathcal{M}$ as

$$\mathrm{mrk}_j(\mathcal{M}) := \mathrm{mrk}_j(\mathcal{C}_*).$$

If furthermore $\mathcal{M}$ is countable, we can choose $\mathcal{C}_*$ such that each C_j for $j \geq 0$ is countable and define the *j-th mean topological dimension of* $\mathcal{M}$ as

$$\mathrm{mdim}_j(\widehat{\mathcal{M}}) := \mathrm{mdim}_j(\widehat{\mathcal{C}_*}).$$

Remark 3.4. $\beta_j^{(2)}(\mathcal{M})$ is the von Neumann–Lück dimension of $\mathrm{tor}_j^{\mathbb{Z}\Gamma}(\mathcal{L}\Gamma, \mathcal{M})$ (see [36, p. 836] for definition). Based on the Comparison Theorem for projective resolutions [36, Theorem 10.46], any two projective resolutions of $\mathcal{M}$ are homotopy equivalent, we know that $\beta_j^{(2)}(\mathcal{M})$ does not depend on the choice of projective resolutions [36, Corollary 10.51]. We refer the reader to [4, Chapter VIII] for discussions on when a $\mathbb{Z}\Gamma$-module admits a "small" projective resolution.

Proposition 3.5. *For any $\mathbb{Z}\Gamma$-module $\mathcal{M}$, we have* $\beta_0^{(2)}(\mathcal{M}) = \mathrm{vrk}(\mathcal{M})$ *and* $\mathrm{mrk}_0(\mathcal{M}) = \mathrm{mrk}_\Sigma(\mathcal{M})$. *When $\mathcal{M}$ is countable, we have* $\mathrm{mdim}_0(\widehat{\mathcal{M}}) = \mathrm{mdim}_\Sigma(\widehat{\mathcal{M}})$.

Proof. From the exactness, we have

$$\mathcal{L}\Gamma \otimes C_0/\operatorname{im} 1 \otimes \partial_1 = \mathcal{L}\Gamma \otimes C_0/\ker 1 \otimes \partial_0 \cong \operatorname{im} 1 \otimes \partial_0 = \mathcal{L}\Gamma \otimes \mathcal{M}$$

and

$$C_0/\operatorname{im} \partial_1 = C_0/\ker \partial_0 \cong \operatorname{im} \partial_0 = \mathcal{M}.$$

So by definition, we have

$$\beta_0^{(2)}(\mathcal{M}) = \dim_{\mathcal{L}\Gamma}(\mathcal{L}\Gamma \otimes C_0/\operatorname{im} 1 \otimes \partial_1) = \dim_{\mathcal{L}\Gamma}(\mathcal{L}\Gamma \otimes \mathcal{M}) = \mathrm{vrk}(\mathcal{M})$$

and

$$\mathrm{mrk}_0(\mathcal{M}) = \mathrm{mrk}_\Sigma(\operatorname{coker} \partial_1) = \mathrm{mrk}_\Sigma(C_0/\operatorname{im} \partial_1) = \mathrm{mrk}_\Sigma(\mathcal{M}).$$

From the exactness, we have $\ker \widehat{\partial_1} = \operatorname{im} \widehat{\partial_0} \cong \widehat{\mathcal{M}}$. So when $\mathcal{M}$ is countable, we have

$$\mathrm{mdim}_0(\widehat{\mathcal{M}}) = \mathrm{mdim}_\Sigma(\ker \widehat{\partial_1}) = \mathrm{mdim}_\Sigma(\widehat{\mathcal{M}}). \qquad \square$$

Example 3.6. Let $\Gamma = \mathbb{F}_2$ be the free group with generators a and b. Set $f = (a-1, b-1)^T \in (\mathbb{Z}\Gamma)^{2\times 1}$. Then $\mathcal{M} := (\mathbb{Z}\Gamma)^{1\times 1}/(\mathbb{Z}\Gamma)^{1\times 2} f$ has the following free resolution:

$$0 \longrightarrow (\mathbb{Z}\Gamma)^{1\times 2} \xrightarrow{R(f)} (\mathbb{Z}\Gamma)^{1\times 1} \longrightarrow \mathcal{M} \longrightarrow 0,$$

where $R(f)$ sends x to xf. Note that $\mathcal{M} \cong \mathbb{Z}$ as the $\mathbb{Z}\Gamma$-modules for the trivial $\mathbb{Z}\Gamma$-module $\mathbb{Z}$. By [33, Lemma 6.36], we know $\mathcal{L}\Gamma \otimes_{\mathbb{Z}\Gamma} \mathbb{Z} = 0$. Thus $\beta_0^{(2)}(\mathcal{M}) = 0$. It follows that $\beta_1^{(2)}(\mathcal{M}) = \dim_{\mathcal{L}\Gamma} \ker 1 \otimes R(f) = \beta_0^{(2)}(\mathcal{M}) - \chi(\mathcal{M}) = 0 - (1-2) = 1$. This example is essentially the same as $0 \to \ker\, \varepsilon \to \mathbb{Z}\Gamma \xrightarrow{\varepsilon} \mathbb{Z} \to 0$, where ε is the argumentation map. See [9, Chapter IV, Theorem 2.12] for a characterization of when $\ker \varepsilon$ is a projective $\mathbb{Z}\Gamma$-module.

4. Main Results

Corollary 1.3 follows when we apply Theorem 1.2 to a deleted projective resolution of a $\mathbb{Z}\Gamma$-module $\mathcal{M}$.

Proof of Theorem 1.2. (1) Note that for any $\mathbb{Z}\Gamma$-module homomorphism $\varphi\colon \mathcal{M} \to \mathcal{N}$ and the inclusion map $i\colon \operatorname{im}\varphi \to \mathcal{N}$, we have $\operatorname{im} 1 \otimes \varphi = \operatorname{im} 1 \otimes i$. Thus $\dim_{\mathcal{L}\Gamma}(\operatorname{im} 1 \otimes \varphi) = \operatorname{vrk}(\operatorname{im}\varphi | \mathcal{N})$. For $\mathbb{Z}\Gamma$-modules $\operatorname{im} \partial_{j+1} \subseteq C_j$, we have

$$\operatorname{vrk}(C_j) = \operatorname{vrk}(\operatorname{im} \partial_{j+1} | C_j) + \operatorname{vrk}(\operatorname{coker} \partial_{j+1}).$$

Since the function $\dim_{\mathcal{L}\Gamma}(\cdot)$ is additive, we have

$$\begin{aligned}
\beta_j^{(2)}(\mathcal{C}_*) &= \dim_{\mathcal{L}\Gamma}(\ker 1 \otimes \partial_j) - \dim_{\mathcal{L}\Gamma}(\operatorname{im} 1 \otimes \partial_{j+1}) \\
&= (\dim_{\mathcal{L}\Gamma}(\mathcal{L}\Gamma \otimes C_j) - \dim_{\mathcal{L}\Gamma}(\operatorname{im} 1 \otimes \partial_j)) - \dim_{\mathcal{L}\Gamma}(\operatorname{im} 1 \otimes \partial_{j+1}) \\
&= (\operatorname{vrk}(\operatorname{im} \partial_{j+1} | C_j) + \operatorname{vrk}(\operatorname{coker} \partial_{j+1})) \\
&\quad - \dim_{\mathcal{L}\Gamma}(\operatorname{im} 1 \otimes \partial_j) - \dim_{\mathcal{L}\Gamma}(\operatorname{im} 1 \otimes \partial_{j+1}) \\
&= \operatorname{vrk}(\operatorname{coker} \partial_{j+1}) - \operatorname{vrk}(\operatorname{im} \partial_j | C_{j-1}).
\end{aligned}$$

(2) For any subgroup H of a discrete abelian group G, denote by $H^\perp$ the elements of $\widehat{G}$ which vanish on H. By Pontryagin duality, we have $H^\perp \cong \widehat{G/H}$. Thus

$$\ker \widehat{\partial_{j+1}} = \{\chi \in \widehat{C_j} \colon \chi \circ \partial_{j+1} = 0\} = (\operatorname{im} \partial_{j+1})^\perp \cong \widehat{C_j / \operatorname{im} \partial_{j+1}} = \widehat{\operatorname{coker} \partial_{j+1}}.$$

By definition, we have $\operatorname{mdim}_\Sigma(\operatorname{im} \widehat{\partial_j} | \widehat{C_{j-1}}) = \operatorname{mdim}_\Sigma(\widehat{\operatorname{im} \partial_j} | \widehat{C_{j-1}})$. Thus by Theorem 2.7, the equalities follow from (1).

(3) By the addition formula of von Neumann–Lück dimension, we first have

$$\mathrm{vrk}(C_{j-1}) = \mathrm{vrk}(\mathrm{im}\,\partial_j | C_{j-1}) + \mathrm{vrk}(\mathrm{coker}\,\partial_j).$$

When $\mathcal{C}_*$ is exact at C_j, we have $\mathrm{coker}\,\partial_{j+1} = \mathrm{im}\,\partial_j$. Thus the first statement follows from (1). The second statement follows from Pontryagin duality and the first statement. □

The follow proposition is an immediate consequence of Theorem 1.2, which can also be proved directly.

Proposition 4.1. *Let $\mathcal{C}_*$ be a chain complex of $\mathbb{Z}\Gamma$-modules*: $0 \to C_k \to \cdots \to C_0 \to 0$ *for some $k \in \mathbb{N}$. If $\mathrm{vrk}(C_j) < \infty$ for all $j \geq 1$, then*

$$\sum_{0\leq j\leq k}(-1)^j \beta_j^{(2)}(\mathcal{C}_*) = \chi(\mathcal{C}_*) = \sum_{0\leq j\leq k}(-1)^j\,\mathrm{mrk}_j(\mathcal{C}_*).$$

In particular, when $\mathcal{C}_*$ is exact, we have

$$\chi(\mathcal{C}_*) = \sum_{0\leq j\leq k}(-1)^j(\mathrm{mrk}_\Sigma(\mathrm{im}\,\partial_j) - \mathrm{mrk}_\Sigma(\mathrm{im}\,\partial_j|C_{j-1})).$$

The following lemma reduces the Juzvinskiĭ formula for mean rank to the finitely generated case.

Lemma 4.2. *Suppose that $\mathrm{mrk}_\Sigma(\mathcal{M}_1|\mathcal{M}_2) = \mathrm{mrk}_\Sigma(\mathcal{M}_1)$ holds when $\mathcal{M}_2$ is a finitely generated free $\mathbb{Z}\Gamma$-module and $\mathcal{M}_1$ is a finitely generated $\mathbb{Z}\Gamma$-submodule of $\mathcal{M}_2$. Then $\mathrm{mrk}_\Sigma(\mathcal{M}_1|\mathcal{M}_2) = \mathrm{mrk}_\Sigma(\mathcal{M}_1)$ holds for any $\mathbb{Z}\Gamma$-modules $\mathcal{M}_1 \subseteq \mathcal{M}_2$. In particular, Γ satisfies Juzvinskiĭ formula for mean rank if and only if $\mathrm{mrk}_\Sigma(\mathcal{M}_1|\mathcal{M}_2) = \mathrm{mrk}_\Sigma(\mathcal{M}_1)$ holds when $\mathcal{M}_2$ is a finitely generated free $\mathbb{Z}\Gamma$-module and $\mathcal{M}_1$ is a finitely generated $\mathbb{Z}\Gamma$-submodule of $\mathcal{M}_2$.*

Proof. CASE 1. $\mathcal{M}_2$ is finitely generated.

Write $\mathcal{M}_2$ as $\mathcal{M}_2 = (\mathbb{Z}\Gamma)^n/\mathcal{N}$ and $\mathcal{M}_1$ as $\mathcal{M}_1 = \mathcal{M}_1'/\mathcal{N}$ for some $n \in \mathbb{N}$ and some $\mathbb{Z}\Gamma$-submodules $\mathcal{N} \subseteq \mathcal{M}_1' \subseteq (\mathbb{Z}\Gamma)^n$. Write $\mathcal{N}$ as the union of an increasing net of finitely generated submodules $\{\mathcal{N}_j\}_{j\in\mathcal{J}}$ of $\mathcal{N}$. Note that for each j, by Proposition 2.8, we have

$$\mathrm{mrk}_\Sigma(\mathcal{N}_j) = \mathrm{mrk}_\Sigma(\mathcal{N}_j|\mathcal{N}_j) \geq \mathrm{mrk}_\Sigma(\mathcal{N}_j|\mathcal{M}_1') \geq \mathrm{mrk}_\Sigma(\mathcal{N}_j|(\mathbb{Z}\Gamma)^n) = \mathrm{mrk}_\Sigma(\mathcal{N}_j).$$

Thus $\mathrm{mrk}_\Sigma(\mathcal{N}_j|(\mathbb{Z}\Gamma)^n) = \mathrm{mrk}_\Sigma(\mathcal{N}_j|\mathcal{M}_1')$ for all j. Moreover, by Proposition 2.8,

$$\mathrm{mrk}_\Sigma(\mathcal{N}|(\mathbb{Z}\Gamma)^n) = \sup_{j\in\mathcal{J}}\mathrm{mrk}_\Sigma(\mathcal{N}_j|(\mathbb{Z}\Gamma)^n) = \sup_{j\in\mathcal{J}}\mathrm{mrk}_\Sigma(\mathcal{N}_j|\mathcal{M}_1') = \mathrm{mrk}_\Sigma(\mathcal{N}|\mathcal{M}_1').$$

Then by Theorem 2.7 and Proposition 2.8, we have

$$\begin{aligned}
&\mathrm{mrk}_\Sigma(\mathcal{M}_1|\mathcal{M}_2)\\
&= \mathrm{mrk}_\Sigma((\mathbb{Z}\Gamma)^n/\mathcal{N}) - \mathrm{mrk}_\Sigma((\mathbb{Z}\Gamma)^n/\mathcal{M}'_1)\\
&= (n - \mathrm{mrk}_\Sigma(\mathcal{N}|(\mathbb{Z}\Gamma)^n)) - (n - \mathrm{mrk}_\Sigma(\mathcal{M}'_1|(\mathbb{Z}\Gamma)^n))\\
&= \mathrm{mrk}_\Sigma(\mathcal{M}'_1) - \mathrm{mrk}_\Sigma(\mathcal{N}|\mathcal{M}'_1)\\
&= \mathrm{mrk}_\Sigma(\mathcal{M}'_1/\mathcal{N}) = \mathrm{mrk}_\Sigma(\mathcal{M}_1).
\end{aligned}$$

CASE 2. $\mathcal{M}_1$ is finitely generated.

Write $\mathcal{M}_2$ as the union of an increasing net of finitely generated submodules $\{\mathcal{M}'_j\}_{j\in\mathcal{J}}$ of $\mathcal{M}_2$ containing $\mathcal{M}_1$. By Proposition 2.8 and the conclusion of Case 1, we have

$$\mathrm{mrk}_\Sigma(\mathcal{M}_1|\mathcal{M}_2) = \inf_{j\in\mathcal{J}} \mathrm{mrk}_\Sigma(\mathcal{M}_1|\mathcal{M}'_j) = \inf_{j\in\mathcal{J}} \mathrm{mrk}_\Sigma(\mathcal{M}_1) = \mathrm{mrk}_\Sigma(\mathcal{M}_1).$$

Now we consider the general case. Write $\mathcal{M}_1$ as the union of an increasing net of finitely generated submodules $\{\mathcal{M}'_j\}_{j\in\mathcal{J}}$ of $\mathcal{M}_1$. Applying Proposition 2.8 and the conclusion of Case 2, we have

$$\mathrm{mrk}_\Sigma(\mathcal{M}_1|\mathcal{M}_2) = \sup_{j\in\mathcal{J}} \mathrm{mrk}_\Sigma(\mathcal{M}'_j|\mathcal{M}_2) = \sup_{j\in\mathcal{J}} \mathrm{mrk}_\Sigma(\mathcal{M}'_j|\mathcal{M}_1) = \mathrm{mrk}_\Sigma(\mathcal{M}_1).$$

Suppose that Γ satisfies Juzvinskiĭ formula for mean rank. Let $\mathcal{M}_2$ be a finitely generated free $\mathbb{Z}\Gamma$-module and $\mathcal{M}_1$ be finitely generated $\mathbb{Z}\Gamma$-submodule of $\mathcal{M}_2$. Consider the quotient map $\mathcal{M}_2 \to \mathcal{M}_2/\mathcal{M}_1$. Then the conclusion follows immediately by Theorem 2.7. The converse direction also follows immediately by Theorem 2.7. □

Remark 4.3. Let L be a function as in [28, Lemma 7.7] satisfying all the properties (i)–(v). Then the corresponding result in Lemma 4.2 still holds without change of proof. In particular, the corresponding statement holds for von Neumann–Lück rank.

Proof of Corollary 1.4. The second statement follows from Theorem 2.7 and the first statement. Suppose Γ satisfies Lück's dimension-flatness, in particular, for any finitely presented $\mathbb{Z}\Gamma$-module $\mathcal{M}$, we have $\beta_1^{(2)}(\mathcal{M}) = 0$. Write $\mathcal{M}$ as $\mathcal{M} = (\mathbb{Z}\Gamma)^n/(\mathbb{Z}\Gamma)^m f$ for some $f \in M_{m,n}(\mathbb{Z}\Gamma)$. Then a projective resolution of $\mathcal{M}$ can be

$$\cdots \xrightarrow{\partial_2} (\mathbb{Z}\Gamma)^m \xrightarrow{\partial_1} (\mathbb{Z}\Gamma)^n \longrightarrow \mathcal{M} \longrightarrow 0,$$

where $\partial_1 = R(f)$. Then by Theorem 1.2, we have

$$\begin{aligned}
\mathrm{vrk}(\mathrm{im}\,\partial_1) - \mathrm{vrk}(\mathrm{im}\,\partial_1|(\mathbb{Z}\Gamma)^n) &= \mathrm{vrk}(\mathrm{coker}\,\partial_2) - \mathrm{vrk}(\mathrm{im}\,\partial_1|(\mathbb{Z}\Gamma)^n)\\
&= \beta_1^{(2)}(\mathcal{M})\\
&= 0.
\end{aligned}$$

By Remark 4.3, we have Γ satisfies Juzvinskiĭ formula for von Neumann–Lück rank.

For the "if" part, by Theorem 1.2, we first have $\beta_1^{(2)}(\mathcal{M}) = 0$ for any finitely presented $\mathbb{Z}\Gamma$-module $\mathcal{M}$. Since both $\dim_{\mathcal{L}\Gamma}(\cdot)$ and $\operatorname{tor}_j^{\mathbb{Z}\Gamma}(\mathcal{L}\Gamma, \cdot)$ commutes with the colimits [33, Theorem 6.7] [36, Proposition 10.99], we have $\beta_1^{(2)}(\mathcal{M}) = 0$ for any $\mathbb{Z}\Gamma$-module $\mathcal{M}$.

Let $0 \to \mathcal{N} \to \mathcal{F} \to \mathcal{M} \to 0$ be an exact sequence of $\mathbb{Z}\Gamma$-modules such that $\mathcal{F}$ is free. Then a projective resolution of $\mathcal{N}$ induces a projective resolution of $\mathcal{M}$. So $\beta_2^{(2)}(\mathcal{M}) = \beta_1^{(2)}(\mathcal{N}) = 0$. Inductively we have $\beta_j^{(2)}(\mathcal{M}) = 0$ for all $j \geq 1$ and $\mathbb{Z}\Gamma$-module $\mathcal{M}$. □

The following proposition was implicitly proven in [33, Conjecture 6.48]. For convenience, we give a proof.

Proposition 4.4. *Let Γ be a discrete group (not necessarily sofic) and H be a subgroup of Γ. Then for any $\mathbb{Z}H$-module $\mathcal{M}$, we have $\beta_j^{(2)}(\mathbb{Z}\Gamma \otimes_{\mathbb{Z}H} \mathcal{M}) = \beta_j^{(2)}(\mathcal{M})$. In particular, taking subgroups respects the property of Lück's dimension-flatness.*

Proof. Let $\mathcal{C}_* \to \mathcal{M}$ be a free resolution of $\mathcal{M}$. Since $\mathbb{Z}\Gamma$ is a flat $\mathbb{Z}H$-module, we get a free resolution $\mathbb{Z}\Gamma \otimes_{\mathbb{Z}H} \mathcal{C}_* \to \mathbb{Z}\Gamma \otimes_{\mathbb{Z}H} \mathcal{M}$ of the $\mathbb{Z}\Gamma$-module $\mathbb{Z}\Gamma \otimes_{\mathbb{Z}H} \mathcal{M}$. Since the induction functor $\mathcal{L}\Gamma \otimes_{\mathcal{L}H} \cdot$ is flat [33, Theorem 6.29 (1)], we have

$$\begin{aligned}\mathcal{L}\Gamma \otimes_{\mathcal{L}H} H_j(\mathcal{L}H \otimes_{\mathbb{Z}H} \mathcal{C}_*) &\cong H_j(\mathcal{L}\Gamma \otimes_{\mathcal{L}H} (\mathcal{L}H \otimes_{\mathbb{Z}H} \mathcal{C}_*)) \\ &\cong H_j(\mathcal{L}\Gamma \otimes_{\mathbb{Z}\Gamma} (\mathbb{Z}\Gamma \otimes_{\mathbb{Z}H} \mathcal{C}_*)).\end{aligned}$$

Thus by [33, Theorem 6.29 (2)], we have

$$\begin{aligned}\beta_j^{(2)}(\mathbb{Z}\Gamma \otimes_{\mathbb{Z}H} \mathcal{M}) &= \dim_{\mathcal{L}\Gamma} H_j(\mathcal{L}\Gamma \otimes_{\mathbb{Z}\Gamma} (\mathbb{Z}\Gamma \otimes_{\mathbb{Z}H} \mathcal{C}_*)) \\ &= \dim_{\mathcal{L}\Gamma}(\mathcal{L}\Gamma \otimes_{\mathcal{L}H} H_j(\mathcal{L}H \otimes_{\mathbb{Z}H} \mathcal{C}_*)) \\ &= \dim_{\mathcal{L}H} H_j(\mathcal{L}H \otimes_{\mathbb{Z}H} \mathcal{C}_*) = \beta_j^{(2)}(\mathcal{M}).\end{aligned}$$
□

As a consequence of Corollary 1.4 and Proposition 4.4, we have:

Corollary 4.5. *If a subgroup H of a sofic group Γ violoates Lück's dimension-flatness, then Γ violoates Juzvinskiĭ formula for mean dimension. In particular, if $\beta_j^{(2)}(H) > 0$ for some $j \geq 1$, then Γ violoates Juzvinskiĭ formula for mean rank and mean dimension.*

We refer the reader to [35, Section 5] for some discussions on L^2-Betti numbers of subgroups. Lück's dimension-flatness for amenable groups [33, Theorem 6.37] can be interpreted in terms of relative sofic mean rank.

Corollary 4.6. *Amenable groups satisfy Lück's dimension-flatness.*

Proof. By [28, Theorem 5.1], we know $\mathrm{mrk}_\Sigma(\mathcal{M}_1|\mathcal{M}_2) = \mathrm{mrk}_\Sigma(\mathcal{M}_1)$ holds for any $\mathbb{Z}\Gamma$-modules $\mathcal{M}_1 \subseteq \mathcal{M}_2$. By Theorem 2.7, Γ satisfies Juzvinskiĭ formula for mean rank. Thus by Corollary 1.4, Γ satisfies Lück's dimension-flatness. □

Corollary 4.7. *Let Γ be a residually finite group and $\{\Gamma_i\}_i$ be a sequence of finite-index decreasing normal subgroups of Γ with the intersection $\{e_\Gamma\}$. Let $\mathcal{C}_*$ be a chain complex of finitely generated free $\mathbb{Z}\Gamma$-modules. Then*

$$\mathrm{mrk}_j(\mathcal{C}_*) = \lim_{i\to\infty} \frac{\mathrm{rk}(H_j(\Gamma_i \setminus \mathcal{C}_*))}{|\Gamma/\Gamma_i|}.$$

Proof. Since residually finite groups satisfy Lück's approximation formula for L^2-Betti numbers [32, Theorem 0.1], by the similar argument as in [33, Lemma 13.4], we have

$$\beta_j^{(2)}(\mathcal{C}_*) = \lim_{i\to\infty} \frac{\mathrm{rk}(H_j(\Gamma_i \setminus \mathcal{C}_*))}{|\Gamma/\Gamma_i|}.$$

Then the conclusion follows from Theorem 1.2. □

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