SYMMETRIES OF LINEARIZED GRAVITY FROM ADJOINT OPERATORS

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Abstract. Using a covariant formulation it is shown that the Teukolsky equation and the Teukolsky-Starobinsky identities for spin-1 and linearized gravity on a vacuum type D background are self-adjoint. This fact is used to construct symmetry operators for each of the four cases. We find both irreducible second order symmetry operators for spin-1, a known fourth order, and a new sixth order symmetry operator for linearized gravity. The results are connected to Hertz and Debye potentials and to the separability of the Teukolsky equation.

1. Introduction

Symmetries and their consequences are important tools in the analysis of partial differential equations. By a symmetry operator of a PDE we mean a differential operator mapping solutions to solutions[1]. In this paper such operators are derived for the test Maxwell fields (spin-1) and linearized gravity (spin-2) on vacuum spacetimes of Petrov type D. These are higher spin analogues of the Carter operator for scalar fields [8].

A vacuum type D background comes with two isometries generated by Killing vectors ξ^a, ζ^a and with a Killing spinor κ_{AB} (or equivalently with a conformal Killing-Yano tensor and its dual). The scalar wave equation admits the Lie derivatives $\mathcal{L}_\xi$ and $\mathcal{L}_\zeta$ along isometries as first order symmetry operators and a second order (Carter) symmetry operator. These operators covariantly characterize the separability of the scalar wave equation and can be associated to the separation constants. They also provide an important tool to prove decay of scalar waves, see [3].

The field equations of non-zero spin have a much more complicated structure. However, for certain frame components of the field-strength (spin-1) or linearized curvature (spin-2) one can derive the Teukolsky master equations (TME) and Teukolsky-Starobinsky identities (TSI) as integrability conditions. They play a crucial role in this paper and further details are discussed below, see also [6] for a covariant form in the spin-1 case and [2] for linearized gravity.

For the spin-1 case all symmetry operators up to second order were classified in [4]. It turned out that the Lie derivatives $\mathcal{L}_\xi$ and $\mathcal{L}_\zeta$ are the first order operators and at second order two symmetry operators, constructed from the Killing spinor κ_{AB}, do exist. In this paper we rederive these two second order operators for spin-1 and with the same method derive two symmetry operators for linearized gravity. The method we are using is an extension of an elegant argument by Wald [17] based on[2]

Theorem 1 ([17]). *Suppose the identity*

$$\mathbf{S}\mathbf{E} = \mathbf{O}\mathbf{T}, \tag{1.1}$$

holds for linear partial differential operators $\mathbf{S}, \mathbf{E}, \mathbf{O}$ *and* $\mathbf{T}$. *Suppose* ψ *satisfies* $\mathbf{O}^\dagger\psi = 0$, *where* $\dagger$ *denotes the adjoint of the operator with respect to some inner product. Then* $\mathbf{S}^\dagger\psi$ *satisfies* $\mathbf{E}^\dagger(\mathbf{S}^\dagger\psi) = 0$. *Thus, in particular, if* $\mathbf{E}$ *is self-adjoint then* $\phi = \mathbf{S}^\dagger\psi$ *is a solution of* $\mathbf{E}(\phi) = 0$.

To apply the theorem, Wald makes two key observations:

(1) *Connecting the TME to covariant field equations*
First use the (self-adjoint) wave equations for the vector potential (spin-1) or the linearized metric (spin-2) to define the $\mathbf{E}$ operator covariantly. Then there exists an operator $\mathbf{S}$ connected to the TME operator $\mathbf{O}$ via (1.1). In fact, the operator $\mathbf{S}$ can be read-off from the source terms of the TME in Teukolsky's original work.

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[1]This is a weaker condition than having a commuting operator.

[2]Proof: The adjoint of (1.1) is given by $\mathbf{E}^\dagger\mathbf{S}^\dagger = \mathbf{T}^\dagger\mathbf{O}^\dagger$. Applied to ψ, we obtain $\mathbf{E}^\dagger(\mathbf{S}^\dagger\psi) = 0$.

(2) *Adjointness property of TME*
The TME operator for one extreme scalar (say ϕ_0 for spin-1 or $\dot{\Psi}_0$ for spin-2) is, up to rescaling by a scalar field, the adjoint of the TME operator for the other extreme scalar (ϕ_2 for spin-1 or $\dot{\Psi}_4$ for spin-2).

Thus from any solution of the spin-1 or spin-2 TME a new vector potential or linearized metric in the kernel of $\mathbf{E}$ can be constructed. This provides the connection to symmetry operators we are interested in. Summarized, Wald provided a re-derivation of the spin-1 Debye potential formulation of Cohen and Kegeles [10] and the first complete proof of the spin-2 Debye potential formulation initiated by Cohen and Kegeles [11] and Chrzanowski [9]. See also [19] for an overview.

We extend the argument as follows. Because of key observation (2) it is natural to group the $\pm$ spin weighted TME into a matrix operator which is self-adjoint by construction. This is inherent to the covariant spinorial formulation we use in this paper and we explicitly prove the self-adjointness of the TME. More remarkable, key observation (1) also works for the TSI, so it can also be cast into the form (1.1). This means there is an operator $\widehat{\mathbf{S}}$ such that $\widehat{\mathbf{S}}\mathbf{E} = \widehat{\mathbf{O}}\mathbf{T}$ with $\widehat{\mathbf{O}}$ the TSI operator. The TSI operator is also self-adjoint, but with respect to a different inner product, see (2.2) below. This way, we are able to generate another symmetry operator for spin-1 and for spin-2 from the TSI[3].

The two second order symmetry operators for spin-1 we get from the TME and TSI are equivalent to the ones found in [4], see remark 5. Furthermore, they can be interpreted in the following sense. One of them characterizes the separability of the TME, similarly to the Carter operator for scalar waves. To see this one needs to make use of the freedom to add/subtract terms vanishing due to the field equations to produce symmetry operators which are "purely angular" or "purely radial" in the sense that after separation of variables the "purely angular" operators will only involve angular variables, and the "purely radial" operators will only involve radial variables. The other one is equivalent to Hertz potentials. See also [1, Chapter 5].

The two symmetry operators for linearized gravity are of higher order than in the spin-1 case. From the TME we get a fourth order operator, which is equivalent to the covariant Hertz potential formalism of Cohen and Kegeles [13]. The operator we get from the TSI is of order six. We show that it is non-trivial and that it is related to the third power of the TME separation constant on a Schwarzschild background.

All calculations in this paper have been done in the *xAct* [14] suit for *Mathematica*, and in particular we have used and developed the *SymManipulator* and *SpinFrames* packages for this work. The typeset ready equations were produced with *TexAct*.

The results of this paper relate various concepts like Hertz and Debye potentials or Teukolsky separability to symmetry operators. We expect these operators to play an important role in the study of decay estimates for spin-1 and spin-2 similar to the scalar wave equation case in [3]. The modifications of the operators with terms vanishing on-shell may open up the possibility to invert certain potential maps and lead to a generalization of the decay results of [5] to a curved background. Finally the new sixth order symmetry operator for linearized gravity may be of interest for the general theory of separation of variables.

1.1. **Overview.** In the preliminaries section 2 we introduce the inner products for the adjoint method and a set of algebraic and differential operators tied to the Petrov type D geometry. In section 3 we study the source-free Maxwell equation (spin-1). The self-adjointness of TME and TSI is shown and afterwards used in Theorem 4 to construct two second order symmetry operators. In subsection 3.2 they are related to Hertz/Debye potentials and separability of the TME. In section 4 we formulate the linearized gravity equations in terms of suitable operators. The TME and TSI are then separately treated in subsections 4.1 and 4.2, respectively. In each case self-adjointness is shown and used to construct a symmetry operator. Finally, in subsection 4.3 we relate the fourth order operator to Hertz/Debye potentials and show the relation of the sixth order operator to TME separability on a Schwarzschild background. In appendix A, we present the adjoints of the operators introduced in this paper. Appendix B contains a list of commutator relations for some of these operators. In appendix C we list the GHP component form of selected operators.

[3]The components of the field strength/linearized curvature are not decoupled in $\widehat{\mathbf{O}}$, but that is not necessary for the discussion of symmetry operators.

2. Preliminaries

We use the 2-spinor formalism and sign convention of Penrose & Rindler [15] in which 24Λ denotes the Ricci scalar, $-2\Phi_{ABA'B'}$ is the trace-free Ricci spinor and Ψ_{ABCD} is the Weyl spinor. All spinors are decomposed into irreducible parts, which are symmetric, so we can work with only symmetric spinors. Throughout the paper we will therefore assume that all spinors are symmetric unless explicitly stated, and that all operators defined here act on general symmetric spinors of the indicated valence. In particular φ will always be used as a general symmetric spinor of indicated valence. We will use the convention to add indices to an operator according to the valence of the spinor on which it acts. If we have for example an operator $\mathbf{A}$ mapping symmetric valence (k,l) to (u,v) spinors, it will be displayed as $\mathbf{A}_{k,l}$. Using this notation, we can in principle write operator identities without spinor indices. We will do this in matrix equations and in Appendix B, but for clarity we will write out spinor indices elsewhere.

Anti-self-dual and self-dual 2-forms correspond to symmetric spinors of valence $(2,0)$ and $(0,2)$ respectively. Anti-self-dual and self-dual 4-tensors with Weyl symmetries correspond to symmetric spinors of valence $(4,0)$ and $(0,4)$ respectively.

Component expressions with respect to a spinor dyad (o_A, ι_A) will be expressed in the GHP notation. For certain operators in GHP form we will talk about "purely angular" or "purely radial" operators. By this we mean that after separation of variables the "purely angular" operators will only involve angular variables, and the "purely radial" operators will only involve radial variables.

Following [6, sec. 2.2], we denote the adjoint of a linear differential operator $\mathbf{A}$ with respect to the bilinear pairing

$$(\phi_{A_1\dots A_k A'_1\dots A'_l}, \psi_{A_1\dots A_k A'_1\dots A'_l}) = \int \phi_{A_1\dots A_k A'_1\dots A'_l} \psi^{A_1\dots A_k A'_1\dots A'_l} d\mu \tag{2.1}$$

by $\mathbf{A}^\dagger$, and the adjoint with respect to the sesquilinear pairing

$$\langle \phi_{A_1\dots A_k A'_1\dots A'_l}, \psi_{A_1\dots A_l A'_1\dots A'_k} \rangle = \int \phi_{A_1\dots A_k A'_1\dots A'_l} \bar{\psi}^{A_1\dots A_k A'_1\dots A'_l} d\mu \tag{2.2}$$

by $\mathbf{A}^\star$. We refer to them as $\dagger$-adjoint and $\star$-adjoint, respectively. The adjoints of certain natural operators with respect to both inner products are listed in appendix A. Also note that $\overline{\mathbf{A}^\dagger(\phi)} = \mathbf{A}^\star(\bar{\phi})$. If we take the $\dagger$-adjoint and $\star$-adjoint of an operator $\mathbf{A}_{k,l}$, mapping symmetric valence (k,l) to (u,v) spinors, we write this as $(\mathbf{A}_{k,l})^\dagger = \mathbf{A}^\dagger_{u,v}$ and $(\mathbf{A}_{k,l})^\star = \mathbf{A}^\star_{v,u}$, respectively. The adjoint operator argument in Theorem 1 holds for both the $\dagger$-adjoint and the $\star$-adjoint.

We will work on vacuum type D spaces, and here we present certain algebraic and differential operators on such spaces which were first introduced (with examples) in [2]. The fundamental operators $\mathscr{C}_{k,l}, \mathscr{C}^\dagger_{k,l}, \mathscr{T}_{k,l}, \mathscr{D}_{k,l}$ acting on symmetric spinors of valence k,l are defined as the irreducible parts of the covariant derivative $\nabla_{AA'}\varphi_{BC\cdots DB'C'\cdots D'}$ of the symmetric spinor $\varphi_{AB\cdots DA'B'\cdots D'}$ of valence k,l. See [4] for a detailed discussion of their properties including commutators. The main feature of the Petrov type D geometry is encoded in the irreducible[4], symmetric Killing spinor κ_{AB} found in [18], satisfying

$$(\mathscr{T}_{2,0}\kappa)_{ABCA'} = 0. \tag{2.3}$$

In a principal dyad the Killing spinor takes the simple form

$$\kappa_{AB} = -2\kappa_1 o_{(A}\iota_{B)}, \tag{2.4}$$

with $\kappa_1 \propto \Psi_2^{-1/3}$. From a commutator it follows that

$$\xi_{AA'} = (\mathscr{C}^\dagger_{2,0}\kappa)_{AA'}, \tag{2.5}$$

is a Killing vector field. Beside the Killing vector field (2.5) another important vector field is defined by

$$U_{AA'} = -\frac{\kappa_{AB}\xi^B{}_{A'}}{3\kappa_1^2} = -\nabla_{AA'}\log(\kappa_1). \tag{2.6}$$

[4]Irreducible here means that it can not be factored in Killing spinors of lower valence.

Using the $U_{AA'}$ vector field, we define the extended fundamental operators with additional (extended) indices n, m,

$$(\mathscr{D}_{k,l,n,m}\varphi)_{A_1\dots A_{k-1}}{}^{A'_1\dots A'_{l-1}} \equiv \left[\nabla^{BB'} + nU^{BB'} + m\bar{U}^{BB'}\right]\varphi_{A_1\dots A_{k-1}B}{}^{A'_1\dots A'_{l-1}}{}_{B'}, \tag{2.7a}$$

$$(\mathscr{C}_{k,l,n,m}\varphi)_{A_1\dots A_{k+1}}{}^{A'_1\dots A'_{l-1}} \equiv \left[\nabla_{(A_1}{}^{B'} + nU_{(A_1}{}^{B'} + m\bar{U}_{(A_1}{}^{B'}\right]\varphi_{A_2\dots A_{k+1})}{}^{A'_1\dots A'_{l-1}}{}_{B'}, \tag{2.7b}$$

$$(\mathscr{C}^\dagger_{k,l,n,m}\varphi)_{A_1\dots A_{k-1}}{}^{A'_1\dots A'_{l+1}} \equiv \left[\nabla^{B(A'_1} + nU^{B(A'_1} + m\bar{U}^{B(A'_1}\right]\varphi_{A_1\dots A_{k-1}B}{}^{A'_2\dots A'_{l+1})}, \tag{2.7c}$$

$$(\mathscr{T}_{k,l,n,m}\varphi)_{A_1\dots A_{k+1}}{}^{A'_1\dots A'_{l+1}} \equiv \left[\nabla_{(A_1}{}^{(A'_1} + nU_{(A_1}{}^{(A'_1} + m\bar{U}_{(A_1}{}^{(A'_1}\right]\varphi_{A_2\dots A_{k+1})}{}^{A'_2\dots A'_{l+1})}. \tag{2.7d}$$

For $n = m = 0$ they coincide with the usual fundamental operators of [4], and we can suppress the rightmost vanishing extended index/indices without notational conflict. Because $U_{AA'}$ is a logarithmic derivative we have

$$(\mathscr{C}_{k,l,n,m}\varphi)_{A_1\dots A_{k+1}}{}^{A'_1\dots A'_{l-1}} = \kappa_1^n\bar{\kappa}_{1'}^m(\mathscr{C}_{k,l}\kappa_1^{-n}\bar{\kappa}_{1'}^{-m}\varphi)_{A_1\dots A_{k+1}}{}^{A'_1\dots A'_{l-1}}, \tag{2.8}$$

and similarly for the other operators. In particular it follows that the commutator of extended fundamental spinor operators with $n_1 = n_2, m_1 = m_2$ reduces to the commutator of the fundamental spinor operators given in [4, Lemma 18]. We can also use this to commute factors of κ_1 or $\bar{\kappa}_{1'}$ in or out of the extended fundamental spinor operators like

$$\kappa_1^n\bar{\kappa}_{1'}^m(\mathscr{C}_{k,l,p,q}\varphi)_{A_1\dots A_{k+1}}{}^{A'_1\dots A'_{l-1}} = (\mathscr{C}_{k,l,p+n,q+m}\kappa_1^n\bar{\kappa}_{1'}^m\varphi)_{A_1\dots A_{k+1}}{}^{A'_1\dots A'_{l-1}}, \tag{2.9}$$

and similarly for the other operators. The fact that $\kappa_1 \propto \Psi_2^{-1/3}$ similarly gives

$$\Psi_2^n(\mathscr{C}_{k,l,p,q}\varphi)_{A_1\dots A_{k+1}}{}^{A'_1\dots A'_{l-1}} = (\mathscr{C}_{k,l,p-3n,q}\Psi_2^n\varphi)_{A_1\dots A_{k+1}}{}^{A'_1\dots A'_{l-1}}. \tag{2.10}$$

Given the Killing spinor (2.4), define the algebraic operators $\mathcal{K}^i_{k,l}, i = 0, 1, 2$ mapping symmetric spinors of valence (k, l) to $(k - 2i + 2, l)$ via

$$(\mathcal{K}^0_{k,l}\varphi)_{A_1\dots A_{k+2}A'_1\dots A'_l} = 2\kappa_1^{-1}\kappa_{(A_1A_2}\varphi_{A_3\dots A_{k+2})A'_1\dots A'_l}, \tag{2.11a}$$

$$(\mathcal{K}^1_{k,l}\varphi)_{A_1\dots A_kA'_1\dots A'_l} = \kappa_1^{-1}\kappa_{(A_1}{}^F\varphi_{A_2\dots A_k)FA'_1\dots A'_l}, \tag{2.11b}$$

$$(\mathcal{K}^2_{k,l}\varphi)_{A_1\dots A_{k-2}A'_1\dots A'_l} = -\tfrac{1}{2}\kappa_1^{-1}\kappa^{CD}\varphi_{A_1\dots A_{k-2}CDA'_1\dots A'_l}. \tag{2.11c}$$

The commutators of the $\mathcal{K}$-operators and the extended fundamental spinor operators are given in appendix B.

Definition 2 (Spin decomposition)**.** *For any symmetric spinor $\varphi_{A_1\dots A_{2s}}$ with integer s, define $s+1$ symmetric valence $2s$ spin-projectors $(\mathcal{P}^i_{2s,0}\varphi)_{A_1\dots A_{2s}}, i = 0\dots s$ solving*

$$\varphi_{A_1\dots A_{2s}} = \sum_{i=0}^{s}(\mathcal{P}^i_{2s,0}\varphi)_{A_1\dots A_{2s}}, \tag{2.12}$$

with $(\mathcal{P}^i_{2s,0}\varphi)_{A_1\dots A_{2s}}$ depending only on the dyad components φ_{s+i} and φ_{s-i}.

These spin-projectors can be expressed in terms of the $\mathcal{K}$-operators and they are independent of the choice of principal frame, see [2, Example 2.7]. The adjoints of the extended fundamental operators, the $\mathcal{K}$-operators and the spin-projectors are given in appendix A.

On a vacuum type D background in a principal dyad, the operator

$$(\mathscr{C}_{n-1,1,c-n}\mathscr{C}^\dagger_{n,0,c}\varphi)_{A_1\dots A_n} \tag{2.13}$$

is diagonal on its dyad components $\varphi_k, k = 0, \dots, n$ for any value of c. It follows that the spin-projectors commute with this diagonal wave operator.

3. Spin-1

In this section we consider second order symmetry operators for the source-free Maxwell equation

$$(\mathscr{C}^\dagger_{2,0}\phi)_{AA'} = 0, \tag{3.1}$$

for the symmetric field strength ϕ_{AB}. The field strength can locally be represented in terms of a real vector potential $\alpha_{AA'}$ via

$$\phi_{AB} = (\mathscr{C}_{1,1}\alpha)_{AB}. \tag{3.2}$$

The symmetry operators of this section naturally lead to complex vector potentials and we refer to remark 7 for the general picture of symmetries for complex Maxwell fields.

3.1. **Maxwell symmetry operators.** Define the operators

$$(\mathbf{E}_{1,1}\varphi)_{AA'} = (\mathscr{C}^{\dagger}_{2,0}\mathscr{C}_{1,1}\varphi)_{AA'}, \tag{3.3a}$$
$$(\mathbf{T}_{1,1}\varphi)_{AB} = (\mathscr{C}_{1,1}\varphi)_{AB}. \tag{3.3b}$$

The first operator is the real Maxwell operator,

$$(\mathbf{E}_{1,1}\varphi)_{AA'} = \tfrac{1}{2}\nabla_{BB'}\nabla_{AA'}\varphi^{BB'} - \tfrac{1}{2}\nabla_{BB'}\nabla^{BB'}\varphi_{AA'}, \tag{3.4}$$

having vector potentials for vacuum Maxwell solutions in its kernel and the second operator is the map (3.2) from a vector potential to its anti-self-dual field strength[5]. Because of reality of $\mathbf{E}_{1,1}$ and properties of the adjoints of the fundamental operators given in appendix A, we find

$$(\mathbf{E}^{\dagger}_{1,1}\varphi)_{AA'} = (\mathbf{E}^{\star}_{1,1}\varphi)_{AA'} = (\overline{\mathbf{E}}_{1,1}\varphi)_{AA'} = (\mathbf{E}_{1,1}\varphi)_{AA'}. \tag{3.5}$$

Define the operators

$$(\mathbf{S}_{1,1}\varphi)_{AB} = \kappa_1^2(\mathcal{K}^1_{2,0}\mathcal{K}^1_{2,0}\mathscr{C}_{1,1,-2}\varphi)_{AB}, \tag{3.6a}$$
$$(\mathbf{O}_{2,0}\varphi)_{AB} = \kappa_1^2(\mathcal{K}^1_{2,0}\mathcal{K}^1_{2,0}\mathscr{C}_{1,1,-2}\mathscr{C}^{\dagger}_{2,0}\varphi)_{AB}, \tag{3.6b}$$
$$(\widehat{\mathbf{S}}_{1,1}\varphi)_{A'B'} = \kappa_1\bar\kappa_{1'}(\overline{\mathcal{K}}^1_{0,2}\mathscr{C}^{\dagger}_{1,1,-2}\mathcal{K}^1_{1,1}\varphi)_{A'B'}, \tag{3.6c}$$
$$(\widehat{\mathbf{O}}_{2,0}\varphi)_{A'B'} = \kappa_1\bar\kappa_{1'}(\overline{\mathcal{K}}^1_{0,2}\mathscr{C}^{\dagger}_{1,1,-2}\mathcal{K}^1_{1,1}\mathscr{C}^{\dagger}_{2,0}\varphi)_{A'B'}. \tag{3.6d}$$

For any solution ϕ_{AB} to the source-free Maxwell equations (3.1) we have $(\mathbf{O}_{2,0}\phi)_{AB} = 0$ and $(\widehat{\mathbf{O}}_{2,0}\phi)_{A'B'} = 0$. These are the covariant TME and TSI, respectively. We collect remarkable properties in

Lemma 3. *The TME operator is †-self-adjoint and the TSI operator is ⋆-self-adjoint,*

$$(\mathbf{O}^{\dagger}_{2,0}\varphi)_{AB} = (\mathbf{O}_{2,0}\varphi)_{AB}, \qquad (\widehat{\mathbf{O}}^{\star}_{2,0}\varphi)_{A'B'} = (\widehat{\mathbf{O}}_{2,0}\varphi)_{A'B'}. \tag{3.7}$$

Furthermore the operators factorize into

$$(\mathbf{O}_{2,0}\varphi)_{AB} = -(\mathbf{F}^{\dagger}_{1,1}\mathbf{F}_{2,0}\varphi)_{AB}, \tag{3.8a}$$
$$(\widehat{\mathbf{O}}_{2,0}\varphi)_{A'B'} = -(\widehat{\mathbf{F}}^{\star}_{1,1}\widehat{\mathbf{F}}_{2,0}\varphi)_{A'B'}, \tag{3.8b}$$

with

$$(\mathbf{F}_{2,0}\varphi)_{AA'} = \big(\mathscr{C}^{\dagger}_{2,0,1}\mathcal{K}^1_{2,0}(\kappa_1\varphi)\big)_{AA'}, \tag{3.9a}$$
$$(\widehat{\mathbf{F}}_{2,0}\varphi)_{AA'} = \big(\mathscr{C}^{\dagger}_{2,0}\mathcal{K}^1_{2,0}(\kappa_1\varphi)\big)_{AA'}. \tag{3.9b}$$

Proof. The proof relies on rescalings of the form (2.9) and $\mathcal{K}$-operator commutators given in appendix B. Commuting $\kappa_1\mathcal{K}^1_{2,0}$ from the left of the diagonal wave operator in (3.6b) to the right yields

$$\kappa_1^2(\mathcal{K}^1_{2,0}\mathcal{K}^1_{2,0}\mathscr{C}_{1,1,-2}\mathscr{C}^{\dagger}_{2,0}\varphi)_{AB} = \kappa_1\big(\mathcal{K}^1_{2,0}\mathscr{C}_{1,1,-1}\mathscr{C}^{\dagger}_{2,0,1}\mathcal{K}^1_{2,0}(\kappa_1\varphi)\big)_{AB},$$

from which (3.8a) follows. Commuting the $\mathcal{K}^1_{1,1}$ and the κ_1 in (3.6d) to the right and arranging extended indices yields

$$\begin{aligned}\kappa_1\bar\kappa_{1'}(\overline{\mathcal{K}}^1_{0,2}\mathscr{C}^{\dagger}_{1,1,-2}\mathcal{K}^1_{1,1}\mathscr{C}^{\dagger}_{2,0}\varphi)_{A'B'} &= \bar\kappa_{1'}\big(\overline{\mathcal{K}}^1_{0,2}\mathscr{C}^{\dagger}_{1,1,-1}\mathscr{C}^{\dagger}_{2,0,1}\mathcal{K}^1_{2,0}(\kappa_1\varphi)\big)_{A'B'}\\ &= \bar\kappa_{1'}\big(\overline{\mathcal{K}}^1_{0,2}\mathscr{C}^{\dagger}_{1,1}\mathscr{C}^{\dagger}_{2,0}\mathcal{K}^1_{2,0}(\kappa_1\varphi)\big)_{A'B'},\end{aligned}$$

from which (3.8b) follows. From these factorizations the self-adjointness (3.7) of TME and TSI is evident. □

[5]The operator defined in (3.3a) differs from [17] by a factor of −2. We also do not restrict the operator given in (3.3b) to depend only on particular dyad components.

Lemma 3 naturally leads to variational principles for the TME and the TSI. These, together with conservation laws, will be discussed in a separate paper. Here however, we use the self-adjointness to prove the main result of this section:

Theorem 4. *Let ϕ_{AB} be a solution to the source-free Maxwell equation* (3.1) *on a vacuum background of Petrov type D.*

(1) *With the operators $\mathbf{S}_{1,1}, \mathbf{E}_{1,1}, \mathbf{O}_{2,0}, \mathbf{T}_{1,1}$ defined above we have the identity*
$$(\mathbf{S}_{1,1}\mathbf{E}_{1,1}\varphi)_{AA'} = (\mathbf{O}_{2,0}\mathbf{T}_{1,1}\varphi)_{AA'}, \tag{3.10}$$
which is of the form (1.1) *and the map*
$$\phi_{AB} \mapsto (\mathbf{S}^{\dagger}_{2,0}\phi)_{AA'} = \big(\mathscr{C}^{\dagger}_{2,0,2}\mathcal{K}^{1}_{2,0}\mathcal{K}^{1}_{2,0}(\kappa_1^2\phi)\big)_{AA'}. \tag{3.11}$$
generates a new complex vector potential for a solution to the vacuum Maxwell equation.

(2) *With the operators $\widehat{\mathbf{S}}_{1,1}, \mathbf{E}_{1,1}, \widehat{\mathbf{O}}_{2,0}, \mathbf{T}_{1,1}$ defined above we have the identity*
$$(\widehat{\mathbf{S}}_{1,1}\mathbf{E}_{1,1}\varphi)_{A'B'} = (\widehat{\mathbf{O}}_{2,0}\mathbf{T}_{1,1}\varphi)_{A'B'}, \tag{3.12}$$
which is of the form (1.1) *and the map*
$$\phi_{AB} \mapsto (\widehat{\mathbf{S}}^{\star}_{2,0}\phi)_{AA'} = \big(\overline{\mathcal{K}}^{1}_{1,1}\mathscr{C}^{\dagger}_{2,0,0,2}\mathcal{K}^{1}_{2,0}(\kappa_1\bar{\kappa}_{1'}\phi)\big)_{AA'}. \tag{3.13}$$
generates a new complex vector potential for a solution to the vacuum Maxwell equation.

Proof. The operators can be viewed as different parts of third order operators,
$$\overbrace{\kappa_1^2\mathcal{K}^1_{2,0}\mathcal{K}^1_{2,0}\mathscr{C}_{1,1,-2}}^{\mathbf{S}_{1,1}}\overbrace{\mathscr{C}^{\dagger}_{2,0}\mathscr{C}_{1,1}}^{\mathbf{E}_{1,1}} = \underbrace{\kappa_1^2\mathcal{K}^1_{2,0}\mathcal{K}^1_{2,0}\mathscr{C}_{1,1,-2}\mathscr{C}^{\dagger}_{2,0}}_{\mathbf{O}_{2,0}}\underbrace{\mathscr{C}_{1,1}}_{\mathbf{T}_{1,1}}, \qquad \overbrace{\kappa_1\bar{\kappa}_{1'}\overline{\mathcal{K}}^1_{0,2}\mathscr{C}^{\dagger}_{1,1,-2}\mathcal{K}^1_{1,1}}^{\widehat{\mathbf{S}}_{1,1}}\overbrace{\mathscr{C}^{\dagger}_{2,0}\mathscr{C}_{1,1}}^{\mathbf{E}_{1,1}} = \underbrace{\kappa_1\bar{\kappa}_{1'}\overline{\mathcal{K}}^1_{0,2}\mathscr{C}^{\dagger}_{1,1,-2}\mathcal{K}^1_{1,1}\mathscr{C}^{\dagger}_{2,0}}_{\widehat{\mathbf{O}}_{2,0}}\underbrace{\mathscr{C}_{1,1}}_{\mathbf{T}_{1,1}}, \tag{3.14}$$
so the identities (3.10) and (3.12) are evident. The rest follows from theorem 1 by using (3.5) and (3.7). □

Remark 5. *In [4], among others, second order symmetry operators for the source-free Maxwell equation* (3.1) *are completely classified. In particular on a vacuum Petrov type D background the list contains, beside second Lie derivatives along isometries, one linear operator and one anti-linear operator. Both operators were presented in terms of complex vector potentials, $A_{AA'}$ and $B_{AA'}$, and a comparison reveals*
$$\begin{aligned}(\mathbf{S}^{\dagger}_{2,0}\phi)_{AA'} &= \big(\mathscr{C}^{\dagger}_{2,0,2}\mathcal{K}^1_{2,0}\mathcal{K}^1_{2,0}(\kappa_1^2\phi)\big)_{AA'}\\ &= \big(\mathcal{K}^1_{1,1}\mathscr{C}^{\dagger}_{2,0,2}\mathcal{K}^1_{2,0}(\kappa_1^2\phi)\big)_{AA'}\\ &= B_{AA'},\end{aligned} \tag{3.15a}$$
$$\begin{aligned}(\widehat{\mathbf{S}}^{\star}_{2,0}\phi)_{AA'} &= \big(\overline{\mathcal{K}}^1_{1,1}\mathscr{C}^{\dagger}_{2,0,0,2}\mathcal{K}^1_{2,0}(\kappa_1\bar{\kappa}_{1'}\phi)\big)_{AA'}\\ &= \tfrac{1}{3}(\mathscr{C}_{0,2}\bar{\kappa})^{B}{}_{A'}\big(\mathcal{K}^1_{2,0}(\kappa_1\phi)\big)_{AB} + \bar{\kappa}_{1'}\big(\overline{\mathcal{K}}^1_{1,1}\mathscr{C}^{\dagger}_{2,0}\mathcal{K}^1_{2,0}(\kappa_1\phi)\big)_{AA'}\\ &= A_{AA'}.\end{aligned} \tag{3.15b}$$
This shows that all irreducible[6] *second order symmetry operators for the Maxwell equation* (3.1) *on vacuum type D backgrounds follow from the adjoint operator argument.*

We collect further properties of the complex vector potentials in

Corollary 6. *Let ϕ_{AB} be a solution to the source-free Maxwell equation* (3.1) *on a vacuum background of Petrov type D.*

(1) *The vector potential* (3.11) *satisfies*
$$(\mathscr{D}_{1,1,2}\mathbf{S}^{\dagger}_{2,0}\phi) = 0, \tag{3.16a}$$
$$(\mathscr{C}_{1,1}\mathbf{S}^{\dagger}_{2,0}\phi)_{AB} = (\mathbf{O}_{2,0}\phi)_{AB} = 0. \tag{3.16b}$$

[6] Irreducible in the sense that they do not factor into first order symmetry operators, i.e. Lie derivatives along isometries.

The first equation can be interpreted as a generalized Lorenz gauge and the second one states that the anti-self-dual field strength of the vector potential vanishes. The self-dual field strength reads

$$\bar{\chi}_{A'B'} = (\mathscr{C}^\dagger_{1,1}\mathbf{S}^\dagger_{2,0}\phi)_{A'B'}. \tag{3.17}$$

This is the anti-linear symmetry operator of [4]. In particular it follows that taking the real part of the complex vector potential (3.11) *does not alter the (self-dual) field strength due to the complex conjugate of* (3.16b).

(2) *The vector potential* (3.13) *satisfies*

$$(\mathscr{C}^\dagger_{1,1}\widehat{\mathbf{S}}^\star_{2,0}\phi)_{A'B'} = (\widehat{\mathbf{O}}_{2,0}\phi)_{A'B'} = 0, \tag{3.18}$$

which states that its self-dual field strength vanishes. The anti-self-dual field strength reads

$$\psi_{AB} = (\mathscr{C}_{1,1}\widehat{\mathbf{S}}^\star_{2,0}\phi)_{AB}. \tag{3.19}$$

This is the linear symmetry operator of [4]. Here again it follows that taking the real part of the complex vector potential (3.11) *does not alter the (anti-self-dual) field strength due to the complex conjugate of* (3.18).

In section 4 we find that the complex potentials (metrics) in the linearized gravity case do have both self-dual and anti-self-dual field strength (curvature) and hence taking the real part does have an effect in that case.

Remark 7. *In general, for a complex vector potential $\alpha_{AA'}$, there is an anti-self-dual and a self-dual field strength*

$$\phi_{AB} = (\mathscr{C}_{1,1}\alpha)_{AB}, \qquad \bar{\pi}_{A'B'} = (\mathscr{C}^\dagger_{1,1}\alpha)_{A'B'}, \tag{3.20}$$

solving the left and right Maxwell equations

$$(\mathscr{C}^\dagger_{2,0}\phi)_{AA'} = 0, \qquad (\mathscr{C}_{0,2}\bar{\pi})_{AA'} = 0. \tag{3.21}$$

The equations are not coupled and the last equation can be read as the complex conjugate of $(\mathscr{C}^\dagger_{2,0}\pi)_{AA'} = 0$, so it is sufficient to analyze one of the equations. Also because $\mathbf{E}_{1,1}$ is a real operator, the above argument goes through for $\bar{\pi}_{A'B'}$ with the symmetry operators being complex conjugates of (3.17) *and* (3.19). *Therefore, with constants c_1, c_2, c_3, c_4, the general irreducible symmetry operator reads (in index-free notation)*

$$\begin{pmatrix} \psi \\ \bar{\chi} \end{pmatrix} = \begin{pmatrix} c_1\mathscr{C}_{1,1}\widehat{\mathbf{S}}^\star_{2,0} & c_2\mathscr{C}_{1,1}\mathbf{S}^\star_{0,2} \\ c_3\mathscr{C}^\dagger_{1,1}\mathbf{S}^\dagger_{2,0} & c_4\mathscr{C}^\dagger_{1,1}\widehat{\mathbf{S}}^\dagger_{0,2} \end{pmatrix} \begin{pmatrix} \phi \\ \bar{\pi} \end{pmatrix}. \tag{3.22}$$

The field $(\mathcal{K}^1_{2,0}\phi)_{AB}$ has components $(\mathcal{K}^1_{2,0}\phi)_0 = \phi_0, (\mathcal{K}^1_{2,0}\phi)_1 = 0, (\mathcal{K}^1_{2,0}\phi)_2 = -\phi_2$. We note the following operator identity for the $\mathbf{S}^\dagger_{2,0}$ acing on the sign-flipped field,

$$(\mathbf{S}^\dagger_{2,0}\mathcal{K}^1_{2,0}\phi)_{AA'} = (\kappa_1^2\mathcal{K}^1_{1,1}\mathscr{C}^\dagger_{2,0}\phi)_{AA'} + (\mathscr{T}_{0,0}\mathcal{K}^2_{2,0}\kappa_1^2\phi)_{AA'}, \tag{3.23}$$

which will be used in the next section. If ϕ_{AB} is a source-free Maxwell field, it states that the lhs is a pure gauge vector potential because the rhs is a gradient (the first term vanishes for Maxwell fields).

Let us finally give an interpretation of the two symmetry operators (3.17) and (3.19).

3.2. **Hertz potentials, Debye potentials and Teukolsky separability.** To discuss the anti-linear symmetry operator (3.17) we briefly recall the Hertz potential construction for spin-1 on a curved background along the lines of [13, Section III]. Let $\bar{P}_{A'B'}$ be a Hertz potential, i.e. a symmetric spinor solving the Hertz equation[7]

$$(\mathscr{C}^\dagger_{1,1}\mathscr{C}_{0,2}\overline{P})_{A'B'} = -2(\mathscr{C}^\dagger_{1,1}\mathcal{G})_{A'B'}, \tag{3.24}$$

with some (arbitrary) Nisbet gauge spinor $\mathcal{G}_{AA'}$. Then $\mathcal{A}_{AA'}$, generated via the Hertz map

$$\mathcal{A}_{AA'} = (\mathscr{C}_{0,2}\overline{P})_{AA'} + 2\mathcal{G}_{AA'}, \tag{3.25}$$

[7]On Minkowski space, setting $\mathcal{G}_{AA'} = 0$, the Hertz equation is given by $\Box\bar{P}_{A'B'} = 0$.

is a complex vector potential and its anti-self-dual field strength

$$\chi_{AB} = (\mathscr{C}_{1,1}\mathcal{A})_{AB}, \tag{3.26}$$

solves the Maxwell equation (3.1) on any background as follows from a commutator. The self-dual field strength $(\mathscr{C}^\dagger_{1,1}\mathcal{A})_{A'B'}$ vanishes identically because of (3.24). This implies that taking the real part of (3.25) does not change the field strength (3.26).

Restricting to a vacuum type D background and choosing

$$\mathcal{G}_{AA'} = \overline{P}_{A'}{}^{B'}\overline{U}_{B'A}, \tag{3.27}$$

the Hertz equation (3.24) becomes

$$(\mathscr{C}^\dagger_{1,1}\mathscr{C}_{0,2,0,-2}\overline{P})_{A'B'} = 0, \tag{3.28}$$

which is a diagonal wave operator, c.f. the complex conjugate of (2.13). It is actually the complex conjugate of the TME (3.6b) if we set

$$\overline{P}_{A'B'} = \bar\kappa^2_{1'}(\overline{\mathcal{K}}^1_{0,2}\overline{\mathcal{K}}^1_{0,2}\bar\phi)_{A'B'}. \tag{3.29}$$

With this choice of $\overline{P}_{A'B'}$ and the gauge (3.27), the field strength (3.26) of the Hertz map is the complex conjugate of the anti-linear symmetry operator (3.17). Note that only the gauge choice (3.27) converts the Hertz equation into the TME and only in this case can the Hertz map be interpreted as a symmetry operator.

Because the Hertz equation in the form (3.28) is diagonal, one can choose $\overline{P}_{A'B'}$ to have only one non-vanishing dyad component. This weighted scalar is called Debye potential and solves (up to rescaling by $\bar\kappa_{1'}$) either one of the scalar TMEs (C.1) in case it is an extreme component, or the Fackerell-Ipser equation $(\mathscr{D}_{1,1}\mathscr{T}_{0,0,-2}\mathcal{K}^2_{2,0}\phi) = 0$ in case it is the middle component. If one chooses the Hertz potential to be a (rescaled) Maxwell field then each of its three components, used as Debye potential, lead to the same new solution (3.25). To see this in terms of the symmetry operator it is important to make use of the freedom to modify the symmetry operator by terms vanishing due to the field equations[8]. We present the GHP components of the anti-linear symmetry operator (3.17) in (C.4) in the appendix and here instead look at the modified form

$$\bar\chi_{A'B'} = (\mathscr{C}^\dagger_{1,1}\mathbf{S}^\dagger_{2,0}\phi)_{A'B'} \pm (\mathscr{C}^\dagger_{1,1}\mathbf{S}^\dagger_{2,0}\mathcal{K}^1_{2,0}\phi)_{A'B'}, \tag{3.30}$$

with the second term vanishing for solutions of the field equations, see (3.23). For the plus sign the components are

$$\bar\chi_{0'} = 2\,\eth'\,\eth'(\kappa_1^2\phi_0), \qquad \bar\chi_{1'} = 2(\text{þ}'\,\eth' + \bar\tau\,\text{þ}')(\kappa_1^2\phi_0), \qquad \bar\chi_{2'} = 2\,\text{þ}'\,\text{þ}'(\kappa_1^2\phi_0), \tag{3.31}$$

depending only on ϕ_0 and for the minus sign they are

$$\bar\chi_{0'} = 2\,\text{þ}\,\text{þ}(\kappa_1^2\phi_2), \qquad \bar\chi_{1'} = 2(\text{þ}\,\eth + \bar\tau'\,\text{þ})(\kappa_1^2\phi_2), \qquad \bar\chi_{2'} = 2\,\eth\,\eth(\kappa_1^2\phi_2), \tag{3.32}$$

depending only on ϕ_2. These are the Debye maps previously discussed e.g. in [13], [17], [16]. The difference of (3.31) and (3.32) is the TSI (C.2) which leads us to interpret the anti-linear symmetry operator (3.17) as a covariant characterization of the Teukolsky-Starobinski constant, see [12] for an explicit proof in the Kerr case. From the above it also follows that the gradient term on the rhs of (3.23) maps between ingoing and outgoing radiation gauge.

The extreme components of the operator (3.17) can alternatively be made "purely angular" or "purely radial" by choosing the modification

$$\bar\chi_{A'B'} = (\mathscr{C}^\dagger_{1,1}\mathbf{S}^\dagger_{2,0}\phi)_{A'B'} \pm (\overline{\mathcal{K}}^1_{0,2}\kappa_1^2\mathscr{C}^\dagger_{1,1,-2}\mathcal{K}^1_{1,1}\mathscr{C}^\dagger_{2,0}\phi)_{A'B'}, \tag{3.33}$$

with the second term vanishing on solutions. For the plus sign the components are

$$\bar\chi_{0'} = 2\,\eth'\,\eth'(\kappa_1^2\phi_0), \tag{3.34a}$$

$$\bar\chi_{1'} = (\text{þ}\,\eth + \bar\tau'\,\text{þ})(\kappa_1^2\phi_2) + (\text{þ}'\,\eth' + \bar\tau\,\text{þ}')(\kappa_1^2\phi_0), \tag{3.34b}$$

$$\bar\chi_{2'} = 2\,\eth\,\eth(\kappa_1^2\phi_2), \tag{3.34c}$$

[8] The situation is similar to the scalar wave operator $\Box$ on Kerr and its Carter symmetry operator $Q = \nabla^a K_{ab}\nabla^b$ with K_{ab} the Killing tensor. The linear combination $Q \pm \Sigma\Box$, for a specific function Σ, is "purely radial" or "purely angular" and leads directly to separation of variables.

while for the minus sign they are

$$\bar{\chi}_{0'} = 2\,\text{þ}\,\text{þ}(\kappa_1^2\phi_2), \tag{3.35a}$$
$$\bar{\chi}_{1'} = (\text{þ}\,\eth + \bar{\tau}'\,\text{þ})(\kappa_1^2\phi_2) + (\text{þ}'\,\eth' + \bar{\tau}\,\text{þ}')(\kappa_1^2\phi_0), \tag{3.35b}$$
$$\bar{\chi}_{2'} = 2\,\text{þ}'\,\text{þ}'(\kappa_1^2\phi_0). \tag{3.35c}$$

Note that the four representations (3.31), (3.32), (3.34), (3.35) of the anti-linear symmetry operator lead to the same field $\bar{\chi}_{A'B'}$. This freedom in the representation may be important in the analysis of further properties of the symmetry operator.

Next, we consider the linear symmetry operator (3.19). Its GHP components are given in (C.3) in the appendix. A modification of the form[9]

$$\psi_{AB} = (\mathscr{C}_{1,1}\widehat{\mathbf{S}}^{\star}_{2,0}\phi)_{AB} \pm \kappa_1\bar{\kappa}_{1'}(\mathcal{K}^1_{2,0}\mathcal{K}^1_{2,0}\mathscr{C}_{1,1,-2}\mathscr{C}^{\dagger}_{2,0}\phi)_{AB}, \tag{3.36}$$

with the second term vanishing on solutions, leads to the "purely angular" extreme components

$$\psi_0 = 2\,\eth\big(\kappa_1\bar{\kappa}_{1'}(\eth' - 2\tau')\big)\phi_0 + \tfrac{1}{3}(\bar{\kappa}_{1'}\mathcal{L}_{\xi} - \kappa_1\mathcal{L}_{\bar{\xi}})\phi_0, \tag{3.37a}$$
$$\psi_2 = 2\,\eth'\big(\kappa_1\bar{\kappa}_{1'}(\eth - 2\tau)\big)\phi_2 - \tfrac{1}{3}(\bar{\kappa}_{1'}\mathcal{L}_{\xi} - \kappa_1\mathcal{L}_{\bar{\xi}})\phi_2, \tag{3.37b}$$

for the plus sign and to the "purely radial" extreme components

$$\psi_0 = 2\,\text{þ}\big(\kappa_1\bar{\kappa}_{1'}(\text{þ}' - 2\rho')\big)\phi_0 - \tfrac{1}{3}(\bar{\kappa}_{1'}\mathcal{L}_{\xi} + \kappa_1\mathcal{L}_{\bar{\xi}})\phi_0, \tag{3.38a}$$
$$\psi_2 = 2\,\text{þ}'\big(\kappa_1\bar{\kappa}_{1'}(\text{þ} - 2\rho)\big)\phi_2 + \tfrac{1}{3}(\bar{\kappa}_{1'}\mathcal{L}_{\xi} + \kappa_1\mathcal{L}_{\bar{\xi}})\phi_2, \tag{3.38b}$$

for the minus sign. See (C.5) for the Lie derivative of weighted scalars. The difference of (3.37) and (3.38) is the TME (C.1), see also [1, Section 5.4]. This leads us to interpret the linear symmetry operator (3.19) as a covariant characterization of the TME separability (and therefore of the Teukolsky separation constant, see [12] for an explicit proof in the Kerr case).

Summarized, we succeeded calculating both irreducible symmetry operators for Maxwell on vacuum type D backgrounds using the adjoint operator method and self-adjointness of the TME and TSI. In the next section we show that these ideas carry over to linearized gravity.

4. Spin-2

The linearized gravity field equations can be derived from a variation of the three irreducible curvature spinors $\Psi_{ABCD}, \Phi_{ABA'B'}$ and Λ. We use the covariant spinor variational operator ϑ developed in [7]. It is invariant under linearized tetrad rotations which allows us to do calculations covariantly. For relations to linearized dyad components (Newman-Penrose scalars), which involve the linearized tetrad, see [7, Remark 6].

Let $\delta g_{ABA'B'} = \delta g_{BAB'A'}$ be the spinorial form of a symmetric tensor field representing a linearized metric and define the irreducible parts[10]

$$G_{ABA'B'} = \delta g_{(AB)(A'B')}, \qquad \not{G} = \delta g^{C}{}_{C}{}^{C'}{}_{C'}. \tag{4.1}$$

It is convenient to introduce a modification of the linearized Weyl spinor $\vartheta\Psi_{ABCD}$[11],

$$\phi_{ABCD} = \tfrac{1}{4}\not{G}\Psi_{ABCD} + \vartheta\Psi_{ABCD}. \tag{4.2}$$

A variation of the Einstein spinor on a vacuum background without sources leads to the spinorial form of the linearized Einstein equation

$$-2\vartheta\Phi_{ABA'B'} - 6\epsilon_{AB}\bar{\epsilon}_{A'B'}\vartheta\Lambda = 0. \tag{4.3}$$

Multiplied by a factor of -2 the operator reads

$$\begin{aligned}4\vartheta\Phi_{ABA'B'} + 12\epsilon_{AB}\bar{\epsilon}_{A'B'}\vartheta\Lambda = & -\nabla_{AA'}\nabla_{BB'}\delta g^{C}{}_{C}{}^{C'}{}_{C'} + \nabla_{CC'}\nabla_{AA'}\delta g_{B}{}^{C}{}_{B'}{}^{C'} \\ & + \nabla_{CC'}\nabla_{BB'}\delta g_{A}{}^{C}{}_{A'}{}^{C'} - \nabla_{CC'}\nabla^{CC'}\delta g_{ABA'B'} \\ & - \epsilon_{AB}\bar{\epsilon}_{A'B'}\nabla_{DD'}\nabla_{CC'}\delta g^{CDC'D'} \\ & + \epsilon_{AB}\bar{\epsilon}_{A'B'}\nabla_{DD'}\nabla^{DD'}\delta g^{C}{}_{C}{}^{C'}{}_{C'}.\end{aligned} \tag{4.4}$$

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[9] Note that the modification is, up to a multiplying function, the TME operator (3.6b), c.f. footnote 8.

[10] Observe that $\delta g_{ABA'B'}$ is not a symmetric spinor, but $G_{ABA'B'}$ is.

[11] In a type D principal frame this modification only affects the middle component.

A computation shows that this operator is self-adjoint and it was therefore used in [17] as **E**. On a Petrov type D background, the irreducible components of (4.4) lead to the matrix equation

$$\begin{pmatrix} \vartheta\Phi \\ 3\vartheta\Lambda \end{pmatrix} = \begin{pmatrix} \tfrac{1}{2}\mathscr{C}^\dagger_{3,1}\mathscr{C}_{2,2} + \tfrac{1}{6}\mathscr{T}_{1,1}\mathscr{D}_{2,2} + \tfrac{1}{2}\Psi_2\mathcal{K}^1_{2,2}\mathcal{K}^1_{2,2} - \tfrac{1}{2}\Psi_2\mathcal{K}^0_{0,2}\mathcal{K}^2_{2,2} & -\tfrac{1}{8}\mathscr{T}_{1,1}\mathscr{T}_{0,0} \\ -\tfrac{1}{8}\mathscr{D}_{1,1}\mathscr{D}_{2,2} & \tfrac{3}{32}\mathscr{D}_{1,1}\mathscr{T}_{0,0} \end{pmatrix}\begin{pmatrix} G \\ \not{G} \end{pmatrix},$$

from which the self-adjointness immediately follows. Furthermore the top-left component operator is self-adjoint on its own and we will use this to define

$$\begin{aligned}(\mathbf{E}_{2,2}G)_{ABA'B'} &= \tfrac{1}{2}(\mathscr{C}^\dagger_{3,1}\mathscr{C}_{2,2}G)_{ABA'B'} + \tfrac{1}{6}(\mathscr{T}_{1,1}\mathscr{D}_{2,2}G)_{ABA'B'} + \tfrac{1}{2}\Psi_2(\mathcal{K}^1_{2,2}\mathcal{K}^1_{2,2}G)_{ABA'B'} \\ &\quad - \tfrac{1}{2}\Psi_2(\mathcal{K}^0_{0,2}\mathcal{K}^2_{2,2}G)_{ABA'B'}. \end{aligned} \tag{4.5}$$

This operator is also real and therefore we have

$$(\mathbf{E}^\dagger_{2,2}G)_{ABA'B'} = (\mathbf{E}^\star_{2,2}G)_{ABA'B'} = (\overline{\mathbf{E}}_{2,2}G)_{ABA'B'} = (\mathbf{E}_{2,2}G)_{ABA'B'}. \tag{4.6}$$

To define the **T**-operator, we consider the map from linearized metric to its Weyl-curvature on a vacuum background, $\vartheta\Psi_{ABCD} = \frac{1}{2}(\mathscr{C}_{3,1}\mathscr{C}_{2,2}G)_{ABCD} - \frac{1}{4}\not{G}\Psi_{ABCD}$. Because the trace term only contributes to the middle component on a type D background in a principal frame, and we are interested in the extreme components only, we define (c.f. (4.2))

$$(\mathbf{T}_{2,2}G)_{ABCD} = \tfrac{1}{2}(\mathscr{C}_{3,1}\mathscr{C}_{2,2}G)_{ABCD} = \phi_{ABCD}. \tag{4.7}$$

Because the equations are considerably more complicated for linearized gravity than in the spin-1 case, we investigate the TME and TSI separately in the following two subsections and interpret the resulting symmetry operators in subsection 4.3.

4.1. The TME and a fourth order symmetry operator.

In this section we derive an operator identity based on the TME, analogous to (3.10) in the spin-1 case. Because (4.5) is the linearized trace-free Ricci spinor of a trace-free metric, $(\mathbf{E}_{2,2}G)_{ABA'B'} = (\vartheta\Phi[G,0])_{ABA'B'}$, we can use the spin-2 TME with sources, derived in [2, eq.(3.13)], given by

$$\kappa_1^4(\mathcal{P}^2_{4,0}\mathcal{K}^1_{4,0}\mathscr{C}_{3,1,-4}\mathscr{C}_{2,2}\vartheta\Phi)_{ABCD} = (\mathscr{C}_{3,1}\mathscr{C}^\dagger_{4,0,4}\mathcal{K}^1_{4,0}\mathcal{P}^2_{4,0}\kappa_1^4\vartheta\Psi)_{ABCD} + 3\Psi_2(\mathcal{K}^1_{4,0}\mathcal{P}^2_{4,0}\kappa_1^4\vartheta\Psi)_{ABCD}. \tag{4.8}$$

Motivated by (4.8), define the operators[12]

$$(\mathbf{S}_{2,2}\varphi)_{ABCD} = (\kappa_1^4\mathcal{P}^2_{4,0}\mathscr{C}_{3,1,-4}\mathscr{C}_{2,2}\varphi)_{ABCD}, \tag{4.9a}$$

$$(\mathbf{O}_{4,0}\varphi)_{ABCD} = (\mathscr{C}_{3,1}\mathscr{C}^\dagger_{4,0,4}\kappa_1^4\mathcal{P}^2_{4,0}\varphi)_{ABCD} + 3(\Psi_2\kappa_1^4\mathcal{P}^2_{4,0}\varphi)_{ABCD}. \tag{4.9b}$$

Then the modified linearized curvature ϕ_{ABCD} of any source-free solution of the linearized Einstein equation (4.3) on a vacuum type D background solves the TME

$$(\mathbf{O}_{4,0}\phi)_{ABCD} = 0. \tag{4.10}$$

Remarkable properties of this operator are summarized in

Lemma 8. *The TME operator is †-self-adjoint,*

$$(\mathbf{O}^\dagger_{4,0}\varphi)_{ABCD} = (\mathbf{O}_{4,0}\varphi)_{ABCD}. \tag{4.11}$$

Furthermore it factorizes, up to a potential term, into

$$(\mathbf{O}_{4,0}\varphi)_{ABCD} = (\mathbf{F}^\dagger_{3,1}\mathbf{F}_{4,0}\varphi)_{ABCD} + 3\Psi_2\kappa_1^4(\mathcal{P}^2_{4,0}\varphi)_{ABCD}, \tag{4.12}$$

with

$$(\mathbf{F}_{4,0}\varphi)_{ABCA'} = (\mathscr{C}^\dagger_{4,0,2}\mathcal{P}^2_{4,0}\kappa_1^2\varphi)_{ABCA'}. \tag{4.13}$$

Proof. Because (4.9b) is diagonal, we can apply another (idempotent) spin-2 projector without altering the result and commute κ_1^2 out to get

$$\begin{aligned}(\mathbf{O}_{4,0}\varphi)_{ABCD} &= \kappa_1^2(\mathcal{P}^2_{4,0}\mathscr{C}_{4,0,-2}\mathscr{C}^\dagger_{4,0,2}\mathcal{P}^2_{4,0}\kappa_1^2\varphi)_{ABCD} + 3\Psi_2\kappa_1^4(\mathcal{P}^2_{4,0}\varphi)_{ABCD} \\ &= (\mathbf{F}^\dagger_{3,1}\mathbf{F}_{4,0}\varphi)_{ABCD} + 3\Psi_2\kappa_1^4(\mathcal{P}^2_{4,0}\varphi)_{ABCD},\end{aligned}$$

From this, the †-self-adjointness of $\mathbf{O}_{4,0}$ is evident. □

[12] The TME (4.8) could have been written without the $\mathcal{K}^1_{4,0}$ operators, see the proof of theorem 9 for details.

Theorem 9. *With the operators* $\mathbf{S}_{2,2}, \mathbf{E}_{2,2}, \mathbf{O}_{4,0}, \mathbf{T}_{2,2}$ *defined above we have, for any symmetric spinor* $G_{ABA'B'}$ *and scalar* $\not{G}$*, the identites*

$$(\mathbf{S}_{2,2}\mathbf{E}_{2,2}G)_{ABCD} = (\mathbf{O}_{4,0}\mathbf{T}_{2,2}G)_{ABCD}, \tag{4.14a}$$
$$(\mathbf{S}_{2,2}\mathscr{T}_{1,1}\mathscr{T}_{0,0}\not{G})_{ABCD} = 0. \tag{4.14b}$$

If $G_{ABA'B'}$ *is the trace-free part of a solution to the source-free vacuum linearized Einstein equation* (4.3) *with modified curvature* $\phi_{ABCD} = (\mathbf{T}_{2,2}G)_{ABCD}$*, then the map*

$$\phi_{ABCD} \mapsto (\mathbf{S}^{\dagger}_{4,0}\phi)_{ABA'B'} = (\mathscr{C}^{\dagger}_{3,1}\mathscr{C}^{\dagger}_{4,0,4}\mathcal{P}^{2}_{4,0}\kappa_1^4\phi)_{ABA'B'}, \tag{4.15}$$

generates a new complex solution to the source-free vacuum linearized Einstein equation.

Proof. Applying another $\mathcal{K}^1_{4,0}$ to (4.8), using $\mathcal{K}^1_{4,0}\mathcal{K}^1_{4,0}\mathcal{P}^2_{4,0} = \mathcal{P}^2_{4,0}$ and identifying the operators (4.9) leads to identity (4.14a) for the trace-free part of the linearized metric. The rhs of (4.8) does not depend on $\not{G}$ and therefore $(\mathbf{S}_{2,2}\vartheta\Phi[0,\not{G}])_{A'B'C'D'} = 0$, where $(\vartheta\Phi[0,\not{G}])_{ABA'B'} = (\mathscr{T}_{1,1}\mathscr{T}_{0,0}\not{G})_{ABA'B'}$. This proves the identity (4.14b).

From theorem 1 together with the self-adjointness of $\mathbf{E}_{2,2}$ and $\mathbf{O}_{4,0}$ given in (4.6) and (4.11), respectively, it follows that (4.15) maps into the kernel of $\mathbf{E}_{2,2}$. The adjoint of (4.14b) yields[13]

$$\mathscr{D}_{1,1}\mathscr{D}_{2,2}\mathbf{S}^{\dagger}_{4,0}\phi = 0, \tag{4.16}$$

which ensures that (4.15) has vanishing Ricci scalar curvature. Hence it generates new solutions to linearized gravity from solutions to the TME (4.10). □

For later reference, we define

$$h_{ABA'B'} = (\mathbf{S}^{\dagger}_{4,0}\phi)_{ABA'B'} \tag{4.17}$$

for the new complex metric generated from TME solutions via (4.15). Analogous to corollary 6 for spin-1, we collect the curvature of the new solution in

Lemma 10. *The complex linearized metric* (4.17) *has self-dual and anti-self-dual curvature*

$$\begin{aligned}\bar{\chi}_{A'B'C'D'} &= (\overline{\mathbf{T}}_{2,2}h)_{A'B'C'D'} \\ &= \tfrac{1}{2}(\mathscr{C}^{\dagger}_{1,3}\mathscr{C}^{\dagger}_{2,2}\mathscr{C}^{\dagger}_{3,1}\mathscr{C}^{\dagger}_{4,0,4}\mathcal{P}^2_{4,0}\kappa_1^4\phi)_{A'B'C'D'},\end{aligned} \tag{4.18a}$$
$$\begin{aligned}\psi_{ABCD} &= (\mathbf{T}_{2,2}h)_{ABCD} \\ &= \tfrac{1}{2}(\mathscr{C}_{3,1}\mathscr{C}_{2,2}\mathscr{C}^{\dagger}_{3,1}\mathscr{C}^{\dagger}_{4,0,4}\mathcal{P}^2_{4,0}\kappa_1^4\phi)_{ABCD} \\ &= \tfrac{1}{2}\Psi_2\kappa_1^3(\mathcal{L}_\xi\mathcal{K}^1_{4,0}\mathcal{P}^2_{4,0}\phi)_{ABCD}.\end{aligned} \tag{4.18b}$$

Proof. The first equation is just an expansion of the operators and (4.18b) follows from

$$\psi_{ABCD} = \tfrac{1}{2}(\mathscr{C}_{3,1}\mathscr{C}^{\dagger}_{4,0}\mathbf{O}_{4,0}\phi)_{ABCD} - \tfrac{1}{2}\Psi_2(\mathbf{O}_{4,0}\phi)_{ABCD} + \tfrac{1}{2}\Psi_2\kappa_1^3(\mathcal{L}_\xi\mathcal{K}^1_{4,0}\mathcal{P}^2_{4,0}\phi)_{ABCD}, \tag{4.19}$$

which is an operator identity when starting with the first equality in (4.18b). This operator identity follows from a lengthy calculation involving the commutator of $\mathscr{C}_{2,2}\mathscr{C}^{\dagger}_{3,1}$. □

Note that the anti-self-dual curvature (4.18b) reduces to first order and (4.19) is the analog of (3.16b). Restricting to the real or imaginary part of (4.15) leads to a mixture of the two curvatures. An interpretation of the symmetry operator is given in subsection 4.3.

4.2. **The TSI and a sixth order symmetry operator.**

In this section we derive an operator identity based on the TSI, analogous to (3.12) in the spin-1 case. The covariant form of the spin-2 TSI can be deduced from the identity, derived in [2, eq.(4.19)],

$$(\mathbf{S}^{\dagger}_{4,0}\mathcal{K}^1_{4,0}\phi)_{ABA'B'} = \tfrac{1}{2}\Psi_2\kappa_1^3(\mathcal{L}_\xi G)_{ABA'B'} + (\widehat{\mathbf{N}}_{2,2}\vartheta\Phi)_{ABA'B'} + (\mathscr{T}_{1,1}\mathcal{A})_{ABA'B'}, \tag{4.20}$$

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[13] Alternatively it follows from a direct commutator calculation.

where the $\mathbf{S}^{\dagger}_{4,0}$ operator is given in (4.15) and

$$\begin{aligned}(\widehat{\mathbf{N}}_{2,2}\varphi)_{ABA'B'} ={}& -\kappa_1^4(\mathcal{K}^0_{0,2}\mathcal{K}^2_{2,2}\mathscr{T}_{1,1,-7}\mathcal{K}^2_{3,1}\mathscr{C}_{2,2}\varphi)_{ABA'B'} - 3\Psi_2\kappa_1^4(\mathcal{K}^1_{2,2}\varphi)_{ABA'B'}\\ &+\kappa_1^4(\mathcal{K}^1_{2,2}\mathscr{C}^{\dagger}_{3,1,-4}\mathscr{C}_{2,2}\varphi)_{ABA'B'} + \tfrac{4}{3}\kappa_1^4(\mathscr{T}_{1,1,-7}\mathcal{K}^2_{3,1}\mathscr{C}_{2,2}\varphi)_{ABA'B'}, \qquad (4.21a)\\ \mathcal{A}_{AA'} ={}& -\tfrac{1}{2}\Psi_2\kappa_1^3\xi^{BB'}(\mathcal{K}^0_{0,2}\mathcal{K}^2_{2,2}G)_{ABA'B'} + \tfrac{2}{3}\kappa_1^3\xi^{B}{}_{A'}(\mathcal{K}^1_{2,0}\mathcal{K}^2_{4,0}\phi)_{AB}\\ &+(\mathcal{K}^1_{1,1}\mathscr{T}_{0,0}\mathcal{K}^2_{2,0}\mathcal{K}^2_{4,0}(\kappa_1^4\phi))_{AA'} - \tfrac{1}{4}\Psi_2\kappa_1^4(\mathcal{K}^1_{1,1}\mathscr{T}_{0,0,2}\not{G})_{AA'}. \qquad (4.21b)\end{aligned}$$

The left (and therefore also the right) hand side of (4.20) can be shown to be a complex, trace-free solution to the linearized Einstein equation, see [2][14]. By applying $\frac{1}{2}\mathscr{C}^{\dagger}_{1,3}\mathscr{C}^{\dagger}_{2,2}$, or equivalently the $\overline{\mathbf{T}}_{2,2}$ operator, we get the corresponding self-dual curvature. It turns out to be convenient to apply $\overline{\mathcal{P}}^2_{0,4}$ to pick out the extreme components and a $\overline{\mathcal{K}}^1_{0,4}$ operator to flip the sign on one of them. This combination of operators on the Lie derivative term just gives a Lie derivative of the complex conjugated curvature. The $(\mathscr{T}_{1,1}\mathcal{A})_{ABA'B'}$ term is the trace-free part of a linearized diffeomorphism, and will therefore not contribute to the gauge independent extreme components of the curvature. As we will see in Theorem 12, the remaining terms can be compactly expressed in terms of the operators

$$\begin{aligned}(\widehat{\mathbf{O}}_{4,0}\varphi)_{A'B'C'D'} &= 2\kappa_1^{-1}\bar\kappa_{1'}^3(\overline{\mathcal{P}}^2_{0,4}\overline{\mathcal{K}}^1_{0,4}\overline{\mathbf{T}}_{2,2}\mathbf{S}^{\dagger}_{4,0}\mathcal{K}^1_{4,0}\varphi)_{A'B'C'D'}\\ &= \kappa_1^3\bar\kappa_{1'}^3(\overline{\mathcal{P}}^2_{0,4}\overline{\mathcal{K}}^1_{0,4}\mathscr{C}^{\dagger}_{1,3,-4}\mathscr{C}^{\dagger}_{2,2,-4}\mathscr{C}^{\dagger}_{3,1,-4}\mathscr{C}^{\dagger}_{4,0}\mathcal{K}^1_{4,0}\mathcal{P}^2_{4,0}\varphi)_{A'B'C'D'}, \qquad (4.22a)\\ (\widehat{\mathbf{S}}_{2,2}\varphi)_{A'B'C'D'} &= 2\kappa_1^{-1}\bar\kappa_{1'}^3(\overline{\mathcal{P}}^2_{0,4}\overline{\mathcal{K}}^1_{0,4}\overline{\mathbf{T}}_{2,2}\widehat{\mathbf{N}}_{2,2}\varphi)_{A'B'C'D'}, \qquad (4.22b)\\ (\widehat{\mathbf{L}}_{0,4}\varphi)_{A'B'C'D'} &= \Psi_2\kappa_1^2\bar\kappa_{1'}^3(\overline{\mathcal{P}}^2_{0,4}\overline{\mathcal{K}}^1_{0,4}\mathcal{L}_{\xi}\varphi)_{A'B'C'D'}. \qquad (4.22c)\end{aligned}$$

In particular, the modified curvature $\phi_{ABCD} = (\mathbf{T}_{2,2}G)_{ABCD}$ of any source-free solution to the linearized Einstein equation (4.4) on a vacuum type D background solves the TSI

$$(\widehat{\mathbf{O}}_{4,0}\phi)_{A'B'C'D'} - (\widehat{\mathbf{L}}_{0,4}\overline{\phi})_{A'B'C'D'} = 0. \qquad (4.23)$$

Remarkable properties of these operators are summarized in

Lemma 11. *The operator* (4.22a) *is* $\star$*-self-adjoint,*

$$(\widehat{\mathbf{O}}^{\star}_{4,0}\varphi)_{A'B'C'D'} = (\widehat{\mathbf{O}}_{4,0}\varphi)_{A'B'C'D'}, \qquad (4.24)$$

and the Lie derivative term (4.22c) *is* †*-self-adjoint,*

$$\big(\widehat{\mathbf{L}}^{\dagger}_{0,4}\varphi\big)_{A'B'C'D'} = (\widehat{\mathbf{L}}_{0,4}\varphi)_{A'B'C'D'}. \qquad (4.25)$$

Furthermore (4.22a) *factorizes into*

$$(\widehat{\mathbf{O}}_{4,0}\varphi)_{A'B'C'D'} = -(\widehat{\mathbf{F}}^{\star}_{2,2}\widehat{\mathbf{F}}_{4,0}\varphi)_{A'B'C'D'}, \qquad (4.26)$$

with

$$(\widehat{\mathbf{F}}_{4,0}\varphi)_{ABA'B'} = (\mathscr{C}^{\dagger}_{3,1}\mathscr{C}^{\dagger}_{4,0}\mathcal{K}^1_{4,0}\kappa_1^3\mathcal{P}^2_{4,0}\varphi)_{ABA'B'}. \qquad (4.27)$$

Proof. Using an extended index identity, we get

$$\begin{aligned}(\widehat{\mathbf{O}}_{4,0}\varphi)_{A'B'C'D'} &= (\overline{\mathcal{P}}^2_{0,4}\bar\kappa_{1'}^3\overline{\mathcal{K}}^1_{0,4}\mathscr{C}^{\dagger}_{1,3}\mathscr{C}^{\dagger}_{2,2}\mathscr{C}^{\dagger}_{3,1}\mathscr{C}^{\dagger}_{4,0}\mathcal{K}^1_{4,0}\kappa_1^3\mathcal{P}^2_{4,0}\varphi)_{A'B'C'D'}\\ &= -(\widehat{\mathbf{F}}^{\star}_{2,2}\widehat{\mathbf{F}}_{4,0}\varphi)_{A'B'C'D'}.\end{aligned}$$

From this, the $\star$-self-adjointness of $\widehat{\mathbf{O}}_{4,0}$ is evident. (4.25) follows directly from the adjoints given in appendix A. □

Theorem 12. *With the operators* $\widehat{\mathbf{S}}_{2,2}, \mathbf{E}_{2,2}, \widehat{\mathbf{O}}_{4,0}, \mathbf{T}_{2,2}$ *defined above we have, for any symmetric spinor* $G_{ABA'B'}$ *and scalar* $\not{G}$*, the identities*

$$\begin{aligned}(\widehat{\mathbf{S}}_{2,2}\mathbf{E}_{2,2}G)_{A'B'C'D'} &= (\widehat{\mathbf{O}}_{4,0}\mathbf{T}_{2,2}G)_{A'B'C'D'} - (\widehat{\mathbf{L}}_{0,4}\overline{\mathbf{T}}_{2,2}G)_{A'B'C'D'}, \qquad (4.28a)\\ (\widehat{\mathbf{S}}_{2,2}\mathscr{T}_{1,1}\mathscr{T}_{0,0}\not{G})_{ABCD} &= 0. \qquad (4.28b)\end{aligned}$$

[14]The analogue of (4.20) in the spin-1 case is (3.23) and there it is a pure gauge vector potential. Here the last term on the rhs of (4.20) is the trace-free part of a linearized diffeomorphism but the first term is not.

If $G_{ABA'B'}$ is the trace-free part of a solution to the source-free vacuum linearized Einstein equation (4.3) *with modified curvature $\phi_{ABCD} = (\mathbf{T}_{2,2}G)_{ABCD}$, then the real part of the map*

$$\phi_{ABCD} \mapsto (\widehat{\mathbf{S}}^{\star}_{4,0}\phi)_{ABA'B'} = -2(\widehat{\mathbf{N}}^{\star}_{2,2}\mathbf{T}^{\dagger}_{4,0}\mathcal{K}^{1}_{4,0}\mathcal{P}^{2}_{4,0}\kappa_1^3\bar{\kappa}_{1'}^{-1}\phi)_{ABA'B'} \tag{4.29}$$

generates a new solution to the source-free vacuum linearized Einstein equation. Here

$$\begin{aligned}(\widehat{\mathbf{N}}^{\star}_{2,2}\varphi)_{ABA'B'} = {}& -\bar{\kappa}_{1'}^4(\mathscr{C}_{1,3,0,-4}\mathscr{C}^{\dagger}_{2,2}\overline{\mathcal{K}}^{1}_{2,2}\varphi)_{ABA'B'} + \tfrac{1}{3}\bar{\kappa}_{1'}^4(\mathscr{C}_{1,3,0,-4}\overline{\mathcal{K}}^{0}_{1,1}\mathscr{D}_{2,2,0,3}\varphi)_{ABA'B'}\\ &-\tfrac{1}{4}\bar{\kappa}_{1'}^4(\mathscr{C}_{1,3,0,-4}\overline{\mathcal{K}}^{0}_{1,1}\mathscr{D}_{2,2,0,3}\overline{\mathcal{K}}^{0}_{2,0}\overline{\mathcal{K}}^{2}_{2,2}\varphi)_{ABA'B'} + 3\bar{\Psi}_2\bar{\kappa}_{1'}^4(\overline{\mathcal{K}}^{1}_{2,2}\varphi)_{ABA'B'},\end{aligned} \tag{4.30a}$$

$$(\mathbf{T}^{\dagger}_{4,0}\varphi)_{ABA'B'} = \tfrac{1}{2}(\mathscr{C}^{\dagger}_{3,1}\mathscr{C}^{\dagger}_{4,0}\varphi)_{ABA'B'}. \tag{4.30b}$$

Proof. Applying the operator $\overline{\mathcal{P}}^{2}_{0,4}\overline{\mathcal{K}}^{1}_{0,4}\overline{\mathbf{T}}_{2,2}$ to the identity (4.20) and identifying the different pieces with the operators (4.22) gives the relation (4.28a) after finding that the $(\mathscr{T}_{1,1}\mathcal{A})_{ABA'B'}$ term can be seen as the trace-free part of a linearized diffeomorphism, which will not contribute to the gauge invariant extreme components of the curvature. The only terms in (4.20) that depends on $\not{G}$ are $(\mathscr{T}_{1,1}\mathcal{A})_{ABA'B'}$ and $(\widehat{\mathbf{N}}_{2,2}\vartheta\Phi)_{ABA'B'}$, but as we have already concluded, the $(\mathscr{T}_{1,1}\mathcal{A})_{ABA'B'}$ term does not contribute to the extreme components of the curvature. Therefore $(\widehat{\mathbf{S}}_{2,2}\vartheta\Phi[0,\not{G}])_{A'B'C'D'} = 0$, where $(\vartheta\Phi[0,\not{G}])_{ABA'B'} = (\mathscr{T}_{1,1}\mathscr{T}_{0,0}\not{G})_{ABA'B'}$. This gives the identity (4.28b).

The $\star$-adjoint of (4.28a), using the self-adjointness of the operators $\widehat{\mathbf{O}}_{4,0}$ (4.24), $\widehat{\mathbf{L}}_{0,4}$ (4.25) and $\mathbf{E}_{2,2}$ (4.6), yields

$$(\mathbf{E}_{2,2}\widehat{\mathbf{S}}^{\star}_{4,0}\varphi)_{ABA'B'} = (\mathbf{T}^{\star}_{0,4}\widehat{\mathbf{O}}_{4,0}\varphi)_{ABA'B'} - (\overline{\mathbf{T}}^{\star}_{4,0}\overline{\widehat{\mathbf{L}}}_{4,0}\varphi)_{ABA'B'}. \tag{4.31}$$

We want to use a solution of the TSI (4.23) to generate new solutions, but the rhs of the last equation is not of that form. However, taking the real part of the "new metric", we find

$$(\mathbf{E}_{2,2}(\widehat{\mathbf{S}}^{\star}_{4,0}\phi + \widehat{\mathbf{S}}^{\dagger}_{0,4}\bar{\phi}))_{ABA'B'} = (\mathbf{T}^{\star}_{0,4}(\widehat{\mathbf{O}}_{4,0}\phi - \widehat{\mathbf{L}}_{0,4}\bar{\phi}))_{ABA'B'} + (\mathbf{T}^{\dagger}_{4,0}(\overline{\widehat{\mathbf{O}}}_{0,4}\bar{\phi} - \overline{\widehat{\mathbf{L}}}_{4,0}\phi))_{ABA'B'}. \tag{4.32}$$

Now the two terms on the rhs contain the rhs of (4.28a) and its complex conjugate. Therefore solutions of the TSI (4.23) generate metrics in the kernel of the $\mathbf{E}$ operator. The adjoint of (4.28b) gives

$$\mathscr{D}_{1,1}\mathscr{D}_{2,2}\widehat{\mathbf{S}}^{\star}_{4,0}\phi = 0. \tag{4.33}$$

As the $\mathscr{D}_{1,1}\mathscr{D}_{2,2}$ operator is real, we see that also the linearized scalar curvature of the new metric vanishes. Hence, the mapping $\phi_{ABCD} \mapsto \mathrm{Re}(\widehat{\mathbf{S}}^{\star}_{4,0}\phi)_{ABA'B'}$ to the real part generates new solutions to linearized gravity from solutions to the TSI equation (4.23). □

For later reference, we define

$$k_{ABA'B'} = \mathrm{Re}(\widehat{\mathbf{S}}^{\star}_{4,0}\phi)_{ABA'B'} \tag{4.34}$$

for the new real metric generated from TSI solutions via (4.29). Since the metric is real, its self-dual curvature is the complex conjugate of the anti-self-dual curvature and hence it is sufficient to calculate the latter via

$$\begin{aligned}\psi_{ABCD} &= (\mathbf{T}_{2,2}k)_{ABCD}\\ &= \tfrac{1}{2}(\mathbf{T}_{2,2}\widehat{\mathbf{S}}^{\star}_{4,0}\phi)_{ABCD} + \tfrac{1}{2}(\mathbf{T}_{2,2}\widehat{\mathbf{S}}^{\dagger}_{0,4}\bar{\phi})_{ABCD}.\end{aligned} \tag{4.35}$$

The second term is the complex conjugate of $(\overline{\mathbf{T}}_{2,2}\widehat{\mathbf{S}}^{\star}_{4,0}\phi)_{A'B'C'D'}$. The first part cannot be reduced by the field equations, but at least the extreme components of the second part can be reduced due to the following identity,

$$\big(\overline{\mathcal{P}}^{2}_{0,4}\mathscr{C}^{\dagger}_{1,3}\mathscr{C}^{\dagger}_{2,2}(\widehat{\mathbf{S}}^{\star}_{4,0}\phi)\big)_{A'B'C'D'} = (\overline{\mathcal{P}}^{2}_{0,4}\mathscr{C}^{\dagger}_{1,3}\mathscr{C}_{0,4}\widehat{\mathbf{O}}_{4,0}\phi)_{A'B'C'D'} - \bar{\Psi}_2(\overline{\mathcal{P}}^{2}_{0,4}\widehat{\mathbf{O}}_{4,0}\phi)_{A'B'C'D'}. \tag{4.36}$$

In the next subsection we give an interpretation of this symmetry operator.

4.3. **Hertz potentials, Debye potentials and Teukolsky separability.** To discuss the complex metric (4.17), we briefly recall the Hertz potential construction for linearized gravity on a vacuum type D background similar to [13, Section V]. Let $\bar{P}_{A'B'C'D'}$ be a Hertz potential with vanishing non-extreme components w.r.t. a principal dyad, i.e. a symmetric spinor solving the Hertz equation

$$(\mathscr{C}^\dagger_{1,3}\mathscr{C}_{0,4,0,4}\bar{P})_{A'B'C'D'} + 3\bar{\Psi}_2\bar{P}_{A'B'C'D'} = 0. \tag{4.37}$$

Then the complex symmetric spinor $H_{ABA'B'}$, generated via the Hertz map

$$H_{ABA'B'} = (\mathscr{C}_{1,3}\mathscr{C}_{0,4,0,4}\bar{P})_{ABA'B'}, \tag{4.38}$$

solves the linearized Einstein equation (4.3).

Remark 13. *In [13, Section V] the more general Hertz equation*

$$(\mathscr{C}^\dagger_{1,3}\mathscr{C}_{0,4}\bar{P})_{A'B'C'D'} + 3\bar{P}_{A'B'C'D'}\bar{\Psi}_2 = (\mathscr{C}^\dagger_{1,3}\mathcal{G})_{A'B'C'D'}, \tag{4.39}$$

and Hertz map

$$H_{ABA'B'} = (\mathscr{C}_{1,3}\mathscr{C}_{0,4}\bar{P})_{ABA'B'} - (\mathscr{C}_{1,3}\mathcal{G})_{ABA'B'}, \tag{4.40}$$

with a "gauge" spinor $\mathcal{G}_{AA'B'C'}$, were proposed. We checked that the linearized Ricci scalar vanishes, $\vartheta\Lambda[H] = 0$, but for the trace-free Ricci spinor components, we find e.g.

$$(\mathbf{E}_{2,2}H)_{00'} = -6\bar{\Psi}_2(\eth + \bar{\tau}')(\tfrac{1}{4}\mathcal{G}_{00'} + \bar{P}_{0'}\bar{\tau}'), \tag{4.41}$$

which fixes $\mathcal{G}_{00'}$. Therefore $\mathcal{G}_{AA'B'C'}$ is not a freely specifiable gauge field on a curved background. For the case $\mathcal{G}_{AA'B'C'} = -4\bar{P}_{A'B'C'D'}\bar{U}^{D'}{}_A$, which has components $\mathcal{G}_{00'} = -4\bar{P}_{0'}\bar{\tau}', \mathcal{G}_{01'} = 0, \mathcal{G}_{02'} = 0$, $\mathcal{G}_{03'} = 4\bar{P}_{4'}\bar{\rho}, \mathcal{G}_{10'} = -4\bar{P}_{0'}\bar{\rho}', \mathcal{G}_{11'} = 0, \mathcal{G}_{12'} = 0, \mathcal{G}_{13'} = 4\bar{P}_{4'}\bar{\tau}$ we end up with (4.37), (4.38). *This choice was made in the course of the proof in [13, Section V].*

The operator in (4.37) is diagonal, c.f. the complex conjugate of (2.13). It is actually the complex conjugate of the TME operator (4.9b) if we set

$$\overline{P}_{A'B'C'D'} = \bar{\kappa}^4_{1'}(\bar{\mathcal{P}}^2_{0,4}\bar{\phi})_{A'B'C'D'}. \tag{4.42}$$

With this choice of $\overline{P}_{A'B'C'D'}$, the Hertz map (4.38) is the complex conjugate of the symmetry operator (4.15) viewed as a map from metric to metric, $H_{ABA'B'} = \bar{h}_{ABA'B'}$. This shows that the spin-2 Hertz potential formalism on vacuum type D backgrounds can be understood as the symmetry operator (4.15).

Because (4.37) is diagonal, one can choose the Hertz potential $\overline{P}_{A'B'C'D'}$ to have only one non-vanishing extreme dyad component. This weighted scalar is called Debye potential and solves (up to rescaling by $\bar{\kappa}_{1'}$) by construction one of the complex conjugated scalar TMEs (C.6).

If we use the linearized curvature as a Hertz potential via (4.42), the extreme components used as Debye potentials generate different new solutions to the linearized Einstein equation. However, the difference is not very complicated and we derive it explicitly. We do this in two steps. First a modification of the symmetry operator analogous to (3.30) is made, but this modification is not pure gauge on a curved background. Then in the second step we add the correction term to have a pure gauge modification and to show that both extreme curvature scalars generate the same new solution to linearized gravity.

A modification of the symmetry operator (4.15) of the form[15]

$$\underline{h}^{\pm}_{ABA'B'} = (\mathbf{S}^\dagger_{4,0}\phi)_{ABA'B'} \pm (\mathbf{S}^\dagger_{4,0}\mathcal{K}^1_{4,0}\phi)_{ABA'B'} \tag{4.43}$$

is again a complex solution to linearized gravity and it depends only on one of the extreme curvature scalars (ϕ_0 for $+$ and ϕ_4 for $-$). These are the Debye potential maps given by the Hertz map (4.38) with one of the extreme components set to zero. However, we will see that $\underline{h}^+ \neq \underline{h}^-$ and contrary to the spin-1 case the difference is not pure gauge. For completeness we present the components of the self-dual curvature $\underline{\bar{\chi}}^{\pm}_{A'B'C'D'} = (\overline{\mathbf{T}}_{2,2}\underline{h}^{\pm})_{A'B'C'D'}$ and anti-self-dual curvature $\underline{\psi}^{\pm}_{ABCD} = (\mathbf{T}_{2,2}\underline{h}^{\pm})_{ABCD}$ of (4.43) for the plus sign in (C.9) and for the minus sign in (C.10) in the appendix.

[15]The second term is a solution on its own due to (4.20), but it is not pure gauge.

For the second step we note that a pure gauge metric can be constructed from (4.20) and in the source-free case we have analogous to (3.23)

$$(\mathbf{S}^{\dagger}_{4,0}\mathcal{K}^{1}_{4,0}\phi)_{ABA'B'} - \tfrac{1}{2}\Psi_2\kappa_1^3(\mathcal{L}_\xi G)_{ABA'B'} = (\mathscr{T}_{1,1}\mathcal{A})_{ABA'B'}, \tag{4.44a}$$
$$-\tfrac{1}{2}\Psi_2\kappa_1^3(\mathcal{L}_\xi \not{G}) = (\mathscr{D}_{1,1}\mathcal{A}), \tag{4.44b}$$

where the second equation is given in [2, eq.(4.13)]. The rhs are the trace-free and trace parts of a linearized diffeomorphism[16], so we can add/subtract this to/from the linearized metric (4.17) without changing the actual perturbation[17],

$$h_{ABA'B'} = (\mathbf{S}^{\dagger}_{4,0}\phi)_{ABA'B'} \pm \big((\mathbf{S}^{\dagger}_{4,0}\mathcal{K}^{1}_{4,0}\phi)_{ABA'B'} - \tfrac{1}{2}\Psi_2\kappa_1^3(\mathcal{L}_\xi G)_{ABA'B'}\big) \tag{4.45a}$$
$$= \underline{h}^{\pm}_{ABA'B'} \mp \tfrac{1}{2}\Psi_2\kappa_1^3(\mathcal{L}_\xi G)_{ABA'B'}, \tag{4.45b}$$
$$\not{h} = \mp \tfrac{1}{2}\Psi_2\kappa_1^3(\mathcal{L}_\xi \not{G}). \tag{4.45c}$$

It follows that the difference between $\underline{h}^{+}$ and $\underline{h}^{-}$ is a Lie derivative and a gauge transformation. Also note that the gauge transformation introduces a trace to the linearized metric.

We now want to study the extreme components of the self-dual and anti-self-dual curvatures of the complex metric $h_{ABA'B'}$. From (4.45b), (C.9) and (C.10) we get

$$\bar{\chi}_{0'} = \eth'\eth'\eth'\eth'(\kappa_1^4\phi_0) - \tfrac{1}{2}\Psi_2\kappa_1^3\mathcal{L}_\xi\bar{\phi}_{0'}, \qquad \bar{\chi}_{4'} = \text{þ}'\text{þ}'\text{þ}'\text{þ}'(\kappa_1^4\phi_0) - \tfrac{1}{2}\Psi_2\kappa_1^3\mathcal{L}_\xi\bar{\phi}_{4'}, \tag{4.46a}$$
$$\psi_0 = \tfrac{1}{2}\Psi_2\kappa_1^3\mathcal{L}_\xi\phi_0, \qquad \psi_4 = -\tfrac{1}{2}\Psi_2\kappa_1^3\mathcal{L}_\xi\phi_4, \tag{4.46b}$$

for the plus case and

$$\bar{\chi}_{0'} = \text{þþþþ}(\kappa_1^4\phi_4) + \tfrac{1}{2}\Psi_2\kappa_1^3\mathcal{L}_\xi\bar{\phi}_{0'}, \qquad \bar{\chi}_{4'} = \eth\eth\eth\eth(\kappa_1^4\phi_4) + \tfrac{1}{2}\Psi_2\kappa_1^3\mathcal{L}_\xi\bar{\phi}_{4'}, \tag{4.47a}$$
$$\psi_0 = \tfrac{1}{2}\Psi_2\kappa_1^3\mathcal{L}_\xi\phi_0, \qquad \psi_4 = -\tfrac{1}{2}\Psi_2\kappa_1^3\mathcal{L}_\xi\phi_4, \tag{4.47b}$$

for the minus case. As the corresponding metrics only differ by a linearized diffeomorphism, these extreme curvature components are the same. Restricting to the real or imaginary part of the metric (4.45a) leads to linear combinations $\frac{1}{2}(\bar{\chi}_n+\overline{\psi}_n)$ or $\frac{1}{2i}(\bar{\chi}_n-\overline{\psi}_n)$ for the self-dual curvature. The Lie derivative terms makes both (4.46) and (4.47) dependent of ϕ_0 and ϕ_4. However, if we study the imaginary part of $h_{ABA'B'}$ we get for the Kerr case $\mathrm{Im}(h)_{ABA'B'} = \mathrm{Im}(\underline{h}^{\pm})_{ABA'B'}$ because $G_{ABA'B'}$ and $\xi_{AA'}$ are real and $\Psi_2\kappa_1^3$ is a real constant in that case. Hence, it reduces down to the Debye potential case with self-dual curvature $\frac{1}{2i}(\underline{\bar{\chi}}^{\pm}_n-\overline{\underline{\psi}}^{\pm}_n)$ given by (C.9) and (C.10).

An alternative point of view can be obtained by noting that $\overline{\mathcal{P}}^{2}_{0,4}\overline{\mathbf{T}}_{2,2}$ on (4.44a) gives

$$(\overline{\mathcal{P}}^{2}_{0,4}\overline{\mathbf{T}}_{2,2}\mathscr{T}_{1,1}\mathcal{A})_{A'B'C'D'} = (\overline{\mathcal{P}}^{2}_{0,4}\overline{\mathbf{T}}_{2,2}\mathbf{S}^{\dagger}_{4,0}\mathcal{K}^{1}_{4,0}\phi)_{A'B'C'D'} - \tfrac{1}{2}\Psi_2\kappa_1^3(\overline{\mathcal{P}}^{2}_{0,4}\mathcal{L}_\xi\bar{\phi})_{A'B'C'D'} \tag{4.48a}$$
$$= \tfrac{1}{2}\kappa_1\bar{\kappa}_{1'}^{-3}\big((\overline{\mathcal{K}}^{1}_{0,4}\widehat{\mathbf{O}}_{4,0}\phi)_{A'B'C'D'} - (\overline{\mathcal{K}}^{1}_{0,4}\widehat{\mathbf{L}}_{0,4}\bar{\phi})_{A'B'C'D'}\big). \tag{4.48b}$$

The lhs vanishes because the extreme curvature components are gauge invariant and the rhs vanishes because of the TSI. Summarized, the extreme components of ϕ_{ABCD} can be used as a Hertz potential via (4.42), or each one of them as a Debye potential. Our analysis shows that the difference between these three possibilities are the TSI and Lie derivatives.

Similar to (3.34) and (3.35) for the spin-1 case, we can write the extreme components of the curvature of (4.17) in "purely angular" or "purely radial" form by adding/subtracting the TSI in a different way,

$$\bar{\chi}_{A'B'C'D'} = (\overline{\mathbf{T}}_{2,2}\mathbf{S}^{\dagger}_{4,0}\phi)_{A'B'C'D'} \pm \tfrac{1}{2}\kappa_1\bar{\kappa}_{1'}^{-3}\big((\widehat{\mathbf{O}}_{4,0}\phi)_{A'B'C'D'} - (\widehat{\mathbf{L}}_{0,4}\bar{\phi})_{A'B'C'D'}\big). \tag{4.49}$$

Then the components read

$$\bar{\chi}_{0'} = \eth'\eth'\eth'\eth'(\kappa_1^4\phi_0) - \tfrac{1}{2}\Psi_2\kappa_1^3\mathcal{L}_\xi\bar{\phi}_{0'}, \tag{4.50a}$$
$$\bar{\chi}_{4'} = \eth\eth\eth\eth(\kappa_1^4\phi_4) + \tfrac{1}{2}\Psi_2\kappa_1^3\mathcal{L}_\xi\bar{\phi}_{4'}. \tag{4.50b}$$

[16] One can discuss the real and imaginary parts separately and deal with real diffeomorphisms.

[17] We understand $(h_{ABA'B'}, \not{h})$ as a representative of gauge equivalent metrics, and therefore the equalities here are up to gauge.

for the plus sign and

$$\bar{\chi}_{0'} = \text{þ}\,\text{þ}\,\text{þ}\,\text{þ}(\kappa_1^4\phi_4) + \tfrac{1}{2}\Psi_2\kappa_1^3\mathcal{L}_\xi\bar{\phi}_{0'}, \tag{4.51a}$$

$$\bar{\chi}_{4'} = \text{þ}'\,\text{þ}'\,\text{þ}'\,\text{þ}'(\kappa_1^4\phi_0) - \tfrac{1}{2}\Psi_2\kappa_1^3\mathcal{L}_\xi\bar{\phi}_{4'}. \tag{4.51b}$$

for the minus sign. These alternative forms of the symmetry operator play an important role if one wants to invert them. This will be analyzed in a separate paper.

Finally, we consider the GHP form of the extreme components of the curvature (4.35) given by the sixth order symmetry operator on a Schwarzschild background to verify that the operator is non-trivial. As stated in (4.35), we have the curvature

$$\psi_{ABCD} = \tfrac{1}{2}(\mathbf{T}_{2,2}\widehat{\mathbf{S}}^{\star}_{4,0}\phi)_{ABCD} + \tfrac{1}{2}(\mathbf{T}_{2,2}\widehat{\mathbf{S}}^{\dagger}_{0,4}\bar{\phi})_{ABCD}. \tag{4.52}$$

Now, we will verify that it is non-trivial by reducing it with respect to the field equations. As we assume that ϕ_{ABCD} satisfies both TME and TSI, we can use the identity (4.36) to reduce the order of the second term due to the TSI. The first term however, will remain sixth order, but we can use the TME to eliminate the $\text{þ}\,\text{þ}'$ derivatives after commutations. The lower order terms from the second term in (4.52) cancel with the lower order terms from the first term in (4.52). After straightforward, but tedious calculations, we end up with

$$\psi_0 = \kappa_1^6(\eth\,\eth' + 2\Psi_2 + 2\rho\rho')(\eth\,\eth' + \Psi_2 + \rho\rho')\,\eth\,\eth'\,\phi_0, \tag{4.53a}$$

$$\psi_4 = \kappa_1^6(\eth'\,\eth + 2\Psi_2 + 2\rho\rho')(\eth'\,\eth + \Psi_2 + \rho\rho')\,\eth'\,\eth\,\phi_4. \tag{4.53b}$$

From this we can conclude that the sixth order operator is indeed non-trivial. Also note that in the Schwarzschild spacetime we have $\kappa_1 = -r/3$ and $\kappa_1^2\Psi_2 + \kappa_1^2\rho\rho' = -1/18$.

Remark 14. *It should be noted that on a generalized Kerr-NUT spacetime (real $\xi_{AA'}$), the operator (similar to* (3.36) *in the spin-1 case)*

$$(\mathcal{S}_{4,0}\phi)_{ABCD} = (\kappa_1\mathcal{P}^2_{4,0}\mathcal{K}^1_{4,0}\mathscr{C}_{3,1,-1,1}\bar{\kappa}_{1'}\overline{\mathcal{K}}^1_{3,1}\mathscr{C}^\dagger_{4,0,-3}\mathcal{P}^2_{4,0}\phi)_{ABCD} - (\kappa_1\mathcal{K}^1_{4,0}\mathcal{P}^2_{4,0}\mathcal{L}_\xi\phi)_{ABCD}. \tag{4.54}$$

is a second order symmetry operator for the TME. In GHP form, it is (c.f. [1, Thm 5.4.1])

$$(\mathcal{S}_{4,0}\phi)_0 = 2\kappa_1\bar{\kappa}_{1'}(\eth - \tau - \bar{\tau}')(\eth' - 4\tau')\phi_0 + (\bar{\kappa}_{1'} - \kappa_1)\mathcal{L}_\xi\phi_0 - \kappa_1^{-3}\bar{\kappa}_{1'}(\mathbf{O}_{4,0}\phi)_0, \tag{4.55a}$$

$$(\mathcal{S}_{4,0}\phi)_1 = 0, \qquad (\mathcal{S}_{4,0}\phi)_2 = 0, \qquad (\mathcal{S}_{4,0}\phi)_3 = 0, \tag{4.55b}$$

$$(\mathcal{S}_{4,0}\phi)_4 = 2\kappa_1\bar{\kappa}_{1'}(\eth' - \bar{\tau} - \tau')(\eth - 4\tau)\phi_4 + (\kappa_1 - \bar{\kappa}_{1'})\mathcal{L}_\xi\phi_4 - \kappa_1^{-3}\bar{\kappa}_{1'}(\mathbf{O}_{4,0}\phi)_4. \tag{4.55c}$$

If we assume that ϕ_0 and ϕ_4 satisfies the TME, then on a Schwarzschild spacetime we get $(\mathcal{S}_{4,0}\phi)_0 = 2\kappa_1^2\,\eth\,\eth'\,\phi_0$ and $(\mathcal{S}_{4,0}\phi)_4 = 2\kappa_1^2\,\eth'\,\eth\,\phi_4$. Therefore, (4.53) *can be written in terms of the $\mathcal{S}_{4,0}$ operator, which gives a relation to the TME separation constants. It is an open question if the sixth order operator can be factored also on the Kerr spacetime. Even though* (4.54) *is a symmetry operator for the TME, it can not be interpreted as a symmetry operator for linearized gravity, but the sixth order operator comes from a linearized metric, and can therefore be cast into a form mapping linearized metrics to linearized metrics.*

Appendix A. Adjoints

In this section we collect the †- and ⋆-adjoints of the algebraic and differential operators introduced in section 2. First of all, for a general composition of operators **A** and **B** one can check

$$(\mathbf{AB})^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger, \qquad (\mathbf{AB})^\star = \mathbf{B}^\star\mathbf{A}^\star. \tag{A.1}$$

The adjoints of the extended fundamental spinor operators (2.7) are given by

$$(\mathscr{D}_{k,l,v,w})^\dagger = -\mathscr{T}_{k-1,l-1,-v,-w}, \qquad (\mathscr{C}_{k,l,v,w})^\dagger = \mathscr{C}^\dagger_{k+1,l-1,-v,-w}, \tag{A.2a}$$

$$(\mathscr{T}_{k,l,v,w})^\dagger = -\mathscr{D}_{k+1,l+1,-v,-w}, \qquad (\mathscr{C}^\dagger_{k,l,v,w})^\dagger = \mathscr{C}_{k-1,l+1,-v,-w}, \tag{A.2b}$$

$$(\mathscr{D}_{k,l,v,w})^\star = -\mathscr{T}_{l-1,k-1,-w,-v}, \qquad (\mathscr{C}_{k,l,v,w})^\star = \mathscr{C}_{l-1,k+1,-w,-v}, \tag{A.2c}$$

$$(\mathscr{T}_{k,l,v,w})^\star = -\mathscr{D}_{l+1,k+1,-w,-v}, \qquad (\mathscr{C}^\dagger_{k,l,v,w})^\star = \mathscr{C}^\dagger_{l+1,k-1,-w,-v}. \tag{A.2d}$$

The adjoints of the algebraic $\mathcal{K}$-operators (2.11) are

$$(\mathcal{K}^0_{k,l})^\dagger = -4\mathcal{K}^2_{k+2,l}, \qquad (\mathcal{K}^1_{k,l})^\dagger = -\mathcal{K}^1_{k,l}, \qquad (\mathcal{K}^2_{k,l})^\dagger = -\tfrac{1}{4}\mathcal{K}^0_{k-2,l}, \tag{A.3a}$$

$$(\overline{\mathcal{K}}^0_{k,l})^\dagger = -4\overline{\mathcal{K}}^2_{k,l+2}, \qquad (\overline{\mathcal{K}}^1_{k,l})^\dagger = -\overline{\mathcal{K}}^1_{k,l}, \qquad (\overline{\mathcal{K}}^2_{k,l})^\dagger = -\tfrac{1}{4}\overline{\mathcal{K}}^0_{k,l-2}, \tag{A.3b}$$

$$(\mathcal{K}^0_{k,l})^\star = -4\overline{\mathcal{K}}^2_{l,k+2}, \qquad (\mathcal{K}^1_{k,l})^\star = -\overline{\mathcal{K}}^1_{l,k}, \qquad (\mathcal{K}^2_{k,l})^\star = -\tfrac{1}{4}\overline{\mathcal{K}}^0_{l,k-2}, \tag{A.3c}$$

$$(\overline{\mathcal{K}}^0_{k,l})^\star = -4\mathcal{K}^2_{l+2,k}, \qquad (\overline{\mathcal{K}}^1_{k,l})^\star = -\mathcal{K}^1_{l,k}, \qquad (\overline{\mathcal{K}}^2_{k,l})^\star = -\tfrac{1}{4}\mathcal{K}^0_{l-2,k}. \tag{A.3d}$$

Multiplication by a scalar, e.g. κ_1 or Ψ_2, is a self-adjoint operation. For the projection operators (2.12) we find

$$(\mathcal{P}^s_{k,0})^\dagger = \mathcal{P}^s_{k,0}, \qquad (\mathcal{P}^s_{k,0})^\star = \overline{\mathcal{P}}^s_{0,k}. \tag{A.4}$$

The Lie derivative of a valence $(0,4)$ spinor $\varphi_{A'B'C'D'}$ is skew-adjoint,

$$\big((\mathcal{L}_\xi)^\dagger\varphi\big)_{A'B'C'D'} = -(\mathcal{L}_\xi\varphi)_{A'B'C'D'}. \tag{A.5}$$

Appendix B. $\mathcal{K}$-operator commutators

To avoid clutter in the notation we present the commutators as operators which can be applied to arbitrary symmetric spinors of valence (k,l), where k and l are large enough so that the combination of operators on the left hand side makes sense. The proof is straightforward but tedious. Examples can be found in [2, Lemma 2.9]. Complex conjugating these identities gives commutators for the $\overline{\mathcal{K}}$ operators.

Lemma 15. *Commuting $\mathcal{K}$-operator outside the extended fundamental spinor operators yields*

$$\mathscr{D}_{k-2,l,v,w}\mathcal{K}^2_{k,l} = \mathcal{K}^2_{k-1,l-1}\mathscr{D}_{k,l,v+1,w}, \tag{B.1a}$$

$$\mathscr{C}^\dagger_{k-2,l,v,w}\mathcal{K}^2_{k,l} = \mathcal{K}^2_{k-1,l+1}\mathscr{C}^\dagger_{k,l,v+1,w}\varphi, \tag{B.1b}$$

$$\mathscr{C}_{k-2,l,v,w}\mathcal{K}^2_{k,l} = \mathcal{K}^2_{k+1,l-1}\mathscr{C}_{k,l,v-1,w} - \tfrac{1}{k+1}\mathcal{K}^1_{k-1,l-1}\mathscr{D}_{k,l,v+k,w}, \tag{B.1c}$$

$$\mathscr{T}_{k-2,l,v,w}\mathcal{K}^2_{k,l} = \mathcal{K}^2_{k+1,l+1}\mathscr{T}_{k,l,v-1,w} - \tfrac{1}{k+1}\mathcal{K}^1_{k-1,l+1}\mathscr{C}^\dagger_{k,l,v+k,w}, \tag{B.1d}$$

$$\mathscr{D}_{k,l,v,w}\mathcal{K}^1_{k,l} = \tfrac{(k-1)(k+2)}{k(k+1)}\mathcal{K}^1_{k-1,l-1}\mathscr{D}_{k,l,v,w} + \tfrac{2}{k}\mathcal{K}^2_{k+1,l-1}\mathscr{C}_{k,l,v-k-1,w}, \tag{B.1e}$$

$$\mathscr{C}^\dagger_{k,l,v,w}\mathcal{K}^1_{k,l} = \tfrac{(k-1)(k+2)}{k(k+1)}\mathcal{K}^1_{k-1,l+1}\mathscr{C}^\dagger_{k,l,v,w} + \tfrac{2}{k}\mathcal{K}^2_{k+1,l+1}\mathscr{T}_{k,l,v-k-1,w}, \tag{B.1f}$$

$$\mathscr{C}_{k,l,v,w}\mathcal{K}^1_{k,l} = \mathcal{K}^1_{k+1,l-1}\mathscr{C}_{k,l,v,w} + \tfrac{1}{2(k+1)}\mathcal{K}^0_{k-1,l-1}\mathscr{D}_{k,l,v+k+1,w}, \tag{B.1g}$$

$$\mathscr{T}_{k,l,v,w}\mathcal{K}^1_{k,l} = \mathcal{K}^1_{k+1,l+1}\mathscr{T}_{k,l,v,w} + \tfrac{1}{2(k+1)}\mathcal{K}^0_{k-1,l+1}\mathscr{C}^\dagger_{k,l,v+k+1,w}, \tag{B.1h}$$

$$\mathscr{D}_{k+2,l,v,w}\mathcal{K}^0_{k,l} = \tfrac{k(k+3)}{(k+1)(k+2)}\mathcal{K}^0_{k-1,l-1}\mathscr{D}_{k,l,v-1,w} - \tfrac{4}{k+2}\mathcal{K}^1_{k+1,l-1}\mathscr{C}_{k,l,v-k-2,w}, \tag{B.1i}$$

$$\mathscr{C}^\dagger_{k+2,l,v,w}\mathcal{K}^0_{k,l} = \tfrac{k(k+3)}{(k+1)(k+2)}\mathcal{K}^0_{k-1,l+1}\mathscr{C}^\dagger_{k,l,v-1,w} - \tfrac{4}{k+2}\mathcal{K}^1_{k+1,l+1}\mathscr{T}_{k,l,v-k-2,w}, \tag{B.1j}$$

$$\mathscr{C}_{k+2,l,v,w}\mathcal{K}^0_{k,l} = \mathcal{K}^0_{k+1,l-1}\mathscr{C}_{k,l,v+1,w}, \tag{B.1k}$$

$$\mathscr{T}_{k+2,l,v,w}\mathcal{K}^0_{k,l} = \mathcal{K}^0_{k+1,l+1}\mathscr{T}_{k,l,v+1,w}. \tag{B.1l}$$

Corollary 16. *Commuting $\mathcal{K}$-operator inside the extended fundamental spinor operators yields*

$$\mathcal{K}^2_{k-1,l-1}\mathscr{D}_{k,l,v,w} = \mathscr{D}_{k-2,l,v-1,w}\mathcal{K}^2_{k,l}, \tag{B.2a}$$

$$\mathcal{K}^2_{k-1,l+1}\mathscr{C}^\dagger_{k,l,v,w} = \mathscr{C}^\dagger_{k-2,l,v-1,w}\mathcal{K}^2_{k,l}, \tag{B.2b}$$

$$\mathcal{K}^2_{k+1,l-1}\mathscr{C}_{k,l,v,w} = \tfrac{(k-1)(k+2)}{k(k+1)}\mathscr{C}_{k-2,l,v+1,w}\mathcal{K}^2_{k,l} + \tfrac{1}{k+1}\mathscr{D}_{k,l,v+k+1,w}\mathcal{K}^1_{k,l}, \tag{B.2c}$$

$$\mathcal{K}^2_{k+1,l+1}\mathscr{T}_{k,l,v,w} = \tfrac{(k-1)(k+2)}{k(k+1)}\mathscr{T}_{k-2,l,v+1,w}\mathcal{K}^2_{k,l} + \tfrac{1}{k+1}\mathscr{C}^\dagger_{k,l,v+k+1,w}\mathcal{K}^1_{k,l}, \tag{B.2d}$$

$$\mathcal{K}^1_{k-1,l-1}\mathscr{D}_{k,l,v,w} = \mathscr{D}_{k,l,v,w}\mathcal{K}^1_{k,l} - \tfrac{2}{k}\mathscr{C}_{k-2,l,v-k,w}\mathcal{K}^2_{k,l}, \tag{B.2e}$$

$$\mathcal{K}^1_{k-1,l+1}\mathscr{C}^\dagger_{k,l,v,w} = \mathscr{C}^\dagger_{k,l,v,w}\mathcal{K}^1_{k,l} - \tfrac{2}{k}\mathscr{T}_{k-2,l,v-k,w}\mathcal{K}^2_{k,l}, \tag{B.2f}$$

$$\mathcal{K}^1_{k+1,l-1}\mathscr{C}_{k,l,v,w} = \tfrac{k(k+3)}{(k+1)(k+2)}\mathscr{C}_{k,l,v,w}\mathcal{K}^1_{k,l} - \tfrac{1}{2(k+1)}\mathscr{D}_{k+2,l,v+k+2,w}\mathcal{K}^0_{k,l}, \tag{B.2g}$$

$$\mathcal{K}^1_{k+1,l+1}\mathscr{T}_{k,l,v,w} = \tfrac{k(k+3)}{(k+1)(k+2)}\mathscr{T}_{k,l,v,w}\mathcal{K}^1_{k,l} - \tfrac{1}{2(k+1)}\mathscr{C}^\dagger_{k+2,l,v+k+2,w}\mathcal{K}^0_{k,l}, \tag{B.2h}$$

$$\mathcal{K}^0_{k-1,l-1}\mathscr{D}_{k,l,v,w} = \mathscr{D}_{k+2,l,v+1,w}\mathcal{K}^0_{k,l} + \tfrac{4}{k+2}\mathscr{C}_{k,l,v-k-1,w}\mathcal{K}^1_{k,l}, \tag{B.2i}$$

$$\mathcal{K}^0_{k-1,l+1}\mathscr{C}^\dagger_{k,l,v,w} = \mathscr{C}^\dagger_{k+2,l,v+1,w}\mathcal{K}^0_{k,l} + \tfrac{4}{k+2}\mathscr{T}_{k,l,v-k-1,w}\mathcal{K}^1_{k,l}, \tag{B.2j}$$
$$\mathcal{K}^0_{k+1,l-1}\mathscr{C}_{k,l,v,w} = \mathscr{C}_{k+2,l,v-1,w}\mathcal{K}^0_{k,l}, \tag{B.2k}$$
$$\mathcal{K}^0_{k+1,l+1}\mathscr{T}_{k,l,v,w} = \mathscr{T}_{k+2,l,v-1,w}\mathcal{K}^0_{k,l}. \tag{B.2l}$$

Lemma 17. *For commuting two $\mathcal{K}$-operators, we have the identities*

$$\mathcal{K}^2_{k+2,l}\mathcal{K}^0_{k,l} = \tfrac{(k-1)(k+2)}{k(k+1)}\mathcal{K}^0_{k-2,l}\mathcal{K}^2_{k,l} - \tfrac{4}{k+2}\mathcal{K}^1_{k,l}\mathcal{K}^1_{k,l}, \tag{B.3a}$$
$$\mathcal{K}^1_{k+2,l}\mathcal{K}^0_{k,l} = \tfrac{k}{k+2}\mathcal{K}^0_{k,l}\mathcal{K}^1_{k,l}, \tag{B.3b}$$
$$\mathcal{K}^2_{k,l}\mathcal{K}^1_{k,l} = \tfrac{k-2}{k}\mathcal{K}^1_{k-2,l}\mathcal{K}^2_{k,l}, \tag{B.3c}$$
$$\mathcal{K}^1_{k,l}\mathcal{K}^1_{k,l} = \mathbf{1} - \tfrac{k-1}{k}\mathcal{K}^0_{k-2,l}\mathcal{K}^2_{k,l}, \tag{B.3d}$$

where $\mathbf{1}$ *is the identity operator.*

Appendix C. GHP form

In this section we collect the GHP components of various covariant operators introduced in previous sections.

C.1. Spin-1.

The components of the TME operator (3.6b) are

$$(\mathbf{O}_{2,0}\phi)_0 = -\frac{\kappa_1}{\bar\kappa_{1'}}\,\text{þ}\big(\kappa_1\bar\kappa_{1'}(\text{þ}'-2\rho')\big)\phi_0 + \frac{\kappa_1}{\bar\kappa_{1'}}\,\eth\big(\kappa_1\bar\kappa_{1'}(\eth'-2\tau')\big)\phi_0 + \tfrac{1}{3}\kappa_1\mathcal{L}_\xi\phi_0, \tag{C.1a}$$
$$(\mathbf{O}_{2,0}\phi)_1 = 0, \tag{C.1b}$$
$$(\mathbf{O}_{2,0}\phi)_2 = -\frac{\kappa_1}{\bar\kappa_{1'}}\,\text{þ}'\big(\kappa_1\bar\kappa_{1'}(\text{þ}-2\rho)\big)\phi_2 + \frac{\kappa_1}{\bar\kappa_{1'}}\,\eth'\big(\kappa_1\bar\kappa_{1'}(\eth-2\tau)\big)\phi_2 - \tfrac{1}{3}\kappa_1\mathcal{L}_\xi\phi_2. \tag{C.1c}$$

The components of the TSI operator (3.6d) are

$$(\widehat{\mathbf{O}}_{2,0}\phi)_{0'} = -\frac{\bar\kappa_{1'}}{\kappa_1}\,\text{þ}\,\text{þ}(\kappa_1^2\phi_2) + \frac{\bar\kappa_{1'}}{\kappa_1}\,\eth'\,\eth'(\kappa_1^2\phi_0), \tag{C.2a}$$
$$(\widehat{\mathbf{O}}_{2,0}\phi)_{1'} = 0, \tag{C.2b}$$
$$(\widehat{\mathbf{O}}_{2,0}\phi)_{2'} = -\frac{\bar\kappa_{1'}}{\kappa_1}\,\text{þ}'\,\text{þ}'(\kappa_1^2\phi_0) + \frac{\bar\kappa_{1'}}{\kappa_1}\,\eth\,\eth(\kappa_1^2\phi_2). \tag{C.2c}$$

The components of the linear symmetry operator (3.19) are

$$\psi_0 = \text{þ}\big(\kappa_1\bar\kappa_{1'}(\text{þ}'-2\rho')\big)\phi_0 + \eth\big(\kappa_1\bar\kappa_{1'}(\eth'-2\tau')\big)\phi_0 - \tfrac{1}{3}\kappa_1\mathcal{L}_{\bar\xi}\phi_0, \tag{C.3a}$$
$$\begin{aligned}\psi_1 = {}& \tfrac{1}{2}(\text{þ}'+\rho')\big(\kappa_1\bar\kappa_{1'}(\eth'-2\tau')\big)\phi_0 + \tfrac{1}{2}(\eth'+\tau')\big(\kappa_1\bar\kappa_{1'}(\text{þ}'-2\rho')\big)\phi_0\\ &+ \tfrac{1}{2}(\text{þ}+\rho)\big(\kappa_1\bar\kappa_{1'}(\eth-2\tau)\big)\phi_2 + \tfrac{1}{2}(\eth+\tau)\big(\kappa_1\bar\kappa_{1'}(\text{þ}-2\rho)\big)\phi_2,\end{aligned} \tag{C.3b}$$
$$\psi_2 = \text{þ}'\big(\kappa_1\bar\kappa_{1'}(\text{þ}-2\rho)\big)\phi_2 + \eth'\big(\kappa_1\bar\kappa_{1'}(\eth-2\tau)\big)\phi_2 + \tfrac{1}{3}\kappa_1\mathcal{L}_{\bar\xi}\phi_2. \tag{C.3c}$$

The components of the anti-linear symmetry operator (3.17) are

$$\bar\chi_{0'} = \text{þ}\,\text{þ}(\kappa_1^2\phi_2) + \eth'\,\eth'(\kappa_1^2\phi_0), \tag{C.4a}$$
$$\bar\chi_{1'} = (\text{þ}\,\eth+\bar\tau'\,\text{þ})(\kappa_1^2\phi_2) + (\text{þ}'\,\eth'+\bar\tau\,\text{þ}')(\kappa_1^2\phi_0), \tag{C.4b}$$
$$\bar\chi_{2'} = \text{þ}'\,\text{þ}'(\kappa_1^2\phi_0) + \eth\,\eth(\kappa_1^2\phi_2). \tag{C.4c}$$

The Lie derivative of ϕ_{AB} components along ξ is given by

$$\mathcal{L}_\xi\phi_0 = 3\Psi_2\kappa_1\phi_0 - 3\kappa_1\rho'\,\text{þ}\,\phi_0 + 3\kappa_1\rho\,\text{þ}'\,\phi_0 + 3\kappa_1\tau'\,\eth\,\phi_0 - 3\kappa_1\tau\,\eth'\,\phi_0, \tag{C.5a}$$
$$\mathcal{L}_\xi\phi_1 = -3\kappa_1\rho'\,\text{þ}\,\phi_1 + 3\kappa_1\rho\,\text{þ}'\,\phi_1 + 3\kappa_1\tau'\,\eth\,\phi_1 - 3\kappa_1\tau\,\eth'\,\phi_1, \tag{C.5b}$$
$$\mathcal{L}_\xi\phi_2 = -3\Psi_2\kappa_1\phi_2 - 3\kappa_1\rho'\,\text{þ}\,\phi_2 + 3\kappa_1\rho\,\text{þ}'\,\phi_2 + 3\kappa_1\tau'\,\eth\,\phi_2 - 3\kappa_1\tau\,\eth'\,\phi_2, \tag{C.5c}$$

C.2. **Spin-2.** The components of the TME operator (4.9b) are

$$(\mathbf{O}_{4,0}\phi)_0 = -\frac{\kappa_1^3}{\bar\kappa_{1'}}\,\text{þ}\big(\kappa_1\bar\kappa_{1'}(\text{þ}'-4\rho')\big)\phi_0 + \frac{\kappa_1^3}{\bar\kappa_{1'}}\,\eth\big(\kappa_1\bar\kappa_{1'}(\eth'-4\tau')\big)\phi_0 + \kappa_1^3\mathcal{L}_\xi\phi_0, \tag{C.6a}$$

$$(\mathbf{O}_{4,0}\phi)_1 = 0, \quad (\mathbf{O}_{4,0}\phi)_2 = 0, \quad (\mathbf{O}_{4,0}\phi)_3 = 0, \tag{C.6b}$$

$$(\mathbf{O}_{4,0}\phi)_4 = -\frac{\kappa_1^3}{\bar\kappa_{1'}}\,\text{þ}'\big(\kappa_1\bar\kappa_{1'}(\text{þ}-4\rho)\big)\phi_4 + \frac{\kappa_1^3}{\bar\kappa_{1'}}\,\eth'\big(\kappa_1\bar\kappa_{1'}(\eth-4\tau)\big)\phi_4 - \kappa_1^3\mathcal{L}_\xi\phi_4. \tag{C.6c}$$

The components of the operator (4.22a) are

$$(\widehat{\mathbf{O}}_{4,0}\phi)_{0'} = -\frac{\bar\kappa_{1'}^3}{\kappa_1}\,\text{þ}\,\text{þ}\,\text{þ}\,\text{þ}(\kappa_1^4\phi_4) + \frac{\bar\kappa_{1'}^3}{\kappa_1}\,\eth'\,\eth'\,\eth'\,\eth'(\kappa_1^4\phi_0), \tag{C.7a}$$

$$(\widehat{\mathbf{O}}_{4,0}\phi)_{1'} = 0, \quad (\widehat{\mathbf{O}}_{4,0}\phi)_{2'} = 0, \quad (\widehat{\mathbf{O}}_{4,0}\phi)_{3'} = 0, \tag{C.7b}$$

$$(\widehat{\mathbf{O}}_{4,0}\phi)_{4'} = -\frac{\bar\kappa_{1'}^3}{\kappa_1}\,\text{þ}'\,\text{þ}'\,\text{þ}'\,\text{þ}'(\kappa_1^4\phi_0) + \frac{\bar\kappa_{1'}^3}{\kappa_1}\,\eth\,\eth\,\eth\,\eth(\kappa_1^4\phi_4). \tag{C.7c}$$

The Lie derivative of ϕ_{ABCD} components along ξ is given by

$$\mathcal{L}_\xi\phi_0 = 6\Psi_2\kappa_1\phi_0 - 3\kappa_1\rho'\,\text{þ}\,\phi_0 + 3\kappa_1\rho\,\text{þ}'\,\phi_0 + 3\kappa_1\tau'\,\eth\,\phi_0 - 3\kappa_1\tau\,\eth'\,\phi_0, \tag{C.8a}$$

$$\mathcal{L}_\xi\phi_1 = 3\Psi_2\kappa_1\phi_1 - 3\kappa_1\rho'\,\text{þ}\,\phi_1 + 3\kappa_1\rho\,\text{þ}'\,\phi_1 + 3\kappa_1\tau'\,\eth\,\phi_1 - 3\kappa_1\tau\,\eth'\,\phi_1, \tag{C.8b}$$

$$\mathcal{L}_\xi\phi_2 = -3\kappa_1\rho'\,\text{þ}\,\phi_2 + 3\kappa_1\rho\,\text{þ}'\,\phi_2 + 3\kappa_1\tau'\,\eth\,\phi_2 - 3\kappa_1\tau\,\eth'\,\phi_2, \tag{C.8c}$$

$$\mathcal{L}_\xi\phi_3 = -3\Psi_2\kappa_1\phi_3 - 3\kappa_1\rho'\,\text{þ}\,\phi_3 + 3\kappa_1\rho\,\text{þ}'\,\phi_3 + 3\kappa_1\tau'\,\eth\,\phi_3 - 3\kappa_1\tau\,\eth'\,\phi_3, \tag{C.8d}$$

$$\mathcal{L}_\xi\phi_4 = -6\Psi_2\kappa_1\phi_4 - 3\kappa_1\rho'\,\text{þ}\,\phi_4 + 3\kappa_1\rho\,\text{þ}'\,\phi_4 + 3\kappa_1\tau'\,\eth\,\phi_4 - 3\kappa_1\tau\,\eth'\,\phi_4. \tag{C.8e}$$

The curvature components of the Debye map given by the metric (4.43) for the plus sign are given by

$$\underline{\overline{\chi}}_{0'}^{+} = \eth'\,\eth'\,\eth'\,\eth'(\kappa_1^4\phi_0), \tag{C.9a}$$

$$\underline{\overline{\chi}}_{1'}^{+} = \big(\text{þ}'(\eth'-\bar\tau) + 3\rho'\bar\tau - 6\bar\rho'\bar\tau + 3\bar\rho'\tau'\big)(\eth'+2\bar\tau)(\eth'+2\bar\tau)(\kappa_1^4\phi_0), \tag{C.9b}$$

$$\underline{\overline{\chi}}_{2'}^{+} = \big((\eth'-\bar\tau)\,\text{þ}'+6\rho'\bar\tau - 12\bar\rho'\bar\tau + 6\bar\rho'\tau'\big)\big((\eth'+2\bar\tau)(\text{þ}'+3\bar\rho') + \rho'\bar\tau - 2\bar\rho'\bar\tau + \bar\rho'\tau'\big)(\kappa_1^4\phi_0), \tag{C.9c}$$

$$\underline{\overline{\chi}}_{3'}^{+} = \big(\eth'(\text{þ}'-\bar\rho') + 3\rho'\bar\tau - 6\bar\rho'\bar\tau + 3\bar\rho'\tau'\big)\,(\text{þ}'+2\bar\rho')(\text{þ}'+2\bar\rho')(\kappa_1^4\phi_0), \tag{C.9d}$$

$$\underline{\overline{\chi}}_{4'}^{+} = \text{þ}'\,\text{þ}'\,\text{þ}'\,\text{þ}'(\kappa_1^4\phi_0), \tag{C.9e}$$

$$\underline{\psi}_0^{+} = \Psi_2\kappa_1^3\mathcal{L}_\xi\phi_0, \qquad \underline{\psi}_1^{+} = 0, \qquad \underline{\psi}_2^{+} = 0, \qquad \underline{\psi}_3^{+} = 0, \qquad \underline{\psi}_4^{+} = 0, \tag{C.9f}$$

and for the minus sign we get

$$\underline{\overline{\chi}}_{0'}^{-} = \text{þ}\,\text{þ}\,\text{þ}\,\text{þ}(\kappa_1^4\phi_4), \tag{C.10a}$$

$$\underline{\overline{\chi}}_{1'}^{-} = \big(\eth(\text{þ}-\bar\rho) + 3\bar\rho\tau + 3\rho\bar\tau' - 6\bar\rho\bar\tau'\big)(\text{þ}+2\bar\rho)(\text{þ}+2\bar\rho)(\kappa_1^4\phi_4), \tag{C.10b}$$

$$\underline{\overline{\chi}}_{2'}^{-} = \big((\eth-\bar\tau')\,\text{þ}+6\bar\rho\tau + 6\rho\bar\tau' - 12\bar\rho\bar\tau'\big)\big((\eth+2\bar\tau')(\text{þ}+3\bar\rho) + \bar\rho\tau + \rho\bar\tau' - 2\bar\rho\bar\tau'\big)(\kappa_1^4\phi_4), \tag{C.10c}$$

$$\underline{\overline{\chi}}_{3'}^{-} = \big(\text{þ}(\eth-\bar\tau') + 3\bar\rho\tau + 3\rho\bar\tau' - 6\bar\rho\bar\tau'\big)(\eth+2\bar\tau')(\eth+2\bar\tau')(\kappa_1^4\phi_4), \tag{C.10d}$$

$$\underline{\overline{\chi}}_{4'}^{-} = \eth\,\eth\,\eth\,\eth(\kappa_1^4\phi_4), \tag{C.10e}$$

$$\underline{\psi}_0^{-} = 0, \qquad \underline{\psi}_1^{-} = 0, \qquad \underline{\psi}_2^{-} = 0, \qquad \underline{\psi}_3^{-} = 0, \qquad \underline{\psi}_4^{-} = -\Psi_2\kappa_1^3\mathcal{L}_\xi\phi_4. \tag{C.10f}$$

Restricting to the real or imaginary part of the metric (4.43) leads to the linear combinations $\frac{1}{2}(\underline{\overline{\chi}}_n^{\pm} + \underline{\overline{\psi}}_n^{\pm})$ or $\frac{1}{2i}(\underline{\overline{\chi}}_n^{\pm} - \underline{\overline{\psi}}_n^{\pm})$ for the self-dual curvature.

We note that on vacuum type D background the identities

$$\eth'\,\eth'\,\eth'\,\eth'(\kappa_1^4\phi_0) = (\eth'-\bar\tau)(\eth'-\bar\tau)(\eth'-\bar\tau)(\eth'+3\bar\tau)(\kappa_1^4\phi_0), \tag{C.11a}$$

$$\text{þ}'\,\text{þ}'\,\text{þ}'\,\text{þ}'(\kappa_1^4\phi_0) = (\text{þ}'-\bar\rho')(\text{þ}'-\bar\rho')(\text{þ}'-\bar\rho')(\text{þ}'+3\bar\rho')(\kappa_1^4\phi_0). \tag{C.11b}$$

and their GHP-primed and c.c. versions follow from the Ricci identities. This connects the more compact form presented here to the equations given in [17], [11], [9].

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