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WITT GROUPS OF SMOOTH PROJECTIVE QUADRICS

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Abstract. Let k be a commutative ring containing $\frac{1}{2}$. In this paper, we construct exact sequences connecting Witt groups of smooth projective quadrics over k and Clifford algebras. The exact sequences have an application to a classical problem in quadratic form theory: the Witt kernel of function fields of quadrics. We also apply the exact sequences to compute Witt groups of certain kinds of smooth projective quadrics.

Contents

1. Introduction

Let F be a field of characteristic $\neq 2$. In the 1970s, researchers in quadratic form theory realized that to understand quadratic forms over F, it is useful to study the quadratic forms over the function field $K(Q_d)$ of the projective quadric Q_d defined by a homogeneous quadratic equation $\varphi = 0$ (cf. [28, Section 3 and Section 4 Chapter X]). Let $W(F)$ be the Witt group of quadratic forms over F. A major problem arises:

Problem 1.1. *Describe the kernel of the restriction homomorphism*

$$W(F) \longrightarrow W(K(Q_d))$$

for an arbitrary projective quadric Q_d.

This problem has been solved for some extreme cases (e.g. Pfister forms [28, Theorem X.4.11] and some low dimensional forms [28, Remark X.4.12]), but is still wide open in general. We show in Proposition 8.1 that the sequence

$$W(C_0(q)_\sigma) \xrightarrow{\mathrm{tr}} W(F) \xrightarrow{p^*} W(Q_d) \tag{1}$$

is exact where $W(C_0(q)_\sigma) \xrightarrow{\mathrm{tr}} W(F)$ is the trace map which sends a Hermitian space $M \times M^{\mathrm{op}} \to C_0(q)$ to the symmetric space $M \times M \to C_0(q) \xrightarrow{\mathrm{tr}} F$ (here $\mathrm{tr} : C_0(q) \to F$ is the algebra trace map). As the map $W(F) \to W(K(Q_d))$ factors $W(F) \longrightarrow W(Q_d) \longrightarrow W(K(Q_d))$, the trace map reveals some information about the Witt kernel.

Let us now explain our main theorem. Let k be a commutative ring containing $\frac{1}{2}$. Let (P, q) be a non-degenerate quadratic form over k of rank $n = d + 2$. Let $S(P^*)$ be the symmetric algebra of $P^* = \mathrm{Hom}(P, k)$. Let $Q_d \subset \mathbb{P}_k^{d+1}$ be the quadric hypersurface $\mathrm{Proj}\ S(P^*)/(q)$ where q is regarded as an element in $S_2(P^*)$. Let $C_0(q)$ be the even part of the Clifford algebra of (P, q), and let σ be the canonical involution of $C_0(q)$. In Section 6, we define a trace map $\mathrm{tr} : W^i(C_0(q)_\sigma) \to W^i(k)$ of shifted Witt groups. Here, $W^i(C_0(q)_\sigma)$ is the i-th shifted Witt group of finitely generated projective modules over $C_0(q)$, which is precisely the group $W^i(C_0(q)_\sigma - \mathrm{proj})$ in Balmer and Preedi's notation [5].

Theorem 1.2. *Let k be a commutative ring containing $\frac{1}{2}$.*
(a). *If $d = \dim Q_d$ is odd, then there is a long exact sequence of abelian groups*

$$\cdots \longrightarrow W^i(C_0(q)_\sigma) \xrightarrow{\mathrm{tr}} W^i(k) \xrightarrow{p^*} W^i(Q_d) \longrightarrow W^{i+1}(C_0(q)_\sigma) \longrightarrow \cdots$$

(b). *If $d = \dim Q_d$ is even, then there is a long exact sequence of abelian groups*

$$\cdots \longrightarrow W^{i-d-1}(k, \det P) \longrightarrow W^i(\mathcal{A}) \longrightarrow W^i(Q_d) \longrightarrow W^{i-d}(k, \det P) \longrightarrow \cdots$$

where $\det(P) = \Lambda^n P$ is the determinant of the finitely generated projective module P and $\mathcal{A}$ is the Swan triangulated category in Section 6. Moreover, the group $W^i(\mathcal{A})$ fits into another long exact sequence of groups

$$\cdots \longrightarrow W^i(C_0(q)_\sigma) \xrightarrow{\mathrm{tr}} W^i(k) \longrightarrow W^i(\mathcal{A}) \longrightarrow W^{i+1}(C_0(q)_\sigma) \longrightarrow \cdots$$

Our method needs the semi-orthogonal decomposition on the derived category $\mathcal{D}^b Q_d$ of bounded complexes of finite rank locally free sheaves on Q_d, which is reviewed in Section 3. The main work of this paper is to manipulate the duality on the finite rank locally free sheaves on quadrics.

If Q_d is isotropic, we prove the following result in Proposition 8.5.

Proposition 1.3. *Let k be a commutative ring with $\frac{1}{2} \in k$. Let Q_d be the quadric defined by an isotropic quadratic form q over k. If d is odd, then*

$$W^i(Q_d) \cong W^i(k) \oplus W^{i+1}(C_0(q)_\sigma)$$

If d is even, then we have a long exact sequence

$$\cdots \to W^{i-d-1}(k, \det P) \to W^i(k) \oplus W^{i+1}(C_0(q)_\sigma) \to W^i(Q_d) \to W^{i-d}(k, \det P) \to \cdots$$

Define the number

$$\delta(n) := \#\{l \in \mathbb{Z} : 0 < l < n, l \equiv 0, 1, 2 \text{ or } 4 \mod 8\}.$$

This number is related to the vector fields on spheres problem; see [1]. By computing the trace map for the case of quadric hypersurfaces defined by quadratic forms $n\langle 1\rangle$, we prove the following result in Theorem 8.9 and Theorem 8.10.

Theorem 1.4. *Let k be a field in which -1 is not a sum of two squares. Let $Q_{0,n}$ be the smooth projective quadric of dimension $d = n-2$ associated to the non-degenerate quadratic form $n\langle 1\rangle$. If $d \not\equiv 0 \mod 4$, then $W^0(Q_{0,n}) \cong W^0(k)/2^{\delta(n)}W^0(k)$. If $d \equiv 0 \mod 4$, then there is an exact sequence*

$$0 \longrightarrow W^0(k)/2^{\delta(n)}W^0(k) \xrightarrow{p^*} W^0(Q_{0,n}) \longrightarrow W^0(k).$$

The map $W^0(Q_{0,n}) \longrightarrow W^0(k)$ in this exact sequence may not be surjective, as the following result (which is proved in Theorem 8.11) shows.

Theorem 1.5. *Let $Q_{0,n}$ be the quadric defined by the anisotropic quadratic form $n\langle 1\rangle$ over $\mathbb{R}$. In other words, $Q_{0,n} = \text{Proj}\left(\mathbb{R}[X_1, \dots, X_n]/(X_1^2 + \cdots + X_n^2)\right)$. Then, we have*

	$W^i(Q_{0,n})$			
	$i=0$	$i=1$	$i=2$	$i=3$
$n \equiv 1 \mod 8$	$\mathbb{Z}/2^{\delta(n)}$	0	0	0
$n \equiv 2 \mod 8$	$\mathbb{Z}/2^{\delta(n)}$	$\mathbb{Z}/2^{\kappa(n)}$	0	0
$n \equiv 3 \mod 8$	$\mathbb{Z}/2^{\delta(n)}$	$\mathbb{Z}/2$	0	0
$n \equiv 4 \mod 8$	$\mathbb{Z}/2^{\delta(n)}$	$(\mathbb{Z}/2)^2$	0	$\mathbb{Z}/2^{\kappa(n)}$
$n \equiv 5 \mod 8$	$\mathbb{Z}/2^{\delta(n)}$	$\mathbb{Z}/2$	0	0
$n \equiv 6 \mod 8$	$\mathbb{Z}/2^{\delta(n)}$	$\mathbb{Z}/2^{\kappa(n)}$	0	0
$n \equiv 7 \mod 8$	$\mathbb{Z}/2^{\delta(n)}$	0	0	0
$n \equiv 8 \mod 8$	$\mathbb{Z}/2^{\delta(n)}$	0	0	$\mathbb{Z}/2^{\kappa(n)}$

where $\kappa(n)$ is an integer in $\mathbb{Z}_{\geq 0}$ depending on n.

We have an immediate application. Let F be a field of characteristic $\neq 2$ and let $K := K(X)$ be the function field of a variety X. Let $\mathcal{V}_K$ be the set of all rank one discrete valuations on K which are trivial on F. We denote $W_{\text{nr}}(X)$ as the *unramified Witt group* of the quadric X which is defined as the subgroup

$$\bigcap_{v \in \mathcal{V}_K} \ker(\partial_v)$$

of $W(K)$ where $\partial_v : W(K) \to W(k(v))$ is the second residue homomorphism, cf. [11]. An alternative definition of unramified Witt groups can be found in [6]. In *loc. cit.*, Balmer and Walter proved that the natural map $W(X) \to W_{\text{nr}}(X)$ is an isomorphism for any regular variety X in $\dim(X) \leq 3$. Applying Colliot-Thélène and Sujatha's computation of $W_{\text{nr}}(Q_{0,n})$ (for $n \leq 8$) in [11], and our computation of $W(Q_{0,n})$ in Theorem 1.5, we present the following result.

Corollary 1.6 (Purity for $Q_{0,n}$ in dimension ≤ 6)**.** *If $\dim(Q_{0,n}) \leq 6$ (i.e. $n \leq 8$), the natural map $W(Q_{0,n}) \to W_{\text{nr}}(Q_{0,n})$ is an isomorphism.*

In Nenashev's work [30], Witt groups of *split* quadrics were studied. In Zibrowius's work [40], Grothendieck-Witt (and Witt) groups of quadrics over $\mathbb{C}$ were computed. This paper focuses on the case of a general projective quadric over an arbitrary commutative ring k with $\frac{1}{2} \in k$. Also, Dell'Ambrogio and Fasel [12] studied Witt groups of certain kinds of quadrics by forgetting the 2 primary torsions. This paper determines the 2 primary torsion for certain cases. Charles Walter gave a talk on Witt groups of quadrics in a conference; cf. [37]. There could be some overlap between our work. However, Walter's work has not been published, and Baptiste Calmès informed me that Walter's approach (presented in [37]) was based on the theory of spectral sequences, which is different from this paper.

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2. Review: Witt groups

2.1. **Classical Witt groups.** Let X be a scheme. A symmetric space $(\mathscr{F}, \phi)$ over X consists of a vector bundle $\mathscr{F}$ and an isomorphism of vector bundles $\phi : \mathscr{F} \to \mathscr{F}^\vee$ such that $\phi = \phi^\vee \circ \mathrm{can}_{\mathscr{F}}$ where $\mathscr{F}^\vee := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$ is the dual bundle of $\mathscr{F}$ and $\mathrm{can}_{\mathscr{F}} : \mathscr{F} \to \mathscr{F}^{\vee\vee} : m \mapsto (\mathrm{can}(m) : f \mapsto f(m))$ is the double dual identification. Two symmetric vector bundles $(\mathscr{F}, \phi)$ and $(\mathscr{G}, \varphi)$ are isometric if there is an isomorphism $i : \mathscr{F} \to \mathscr{G}$ of vector bundles such that $i^\vee \varphi i = \phi$. The orthogonal sum $(\mathscr{F}, \phi) \perp (\mathscr{G}, \varphi)$ is defined to be $(\mathscr{F} \oplus \mathscr{G}, \phi \oplus \varphi)$. A symmetric space $(\mathscr{M}, \mu)$ is metabolic if there is a short exact sequence of vector bundles

$$0 \to \mathscr{E} \xrightarrow{j} \mathscr{M} \xrightarrow{j^\vee \mu} \mathscr{E}^\vee \to 0$$

The bundle $\mathscr{E}$ is usually called a Lagrangian of $\mathscr{M}$.

Definition 2.1 (Section I.5 [23])**.** The Witt group $W(X)$ of a scheme X is the free abelian group on isometry classes of symmetric spaces over X modulo the following relations:

- $[(\mathscr{F}, \phi) \perp (\mathscr{G}, \varphi)] = [(\mathscr{F}, \phi)] + [(\mathscr{G}, \varphi)]$
- $[(\mathscr{M}, \phi)] = 0$ for any metabolic space $(\mathscr{M}, \phi)$.

2.2. **Triangular Witt groups.** The Witt group $W(X)$ above sits in the framework of triangulated categories with dualities (see below in this section) or the framework of dg categories with duality (see Appendix 2.4 for a review of dg categories with duality). A triangulated category with duality was first defined by Balmer [3]. We use a slight generalization of this by Schlichting, cf. Definition 3.1 [34], because the definition of Schlichting fits into the theory of closed symmetric monoindal categories [22]. This will be more convenient for us to check the commutativity of certain diagrams.

Definition 2.2 (Definition 3.1 [34])**.** A *triangulated category with duality* is a triangulated category $\mathcal{T}$ (with the translation functor $T : \mathcal{T} \to \mathcal{T}$ only assumed to be an auto-equivalence) together with an additive functor $\# : \mathcal{T}^{\mathrm{op}} \to \mathcal{T}$ and natural isomorphisms $\lambda : \# \to T\#T$ and $\varpi : 1 \to \#\#$ satisfying the following conditions:

- The diagram commutes

$$\begin{array}{ccc} T & \xrightarrow{\varpi_T} & \#\#T \\ {\scriptstyle T\varpi}\downarrow & & \downarrow{\scriptstyle \lambda_{\#T}} \\ T\#\# & \xleftarrow[T\#\lambda]{} & T\#T\#T \end{array}$$

- For all objects X of $\mathcal{T}$, we have $\varpi_X^{\#} \circ \varpi_{X^{\#}} = 1_{X^{\#}}$.
- If
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$$
is a distinguished triangle in $\mathcal{T}$. Then, the following triangle is also distinguished in $\mathcal{T}$
$$Z^{\#} \xrightarrow{g^{\#}} Y^{\#} \xrightarrow{f^{\#}} X^{\#} \xrightarrow{T(h^{\#})\circ\lambda_X} T(Z^{\#})$$

Remark 2.3. The difference between Definition 2.2 and Balmer's triangulated category with δ-exact duality (cf. Definition 2.2 [3] or Definition 1.4.1 [4]) is that

- Balmer assumes the translation functor $T : \mathcal{T} \to \mathcal{T}$ to be an automorphism, but Definition 2.2 only assumes it to be an auto-equivalence. Even though every triangulated category considered in this paper has the translation functor which is an automorphism, it is better to keep this general definition.
- The condition $\#T = T^{-1}\#$ is assumed everywhere in Balmer's work of Witt groups. Sometimes, the requirement of this condition does not behave well. For instance, the triangulated category with duality (4) in Section 2.3 below which comes naturally from a closed symmetric monoidal category can never be a δ-exact duality (see also Remark 2.13). To remedy this, Schlichting [34] introduced the natural transformation $\lambda : \# \to T\#T$ in order to capture the data of the compatibility of the duality and the translation which requires Condition (1) in Definition 2.2.

Let $\mathcal{T} = \left(\mathcal{T}, \#, \varpi, \lambda\right)$ be a triangulated category with duality. In particular, $\left(\mathcal{T}, \#, \varpi\right)$ is a category with duality. An *inner product space* (or *symmetric space*) is a pair (X, ϕ) such that $X \in \mathcal{T}$ and $\varphi : X \xrightarrow{\cong} X^{\#}$ is an isomorphism with $\varphi^{\#} \circ \varpi_X = \varphi$. Orthogonal sum and isometries of inner product spaces are defined as usual.

Definition 2.4. The *Witt group* $W^0(\mathcal{T})$ of a triangulated category with duality $\mathcal{T} = \left(\mathcal{T}, \#, \varpi, \lambda\right)$ is the free abelian group generated by isometry classes $[X, \varphi]$ of inner product spaces (X, φ) in $\mathcal{T}$ subject to the following relations

- $[X, \varphi] + [Y, \psi] = [X \oplus Y, \varphi \oplus \psi]$
- If we have a map of exact triangles
$$\begin{array}{ccccccc} X_{-1} & \xrightarrow{f} & X_0 & \xrightarrow{g} & X_1 & \xrightarrow{h} & T(X_{-1}) \\ \varphi_{-1}\downarrow\cong & & \varphi_0\downarrow\cong & & \varphi_1\downarrow\cong & & T\varphi_{-1}\downarrow\cong \\ X_1^{\#} & \xrightarrow{g^{\#}} & X_0^{\#} & \xrightarrow{f^{\#}} & X_{-1}^{\#} & \xrightarrow{T(h^{\#})\circ\lambda_{X_{-1}}} & T(X_1^{\#}) \end{array}$$
with $\varphi_i = \varphi_{-i}^{\#} \circ \varpi$ is an isomorphism for each $i \in \{-1, 0, 1\}$, then $[X_0, \varphi_0] = 0$.

Remark 2.5. In Remark 3.14 [34], Schlichting showed that the Witt group defined in the way of Definition 2.4 agrees with Balmer's in [3]. More precisely, assume that $\left(\mathcal{T}, \#, \varpi, \delta\right)$ is a triangulated category with δ-exact duality in the sense of Balmer's Definition 1.4.3 [4]. We can get a well-defined triangulated category with duality $(\mathcal{T}, \#, \varpi, \lambda)$ in the sense of Schlichting's Definition 2.2 by defining $\lambda : \# \xrightarrow{\delta\cdot\mathrm{id}} T\#T$. Denote $W^S(\mathcal{T})$ (resp. $W^B(\mathcal{T})$) to be Schlichting's (Balmer's) Witt group. One checks that the map $W^S(\mathcal{T}) \to W^B(\mathcal{T}) : [X, \varphi] \mapsto [X, \varphi]$ is an isomorphism of groups. Therefore, we can identify Schlichting's triangular Witt groups with Balmer's by this isomorphism.

Definition 2.6 (Definition 3.10 [34])**.** If $\mathcal{T} = (\mathcal{T}, \#, \mathrm{can}, \lambda)$ is a triangulated category with duality, we define the translated duality on $\mathcal{T}$ to be
$$\mathcal{T}^{[1]} = \left(\mathcal{T}, \#^{[1]}, \mathrm{can}^{[1]}, \lambda^{[1]}\right) := \left(\mathcal{T}, T\#, -\lambda_{\#} \circ \mathrm{can}, -T(\lambda)\right)$$

Similarly, we define

$$\mathcal{T}^{[-1]} = \left(\mathcal{T}, \#^{[-1]}, \mathrm{can}^{[-1]}, \lambda^{[-1]}\right) := \left(\mathcal{T}, \#T, (\#\lambda)\circ \mathrm{can}, -\lambda_T\right)$$

Iterating, we get triangulated categories with duality $\mathcal{T}^{[j]}$ for $j \in \mathbb{Z}$:

$$\mathcal{T}^{[j]} = \begin{cases} \mathcal{T} & \text{for } j = 0 \\ (\mathcal{T}^{[j-1]})^{[1]} & \text{for } j > 0 \\ (\mathcal{T}^{[j+1]})^{[-1]} & \text{for } j < 0 \end{cases}$$

Definition 2.7 (Definition 3.11 [34])**.** The *shifted Witt group for $j \in \mathbb{Z}$* is

$$W^j(\mathcal{T}) := W^0(\mathcal{T}^{[j]}).$$

Remark 2.8. It is also worth mentioning that Balmer's shifted Witt groups (cf. Definition 1.4.5 [4]) agree with Schlichting's. Note that Witt groups are 4-periodicity, i.e., $W^j(\mathcal{T}) \cong W^{j+4}(\mathcal{T})$, cf. Proposition 2.14 [3].

Definition 2.9. A *semi-orthogonal decomposition* of a triangulated category $\mathcal{T}$, denoted by

$$\mathcal{T} = \left\langle \mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_n \right\rangle$$

consists of the following data:

- $\mathcal{T}_i$ are full triangulated subcategories of $\mathcal{T}$;
- $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_n$ generate $\mathcal{T}$;
- $\mathrm{Hom}(\mathcal{T}_i, \mathcal{T}_j) = 0$ for all $j < i$.

The following theorem was proved in [38], or see Proposition 6.8 [34].

Theorem 2.10 (Additivity)**.** *Let $(\mathcal{T}, \#)$ be a triangulated category with duality, and let $\mathcal{T}_1, \mathcal{A}, \mathcal{T}_2$ be full triangulated subcategories of $\mathcal{T}$. Assume $\frac{1}{2} \in \mathcal{T}$. If*

- *$\mathcal{A}$ is fixed by the duality (that is $\#(\mathcal{A}) \subset \mathcal{A}$),*
- *$\mathcal{T}_1$ and $\mathcal{T}_2$ are exchanged by the duality, i.e. $\#(\mathcal{T}_1) \subset \mathcal{T}_2$ and $\#(\mathcal{T}_2) \subset \mathcal{T}_1$,*
- *$\mathcal{T} = \left\langle \mathcal{T}_1, \mathcal{A}, \mathcal{T}_2 \right\rangle$ is a semi-orthogonal decomposition.*

then the inclusion $\mathcal{A} \subset \mathcal{T}$ induces isomorphisms of Witt groups

$$W^j(\mathcal{A}) \cong W^j(\mathcal{T}).$$

A sequence $\mathcal{T}_1 \to \mathcal{T}_2 \to \mathcal{T}_3$ of triangulated categories is called *exact* if the composition is trivial, $\mathcal{T}_1 \to \mathcal{T}_2$ makes $\mathcal{T}_1$ a full subcategory which is closed under direct factors, and the induced functor $\mathcal{T}_2/\mathcal{T}_1 \to \mathcal{T}_3$ is an equivalence, where $\mathcal{T}_2/\mathcal{T}_1$ is the Verdier quotient.

Theorem 2.11 (Localization [3] or Theorem 6.6 [34])**.** *Let $\frac{1}{2} \in \mathcal{T}_i$, and let $\mathcal{T}_2 = (\mathcal{T}_2, *)$ be a triangulated category with duality. Assume that $\mathcal{T}_1 \to \mathcal{T}_2 \to \mathcal{T}_3$ is an exact sequence of triangulated categories and $\mathcal{T}_1$ is invariant under the duality of $\mathcal{T}_2$. Then, there is a long exact sequence of Witt groups*

$$\cdots \to W^{j-1}(\mathcal{T}_3) \xrightarrow{\partial} W^j(\mathcal{T}_1) \longrightarrow W^j(\mathcal{T}_2) \longrightarrow W^j(\mathcal{T}_3) \xrightarrow{\partial} W^{j+1}(\mathcal{T}_1) \to \cdots$$

Remark 2.12. For Balmer's description of the connecting homomorphism $\partial : W^{i-1}(\mathcal{T}_3) \to W^i(\mathcal{T}_1)$, we may refer to Corollary 5.16 [3]. In *loc. cit.*, Balmer described the boundary map ∂ in the localization sequence above using a 'cone construction'. Here we interpret it in Schlichting's framework. Let $(\mathcal{K}, *, \varpi, \lambda)$ be a triangulated category with duality and let $u : A \to A^\#$ be a symmetric form, i.e. $u = u^\# \circ \varpi_A$. Choose an exact triangle of u:

$$A \xrightarrow{u} A^\# \xrightarrow{u_1} C \xrightarrow{u_2} TA$$

The proof of Theorem 2.6 [3] shows that there exists an isomorphism ψ such that the diagram

$$
\begin{array}{ccccccc}
A & \xrightarrow{u} & A^{\#} & \xrightarrow{u_1} & C & \xrightarrow{u_2} & T(A) \\
\downarrow{\scriptstyle \varpi_A} & \nearrow{\scriptstyle u^{\#}} & \downarrow{\scriptstyle \lambda_A} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle T\varpi_A} \\
A^{**} & \xrightarrow[T((Tu)^{\#})\circ\lambda_{A^{\#}}]{} & T((TA)^{\#}) & \xrightarrow[-Tu_2^{\#}]{} & T(C^{\#}) & \xrightarrow[Tu_1^{\#}]{} & T(A^{**})
\end{array}
\tag{2}
$$

commutes and ψ is symmetric in $(\mathcal{K}, \#^{[1]})$. Note that the upper triangle diagram on the left square is commutative by the symmetry of u and the lower triangle diagram on the left square is commutative by the naturality of $\lambda : \# \to T\#T$. Define cone(u) to be the symmetric space (C, ψ) on $(\mathcal{K}, \#^{[1]})$. The map ∂ can be described as follows: Any symmetric space $u : A \to A^{\#^{[j-1]}}$ on $(\mathcal{T}_3, \#^{[j-1]})$ can be lifted to be a symmetric form $u' : A' \to (A')^{\#^{[j-1]}}$ on $(\mathcal{T}_2, \#^{[j-1]})$, and $\partial([A, u]) = [\text{cone}(u')]$.

2.3. **Derived Witt groups of a scheme.** Let X be a scheme. Let $\mathcal{D}^b X$ be the derived category of bounded (cochain) complexes of finite rank locally free sheaves over X. The tensor product $M \otimes N$ of two complexes M and N in $\mathcal{D}^b X$ is given by

$$(M \otimes N)^n = \bigoplus_{i+j=n} M^i \otimes_{\mathscr{O}_X} N^j$$

with differential $d(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy$. The mapping complex $[M, N]$ is given by

$$[M, N]^n = \prod_{-i+j=n} \mathscr{H}om_{\mathscr{O}_X}(M^i, N^j)$$

with differential

$$df = d \circ f - (-1)^{|f|} f \circ d. \tag{3}$$

The triangulated category with duality playing an important role in this paper is the following quadruple

$$(\mathcal{D}^b X, \#_{\mathscr{L}[i]}, \text{can}_{\mathscr{L}[i]}, \lambda_{\mathscr{L}[i]}) \tag{4}$$

where $\mathscr{L}[i]$ is a complex with $\mathscr{L}$ a line bundle in the degree $-i$, $\#_{\mathscr{L}[i]} : \mathcal{D}^b X^{\text{op}} \to \mathcal{D}^b X$ is the functor sending M to $[M, \mathscr{L}[i]]$, $\text{can}_{\mathscr{L}[i]}$ is the natural isomorphism $1 \to \# \circ \#$ given by

$$\text{can}_M : M \to [[M, \mathscr{L}[i]], \mathscr{L}[i]] : x \mapsto (\hat{x} : f \mapsto (-1)^{|x||f|} f(x)),$$

and $\lambda : \# \to T\#T$ is the natural isomorphism with λ_M defined by the inverse of the isomorphism

$$\alpha_M : T[TM, \mathscr{L}[i]] \to [M, \mathscr{L}[i]] : \alpha(b \otimes f)(x) = (-1)^{|b||f|} f(b \otimes x).$$

To simplify the notations, we usually write $\left(\mathcal{D}^b X, \#_{\mathscr{L}[i]}\right)$ instead of quadruple (4) in this paper. In fact, $\left(\mathcal{D}^b X, \#_{\mathscr{L}[i]}\right)$ fits into the framework of closed symmetric mononidal categories, because it comes naturally from the dg category with weak equivalences and duality

$$\left(\text{Ch}^b(X), \text{quis}, \#_{\mathscr{L}[i]}, \text{can}_{\mathscr{L}[i]}\right)$$

by taking associated triangulated categories (see Section 2.4 for more details).

Remark 2.13. Note that our sign choice on the differential (3) of $[M, N]$ is different from [3], [6], [15] and [16], where they use

$$df = f \circ d - (-1)^{|f|} d \circ f \tag{5}$$

Both sign choices are compatible, cf. Section 6 [32]. However, the sign choice of the differential (5) of $[M, N]$ forces the evaluation map $[M, N] \otimes M \to N : f \otimes m \mapsto (-1)^{\frac{(|f|+|m|)}{2}} f(m)$ of complexes to contain a non-natural sign $((-1)^{\frac{(|f|+|m|)}{2}})$. This provides non-naturality for us to check strict commutative diagrams. Therefore, in order to use the evaluation map $[M, N] \otimes M \to N : f \otimes m \mapsto f(m)$ containing no sign, we adopt the differential (3) in this paper. Unfortunately, the differential (3) used in this paper does not satisfy the axiom "$T\#_{\mathscr{O}} \neq \#_{\mathscr{O}}T$" in Balmer's definition of the δ-exact duality, but it fits into Schlichting's definition of triangulated category (cf. Definition 2.2). Therefore, we choose Schlichting's framework in this paper.

Definition 2.14. For a scheme X and $i \in \mathbb{Z}$, we define

$$W^i(X,\mathscr{L}) := W^i(\mathcal{D}^bX, \#_{\mathscr{L}}).$$

If $\mathscr{L} = \mathscr{O}$, we usually write $W^i(X) := W^i(X,\mathscr{O})$.

Lemma 2.15 (Lemma 3.12 [34])**.** *There is an isomorphism of triangulated categories with duality* $\left(\mathcal{D}^bX, \#_{\mathscr{L}[i]}[j]\right) \cong \left(\mathcal{D}^bX, \#_{\mathscr{L}[i+j]}\right)$, *which induces an isomorphism* $W^j(\mathcal{D}^bX, \#_{\mathscr{L}[i]}) \cong W^{i+j}(X,\mathscr{L})$.

The following result shows that the classical Witt group fits into the framework of triangular Witt groups.

Lemma 2.16 (Theorem 1.4.11 [4] or Proposition 3.8 [34])**.** *Let* $\frac{1}{2} \in X$. *The obvious functor* $\mathrm{Vect}(X) \to \mathcal{D}^bX$, *sending everything in degree zero, induces an isomorphism* $W(X) \to W^0(X)$.

2.4. **Dg categories with duality.** We collect necessary definitions and results from Section 1 of Schlichting's work [34]. Let k be a commutative ring containing $\frac{1}{2}$. Let $C(k)$ denote the symmetric monoidal category of complexes of k-modules. Let $\mathcal{A}$ be a small dg category (that is a small category enriched in $C(k)$). The category $Z^0\mathcal{A}$ of zero cycles (resp. $H^0\mathcal{A}$ of zero cohomology) is called the underlying category (resp. the homotopy category) of the dg category $\mathcal{A}$. A morphism in the dg category $\mathcal{A}$ means a morphism in $Z^0\mathcal{A}$.

A *pointed dg category* is a dg category equipped with a choice of zero object called base-point. Let $\mathscr{C}_k$ be the category of bounded complexes of finitely generated free k-modules. We can consider $\mathscr{C}_k$ as a dg category in a canonical way, cf. Example 1.2 [34]. A *pretriangulated dg category* is a pointed dg category $\mathcal{A}$ such that $\mathcal{A}$ is exact (cf. Definition 1.3 [34]) and such that the dg functor $\mathcal{A} \to \mathscr{C}_k \otimes \mathcal{A} : A \mapsto \mathbb{1} \otimes A$ gives an equivalence of dg categories. If $\mathcal{A}$ is a pretriangulated category, then $Z^0\mathcal{A}$ is an exact category and $H^0\mathcal{A}$ is triangulated. Every dg category considered in this paper is pretriangulated.

A *dg category with duality* is a triplet $\left(\mathcal{A}, \#, \mathrm{can}\right)$ consisting of a (pretriangulated) dg category $\mathcal{A}$, a dg functor $\# : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$ and a natural transformation $\mathrm{can} : 1 \to \# \circ \#^{\mathrm{op}}$ such that $\mathrm{can}_A^{\#} \circ \mathrm{can}_{A^\#} = 1_{A^\#}$. A *dg form functor* $\left(\mathcal{A}, \#_{\mathcal{A}}, \mathrm{can}_{\mathcal{A}}\right) \to \left(\mathcal{B}, \#_{\mathcal{B}}, can_{\mathcal{B}}\right)$ is a pair (F,φ) consisting of a dg functor $F : \mathcal{A} \to \mathcal{B}$ and a natural transformation $\varphi : F \circ \#_{\mathcal{A}} \to \#_{\mathcal{B}} \circ F^{\mathrm{op}}$ (called *duality compatibility morphism*) such that the diagram

$$\begin{array}{ccc} FA & \xrightarrow{F(\mathrm{can}_A)} & F(A^{\#_{\mathcal{A}}\#_{\mathcal{A}}}) \\ \downarrow{\scriptstyle \mathrm{can}_{FA}} & & \downarrow{\scriptstyle \varphi_{A^\#}} \\ (FA)^{\#_{\mathcal{B}}\#_{\mathcal{B}}} & \xrightarrow{\varphi_A^{\#}} & F(A^{\#_{\mathcal{A}}})^{\#_{\mathcal{B}}} \end{array}$$

commutes. A *dg category with weak equivalences* is a pair $(\mathcal{A}, w)$ consisting of the following data: a full dg subcategory $\mathcal{A}^w \subset \mathcal{A}$ and a set of morphisms w in $\mathcal{A}$ such that $f \in w$ if and only if f is an isomorphism in the quotient triangulated category $\mathcal{T}(\mathcal{A}, w) := H^0\mathcal{A}/H^0(\mathcal{A}^w)$. An *exact dg functor* $F : (\mathcal{A}, w) \to (\mathcal{C}, v)$ of (pretriangulated) dg categories with weak equivalences is a dg functor $F : \mathcal{A} \to \mathcal{C}$ preserving weak equivalences, i.e. $F(\mathcal{A}^w) \subset \mathcal{C}^v$. A quadruple $\mathscr{A} = \left(\mathcal{A}, w, \#, \mathrm{can}\right)$ is called a *dg category with weak equivalences and duality* if $(\mathcal{A}, w)$ is a dg category with weak equivalences, $\left(\mathcal{A}, \#, \mathrm{can}\right)$ is a (pretriangulated) dg category with duality, $\mathcal{A}^w$ is invariant under the duality, and $\mathrm{can}_A \in w$ for all objects A in $\mathcal{A}$. An *exact dg form functor* $(F,\varphi) : (\mathcal{A}, w, \#, \mathrm{can}) \to (\mathcal{C}, v, \#, \mathrm{can})$ consists of a dg form functor $(F,\varphi) : (\mathcal{A}, \#, \mathrm{can}) \to (\mathcal{C}, \#, \mathrm{can})$ such that the dg functor $F : (\mathcal{A}, w) \to (\mathcal{C}, v)$ is exact. An exact dg form functor $(F,\varphi) : \Big(\mathcal{A}, w, \#, \mathrm{can}\Big) \to \Big(\mathcal{C}, v, \#, \mathrm{can}\Big)$ is called *non-singular* if $\varphi_A : F(A^{\#}) \to F(A)^{\#}$ is a weak equivalence in v for any object $A \in \mathcal{A}$. For every (pretriangulated) dg category $\mathscr{A}$ with weak equivalences and duality, there is an associated triangulated category $\mathcal{T}\mathscr{A} = \Big(\mathcal{T}(\mathcal{A}, w), \#, \mathrm{can}, \lambda\Big)$ with duality, cf. Section 3 [34]. Any non-singular exact dg form functor $\mathscr{A} \to \mathscr{C}$ induces a duality preserving functor $\mathcal{T}\mathscr{A} \to \mathcal{T}\mathscr{C}$ of triangulated categories with duality.

Let $\Big(\mathcal{A}, \#_{\mathcal{A}}, \mathrm{can}_{\mathcal{A}}\Big)$ and $\Big(\mathcal{B}, \#_{\mathcal{B}}, \mathrm{can}_{\mathcal{B}}\Big)$ be dg categories with duality. The tensor product dg category

$$\Big(\mathcal{A}, \#_{\mathcal{A}}, \mathrm{can}_{\mathcal{A}}\Big) \otimes \Big(\mathcal{B}, \#_{\mathcal{B}}, \mathrm{can}_{\mathcal{B}}\Big)$$

with duality is the dg category with duality $\Big(\mathcal{A} \otimes \mathcal{B}, \#_{\mathcal{A}} \otimes \#_{\mathcal{B}}, \mathrm{can}_{\mathcal{A}} \otimes \mathrm{can}_{\mathcal{B}}\Big)$ defined in Section 1 [34].

Lemma 2.17. *Assume that* $\Big(\mathcal{A}, \#_{\mathcal{A}}, \mathrm{can}_{\mathcal{A}}\Big) \otimes \Big(\mathcal{B}, \#_{\mathcal{B}}, \mathrm{can}_{\mathcal{B}}\Big) \to \Big(\mathcal{C}, \#_{\mathcal{C}}, \mathrm{can}_{\mathcal{C}}\Big)$ *is a dg form functor. Any symmetric form* (A, φ) *on* $\Big(\mathcal{A}, \#_{\mathcal{A}}, \mathrm{can}_{\mathcal{A}}\Big)$ *induces a dg form functor* $(A,\varphi) \otimes - : \Big(\mathcal{B}, \#_{\mathcal{B}}, \mathrm{can}_{\mathcal{B}}\Big) \longrightarrow \Big(\mathcal{C}, \#_{\mathcal{C}}, \mathrm{can}_{\mathcal{C}}\Big)$.

Proof. Recall that a dg form functor $\Big(\mathcal{A}, \#_{\mathcal{A}}, \mathrm{can}_{\mathcal{A}}\Big) \otimes \Big(\mathcal{B}, \#_{\mathcal{B}}, \mathrm{can}_{\mathcal{B}}\Big) \to \Big(\mathcal{C}, \#_{\mathcal{C}}, \mathrm{can}_{\mathcal{C}}\Big)$ consists of the following data: a dg functor $F : \mathcal{A} \otimes \mathcal{B} \to \mathcal{C} : (A, B) \mapsto A \otimes B$ and a duality compatibility morphism $\epsilon_{A,B} : A^{\#} \otimes B^{\#} \to (A \otimes B)^{\#}$ such that the diagram

$$\begin{array}{ccc} A \otimes B & \longrightarrow & A^{\#\#} \otimes B^{\#\#} \\ \big\downarrow & & \big\downarrow \\ (A \otimes B)^{\#\#} & \longrightarrow & (A^{\#} \otimes B^{\#})^{\#} \end{array} \tag{6}$$

commutes. Given a symmetric form (A, φ) on $(\mathcal{A}, \#_{\mathcal{A}}, \mathrm{can}_{\mathcal{A}})$, we can define a dg form functor

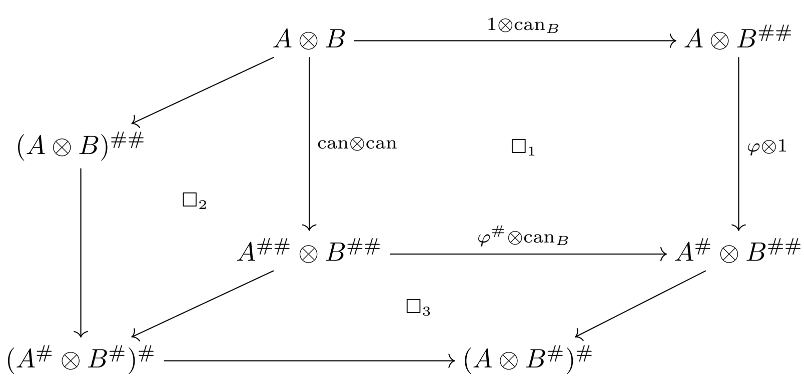
$$(A, \varphi) \otimes - : \Big(\mathcal{B}, \#_{\mathcal{B}}, \mathrm{can}_{\mathcal{B}}\Big) \longrightarrow \Big(\mathcal{C}, \#_{\mathcal{C}}, \mathrm{can}_{\mathcal{C}}\Big)$$

by the data: a dg functor $A \otimes - : \mathcal{B} \to \mathcal{C} : B \mapsto A \otimes B$ and a duality compatibility morphism

$$\varphi : A \otimes B^{\#} \xrightarrow{\varphi \otimes 1} A^{\#} \otimes B^{\#} \xrightarrow{\epsilon_{A,B}} (A \otimes B)^{\#}$$

Consider the diagram

(7)



We conclude the commutativity of $\square_1$ by the symmetry of the form φ, the commutativity of $\square_2$ which follows from the commutative diagram (6), and the commutativity of $\square_3$ which holds due to the naturality of ϵ. $\square$

2.5. **Products from dg categories.** Let A, B be dg k-algebras. Recall from [32, Section 7.2] the definition of dg A-modules. We denote by A-Mod $:= A$-Mod-k, Mod-$B := k$-Mod-B and A-Mod-B the dg category of dg left A-modules, dg right B-modules, and of dg left A-modules and right B-modules. If A is a dg algebra with involution, then for any dg left A-module M we have a dg right A-module M^{op}, cf. [32, Section 7.3]. Let (I, i) denote an A-bimodule I together with an A-bimodule isomorphism $i : I \to I^{\mathrm{op}}$ such that $i^{\mathrm{op}} \circ i = 1$. By abuse of the notation, we write I for (I, i) if the isomorphism i is understood. In fact, I is called a *duality coefficient* in [32, Section 7.3]. There is a dg category with duality

$$\Big(A\text{-Mod-}B, \#_I, \mathrm{can}_I\Big)$$

where the dg functor $\#_I^i : (A\text{-Mod-}B)^{\mathrm{op}} \to A\text{-Mod-}B$ (or $\#_I$ if i is understood) is defined by $M^{\#_I} = [M^{\mathrm{op}}, I]_A$ (Here, $[M^{\mathrm{op}}, I]_A$ is an object in A-Mod-B consisting maps of right A-modules, cf. [32, Section 7]) and $\mathrm{can}_I : M \to M^{\#\#}$ by

$$\mathrm{can}(x)(f^{\mathrm{op}}) = (-1)^{|f||x|} i(f(x^{\mathrm{op}})). \tag{8}$$

Remark 2.18. The trick to memorize the sign is that whenever we want to swap the position of two symbols x and f, we introduce a sign $(-1)^{|x||f|}$. For instance, in the above formula (8), we swapped the order of the symbols x, f on the left-hand side to f, x on the right-hand side. For another example about the sign trick, see Lemma 2.19. This sign trick is a consequence of the coherence theorem, cf. [22].

By abuse of the notation, we write $(A\text{-Mod-}B, \#_I)$ for this dg category with duality.

Lemma 2.19. *Let A, B be dg algebras with involution. There is a dg form functor*

$$\Big(A\text{-Mod-}B, \#_I\Big) \otimes \Big(B\text{-Mod}, \#_B\Big) \to \Big(A\text{-Mod}, \#_I\Big)$$

sending (M, N) to $M \otimes_B N$ with the duality compatibility map

$$\gamma : [M^{\mathrm{op}}, I]_A \otimes_B [N^{\mathrm{op}}, B]_B \to [(M \otimes_B N)^{\mathrm{op}}, I]_A \tag{9}$$

defined by

$$\gamma(f \otimes g)((m \otimes n)^{\mathrm{op}}) = (-1)^{|m||n|} f(g(n^{\mathrm{op}})m^{\mathrm{op}}).$$

Proof. By the definition of dg form functor above, we only need to check that the following diagram commutes.

$$\begin{array}{ccc} M \otimes_B N & \xrightarrow{\mathrm{can}\otimes\mathrm{can}} & M^{\#_I\#_I} \otimes_B N^{\#_B\#_B} \\ \Big\downarrow{\scriptstyle \mathrm{can}} & & \Big\downarrow{\scriptstyle \gamma\#} \\ (M \otimes_B N)^{\#_I\#_I} & \xrightarrow{\#\gamma} & (M^{\#_I} \otimes_B N^{\#_B})^{\#_I} \end{array}$$

The readers are invited to do this exercise (The trick is to use Remark 2.18). ∎

2.6. **Notations on the duality.** Since we are going to use different dualities on various categories, this may lead to confusions. We summarize our notations on dualities used in this paper in the following table.

	A-Mod-B (Section 2.5)
I	duality coefficient on A
$\#_I$	$[(-)^{\mathrm{op}_A}, I]_A$

	$\mathrm{Ch}^b(k)$ (Section 2.3)	$\mathrm{Ch}^b(X)$ (Section 2.3)
L	rank one free k-module	line bundle
$\#_{L[i]}$	$[-, L[i]]_k$	$[-, L[i]]_{\mathscr{O}}$
$\#_P^d$	$[-, \det(P)[d]]_k$	$[-, \mathscr{O} \otimes \det(P)[d]]_{\mathscr{O}}$
$\#^d$	$[-, k[d]]_k$	$[-, \mathscr{O}[d]]_{\mathscr{O}}$
$\#_P$	$[-, \det(P)[0]]_k$	$[-, \mathscr{O} \otimes \det(P)[0]]_{\mathscr{O}}$

Throughout the paper, we also write $M^* := \mathrm{Hom}_k(M,k)$ (resp. $\mathscr{F}^\vee := \mathscr{H}om_{\mathscr{O}_X}(M,\mathscr{O}_X)$) for a k-module M (resp. an $\mathscr{O}_X$-module $\mathscr{F}$).

3. Review: Semi-orthogonal decomposition on $\mathcal{D}^bQ$

Let (P,q) be a non-degenerate quadratic form of rank n over k. One has a graded ring $A = S(P^*)/(q)$ by considering q as an element in $S^2(P^*)$ where $P^* := \mathrm{Hom}_k(P,k)$, cf. [36, Section 2]. Define Q (or Q_d) to be the projective variety Proj A which is smooth of relative dimension $d = n-2$ over k, cf. [36, Proposition 2.2].

Let $\mathcal{D}^bQ$ be the derived category of bounded chain complexes of finite rank locally free sheaves over Q. It is well-known that Swan's computation of K-theory of quadric hypersurfaces [36] can be adapted to deduce a semi-orthogonal decomposition of $\mathcal{D}^bQ$. Meanwhile, $\mathcal{D}^bQ$ has been extensively studied, cf. [21], [26]. Since Swan's version is most related to what we are doing, I will explain how to adapt Swan's computation of K-theory of quadrics to a semi-orthogonal decomposition of $\mathcal{D}^bQ$. I thank Marco Schlichting for sharing with me his personal notes on this.

Define $\mathscr{O}(i) := \widetilde{A(i)}$. Recall from [36, Section 8] that there is a pairing $\mathscr{O}(i)\otimes P^* \to \mathscr{O}(i+1)$ induced by the multiplication in the symmetric algebra. It follows that one has a map $\mathscr{O}(i) \to \mathscr{O}(i+1)\otimes P$ as the following composition

$$\mathscr{O}(i) \to \mathrm{Hom}_k(P^*,\mathscr{O}(i+1)) \xrightarrow{\cong} \mathscr{O}(i+1)\otimes \mathrm{Hom}_k(P^*,k) \to \mathscr{O}(i+1)\otimes P$$

where the first map is induced by the pairing, the middle map is the canonical isomorphism, and the last map is induced by the inverse of the double dual identification $P \to \mathrm{Hom}_k(P^*,k)$.

Definition 3.1. If (P,q) is a quadratic form over k, the Clifford algebra $C(q)$ is defined as $T(P)/I(q)$ where $T(P)$ is the tensor algebra of P over k and where $I(q)$ is the two-sided ideal generated by $v\otimes v - q(v)$ for all $v \in P$.

The Clifford algebra $C(q) = C_0(q)\oplus C_1(q)$ has a $\mathbb{Z}/2\mathbb{Z}$-grading induced by the grading on $T(P)$. Let $\iota : P \to C(q)$ be the inclusion. The Clifford algebra $C(q)$ has the following universal property: for any k-algebra B and any k-linear map $\varphi : P \to B$ with $\varphi(v)^2 = q(v)$, there is a unique k-algebra homomorphism $\tilde{\varphi} : C(q) \to B$ such that $\tilde{\varphi}\circ\iota = \varphi$.

If $M = M_0 \oplus M_1$ is a $\mathbb{Z}/2\mathbb{Z}$-graded *left* $C(q)$-module, then there are natural maps $P\otimes M_j \to M_{j+1}$ which induce maps

$$\ell = \ell_{i,j} : \mathscr{O}(i)\otimes M_j \to \mathscr{O}(i+1)\otimes M_{j+1}$$

by the composition

$$\mathscr{O}(i)\otimes M_j \to \mathscr{O}(i+1)\otimes P\otimes M_j \to \mathscr{O}(i+1)\otimes M_{j+1}.$$

It follows that there is a sequence ($s_i = i+d+1 \in \mathbb{Z}/2\mathbb{Z}$)

$$\cdots \longrightarrow \mathscr{O}(-i-1)\otimes M_{s_{i+1}} \longrightarrow \mathscr{O}(-i)\otimes M_{s_i} \longrightarrow \mathscr{O}(-i+1)\otimes M_{s_{i-1}} \longrightarrow \cdots \tag{10}$$

which is called *Clifford sequence* in [36, Section 8].

Definition 3.2. The *Swan bundle* is defined as

$$\mathscr{U}_i(M) := \mathrm{coker}\Big[\mathscr{O}(-i-2)\otimes M_{s_{i+2}} \to \mathscr{O}(-i-1)\otimes M_{s_{i+1}}\Big]$$

If $M = C(q)$, then we define $\mathscr{U}_i := \mathscr{U}_i(C(q))$.

Swan proved that $\mathrm{End}(\mathscr{U}_i) \cong C_0(q)$ for any $i \in \mathbb{Z}$, cf. [36, Lemma 8.7]. Note that $\mathscr{U}_i(M) = \mathscr{U}_i \otimes_{C_0} M_0$.

Let $\Lambda^m := \Lambda^m(P^*)$ be the m-th exterior power. Let $\Lambda := \bigoplus_m \Lambda^m$ and $\Lambda^{(i)} := \bigoplus_m \Lambda^{2m+i}$ for $i \in \mathbb{Z}/2\mathbb{Z}$. Then, $\Lambda = \Lambda^{(0)}\oplus\Lambda^{(1)}$ can be viewed as a $\mathbb{Z}/2\mathbb{Z}$-graded left $C(q)$-module. To illustrate, choose $\beta \in P^*\otimes P^*$ lifting $q \in S^2(P^*)$. Let $x \in P$, and let β_x be the image of β under the map $P^*\otimes P^* \xrightarrow{\delta_x\otimes 1} k\otimes P^* = P^*$ where $\delta_x : P^* \to k : f \mapsto f(x)$. Let $L_x : \Lambda \to \Lambda : w \mapsto \beta_x \wedge w$. Note that δ_x extends uniquely to a derivation δ_x on Λ. Define a map $\varphi : P \to \mathrm{End}(\Lambda) : x \mapsto \delta_x + L_x$. This map satisfies $\varphi(x)^2 = q(x)$, hence extends to $C(q) \to \mathrm{End}(\Lambda)$ which gives the $\mathbb{Z}/2\mathbb{Z}$-graded $C(q)$-action on Λ, cf. [36, The paragraph after Proposition 8.2].

The inclusion $\det(P^*) \subset \Lambda$ induces a $\mathbb{Z}/2\mathbb{Z}$-graded $C(q)$-module isomorphism $C(q)\otimes \det P^* \to \Lambda(d)$ where $\Lambda(d)^{(i)} = \Lambda^{(d+i)}$, cf. [36, Lemma 8.3]. Then, one sees that $\mathscr{U}_i(\Lambda(d)) = \mathscr{U}_i \otimes \det P^*$ and that $\mathrm{End}(\mathscr{U}_i(\Lambda)) = \mathrm{End}(\mathscr{U}_i) = C_0(q)$, cf. [36, Corollary 8.8].

Lemma 3.3. $\mathrm{Ext}^p(\mathscr{U}_i, \mathscr{U}_i) = 0$ *for any* $p > 0$ *and any* i.

Proof. It is enough to verify this locally on Spec k. We assume P is free and $\mathscr{U}_i(\Lambda(d)) \cong \mathscr{U}_i$. Moreover, there is a vector x such that $q(x) \in k^\times$. Multiplying x on the right yields an isomorphism of k-modules $C_0 \cong C_1$ which induces an isomorphism of $\mathscr{O}_X$-modules $\mathscr{U}_i \otimes \mathscr{O}(1) \cong \mathscr{U}_{i-1}$. So, it suffices to check $\mathrm{Ext}^p(\mathscr{U}_{d-1}, \mathscr{U}_{d-1}) = 0$ for any $p > 0$. This follows from Lemma 6.1 [36] because $\mathscr{U}_{d-1} \cong \mathscr{Z}_{d-1}(\mathscr{O}(1))$ (cf. Corollary 8.6 [36]) and $\mathscr{U}_{d-1} \in \mathscr{P}_0 \subset \mathscr{R}_d$ (cf. the proof of Lemma 9.3 [36]). □

Replacing $M = M_0 \oplus M_1$ with $\Lambda(d)$ in the sequence (10), we get a sequence

$$\cdots \to \mathscr{O}(-i-1) \otimes \Lambda(d)^{(s_{i+1})} \to \mathscr{O}(-i) \otimes \Lambda(d)^{(s_i)} \to \mathscr{O}(-i+1) \otimes \Lambda(d)^{(s_{i-1})} \to \cdots \tag{11}$$

Recall the *Tate resolution*

$$\ldots \to \mathscr{O}(-2) \otimes (\Lambda^1 \oplus \Lambda^3 \oplus \Lambda^5) \to \mathscr{O}(-1) \otimes (\Lambda^0 \oplus \Lambda^2) \to \mathscr{O} \otimes \Lambda^1 \to \mathscr{O}(1), \tag{12}$$

for $\mathscr{O}(1)$, cf. [36, Section 7] or [36, Proof of Lemma 8.4]. Precisely, the resolution (12) is given by

$$T_{-i} := \mathscr{O}(-i) \otimes \big(\bigoplus_{d\geq 0} \Lambda^{i+1-2d}\big)$$

where $\Lambda^i := 0$ whenever $i > n$ or $i < 0$. The differential

$$\partial_i : \mathscr{O}(-i) \otimes \big(\bigoplus_{d\geq 0} \Lambda^{i+1-2d}\big) \to \mathscr{O}(-i+1) \otimes \big(\bigoplus_{d\geq 0} \Lambda^{i-2d}\big)$$

is defined by $\partial_i' + \partial_i''$ where

$$\partial_i' : f \otimes (p_1 \wedge \cdots \wedge p_k) \mapsto \sum_s (-1)^{s+1} f p_s \otimes (p_1 \wedge \cdots \wedge \hat{p}_s \wedge \cdots \wedge p_k)$$

and

$$\partial_i'' : f \otimes w \mapsto \gamma \wedge (f \otimes w) = \sum_i f\xi_i \otimes (\beta_i \wedge w)$$

where the element $\gamma = \sum \xi_i \otimes \beta_i \in P^* \otimes P^*$ lifts $q \in S_2(P^*)$ via the natural surjective map $P^* \otimes P^* \to S_2(P^*)$.

The sequence (11) is exact, since Swan showed that when $i \geq d$ the sequence (11) and the Tate resolution (12) coincide, and the maps in the sequence (11) have '2-periodicity', so when $i < d$ the exactness of the sequence (11) is analogous to the case $i \geq d$, cf. [36, Lemma 8.4].

Proposition 3.4. *For any* $i \in \mathbb{Z}$. *The functors*

$$\mathscr{O}(i) \otimes - : \mathcal{D}^b k \to \mathcal{D}^b Q \text{ and } \mathscr{U}_i \otimes_{C_0(q)} - : \mathcal{D}^b C_0(q) \to \mathcal{D}^b Q$$

are fully faithful, where $\mathcal{D}^b C_0(q)$ *is the derived category of bounded complexes of finitely generated projective left* $C_0(q)$*-modules.*

Proof. Note that

$$\mathrm{Hom}_{\mathcal{D}^b Q}(\mathscr{O}(i), \mathscr{O}(i)[p]) = H^p(Q, \mathscr{O}) = \begin{cases} 0 & \text{if } p > 0 \\ k & \text{if } p = 0 \end{cases}$$

by [36, Lemma 5.2] and that

$$\mathrm{Ext}^p(\mathscr{U}_i, \mathscr{U}_i) = \begin{cases} 0 & \text{if } p > 0 \\ C_0(q) & \text{if } p = 0 \end{cases}$$

by [36, Corollary 8.8] and Lemma 3.3. The result follows. □

Definition 3.5. Let Q_d be a smooth projective quadric of dimension d. We introduce the following notations in this paper.

- $\mathcal{U}_i$ is the smallest full idempotent complete triangulated subcategory of $\mathcal{D}^bQ_d$ containing the bundle $\mathscr{U}_i$, that is the essential image of the functor $\mathscr{U}_i \otimes_{C_0} - : \mathcal{D}^bC_0(q) \to \mathcal{D}^bQ_d$.
- $\mathcal{A}_i$ is the essential image of the functor $\mathscr{O}(i) \otimes - : \mathcal{D}^bk \to \mathcal{D}^bQ$.
- $\mathscr{V}_i$ is the bundle $\mathscr{U}_{m+i} \otimes \mathscr{O}(m)$ where $m := \lfloor \frac{d}{2} \rfloor$.
- $\mathcal{V}_i$ is the essential image of the functor $\mathscr{V}_i \otimes_{C_0} - : \mathcal{D}^bC_0(q) \to \mathcal{D}^bQ_d$.

Remark 3.6. If k is a local ring, then P is free. Moreover, there is a vector x such that $q(x) \in k^\times$. Multiplying x on the right gives an isomorphism $C_0 \cong C_1$ which induces an isomorphism of $\mathscr{O}_X$-modules $\mathscr{U}_i \otimes \mathscr{O}(1) \cong \mathscr{U}_{i-1}$, so $\mathscr{V}_i \cong \mathscr{U}_i$.

Theorem 3.7. *There is a semi-orthogonal decomposition*

$$\mathcal{D}^bQ = \Big\langle \mathcal{U}_{d-1}, \mathcal{A}_{1-d}, \dots, \mathcal{A}_{-1}, \mathcal{A}_0 \Big\rangle.$$

Proof. Firstly, we show that the set

$$\Sigma = \Big\{ \mathscr{U}_{d-1} \otimes \det(P^*), \mathscr{O}(1-d), \dots, \mathscr{O} \Big\}$$

generates $\mathcal{D}^bQ$ as an idempotent complete triangulated category. Let $\Big\langle \Sigma \Big\rangle \subset \mathcal{D}^bQ$ denote the full triangulated subcategory generated by Σ. Note that Q is a projective scheme and thus a scheme with an ample line bundle $\mathscr{O}(1)$. By [33, Lemma 3.5.2 or Lemma A.4.7], the triangulated category $\mathcal{D}^bQ$ is generated (as an idempotent complete triangulated category) by the line bundles $\mathscr{O}(i)$ for $i \leq 0$. Taking duals, we see that $\mathcal{D}^bQ$ is generated (as an idempotent complete triangulated category) by $\mathscr{O}(i)$ for $i \geq 0$. The Tate resolution (12) implies that $\mathscr{O}(1)$ is in $\Big\langle \Sigma \Big\rangle$. Moreover, note that, for $i \geq 2$, $\mathscr{O}(i)$ is in $\mathfrak{R}_{-1}$ (i.e. (-1)-regular), cf. Lemma 5.2 [36]. Putting $\mathscr{O}(i)$ into the canonical resolution (cf. [36, p. 126 Section 6]), we get an acyclic complex

$$0 \to \mathscr{U} \otimes_{\mathrm{End}_X(\mathscr{U})} T(\mathscr{O}(i)) \to \mathscr{O}(1-d) \otimes T_{d-1}(\mathscr{O}(i)) \to \cdots \to \mathscr{O} \otimes T_0(\mathscr{O}(i)) \to \mathscr{O}(i) \to 0$$

for $i \geq 2$ where $\mathscr{U} := \mathscr{Z}_{d-1}(\mathscr{O}(1))$ is isomorphic to $\mathscr{U}_{d-1} \otimes \det P^*$, cf. Corollary 8.6 [36]. Note that $\mathrm{End}_X(\mathscr{U}) = C_0(q)$ by Corollary 8.8 [36]. Since $T(\mathscr{O}(i)) \in \mathcal{D}^bC_0(q)$ and $T_l(\mathscr{O}(i)) \in \mathcal{D}^b(k)$ (by their definitions in Section 6 [36]), we see that $\mathscr{O}(i)$ is in $\Big\langle \Sigma \Big\rangle$ for $i \geq 2$.

Furthermore, the set

$$\Sigma' = \Big\{ \mathcal{U}_{d-1}, \mathcal{A}_{1-d}, \dots, \mathcal{A}_0 \Big\}$$

also generates $\mathcal{D}^bQ$ as an idempotent complete triangulated category. This is because

$$\det(P) \otimes \Sigma = \Big\{ \mathscr{U}_{d-1}, \mathscr{O}(1-d) \otimes \det(P), \dots, \mathscr{O} \otimes \det(P) \Big\}$$

is an idempotent complete generating set of $\mathcal{D}^bQ$. It is also true that $\mathcal{A}_i \otimes \det(P) = \mathcal{A}_i$ since $\mathcal{D}^b(k) \cong \mathcal{A}_i$ by the definition of $\mathcal{A}_i$. Therefore, Σ' is an idempotent complete generating set of $\mathcal{D}^bQ$.

Note that

$$\mathrm{Hom}_{\mathcal{D}^bQ}(\mathscr{O}(i), \mathscr{O}(j)[p]) = H^p(Q, \mathscr{O}(j-i)) = 0$$

for $1-d \leq j < i \leq 0$, cf. [36, Lemma 5.2]. Moreover, we have

$$\mathrm{Hom}_{\mathcal{D}^bQ}(\mathscr{O}(i), \mathscr{U}_{d-1}[p]) = H^p(Q, \mathscr{U}_{d-1}(-i)) = 0$$

for $1-d \leq i \leq 0$, cf. [36, Proof of Lemma 9.3] for $p > 0$ and [36, Lemma 9.5] for $p = 0$. Thus, we conclude that $\mathrm{Hom}(\mathcal{A}_i, \mathcal{A}_j) = 0$ for $j < i$ and $\mathrm{Hom}(\mathcal{A}_i, \mathcal{U}_{d-1}) = 0$. □

Corollary 3.8. *There is a semi-orthogonal decomposition*

$$\mathcal{D}^bQ_d = \begin{cases} \Big\langle \mathcal{A}_{-m}, \dots, \mathcal{A}_{-1}, \mathcal{V}_0, \mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_m \Big\rangle & \text{if } d = 2m+1; \\ \Big\langle \mathcal{A}_{1-m}, \dots, \mathcal{A}_{-1}, \mathcal{V}_0, \mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_{m-1}, \mathcal{A}_m \Big\rangle & \text{if } d = 2m. \end{cases}$$

Proof. Let $d = 2m+1$. By taking the tensor product $\mathscr{O}(m) \otimes E$ for every element $E \in \Sigma$, we get another set

$$\Big\{ \mathscr{V}_m, \mathscr{O}(-m), \ldots, \mathscr{O}, \ldots, \mathscr{O}(m) \Big\}.$$

This set generates $\mathcal{D}^b Q$ as an idempotent complete triangulated category. Note that

$$\Sigma' = \Big\{ \mathscr{O}(-m), \ldots, \mathscr{O}(-1), \mathscr{V}_0, \mathscr{O}, \mathscr{O}(1), \ldots, \mathscr{O}(m) \Big\}$$

also generates $\mathcal{D}^b Q$ as an idempotent complete triangulated category. The exact sequence

$$0 \longrightarrow \mathscr{V}_m \longrightarrow \mathscr{O}(-m) \otimes C_{s_{2m}} \longrightarrow \cdots\cdots \longrightarrow \mathscr{O}(-1) \otimes C_{s_{m+1}} \longrightarrow \mathscr{V}_0 \longrightarrow 0$$

shows that $\mathscr{V}_m$ is in $\Big\langle \Sigma' \Big\rangle$.

Further, we want to check $\mathrm{Hom}(\mathscr{V}_0, \mathscr{O}(i)[p]) = 0$ and $\mathrm{Hom}(\mathscr{O}(j), \mathscr{V}_0[p]) = 0$ for $-m \leq i \leq -1, 0 \leq j \leq m$. This can be checked locally on $\mathrm{Spec}\, k$. We have $H^p(Q, \mathscr{F} \otimes \mathscr{V}_0^\vee) = \mathrm{Ext}^p(\mathscr{V}_0, \mathscr{F})$, and $\mathscr{V}_i \cong \mathscr{U}_i$ by the proof of Lemma 3.3. By Theorem 5.12 below, $\mathrm{Hom}(\mathscr{V}_0, \mathscr{O}(i)[p]) = H^p(Q, \mathscr{V}_0^\vee(i))$ is isomorphic to $H^p(Q, \mathscr{U}_{-1}(i)) = H^p(Q, \mathscr{U}_{d-1}(d+i))$ which is zero for $-m \leq i \leq -1$ and $p \geq 0$ by [36, Proof of Lemma 9.3 and Lemma 9.5]. Moreover, $\mathrm{Hom}(\mathscr{O}(j), \mathscr{V}_0[p]) = H^p(Q, \mathscr{U}_0(-j)) = H^p(Q, \mathscr{U}_{d-1}(d-j-1))$ which is also zero for $0 \leq j \leq m$ and $p \geq 0$ by *loc. cit.*. For the case of $d = 2m$, one uses a similar procedure. □

Definition 3.9. In this paper, we introduce the following notations.

- $\mathcal{A}$ is the idempotent complete full triangulated subcategory $\Big\langle \mathcal{V}_0, \mathcal{A}_0 \Big\rangle$ of $\mathcal{D}^b Q$.
- $\mathcal{A}_{[i,j]}$ is the idempotent complete full triangulated subcategory $\Big\langle \mathcal{A}_i, \ldots, \mathcal{A}_j \Big\rangle$ of $\mathcal{D}^b Q$ for $i \leq j$.
- $\mathcal{A}'$ is defined to be $\Big\langle \mathcal{A}_{[1-\lfloor \frac{d}{2} \rfloor, -1]}, \mathcal{A}, \mathcal{A}_{[1, \lfloor \frac{d}{2} \rfloor - 1]} \Big\rangle$.

We call the triangulated category $\mathcal{A}$ the *Swan triangulated category.*

Corollary 3.10. *There is a semi-orthogonal decomposition*

$$\mathcal{D}^b Q_d = \begin{cases} \Big\langle \mathcal{A}_{[-m,-1]}, \mathcal{A}, \mathcal{A}_{[1,m]} \Big\rangle & \text{if } d = 2m+1; \\ \Big\langle \mathcal{A}_{[1-m,-1]}, \mathcal{A}, \mathcal{A}_{[1,m-1]}, \mathcal{A}_m \Big\rangle = \Big\langle \mathcal{A}', \mathcal{A}_m \Big\rangle & \text{if } d = 2m. \end{cases}$$

4. An exact sequence

Consider the following maps of exact sequences

(13)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \ker(l) & \xrightarrow{\cong} & \ker(v) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
Cl_{-d-1}(\Lambda) & \longrightarrow & Cl_{-d}(\Lambda) & \longrightarrow & Cl_{-d+1}(\Lambda) & \longrightarrow & \mathscr{U}_{d-2}(\Lambda(d)) & \longrightarrow & 0 \\
\downarrow{=} & & \downarrow{=} & & \downarrow{l} & & \downarrow{\exists v} & & \\
T_{-d-1} & \longrightarrow & T_{-d} & \xrightarrow{\varphi_d} & T_{-d+1} & \longrightarrow & \mathrm{coker}(\varphi_d) & \longrightarrow & 0
\end{array}$$

where the middle row is the exact sequence (11) with $Cl_{-i}(\Lambda) := \mathscr{O}(-i) \otimes \Lambda(d)^{(s_i)}$, the bottom row is the Tate resolution (12), and the map l is the projection taking the component $\mathscr{O}(1-d) \otimes \Lambda^n(P^*)$ to 0. By the universal property of cokernels, we observe that there is a surjective map $v : \mathscr{U}_{d-2}(\Lambda) \to \mathrm{coker}(\varphi_d)$ such that the lower-third square in Diagram (13) is commutative. By taking kernels of all the maps between the bottom and the middle exact sequences of Diagram (13), we obtain the exact sequence in the top row. In particular, we see that

$$\mathscr{O}(1-d) \otimes \det(P^*) \cong \ker(l) \cong \ker(v).$$

Let s denote the composition $\mathscr{O}(1-d)\otimes\det(P^*)\to\ker(l)\to\ker(v)\to\mathscr{U}_{d-2}(\Lambda(d))$. Then, we have proved the following result.

Proposition 4.1. *There is an exact sequence*

$$0\to\mathscr{O}(1-d)\otimes\det(P^*)\xrightarrow{s}\mathscr{U}_{d-2}(\Lambda(d))\to T_{-d+2}\to\cdots\to T_{-1}\to T_0\to\mathscr{O}(1)\to 0$$

where the sequence of maps

$$T_{-d+2}\to\cdots\to T_{-1}\to T_0\to\mathscr{O}(1)\to 0$$

is the Tate resolution truncated from T_{-d+2}.

Corollary 4.2. *Let $d=2m$ be an even number. Tensoring the exact sequence of Proposition 4.1 with the line bundle $\mathscr{O}(m-1)$, we obtain another exact sequence*

$$0\to\mathscr{O}(-m)\otimes\det(P^*)\xrightarrow{s}\mathscr{U}_{d-2}(\Lambda(d))(m-1)\to T_{-d+2}(m-1)\to\cdots\to\mathscr{O}(m)\to 0. \tag{14}$$

The exact sequence (14) is denoted by $T_{[-m,m]}$.

5. The canonical involution and the trace

5.1. **Symmetric forms on Clifford algebras.** Let A be a k-algebra. Recall that an involution $\tau: A\to A^{\mathrm{op}}$ is a k-algebra homomorphism such that $\tau^2=1$.

Definition 5.1. The inclusion $\varphi: P\hookrightarrow C(q)^{\mathrm{op}}$ satisfies $\varphi(v)^2=q(v)$, hence it provides a k-algebra homomorphism $\sigma: C(q)\to C(q)^{\mathrm{op}}, x\mapsto\bar{x}$ which is an involution. We call σ the *canonical involution* on $C(q)$.

Remark 5.2. Note that the canonical involution preserves the $\mathbb{Z}/2\mathbb{Z}$-grading on $C(q)$, so it provides an involution $\sigma_0: C_0(q)\to C_0(q)$.

Remark 5.3. Assume that (P,q) is free of rank n over k. Let $\{e_1,\dots,e_n\}$ be a basis of P. The Clifford algebra $C(q)$ is free of rank 2^n over k with a basis $\{e^\Delta:\Delta\in\mathbb{F}_2^n\}$ where $e^\Delta := e_1^{b_1}\cdot\dots\cdot e_n^{b_n}$ with $\Delta=(b_1,\dots,b_n)\in\mathbb{F}_2^n$, cf. [24, Theorem IV.1.5.1]. It follows that $\sigma(e_1^{b_1}\cdot\dots\cdot e_n^{b_n})=e_n^{b_n}\cdot\dots\cdot e_1^{b_1}$.

Definition 5.4 (Trace map for algebras)**.** Assume that M is a finite rank free k-module. For any $f\in\mathrm{End}(M)$, we define $\mathrm{tr}(f)$ as the trace of the matrix induced by f in terms of a choice of finite k-basis of M. This is independent of the choice of basis. Let A be a k-algebra. Every element $a\in A$ induces a k-module homomorphism $A\to A$ by left multiplication, denoted by δ_a. If A is finite projective as a k-module, then we choose a k-module N such that $A\oplus N$ is finite free. The *trace* can be defined as the composition $A\hookrightarrow A\oplus N\to k$ where the second map is the trace defined in the case of free modules. In other words, $\mathrm{tr}(a):=\mathrm{tr}(\delta_a\oplus 0_N)$ for any $a\in A$. This is independent of the choice of N, see the second paragraph of [29, Chapter 3.6] for instance.

Now, consider the case when A is the Clifford algebra $C(q)$. Recall that $C(q)$ is a k-algebra which is projective as a k-module, cf. [36].

Remark 5.5. (a). The trace map on $C(q)$ restricts to the even part $C_0(q)$.
(b). If P is free of rank n with a basis $\{e_1,\dots,e_n\}$, the trace (cf. Definition 5.4) can be equivalently defined to be the k-module homomorphism $\mathrm{tr}: C(q)\to k$ given by $\mathrm{tr}(ae^0)=2^n\cdot a$ (for $a\in k$) and $\mathrm{tr}(e^\Delta)=0$ for any $\Delta\neq 0\in\mathbb{F}_2^n$.

Lemma 5.6. *Let k be a commutative ring containing $\frac{1}{2}$. Then the trace* $\mathrm{tr}: C(q)\to k$ *has the property that* $\mathrm{tr}(uv)=\mathrm{tr}(vu)$ *and* $\mathrm{tr}(z)=\mathrm{tr}(\bar{z})$ *for any $u,v,z\in C(q)$.*

Proof. Assume (P,q) is free over k. Then, $\mathrm{tr}(uv)=\mathrm{tr}(vu)$ because $C(q)$ is free. Moreover, and $\mathrm{tr}(z)=\mathrm{tr}(\bar{z})$ by Remark 5.5 (b). If (P,q) is not free over k, one can check this locally. □

The trace map together with the canonical involution induces a map

$$B: C(q)\times C(q)\to k:(x,y)\mapsto\mathrm{tr}(\bar{x}y)$$

Note that $\mathrm{tr}(\bar{x}y)=\mathrm{tr}(x\bar{y})$ by Lemma 5.6.

Lemma 5.7. *Let k be a commutative ring with $\frac{1}{2}$. Assume (P,q) is non-degenerate. Then B is a non-degenerate symmetric bilinear form over k.*

Proof. Firstly, B is bilinear (note that $\mathrm{tr}(\overline{ay}) = a\mathrm{tr}(\bar{y})$ for any $a \in k$). Moreover B is symmetric by Lemma 5.6. By definition, B is non-degenerate if the k-module homomorphism

$$\hat{B} : C(q) \to \mathrm{Hom}(C(q), k), a \mapsto (y \mapsto \mathrm{tr}(\bar{a}y))$$

is an isomorphism. This can be checked locally. We assume k is local. Every projective module over k is free, so $C(q)$ is free. Moreover, (P,q) over the local ring k has an orthogonal basis $\{e_1, \cdots, e_n\}$. We realize that

$$\mathrm{tr}(\overline{e^{\Delta}} e^{\Delta'}) = \begin{cases} 0 & \text{if } \Delta \neq \Delta'; \\ 2^n q(e^{\Delta}) & \text{if } \Delta = \Delta' \end{cases} \tag{15}$$

where $q(e^{\Delta}) = q(e_1)^{a_1} \cdots q(e_n)^{a_n}$ for $e^{\Delta} = e_1^{a_1} \cdots e_n^{a_n}$ and $\Delta = (a_1, \ldots, a_n) \in \mathbb{F}_2^n$. Since 2 and $q(e_i)$ are all units in k, $\hat{B}$ is an isomorphism. □

Let A be any k-algebra. If M is a right A-module, then M^* may be considered as a left A-module via the right multiplication. If we endow A with an involution $a \mapsto \bar{a}$, then its opposite module $M^{\mathrm{op}} := \{m^{\mathrm{op}} : m \in M\}$ equals M as a k-module and can be viewed as a left A-module: $a \cdot m^{\mathrm{op}} = (m\bar{a})^{\mathrm{op}}$ for any $a \in A$. It follows that $M^* := \mathrm{Hom}_k(M^{\mathrm{op}}, k)$ can be considered as a right A-module.

Lemma 5.8. *Let k be a commutative ring with $\frac{1}{2}$. Consider $C(q)$ as a right $C(q)$-module. Then the map*

$$\vartheta : C(q) \to \mathrm{Hom}_k(C(q)^{\mathrm{op}}, k) : x \mapsto \big(\vartheta(x) : y^{\mathrm{op}} \mapsto \mathrm{tr}(\bar{x}y)\big)$$

is a right $C(q)$-module isomorphism such that $\vartheta^* \circ \mathrm{can} = \vartheta$ where $\mathrm{can} : C(q) \to C(q)^{**} : m \mapsto (\mathrm{can}(m) : f^{\mathrm{op}} \mapsto f^{\mathrm{op}}(m^{\mathrm{op}}))$ is the double dual identification.

Proof. The map ϑ is a k-linear isomorphism by Lemma 5.7. We check that $\vartheta(xa)(y^{\mathrm{op}}) = [\vartheta(x)a](y^{\mathrm{op}})$ for all $a \in C(q)$ and $y^{\mathrm{op}} \in C(q)^{\mathrm{op}}$. Observe that $\vartheta(xa)(y^{\mathrm{op}}) = \mathrm{tr}(\overline{xa} \cdot y) = \mathrm{tr}(\bar{a}\bar{x}y)$ and $[\vartheta(x)a](y^{\mathrm{op}}) = (\vartheta(x))(a \cdot y^{\mathrm{op}}) = \vartheta(x)((y\bar{a})^{\mathrm{op}}) = \mathrm{tr}(\bar{x}y\bar{a})$. Using the identity $\mathrm{tr}(uv) = \mathrm{tr}(vu)$ for any $u, v \in C(q)$, we conclude the result. □

Remark 5.9. The map ϑ induces an isomorphism $\vartheta_i : C_i(q) \to \mathrm{Hom}_k(C_i(q)^{\mathrm{op}}, k)$ of right $C_0(q)$-modules for any $i \in \mathbb{Z}$.

5.2. **Symmetric forms on Swan bundles.** Let k be a commutative ring with $\frac{1}{2}$. Let Q_d be a smooth quadric of dimension d. Let $m := \lfloor \frac{d}{2} \rfloor$, and let $r = m+d+1$. Recall from Definition 3.2 for the definition of $\mathscr{U}_i$ and from Definition 3.5 for $\mathscr{V}_i := \mathscr{U}_{m+i} \otimes \mathscr{O}(m)$. Then, $\mathscr{V}_i$ can be considered as a right $C_0(q)$-module with the right $C_0(q)$-action coming from $\mathscr{U}_{m+i}$. Recall also that $\mathscr{F}^{\vee} := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$.

Let C_s denote $C_s(q)$ for $s \in \mathbb{Z}/2\mathbb{Z}$ for simplicity. Define a map

$$\eta_i : \mathscr{O}(-i) \otimes C_s \longrightarrow (\mathscr{O}(i) \otimes C_{-s}^{\mathrm{op}})^{\vee}$$

by $\eta_i(f \otimes x)(g \otimes y^{\mathrm{op}}) = fg \cdot \mathrm{tr}(\bar{x}y)$. It is evident that η_i is an isomorphism of $\mathscr{O}_X$-modules, because η_i is the composition

$$\mathscr{O}(-i) \otimes C_s \xrightarrow{\psi \otimes \vartheta} \mathscr{O}(i)^{\vee} \otimes (C_{-s}^{\mathrm{op}})^* \longrightarrow (\mathscr{O}(i) \otimes C_{-s}^{\mathrm{op}})^{\vee}$$

where $\psi : \mathscr{O}(-i) \to \mathscr{O}(i)^{\vee}$ is the natural isomorphism, ϑ is defined as in Lemma 5.8 and the last map is just the canonical isomorphism. Note that $C_s = C_{-s}$. By Lemma 5.8 and Remark 5.9, the maps η_i are well-defined right $C_0(q)$-homomorphisms.

Lemma 5.10. *The diagram*

$$\begin{array}{ccc} \mathscr{O}(-i-2)\otimes C_{r+i+2} & \xrightarrow{\ell} & \mathscr{O}(-i-1)\otimes C_{r+i+1} \\ \downarrow \eta & & \downarrow \eta \\ (\mathscr{O}(i+2)\otimes C^{\mathrm{op}}_{-r-i-2})^\vee & \xrightarrow{\ell^\vee} & (\mathscr{O}(i+1)\otimes C^{\mathrm{op}}_{-r-i-1})^\vee \end{array}$$

is commutative.

Proof. This can be checked locally. Assume that k is a local ring. Let $f\otimes m \in \mathscr{O}(-i-2)\otimes C_{r+i+2}$. We realize that the map ℓ may be interpreted as $\ell(f\otimes m)=\sum_{i=1}^n \beta_i f\otimes \xi_i m$ where $\{\xi_1,\dots,\xi_n\}$ is a basis of P and $\{\beta_1,\dots,\beta_n\}$ is the corresponding dual basis of P^*. Let $g\otimes n^{\mathrm{op}}\in \mathscr{O}(i+1)\otimes C^{\mathrm{op}}_{-r-i-1}$. It reduces to check that

$$\eta(\sum \beta_i f\otimes \xi_i m)(g\otimes n^{\mathrm{op}}) = \eta(f\otimes m)(\sum \beta_i g\otimes(\xi_i n)^{\mathrm{op}}).$$

The left-hand side equals $\sum \beta_i fg\cdot \mathrm{tr}(\overline{\xi_i m}n) = \sum \beta_i fg\cdot \mathrm{tr}(\bar m\bar\xi_i n)$, while the right-hand side equals $\sum \beta_i fg\cdot \mathrm{tr}(\bar m \xi_i n)$. Note that $\bar\xi=\xi\in P\subset C(q)$. This proves the commutativity. □

Definition 5.11. By definition of $\mathscr{V}_i$ (cf. Definition 3.5), we define

$$h_0:\mathscr{V}_0\longrightarrow (\mathscr{V}^{\mathrm{op}}_{-1})^\vee \text{ and } h_{-1}:\mathscr{V}_{-1}\longrightarrow (\mathscr{V}^{\mathrm{op}}_0)^\vee$$

to be induced maps from η_i by taking cokernels of ℓ and $\ell^\vee$ in Lemma 5.10.

Since η are isomorphisms of $\mathscr{O}_X$-modules and right $C_0(q)$-modules, so are the maps h_i.

Lemma 5.12. *There is an inner product of exact sequences*

$$\begin{array}{ccccc} \mathscr{V}_0 & \xrightarrow{\ell} & \mathscr{O}\otimes C_r & \xrightarrow{\ell} & \mathscr{V}_{-1} \\ \downarrow h_0 & & \downarrow \eta_0 & & \downarrow h_{-1} \\ (\mathscr{V}^{\mathrm{op}}_{-1})^\vee & \xrightarrow{\ell^\vee} & (\mathscr{O}\otimes C^{\mathrm{op}}_r)^\vee & \xrightarrow{\ell^\vee} & (\mathscr{V}^{\mathrm{op}}_0)^\vee \end{array}$$

of $\mathscr{O}$-modules and right $C_0(q)$-modules, i.e. the equations $\eta_0=(\eta_0)^\vee\circ \mathrm{can}_{\mathscr{O}\otimes C_r}$, $h_{-1}=(h_0)^\vee\circ \mathrm{can}_{\mathscr{V}_{-1}}$ and $h_0=(h_{-1})^\vee\circ\mathrm{can}_{\mathscr{V}_0}$ hold.

Proof. By Lemma 5.10, we conclude the commutativity of the diagrams. Since ϑ is symmetric by Lemma 5.8, we see that η_0 is symmetric, i.e. $\eta_0=(\eta_0)^\vee\circ\mathrm{can}_{\mathscr{O}\otimes C_r}$. By taking images of ℓ, we conclude that $h_{-1}=(h_0)^\vee\circ\mathrm{can}_{\mathscr{V}_{-1}}$ and $h_0=(h_{-1})^\vee\circ\mathrm{can}_{\mathscr{V}_0}$. □

Corollary 5.13. *The map $\ell:\mathscr{V}_0\longrightarrow\mathscr{O}\otimes C_r$ is a Lagrangian of the symmetric space*

$$\eta_0:\mathscr{O}\otimes C_r\longrightarrow(\mathscr{O}\otimes C^{\mathrm{op}}_r)^\vee$$

of $\mathscr{O}_X$-modules and right $C_0(q)$-modules.

6. Witt groups of the Swan triangulated category $\mathcal{A}$

Recall the *Swan triangulated category* $\mathcal{A}=\left\langle \mathcal{V}_0,\mathcal{A}_0\right\rangle$ defined in Definition 3.9. Our aim in this section is to prove the following result.

Theorem 6.1. *Let k be a commutative ring containing $\frac{1}{2}$. There is a long exact sequence of abelian groups*

$$\cdots\longrightarrow W^i(C_0(q)_\sigma)\xrightarrow{\mathrm{tr}} W^i(k)\longrightarrow W^i(\mathcal{A})\longrightarrow W^{i+1}(C_0(q)_\sigma)\longrightarrow\cdots$$

Lemma 6.2. *The triangulated subcategory $\mathcal{A}\subset\mathcal{D}^b(Q_d)$ is fixed by the duality $\#_{\mathscr{O}}$.*

Proof. Let $r = \lfloor \frac{d}{2} \rfloor + d + 1$. By the definition of $\mathscr{V}_i$, we have an exact sequence

$$\mathscr{V}_0 \rightarrowtail \mathscr{O} \otimes C_r(q) \twoheadrightarrow \mathscr{V}_{-1} \tag{16}$$

It follows that $\mathscr{V}_{-1} \in \mathcal{A}$. By Lemma 5.12, we see $\mathscr{V}_0^\vee \cong \mathscr{V}_{-1}$ as $\mathscr{O}_X$-modules. □

Hence, we obtain a triangulated category with duality $(\mathcal{A}, \#_{\mathscr{O}})$. Note that the sequence

$$\mathcal{A}_0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{A}_0$$

of triangulated categories is exact where $\mathcal{A}/\mathcal{A}_0$ is the Verdier quotient. Since $\mathcal{A}_0$ is invariant under the duality $\#_{\mathscr{O}}$, we obtain an exact sequence of abelian groups

$$\cdots \longrightarrow W^i(\mathcal{A}_0) \longrightarrow W^i(\mathcal{A}) \longrightarrow W^i(\mathcal{A}/\mathcal{A}_0) \xrightarrow{\partial} W^{i+1}(\mathcal{A}_0) \longrightarrow \cdots$$

by Theorem 2.11. Observe that the equivalence $\mathscr{O} \otimes - : \mathcal{D}^b(k) \xrightarrow{\sim} \mathcal{A}_0$ of triangulated categories induces an isomorphism of abelian groups $W^i(k) \xrightarrow{\sim} W^i(\mathcal{A}_0)$. We deduce an exact sequence of abelian groups

$$\cdots \longrightarrow W^i(k) \longrightarrow W^i(\mathcal{A}) \longrightarrow W^i(\mathcal{A}/\mathcal{A}_0) \xrightarrow{\partial} W^{i+1}(k) \longrightarrow \cdots . \tag{17}$$

6.1. **On the Witt group $W^i(\mathcal{A}/\mathcal{A}_0)$.** Let $\#^{-1}$ denote the duality $\#_{\mathscr{O}[-1]}$. Truncate the component $\mathscr{V}_{-1}$ of the exact sequence (16). We obtain the following (cochain) complex concentrated in degree $[0, 1]$ denoted by Cl.

$$Cl := \big(\cdots \longrightarrow 0 \longrightarrow \mathscr{V}_0 \xrightarrow{\ell} \mathscr{O} \otimes C_r(q) \longrightarrow 0 \longrightarrow \cdots \big).$$

Let $t : Cl \to \mathscr{V}_0$ and $s : Cl \to (\mathscr{V}_0^{\mathrm{op}})^{\#^{-1}}$ be the following morphisms of complexes respectively.

$$\begin{array}{ccccccc} Cl & \quad & \mathscr{V}_0 & \xrightarrow{\ell} & \mathscr{O} \otimes C_r \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 0} \\ \mathscr{V}_0 & & \mathscr{V}_0 & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccccccc} Cl & \quad & \mathscr{V}_0 & \xrightarrow{\ell} & \mathscr{O} \otimes C_r \\ \downarrow{\scriptstyle s} & & \downarrow & & \downarrow{\scriptstyle h_{-1} \circ \ell} \\ (\mathscr{V}_0)^{\#^{-1}} & & 0 & \longrightarrow & (\mathscr{V}_0)^\vee \end{array} \tag{18}$$

Recall the composition $\mathscr{O} \otimes C_r(q) \xrightarrow{\ell} \mathscr{V}_{-1} \xrightarrow{h_{-1}} (\mathscr{V}_0)^\vee$ in Lemma 5.12. Note that s is a quasi-isomorphism of $\mathscr{O}_X$-modules. Moreover, we observe that cone(t) is in $\mathcal{A}_0$ so that t is an isomorphism in $\mathcal{A}/\mathcal{A}_0$. Therefore, the morphism $\mu = s \circ t^{-1}$, i.e., the left roof

$$\mu : \ \mathscr{V}_0 \xleftarrow{t} Cl \xrightarrow{s} (\mathscr{V}_0)^{\#^{-1}}$$

is an isomorphism in the triangulated category $\mathcal{A}/\mathcal{A}_0$.

Lemma 6.3. *The pair $(\mathscr{V}_0, \mu)$ is a symmetric space in the category with duality $\Big(\mathcal{A}/\mathcal{A}_0, \#^{-1}\Big)$.*

Proof. We only need to show that μ is symmetric with respect to the duality $\#^{-1}$, i.e. $\mu^{\#^{-1}} \circ \mathrm{can}_{\mathscr{V}_0} = \mu$. Observe that $\mu^{\#^{-1}} \circ \mathrm{can}_{\mathscr{V}_0} = (t^{\#^{-1}})^{-1} \circ s^{\#^{-1}} \circ \mathrm{can}_{\mathscr{V}_0}$ is represented by a right roof. Thus, it suffices to prove $s^{\#^{-1}} \circ \mathrm{can}_{\mathscr{V}_0} \circ t = t^{\#^{-1}} \circ s$ in $\mathcal{A}/\mathcal{A}_0$. It is enough to show that the morphism of complexes $s^{\#^{-1}} \circ \mathrm{can}_{\mathscr{V}_0} \circ t - t^{\#^{-1}} \circ s$ from $Cl \to (Cl^{\mathrm{op}})^{\#^{-1}}$ is null-homotopic. The following diagram illustrates the situation.

$$\begin{array}{ccccc} Cl & \quad & \mathscr{V}_0 & \xrightarrow{\ell} & \mathscr{O} \otimes C_r \\ \downarrow{\scriptstyle s^* \circ \mathrm{can}_{\mathscr{V}_0} \circ t - t^{\#} \circ s} & & \downarrow{\scriptstyle \ell^\vee \circ h_0} & \swarrow{\scriptstyle \eta_0} & \downarrow{\scriptstyle -h_{-1} \circ \ell} \\ Cl^{\#} & & (\mathscr{O} \otimes C_r)^\vee & \xrightarrow{-\ell^\vee} & \mathscr{V}_0^\vee \end{array}$$

Here, all the maps are defined in Section 5.2. The map η_0 gives the desired homotopy by Lemma 5.12. □

Define $\mathscr{A}_i$ and $\mathscr{A}$ to be the full dg subcategories of $\mathrm{Ch}^b(Q_d)$ corresponding to the full triangulated subcategories $\mathcal{A}_i$ and $\mathcal{A}$ of $\mathcal{D}^bQ_d$ respectively. Explicitly, $\mathscr{A}_i$ (resp. $\mathscr{A}$) has objects lying in $\mathcal{A}_i$ (resp. $\mathcal{A}$). Let w be the set of morphisms in $\mathscr{A}$ that become isomorphisms in the quotient $\mathcal{A}/\mathcal{A}_0$. Therefore, the associated triangulated category $\mathcal{T}(\mathscr{A}, w)$ is just $\mathcal{A}/\mathcal{A}_0$.

The proof of [34, Lemma 3.9] tells us that the symmetric space $(\mathscr{V}_0, \mu)$ in $\mathcal{A}/\mathcal{A}_0$ (in Lemma 6.3) can be lifted to a symmetric form (Cl, μ) in the dg category

$$\left(\mathscr{A}, w, \#^{-1}, \mathrm{can}\right)$$

with weak equivalences and duality, such that the morphism μ is in w and that (Cl, μ) is isometric to $(\mathscr{V}_0, \mu)$ in $\left(\mathcal{A}/\mathcal{A}_0, \#^{-1}\right)$. The form (Cl, μ) is displayed as follows.

$$\begin{array}{ccc} Cl & & \mathscr{V}_0 \xrightarrow{\ \ell\ } \mathscr{O}\otimes C_r \\ \downarrow{\scriptstyle \mu} & \quad & \downarrow{\scriptstyle \frac{\eta_0\circ\ell}{2}} \qquad\qquad \downarrow{\scriptstyle \frac{\ell^\vee\circ\eta_0}{2}} \\ Cl^{\#^{-1}} & & (\mathscr{O}\otimes C_r)^\vee \xrightarrow{\ -\ell^\vee\ } \mathscr{V}_0^\vee \end{array} \tag{19}$$

Note that $\mu : Cl \to Cl^{\#^{-1}}$ is also a symmetric form in the dg category with duality $\left(\mathscr{O}\text{-Mod}, \#^{-1}\right)$. Consider now the the dg category with duality $\left(\mathscr{O}\text{-Mod-}C_0(q), \#^{-1}\right)$ where the duality functor $\#^{-1}$ sends M to $[M^{\mathrm{op}}, \mathscr{O}[-1]]_{\mathscr{O}}$. We further observe the following result.

Lemma 6.4. *Consider the right $C_0(q)$-module structure of Cl. The map $\mu : Cl \to [Cl^{\mathrm{op}}, \mathscr{O}[-1]]_{\mathscr{O}}$ is a symmetric form in $\left(\mathscr{O}\text{-Mod-}C_0(q), \#^{-1}\right)$.*

Proof. We only need to show μ is a right $C_0(q)$-module map. This has already been proved by Lemma 5.12. □

Lemma 6.5. *Let $\mathrm{Ch}^b(C_0(q))$ be the dg category of bounded complexes of finitely generated projective left $C_0(q)$-modules. Tensoring with $\mu : Cl \to [Cl^{\mathrm{op}}, \mathscr{O}[-1]]$ gives a nonsingular exact dg form functor*

$$Cl \otimes_{C_0} - : \left(\mathrm{Ch}^b(C_0(q)), \mathrm{quis}, \#_{C_0(q)}\right) \longrightarrow \left(\mathscr{A}, w, \#^{-1}\right) \tag{20}$$

Proof. Firstly, applying $A = (\mathscr{O}, 1)$, $I = (\mathscr{O}[-1], 1)$ and $B = (C_0(q), \sigma)$ to Lemma 2.19, we deduce a dg form functor

$$\left(\mathscr{O}\text{-Mod-}C_0(q), \#^{-1}\right) \otimes \left(C_0(q)\text{-Mod}, \#_{C_0(q)}\right) \to \left(\mathscr{O}\text{-Mod}, \#^{-1}\right). \tag{21}$$

By Lemma 6.4, the map $\mu : Cl \to [Cl^{\mathrm{op}}, \mathscr{O}[-1]]$ is a symmetric form in the dg category with duality $\left(\mathscr{O}\text{-Mod-}C_0(q), \#_{\mathscr{O}[-1]}\right)$. By Lemma 2.17, we get a dg form functor

$$Cl \otimes_{C_0} - : \left(C_0(q)\text{-Mod}, \#_{C_0(q)}\right) \to \left(\mathscr{O}\text{-Mod}, \#^{-1}\right).$$

Note that $\mathrm{Ch}^b(C_0(q)) \subset C_0(q)\text{-Mod}$ and $\mathscr{A} \subset \mathscr{O}\text{-Mod}$ are full dg subcategories. Since $Cl \in \mathscr{A}$, we observe that $Cl \otimes M \in \mathscr{A}$ for any $M \in \mathrm{Ch}^b(C_0(q))$. This implies that, by restriction, we have a dg form functor

$$Cl \otimes_{C_0} - : \left(\mathrm{Ch}^b(C_0(q)), \#_{C_0(q)}\right) \longrightarrow \left(\mathscr{A}, \#^{-1}\right).$$

Taking weak equivalences into account, we obtain an exact dg form functor

$$Cl \otimes_{C_0} - : \left(\mathrm{Ch}^b(C_0(q)), \mathrm{quis}, \#_{C_0(q)}\right) \longrightarrow \left(\mathscr{A}, w, \#^{-1}\right) \tag{22}$$

This is because that the functor $Cl \otimes_{C_0} - : \mathrm{Ch}^b(C_0(q)) \to \mathscr{A}$ sends quasi-isomorphisms to quasi-isomorphisms and w contains all quasi-isomorphisms in $\mathscr{A}$. To show $Cl \otimes_{C_0} -$ is non-singular, we have to prove the duality compatability map of $Cl \otimes -$

$$Cl \otimes_{C_0(q)} M^{\#_{C_0(q)}} \longrightarrow Cl^{\#^{-1}} \otimes_{C_0(q)} M^{\#_{C_0(q)}} \longrightarrow (Cl \otimes_{C_0(q)} M)^{\#^{-1}}$$

is in w. Note that $Cl \to Cl^{\#^{-1}} \in w$ which implies the first map is in w. Moreover, the second map is an isomorphism, so is in w. This is clear for $M = C_0(q)$ and the property is preserved by direct sums and inherited by direct summands. □

Proposition 6.6. *Tensoring with $\mu : Cl \to Cl^{\#^{-1}}$ induces an isomorphism of abelian groups*

$$Cl \otimes_{C_0} - : W^i(C_0(q)_\sigma) \longrightarrow W^{i-1}(\mathcal{A}/\mathcal{A}_0).$$

Proof. By Lemma 6.5, we obtain a duality preserving functor

$$Cl \otimes_{C_0} - : \Big(\mathcal{D}^b C_0(q), \#_{C_0(q)}\Big) \longrightarrow \Big(\mathcal{A}/\mathcal{A}_0, \#^{-1}\Big)$$

by taking associated triangulated categories. This induces the map

$$Cl \otimes_{C_0} - : W^i(C_0(q)_\sigma) \longrightarrow W^{i-1}(\mathcal{A}/\mathcal{A}_0).$$

Note that $Cl \otimes_{C_0} - : \mathcal{D}^b C_0(q) \to \mathcal{A}/\mathcal{A}_0$ is an equivalence of triangulated categories because $\mathscr{V}_0 \otimes_{C_0} - : \mathcal{D}^b C_0(q) \approx \mathcal{D}^b \mathrm{End}(\mathscr{V}_0) \to \mathcal{A}/\mathcal{A}_0$ is an equivalence of triangulated category (cf. Theorem 3.2 [9] and Proposition 1.6 [10]) and $t : Cl \to \mathscr{V}_0$ (See Diagram (18)) is an isomorphism in $\mathcal{A}/\mathcal{A}_0$. Therefore, the map $Cl \otimes_{C_0} -$ is an isomorphism. □

Corollary 6.7. *There is an exact sequence of abelian groups*

$$\cdots \longrightarrow W^{i-1}(k) \longrightarrow W^{i-1}(\mathcal{A}) \longrightarrow W^i(C_0(q)) \xrightarrow{\partial} W^i(k) \longrightarrow \cdots \tag{23}$$

6.2. **The trace map.** Let k be a commutative ring containing $\frac{1}{2}$. In this section, we introduce a trace map of Witt groups $\mathrm{tr} : W^i(C_0(q)_\sigma) \to W^i(k)$. We show that the trace map is precisely the connecting homomorphism $\partial : W^i(C_0(q)) \to W^i(k)$ in the exact sequence (23).

Firstly, applying $A = I = (k, 1)$ and $B = (C_0(q), \sigma)$ to Lemma 2.19, we deduce a dg form functor

$$\Big(k\text{-Mod-}C_0(q), \#_k\Big) \otimes \Big(C_0(q)\text{-Mod}, \#_{C_0(q)}\Big) \to \Big(k\text{-Mod}, \#_k\Big) \tag{24}$$

Recall the symmetric form $\vartheta_r : C_r \to [(C_r)^{\mathrm{op}}, k]_k$ of right $C_0(q)$-modules in Remark 5.9.

Lemma 6.8. *Tensoring with (C_r, ϑ_r) gives a nonsingular exact dg form functor*

$$(C_r, \vartheta_r) \otimes_{C_0} - : \Big(\mathrm{Ch}^b(C_0(q)), \mathrm{quis}, \#_{C_0(q)}\Big) \longrightarrow \Big(\mathrm{Ch}^b(k), \mathrm{quis}, \#_k\Big)$$

Proof. Note that (C_r, ϑ_r) is a symmetric form in $(k\text{-Mod-}C_0(q), \#_k)$. By Lemma 2.17 and the dg form functor (24), we get a dg form functor

$$(C_r, \vartheta_r) \otimes_{C_0} - : \Big(C_0(q)\text{-Mod}, \#_{C_0(q)}\Big) \to \Big(k\text{-Mod}, \#_k\Big).$$

Note that $\mathrm{Ch}^b(C_0(q)) \subset C_0(q)$-Mod and $\mathrm{Ch}^b(k) \subset k$-Mod are full dg subcategories. Note that $C_r \otimes_{C_0} M \in \mathrm{Ch}^b(k)$ for any $M \in \mathrm{Ch}^b(C_0(q))$. This implies that, by restriction, we have a dg form functor

$$(C_r, \vartheta_r) \otimes_{C_0} - : \Big(\mathrm{Ch}^b(C_0(q)), \#_{C_0(q)}\Big) \longrightarrow \Big(\mathrm{Ch}^b(k), \mathrm{quis}, \#_k\Big).$$

Taking weak equivalences into account, we obtain an exact dg form functor

$$(C_r, \vartheta_r) \otimes_{C_0} - : \Big(\mathrm{Ch}^b(C_0(q)), \mathrm{quis}, \#_{C_0(q)}\Big) \longrightarrow \Big(\mathrm{Ch}^b(k), \mathrm{quis}, \#_k\Big) \tag{25}$$

This is becauce that the functor $(C_r, \vartheta_r) \otimes - : \mathrm{Ch}^b(C_0(q)) \to \mathrm{Ch}^b(k)$ sends quasi-isomorphisms to quasi-isomorphisms. To show $(C_r, \vartheta_r) \otimes -$ is non-singular, we have to prove the duality compatability map of $C_r \otimes -$

$$C_r \otimes_{C_0(q)} M^{\#_{C_0(q)}} \longrightarrow C_r^{\#_k} \otimes_{C_0(q)} M^{\#_{C_0(q)}} \longrightarrow (C_r \otimes_{C_0(q)} M)^{\#_k}$$

is a quasi-isomorphism. Note that $C_r \to C_r^{\#_k}$ is an isomorphism which implies the first map is an isomorphism. Moreover, the second map is an isomorphism. This is clear for $M = C_0(q)$ and the property is preserved by direct sums and inherited by direct summands. □

Definition 6.9. The *trace map of the even part Clifford algebra*

$$\mathrm{tr}: W^i(C_0(q)_\sigma) \to W^i(k)$$

is the map of Witt groups induced by the duality preserving functor

$$(C_r, \vartheta_r) \otimes_{C_0} - : \left(\mathcal{D}^b(C_0(q)), \#_{C_0(q)}\right) \to \left(\mathcal{D}^b(k), \#_k\right)$$

obtained from the nonsingular exact dg form functor in Lemma 6.8 by taking associated triangulated categories.

Remark 6.10. In Definition 6.9, the map $\mathrm{tr}: W^i(C_0(q)) \to W^i(k)$ is called the trace map, because the duality compatibility map used to construct the duality preserving functor $(C_r, \vartheta_r) \otimes_{C_0} - : (\mathcal{D}^b(C_0(q)), \#_{C_0(q)}) \to (\mathcal{D}^b(k), \#_k)$ depends on the trace map $\mathrm{tr}: C_0(q) \to k$ of the algebra $C_0(q)$. For example, the map $\mathrm{tr}: W^0(C_0(q)_\sigma) \to W^0(k)$ sends a symmetric space $M \times M^{\mathrm{op}} \to C_0(q)$ to the symmetric space $M \times M \to C_0(q) \xrightarrow{\mathrm{tr}} k$.

Theorem 6.11. *Let k be a commutative ring with $\frac{1}{2} \in k$. The diagram*

$$\begin{array}{ccc} W^i(C_0(q)_\sigma) & \xrightarrow{\mathrm{tr}} & W^i(k) \\ {\scriptstyle (Cl,\mu)\otimes_{C_0}-}\downarrow \cong & & {\scriptstyle (\mathscr{O},1)\otimes -}\downarrow \cong \\ W^{i-1}(\mathcal{A}/\mathcal{A}_0) & \xrightarrow{\partial} & W^i(\mathcal{A}_0) \end{array}$$

of abelian groups commutes.

Proof. Define $\mathscr{A}_0$ and $\mathscr{A}$ to be the full dg subcategories of the dg category $\mathrm{Ch}^b(Q_d)$ corresponding to the full triangulated subcategories $\mathcal{A}_0$ and $\mathcal{A}$ of $\mathcal{D}^b Q_d$ respectively. Explicitly, $\mathscr{A}_0$ (resp. $\mathscr{A}$) has objects lying in $\mathcal{A}_0$ (resp. $\mathcal{A}$). Let quis (resp. w) be the set of quasi isomorphisms in $\mathscr{A}$ (resp. the set of morphisms in $\mathscr{A}$ that become isomorphisms in the Verdier quotient $\mathcal{T}(\mathscr{A}, \mathrm{quis})/\mathcal{T}(\mathscr{A}_0, \mathrm{quis}) = \mathcal{A}/\mathcal{A}_0$).

Step I. Let $\mathrm{Fun}([1], \mathscr{A})$ be the dg category consisting of objects $A = (A_0 \to A_1) \in \mathscr{A}$ and morphisms

$$\begin{array}{ccccc} A & \quad & A_0 & \longrightarrow & A_1 \\ \downarrow f & & \downarrow f_0 & & \downarrow f_1 \\ B & & B_0 & \longrightarrow & B_1 \end{array}$$

(cf. Section 2.1 [34]). A quasi-isomorphism $f: A \to B$ in $\mathrm{Fun}([1], \mathscr{A})$ means that f_0 and f_1 are both quasi-isomorphisms of complexes. Let $\mathrm{Fun}_w([1], \mathscr{A})$ be the dg full subcategory of $\mathrm{Fun}([1], \mathscr{A})$ consisting of objects $A_0 \to A_1 \in w$.

We establish a dg form functor

$$\left(\mathrm{Fun}([1], \mathscr{O}\text{-Mod-}C_0(q)), \#^{-1}\right) \otimes \left(C_0(q)\text{-Mod}, \#_{C_0(q)}\right) \to \left(\mathrm{Fun}([1], \mathscr{O}\text{-Mod}), \#^{-1}\right) \tag{26}$$

induced by Formula (21). In light of Lemma 5.12, we consider $\mu: Cl \to Cl^{\#^{-1}}$ as an object in the dg category $\mathrm{Fun}([1], \mathscr{O}\text{-Mod-}C_0(q))$. We construct the following symmetric bilinear form (μ, Σ) in $\left(\mathrm{Fun}([1], \mathscr{O}\text{-Mod-}C_0(q)), \#^{-1}\right)$.

$$\begin{array}{ccccc} \mu & \quad & Cl & \xrightarrow{\mu} & Cl^{\#^{-1}} \\ \Sigma\downarrow & & \mathrm{can}\downarrow & & 1\downarrow \\ \mu^{\#^{-1}} & & Cl^{\#^{-1}\#^{-1}} & \xrightarrow{\mu^{\#^{-1}}} & Cl^{\#^{-1}} \end{array}$$

By Lemma 2.17 and the dg form functor (26), the form (μ, Σ) induces a non-singular exact dg form functor

$$(\mu, \Sigma) \otimes_{C_0} - : \left(\mathrm{Ch}^b(C_0(q)), \mathrm{quis}, \#_{C_0(q)}\right) \longrightarrow \left(\mathrm{Fun}_w([1], \mathscr{A}), \mathrm{quis}, \#^{-1}\right)$$

(Note that $\mu : Cl \to Cl^{\#^{-1}} \in w$).

Step II. Define a dg functor $P : \mathrm{Fun}_w([1], \mathscr{A}) \to \mathscr{A}$ by sending $A_0 \to A_1$ to A_0 and (f_0, f_1) to f_0. It provides a dg form functor $(P, \varphi) : \left(\mathrm{Fun}_w([1], \mathscr{A}), \#^{-1}\right) \to \left(\mathscr{A}, \#^{-1}\right)$ with the duality compatibility map by $\varphi_f = f^{\#}$. Consider the following diagram of non-singular exact dg form functors

$$
\begin{array}{ccccc}
 & & \left(\mathrm{Ch}^b(C_0(q)), \mathrm{quis}, \#_{C_0(q)}\right) & \xrightarrow{(C_r,\vartheta_r)\otimes_{C_0}-} & \left(\mathrm{Ch}^b(k), \mathrm{quis}, \#_k\right) \\
 & \swarrow^{(Cl,\mu)\otimes_{C_0}-} & \Big\downarrow{\scriptstyle (\mu,\Sigma)\otimes_{C_0}-} & & \Big\downarrow{\scriptstyle (\mathscr{O},1)\otimes -} \\
\left(\mathscr{A}, w, \#^{-1}\right) & \xleftarrow{P} & \left(\mathrm{Fun}_w([1], \mathscr{A}), \mathrm{quis}, \#^{-1}\right) & \xrightarrow{\mathrm{cone}} & \left(\mathscr{A}_0, \mathrm{quis}, \#_{\mathscr{O}}\right)
\end{array}
\tag{27}
$$

The left triangle diagram is strictly commutative on the level of dg categories. We want to show that the right square is commutative up to homotopy, so it gives the commutativity of the square diagram of associated Witt groups. We only need to construct a homotopy (That is a natural transformation of dg form functors which is a weak equivalence object-wise (see Section 2.1 [32] for the definition))

$$F : ((\mathscr{O}, 1) \otimes -) \circ ((C_r, \vartheta_r) \otimes_{C_0} -) \longrightarrow \mathrm{cone} \circ ((\mu, \Sigma) \otimes_{C_0} -)$$

and check the commutativity for the generator $C_0(q) \in \mathrm{Ch}^b(C_0(q))$, because we have the pretriangulated hull construction (cf. Section 1.6 [34]). We first prove the following auxiliary lemma.

Lemma 6.12. *Let $\psi : V \to V^\vee$ be an inner product space with a Lagrangian $\mathcal{L}$ given by $\iota : L \to V$ in an exact category with duality $(\mathcal{E}, \vee)$. Consider $\mathcal{L}$ as a cochain complex with L in degree zero and V in degree one. Then, the map of complexes*

$$
\begin{array}{c}
\mathcal{L} \\
\Big\downarrow{\scriptstyle \Psi} \\
\mathcal{L}^{\#^{-1}}
\end{array}
\qquad
\begin{array}{ccc}
L & \xrightarrow{\iota} & V \\
\Big\downarrow{\scriptstyle \frac{\psi\circ\iota}{2}} & & \Big\downarrow{\scriptstyle \frac{\iota^\vee\circ\psi}{2}} \\
V^\vee & \xrightarrow{-\iota^\vee} & L^\vee
\end{array}
$$

is a symmetric form in $(\mathrm{Ch}^b(\mathcal{E}), \#^{-1})$. Moreover, the inner product space (V, ψ) is isometric to $\mathrm{cone}(\mathcal{L}, \Psi)$ in $(\mathrm{Ch}^b(\mathcal{E}), \mathrm{quis}, \#)$.

Proof. By construction, $(\mathcal{L}, \Psi)$ is symmetric in $(\mathrm{Ch}^b(\mathcal{E}), \#^{-1})$. Construct the following map

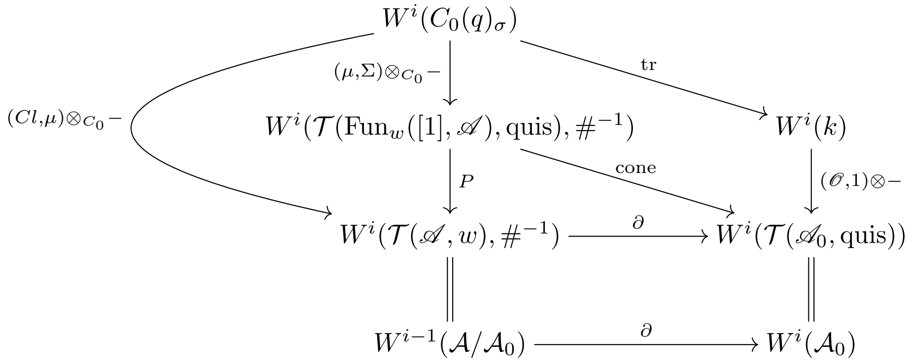
$$
\begin{array}{c}
V \\
\Big\downarrow{\scriptstyle \Phi} \\
\mathrm{cone}(\mathcal{L}, \Psi)
\end{array}
\qquad
\begin{array}{ccccc}
0 & \longrightarrow & V & \longrightarrow & 0 \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \begin{pmatrix} 1 \\ \psi/2 \end{pmatrix}} & & \Big\downarrow \\
L & \xrightarrow{\begin{pmatrix} -\iota \\ \frac{\psi\circ\iota}{2} \end{pmatrix}} & V \oplus V^\vee & \xrightarrow{\begin{pmatrix} \frac{\iota^\vee\circ\psi}{2} & -\iota^\vee \end{pmatrix}} & L^\vee
\end{array}
\tag{28}
$$

One checks that Φ is a desired isometry. ∎

Let's go back to the construction of the homotopy F in the proof of Step II. Note that Corollary 5.13 fits into the the "Lagrangian" situation (Lemma 6.12) by setting $V := \mathscr{O} \otimes C_r(q)$, $\psi := \eta_0$, $L = \mathscr{V}_0$ and $\iota := \ell$. In light of the proof of Lemma 6.12, we define the homotopy $F_{C_0(q)} : \mathscr{O} \otimes C_r \to \mathrm{cone}(\mu)$ by identifying it with the isometry Φ in Diagram (28). Lemma 6.12 also implies that $F_{C_0(q)}$ is a quasi isomorphism. This confirms that the right square of Diagram (27) is commutative up to homotopy.

Step III. Recall that $\mathcal{A}/\mathcal{A}_0$ is just the associated triangulated category $\mathcal{T}(\mathscr{A}, w)$. Take associated Witt groups of Diagram (27). Relate the connecting homomorphism $\partial : W^{i-1}(\mathcal{A}/\mathcal{A}_0) \to W^i(\mathcal{A}_0)$ (cf.

Remark 2.12) with the morphisms P and cone (cf. Diagram (27)) by the following diagram.



This finishes the proof. □

7. Proof of Theorem 1.2

7.1. **Proof of Theorem 1.2 (a).** Let Q_d be the quadric hypersurface of dimension d corresponding to a non-degenerate form (P,q) of rank $n=d+2$ with d odd.

Lemma 7.1. *Let $d=2m+1$ be an odd number. Let k be a commutative ring with $\frac{1}{2}\in k$. There is an isomorphism $W^i(\mathcal{A})\cong W^i(Q_d)$.*

Proof. By Corollary 3.10, we know that $\mathcal{D}^bQ_d$ has a semi-orthogonal decomposition

$$\left\langle \mathcal{A}_{[-m,-1]},\mathcal{A},\mathcal{A}_{[1,m]}\right\rangle$$

if $d=2m+1$. Since $\mathcal{A}_{[-m,-1]}$ is switched into $\mathcal{A}_{[1,m]}$ by the duality $\#_{\mathscr{O}}$ and $\mathcal{A}$ is fixed by the duality $\#_{\mathscr{O}}$ (cf. Lemma 6.2), the result is just a consequence of the additivity theorem. □

Theorem 7.2 (Theorem 1.2 (a)). *Let $d=2m+1$ be an odd number. There is a long exact sequence of Witt groups*

$$\cdots\to W^i(k)\longrightarrow W^i(Q_d)\longrightarrow W^{i+1}(C_0(q)_\sigma)\xrightarrow{\mathrm{tr}} W^{i+1}(k)\to\cdots$$

Proof. Apply Lemma 7.1 to Theorem 6.1. The result follows. □

7.2. **Proof of Theorem 1.2 (b).** Let $d=2m$ be an even integer. Recall $\mathcal{A}'$ in Definition 3.9. There is an exact sequence of triangulated categories

$$\mathcal{A}'\longrightarrow\mathcal{D}^bQ_d\longrightarrow\mathcal{D}^bQ_d/\mathcal{A}'$$

Note that $\mathcal{A}'$ is fixed by the duality $\#_{\mathscr{O}}$ on $\mathcal{D}^bQ$. By the localizaition theorem, we have a long exact sequence

$$\cdots\to W^i(\mathcal{A}')\to W^i(Q_d)\to W^i(\mathcal{D}^bQ/\mathcal{A}')\to W^{i+1}(\mathcal{A}')\to\cdots \tag{29}$$

By the additivity theorem, we conclude $W^i(\mathcal{A}')\cong W^i(\mathcal{A})$. Hence we can replace $\mathcal{A}'$ by $\mathcal{A}$ in the long exact sequence (29), and we get the following long exact sequence.

$$\cdots\to W^i(\mathcal{A})\to W^i(Q_d)\to W^i(\mathcal{D}^bQ/\mathcal{A}')\to W^{i+1}(\mathcal{A})\to\cdots \tag{30}$$

It is now important to study the group $W^i(\mathcal{D}^bQ/\mathcal{A}')$. Consider the exact sequence $T_{[-m,m]}$ in Corollary 4.2. Truncate the component $\mathscr{O}(-m)\otimes\det(P^*)$ in $T_{[-m,m]}$. We obtain a complex

$$0\to\mathscr{U}_{d-2}(\Lambda(d))(m-1)\to T_{-d+2}(m-1)\to\cdots\to T_0(m-1)\to\mathscr{O}(m)\to 0 \tag{31}$$

concentrated in degree $[-d,0]$. Denote this new complex by $T_{[1-m,m]}$. Note that

$$T_{-d+2}(m-1)=\mathscr{O}(1-m)\otimes\big(\bigoplus_{j\geq 0}\Lambda^{(-d+2)+1-2j}(P^*)\big).$$

Let ψ denote the right roof $s^{-1} \circ \iota$

$$\mathscr{O}(m) \xrightarrow{\iota} T_{[1-m,m]} \xleftarrow{s} \mathscr{O}(m)^{\#^d_{P^*}}$$

The only non-trivial component of $\iota : \mathscr{O}(m) \to T_{[1-m,m]}$ (resp. $s : \mathscr{O}(m)^{\#^d_{P^*}} \to T_{[1-m,m]}$) is the map $1 : \mathscr{O}(m) \to \mathscr{O}(m)$ (resp. the composition $\mathscr{O}(m)^\vee \otimes \det(P^*) \xrightarrow{\sim} \mathscr{O}(-m) \otimes \det(P^*) \xrightarrow{s} \mathscr{U}_{d-2}(\Lambda(d))$ in the degree 0 (resp. degree $-d$). Recall from Section 2.6 that $\#^d_{P^*}$ is the duality given by the functor $[-, \mathscr{O} \otimes \det(P^*)[d]]_{\mathscr{O}} : (\mathcal{D}^bQ/\mathcal{A}')^{\mathrm{op}} \longrightarrow \mathcal{D}^bQ/\mathcal{A}'$.

Lemma 7.3. *Let $d = 2m$ be an even integer. The right roof $(\mathscr{O}(m), \psi)$ is a symmetric space in the category with duality*

$$\Big(\mathcal{D}^bQ/\mathcal{A}', \#^d_{P^*}, \epsilon \cdot \mathrm{can}\Big)$$

for some $\epsilon \in k^\times$ such that $\epsilon^2 = 1$. If k is a local ring containing $\frac{1}{2}$, then $\epsilon = \pm 1$.

Proof. It is enough to show that ι and s are both isomorphisms in $\mathcal{D}^bQ/\mathcal{A}'$ and that ψ is ϵ-symmetric. Indeed, s is a quasi-isomorphism, hence also an isomorphism in $\mathcal{D}^bQ/\mathcal{A}'$, cf. Corollary 4.2. Furthermore, the cone of the morphism ι in $\mathcal{D}^bQ$ is already in the triangulated category $\mathcal{A}_{[1-m,m-1]}$ by the exact sequence (12). Thus, ι is an isomorphism in $\mathcal{D}^bQ/\mathcal{A}'$ and we conclude that ψ is an isomorphism in

$$\mathrm{Hom}_{\mathcal{D}^bQ/\mathcal{A}'}(\mathscr{O}(m), \mathscr{O}(m)^{\#^d_{P^*}}).$$

Define $\psi^t := \psi^{\#^d_{P^*}} \circ \mathrm{can}_{\mathscr{O}(m)}$. Note that $\psi^{tt} = \psi$. Next, we show that $\psi = \epsilon\psi^t$ for some $\epsilon \in k^\times$ such that $\epsilon^2 = 1$. In fact, the morphism ψ can also induce an isomorphism

$$\mathrm{Hom}_{\mathcal{D}^bQ/\mathcal{A}'}(\mathscr{O}(m), \mathscr{O}(m)^{\#^d_{P^*}}) \cong \mathrm{Hom}_{\mathcal{D}^bQ/\mathcal{A}'}\big(\mathscr{O}(m), \mathscr{O}(m)\big)$$

The right-hand side is isomorphic to

$$\mathrm{Hom}_{\mathcal{D}^bQ}(\mathscr{O}(m), \mathscr{O}(m)) \cong k,$$

because $\mathscr{O}(m) \otimes - : \mathcal{D}^bk \to \mathcal{D}^bQ \to \mathcal{D}^bQ/\mathcal{A}'$ is fully faithful. Thus, we conclude $\psi^t = \epsilon\psi$ for some $\epsilon \in k^\times$. Observe that $\psi = \psi^{tt} = (\epsilon\psi)^t = \epsilon^2\psi$. This implies $\epsilon^2 = 1$, because ψ is an isomorphism.

If k is a local ring, then every projective module over k is free. In particular, the projective module P involved in the duality $\#^d_{P^*}$ is free. Thus, we can conclude $\epsilon = \pm 1$ from $\epsilon^2 - 1 = 0$ (Since $2 \in k^\times$, either $\epsilon + 1$ or $\epsilon - 1$ is a unit. It follows that either $\epsilon + 1 = 0$ or $\epsilon - 1 = 0$). □

Let $\mathscr{A}'$ be the full dg subcategory of $\mathrm{Ch}^b(Q_d)$ associated to the triangulated category $\mathcal{A}'$. Let v be the set of morphisms in $\mathrm{Ch}^b(Q_d)$ which become isomorphisms in the Verdier quotient $\mathcal{D}^bQ_d/\mathcal{A}'$.

The proof of [34, Lemma 3.9] tells us that the form $(\mathscr{O}(m), \psi)$ (in Lemma 7.3) can be lifted to a symmetric form $(B_{\mathscr{O}(m)}, B_\psi)$ in the dg category

$$\Big(\mathrm{Ch}^b(Q_d), v, \#^d_{P^*}, \epsilon \cdot \mathrm{can}\Big)$$

with weak equivalences and duality, such that the morphism B_ψ is in v and that $(B_{\mathscr{O}(m)}, B_\psi)$ is isometric to $(\mathscr{O}(m), \psi)$ in $\Big(\mathcal{D}^bQ_d, \#^d_{P^*}, \epsilon \cdot \mathrm{can}\Big)$ for some $\epsilon \in k^\times$ such that $\epsilon^2 = 1$.

Lemma 7.4. *Let $d = 2m$ be an even integer. Tensoring with $(B_{\mathscr{O}(m)}, B_\psi)$ gives a non-singular exact dg form functor*

$$(B_{\mathscr{O}(m)}, B_\psi) \otimes - : \Big(\mathrm{Ch}^b(k), \mathrm{quis}, \#_P, \mathrm{can}\Big) \longrightarrow \Big(\mathrm{Ch}^b(Q_d), v, \#^d, \epsilon \cdot \mathrm{can}\Big)$$

of dg categories with weak equivalences and duality for some $\epsilon \in k^\times$ such that $\epsilon^2 = 1$.

Proof. Construct a dg form functor

$$\Big(\mathscr{O}\text{-Mod-}k, \#^d_{P^*}\Big) \otimes \Big(k\text{-Mod}, \#_P\Big) \to \Big(\mathscr{O}\text{-Mod}, \#^d\Big)$$

with the duality compatibility map

$$\gamma : [M, \mathscr{O} \otimes \det(P^*)[d]]_{\mathscr{O}} \otimes [N, \det(P)]_k \to [M \otimes N, \mathscr{O}[d]]_{\mathscr{O}}$$

given by $\gamma(f \otimes g)(m \otimes n) = (-1)^{|g||m|}\kappa(f(m) \otimes g(n))$, where the map $\kappa : \mathscr{O} \otimes \det(P^*) \otimes \det(P) \to \mathscr{O}$ defined by $a \otimes f \otimes p \mapsto af(p)$ is an isomorphism. To see this dg form functor is well-defined, one can go through the same argument as in the proof of Lemma 2.19. The rest of the proof is very similar to the situation in Lemma 6.5 and 6.8. □

In the next proposition, we determine that $\epsilon = 1$.

Proposition 7.5. *Let $d = 2m$ be an even integer. Tensoring with $(B_{\mathscr{O}(m)}, B_\psi)$ gives an isomorphism of groups*

$$(B_{\mathscr{O}(m)}, B_\psi) \otimes - : W^i(k, \det(P)) \xrightarrow{\sim} W^{i+d}(\mathcal{D}^b Q/\mathcal{A}').$$

Proof. By Lemma 7.4, we obtain a duality preserving functor

$$(B_{\mathscr{O}(m)}, B_\psi) \otimes - : \left(\mathcal{D}^b k, \#_P, \mathrm{can}\right) \longrightarrow \left(\mathcal{D}^b Q_d, \#^d, \epsilon \cdot \mathrm{can}\right)$$

by taking associated triangulated categories for some $\epsilon \in k^\times$ such that $\epsilon^2 = 1$. This induces the map

$$(B_{\mathscr{O}(m)}, B_\psi) \otimes - : W^i(k, \det(P)) \longrightarrow {}_\epsilon W^{i+d}(\mathcal{D}^b Q/\mathcal{A}')$$

Note that $B_{\mathscr{O}(m)} \otimes - : \mathcal{D}^b k \to \mathcal{D}^b Q_d/\mathcal{A}'$ is an equivalence of triangulated categories. Therefore, the map $(B_{\mathscr{O}(m)}, B_\psi) \otimes - : W^i(k, \det(P)) \longrightarrow {}_\epsilon W^{i+d}(\mathcal{D}^b Q/\mathcal{A}')$ is an isomorphism of groups for some $\epsilon \in k^\times$ such that $\epsilon^2 = 1$.

Next, we show that $\epsilon = 1$. If k is a local ring containing $\frac{1}{2}$, then $\epsilon = \pm 1$ by Lemma 7.3. Let F be the residue field of k. Base changing of the form $(B_{\mathscr{O}(m)}, B_\psi)$ via the quotient map $k \to F$, we may assume that k is a field of characteristic $\neq 2$ in the following argment. If k is a field of characteristic $\neq 2$, the duality given by $\det(P) \cong k$ is the trivial duality. Recall that $d = 2m$. If k is a field containing $\frac{1}{2}$, I claim $\epsilon = -1$ is impossible, hence $\epsilon = 1$. Note that by our notation ${}_{-1}W^i(\mathcal{D}^b Q/\mathcal{A}') \cong W^{i+2}(\mathcal{D}^b Q/\mathcal{A}')$. If m is odd and $\epsilon = -1$, one has an exact sequence

$$W^2(\mathcal{A}') \longrightarrow W^2(Q) \longrightarrow W^2(k)$$

By the base change of the form $(B_{\mathscr{O}(m)}, B_\psi)$ via the obvious map $k \to \bar{k}$, an exact sequence

$$W^2(\mathcal{A}'_{\bar{k}}) \longrightarrow W^2(Q_{\bar{k}}) \longrightarrow W^2(\bar{k})$$

appears. Note that $W^2(\mathcal{A}'_{\bar{k}}) \cong W^2(\mathcal{A}_{\bar{k}}) = 0$ by the additivity theorem (cf. Theorem 2.10) and Proposition 8.7. Moreover, $W^2(\bar{k}) = 0$ is well-known. This implies $W^2(Q_{\bar{k}}) = 0$. This is a contradiction because $W^2(Q_{\bar{k}}) \cong W^2(Q_{\mathbb{C}})$ by [39, Corollary 4.1] and $W^2(Q_{\mathbb{C}}) \cong \mathbb{Z}/2\mathbb{Z}$ by [40, Theorem 4.9]. If m is even and $\epsilon = -1$, then we have an exact sequence

$$W^0(\mathcal{A}'_{\bar{k}}) \longrightarrow W^0(Q_{\bar{k}}) \longrightarrow W^2(\bar{k}).$$

By the additivity theorem (cf. Theorem 2.10) and Proposition 8.7, we deduce that there is a surjection $\mathbb{Z}/2\mathbb{Z} \cong W^0(\bar{k}) \twoheadrightarrow W^0(\mathcal{A}_{\bar{k}}) \cong W^0(\mathcal{A}'_{\bar{k}})$. Note that $W^2(\bar{k}) = 0$ gives a surjective map $W^0(\bar{k}) \to W^0(Q_{\bar{k}})$, so $W^0(Q_{\bar{k}}) = 0$ or $\mathbb{Z}/2\mathbb{Z}$. This is also a contradiction because $W^0(Q_{\bar{k}}) \cong W^0(Q_{\mathbb{C}}) = (\mathbb{Z}/2\mathbb{Z})^2$ by [30, Theorem 6.2] or [40, Theorem 4.9]. Thus, we have proved that $\epsilon = 1$ for k is a field of characteristic $\neq 2$.

If k is a commutative ring containing $\frac{1}{2}$, we let $k_{\mathfrak{p}}$ be the localization of k at a prime ideal $\mathfrak{p}$, and let $\epsilon_{\mathfrak{p}}$ be the image of ϵ under the map $k \to k_{\mathfrak{p}}$. It follows that $\epsilon_{\mathfrak{p}} = 1$ in $k_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p}$ of k by the local case. This implies $\epsilon = 1$ in k. □

Corollary 7.6. *Let $d = 2m$ be an even integer. There is a long exact sequence*

$$\cdots \to W^i(\mathcal{A}) \to W^i(Q_d) \to W^{i-d}(k, \det(P)) \to W^{i+1}(\mathcal{A}) \to \cdots \tag{32}$$

The proof of Theorem 1.2 (b) is finished by Theorem 6.1 and Corollary 7.6.

8. On the application of Theorem 1.2

8.1. **The kernel of the map $p^* : W(F) \to W(Q_d)$.** Let F be a field of characteristic $\neq 2$. Let Q_d be the quadric defined by the quadratic form $q = \langle a_1, \dots, a_n \rangle$ with $a_i \in k^\times$.

Proposition 8.1. *The sequence (1)*

$$W(C_0(q)_\sigma) \xrightarrow{\text{tr}} W(F) \xrightarrow{p^*} W(Q_d)$$

mentioned in the introduction is exact.

Proof. If d is odd, the sequence is exact by Theorem 1.2 (a). If d is even, we note that the map $W^0(\mathcal{A}) \to W^0(Q)$ is injective by Theorem 1.2 (b), because $W^{-d-1}(F) = 0$, cf. [3]. □

Proposition 8.2. *Then, the map $p^* : W^0(F) \to W^0(Q_d)$ induces a well-defined map*

$$W^0(F)/\langle\langle a_1a_2, \dots, a_1a_n \rangle\rangle W^0(F) \to W^0(Q_d)$$

where $\langle\langle a_1a_2, \dots, a_1a_n \rangle\rangle := \langle 1, a_1a_2 \rangle \otimes \cdots \otimes \langle 1, a_1a_n \rangle$ is the Pfister form. If d is odd, this map is surjective.

Proof. There is an element $[\beta]$ in $W^0(C_0(q)_\sigma)$ represented by the form

$$\beta : C_r(q) \times C_r(q)^{\text{op}} \to C_0(q) : (x, y) \mapsto x\bar{y}.$$

The result follows by invoking the exact sequence

$$W^0(C_0(q)_\sigma) \xrightarrow{\text{tr}} W^0(F) \xrightarrow{p^*} W^0(Q_d)$$

(cf. Proposition 8.1) and showing that $\text{tr}([\beta]) = \langle\langle a_1a_2, \dots, a_1a_n \rangle\rangle$. Let $\{e_1, \dots, e_n\}$ be an orthogonal basis of the quadratic form q. The Clifford algebra $C(q)$ has a basis $\{e^\Delta : \Delta \in \mathbb{F}_2^n\}$ where $e^\Delta := e_1^{b_1} \cdot \dots \cdot e_n^{b_n}$ with $\Delta = (b_1, \dots, b_n) \in \mathbb{F}_2^n$. Let $|\Delta| = \sum b_i$. Then, $C_r(q)$ has a k-basis $\{e^\Delta : \Delta \in \Omega_r\}$ where Ω_r is the set $\{\Delta \in \mathbb{F}_2^n : |\Delta| \equiv r \mod 2\}$. Precisely, $\text{tr}([\beta])$ is the form

$$C_r(q) \times C_r(q) \xrightarrow{\beta} C_0(q) \xrightarrow{\text{tr}} F$$

cf. Remark 6.10.

By Equation (15), we see that $\text{tr}([\beta]) = \perp_{\Delta \in \Omega_r} q(e^\Delta) \in W^0(F)$ where $q(e^\Delta) := q(e_1)^{b_1} \cdots q(e_n)^{b_n} = a_1^{b_1} \cdots a_n^{b_n}$. (Note that $2 \in F^\times$. It follows that $\langle 2^n \rangle$ is isometric to $\langle 1 \rangle$ for $n \geq 2$.) Observe that $\perp_{\Delta \in \Omega_r} q(e^\Delta) \cong \langle a_1^r \rangle \otimes \langle\langle a_1a_2, \dots, a_1a_n \rangle\rangle$ in $W^0(F)$, because

$$q(e^\Delta) = \langle a_1^{|\Delta|+r} a_1^{b_1} \cdots a_n^{b_n} \rangle \cong \langle a_1^{2b_1+r}(a_1a_2^{b_2}) \cdots (a_1a_n^{b_n}) \rangle \cong \langle a_1^r \cdot (a_1a_2^{b_2}) \cdots (a_1a_n^{b_n}) \rangle$$

if $\Delta = (b_1, \cdots, b_n) \in \Omega_r$. If d is odd, the sequence

$$W^0(C_0(q)_\sigma) \xrightarrow{\text{tr}} W^0(F) \xrightarrow{p^*} W^0(Q_d) \longrightarrow W^1(C_0(q)_\sigma) = 0$$

is exact where $W^1(C_0(q)_\sigma) = 0$ by Theorem A.3. The result follows. □

Remark 8.3. Assume $C_0(q)$ is a division algebra. See [35, Exercise 3.16] for a discussion of Clifford division algebras. If $n = 3$, then $\text{tr}([\beta]) = \langle\langle a_1a_2, a_1a_3 \rangle\rangle$ by the previous proposition. The involution center of $C_0(q)$ is the base F, and $\text{tr}([\beta])W(F) = \ker p^*$. It follows that the map

$$p^* : W^0(F)/\langle\langle a_1a_2, a_1a_3 \rangle\rangle W^0(F) \to W^0(Q_d)$$

is an isomorphism. For $n > 3$, the description of $\ker(p^*)$ is still open.

8.2. **Witt groups of isotropic quadrics.** Let k be a commutative ring with $\frac{1}{2} \in k$. Let Q_d be the quadric defined by an isotropic quadratic form q over k.

Lemma 8.4. *The pullback $p^* : W^0(k) \to W^0(Q_d)$ is split injective.*

Proof. Since q is isotropic, the quadric Q_d has a rational point $s : \operatorname{Spec} k \to Q_d$ which induces a map $s^* : W^0(Q_d) \to W^0(k)$. The map s^* is a splitting of p^*. □

Proposition 8.5. *Let k be a commutative ring with $\frac{1}{2} \in k$. Let Q_d be the quadric defined by an isotropic quadratic form q over k. If d is odd, then*

$$W^i(Q_d) \cong W^i(k) \oplus W^{i+1}(C_0(q)_\sigma)$$

If d is even, then we have a long exact sequence

$$\cdots \to W^{i-d-1}(k, \det P) \to W^i(k) \oplus W^{i+1}(C_0(q)_\sigma) \to W^i(Q_d) \to W^{i-d}(k, \det P) \to \cdots$$

Proof. Applying Lemma 8.4 to Theorem 7.2, we conclude the result if d is odd. If d is even, we have an exact sequence

$$\cdots \longrightarrow W^i(C_0(q)_\sigma) \xrightarrow{\text{tr}} W^i(k) \longrightarrow W^i(\mathcal{A}) \longrightarrow W^{i+1}(C_0(q)_\sigma) \longrightarrow \cdots$$

by Theorem 6.1. Note that the map $W^i(k) \to W^i(\mathcal{A})$ is split injective if Q_d has a rational point $s : \operatorname{Spec} k \to Q_d$. A splitting is given by $W^i(\mathcal{A}) \to W^i(Q_d) \xrightarrow{s^*} W^i(k)$ where the first map is induced by the inclusion $\mathcal{A} \hookrightarrow \mathcal{D}^b Q_d$. □

8.3. **Witt groups of quadrics over a commutative semi-local ring.** Let k be a commutative semi-local ring with $\frac{1}{2}$ in this subsection.

Definition 8.6. In this paper, we introduce the following notation

- $\mathrm{k}W(C_0(q)_\sigma)$ is the kernel of the trace map $\text{tr} : W^0(C_0(q)_\sigma) \to W^0(k)$.
- $\mathrm{c}W(k)$ is the cokernel of the trace map $\text{tr} : W^0(C_0(q)_\sigma) \to W^0(k)$.

Proposition 8.7. *Let k be a commutative semilocal ring with $\frac{1}{2} \in k$. Assume that the odd indexed Witt groups $W^{2l+1}(C_0(q)_\sigma) = 0$ (see Theorem A.3 for a vanishing result). Then,*

$$W^i(\mathcal{A}) = \begin{cases} \mathrm{c}W(k) & \text{if } i = 0 \mod 4 \\ W^2(C_0(q)_\sigma) & \text{if } i = 1 \mod 4 \\ 0 & \text{if } i = 2 \mod 4 \\ \mathrm{k}W(C_0(q)_\sigma) & \text{if } i = 3 \mod 4 \end{cases}$$

Proof. Note that if k is a commutative semilocal ring containing $\frac{1}{2}$, then $W^i(k) = 0$ for $i \not\equiv 0 \mod 4$ by [5] and $W^{2l+1}(C_0(q)_\sigma) = 0$ by the assumption. The result follows by the 4-periodicity of Witt groups and by applying these vanishing results to Theorem 6.1. □

Proposition 8.8. *Let k be a commutative semilocal ring with $\frac{1}{2} \in k$. Assume that the odd indexed Witt groups $W^{2l+1}(C_0(q)_\sigma) = 0$ (see Theorem A.3 for a vanishing result). If d is odd, then*

$$W^i(Q_d) = \begin{cases} \mathrm{c}W(k) & \text{if } i = 0 \mod 4 \\ W^2(C_0(q)_\sigma) & \text{if } i = 1 \mod 4 \\ 0 & \text{if } i = 2 \mod 4 \\ \mathrm{k}W(C_0(q)_\sigma) & \text{if } i = 3 \mod 4 \end{cases}$$

If d is even, then

$$W^i(Q_d) = \begin{cases} \mathrm{c}W(k) & \text{if } d \equiv 2 \mod 4 \text{ and if } i \equiv 0 \mod 4 \\ W^2(C_0(q)_\sigma) & \text{if } d \equiv 2 \mod 4 \text{ and if } i \equiv 1 \mod 4 \\ 0 & \text{if } d \equiv 0 \mod 4 \text{ and if } i \equiv 2 \mod 4 \\ \mathrm{k}W(C_0(q)_\sigma) & \text{if } d \equiv 0 \mod 4 \text{ and if } i \equiv 3 \mod 4 \end{cases}$$

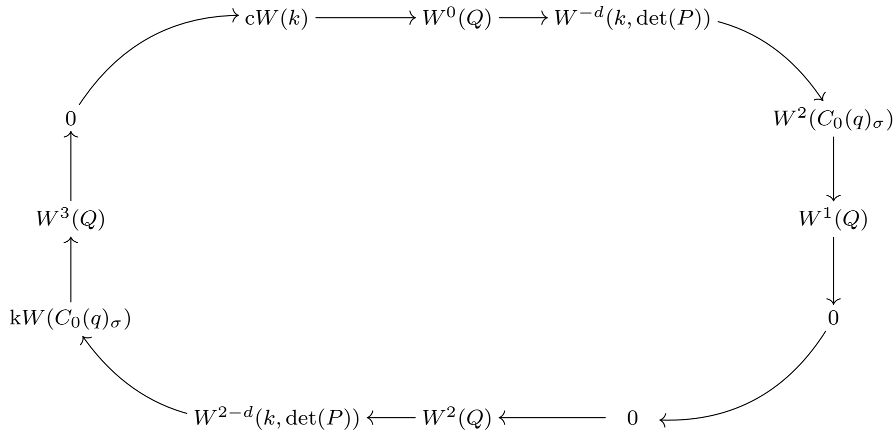
If $d \equiv 2 \mod 4$, then there is an exact sequence

$$0 \to W^2(Q_d) \to W^0(k, \det(P)) \to \mathrm{k}W(C_0(q)_\sigma) \to W^3(Q_d) \to 0.$$

If $d \equiv 0 \mod 4$*, then there is an exact sequence*

$$0 \to \mathrm{c}W(k) \to W^0(Q_d) \to W^0(k, \det(P)) \to W^2(C_0(q)_\sigma) \to W^1(Q_d) \to 0.$$

Proof. If d is odd, the results follow by applying Proposition 8.7 to Lemma 7.1. If d is even, we note that $W^{2i+1}(k, \det(P)) = 0$ for any semi-local ring k (because any projective module over a connected semi-local ring is free, and $W^i(R_1 \times R_2, L_1 \times L_2) = W^i(R_1, L_1) \oplus W^i(R_2, L_2)$). Applying Proposition 8.7 to Corollary 7.6, we obtain a 12-term long exact sequence



If $d \equiv 0 \mod 4$ (resp. $d \equiv 2 \mod 4$), then $W^{2-d}(k, \det(P)) = 0$ (resp. $W^d(k, \det(P)) = 0$). The results follow. $\square$

8.4. **Witt groups of quadrics defined by** $m\langle -1\rangle \perp n\langle 1\rangle$**.** Let k be a field in which -1 is not a sum of two squares. Let $Q_{m,n}$ be the quadric hypersurface defined by the quadratic form $q_{m,n} = m\langle -1\rangle \perp n\langle 1\rangle$. In this section we compute $W^i(Q_{m,n})$.

Let Y denote the quaternion algebra $(\frac{-1,-1}{k})$. Note that the algebra Y is a division algebra. Following Karoubi (cf. [20, Chapter III.3] and [28, Chapter V]), we write $C_0^{m,n} := C_0(q_{m,n})$ and $C^{m,n} := C(q_{m,n})$ for convenience. Assume $n + m \geq 3$. There are, therefore, three cases to consider:

Case 1. $m \geq 1$ and $n \geq 1$. In this case, the quadric $Q_{m,n}$ is defined by an isotropic quadratic form. So, Corollary 8.5 applies.

Case 2. $m = 0$ or $n = 0$, and $m + n$ is odd. We only need to consider $Q_{0,n}$, since $Q_{0,n} \cong Q_{n,0}$ as algebraic varieties. Note that $C_0^{0,n} \cong C^{n-1,0}$. We copy the relevant information on the table in [28, p. 125] for this case.

Table 1	n=3	n=5	n=7	n=9
$C^{n-1,0}$	Y	$M_2(Y)$	$M_8(k)$	$M_{16}(k)$
Type of σ	symplectic		orthogonal	

The type of the canonical involution is determined in [25, Proposition 8.4]. The Clifford algebra is 8-periodic in the sense that $C^{n+8,0} \cong M_{16}(C^{n,0})$. By Morita equivalence (cf. Chapter I.9 [24]), we see that

$$W^0((C_0^{0,n})_\sigma) \cong \begin{cases} W^0(Y_{\sigma_s}) & \text{if } n \equiv 3,5 \mod 8 \\ W^0(k) & \text{if } n \equiv 1,7 \mod 8 \end{cases}$$

where σ_s is the involution which is 1 on the center and -1 outside the center.

Theorem 8.9. *Let* $n = d+2$ *be an odd number. Assume* k *is a field in which* -1 *is not a sum of two squares. Then,*

$$W^0(Q_{0,n}) \cong W^0(k)/2^{\delta(n)}W^0(k)$$

where $\delta(n)$ is the cardinality of the set $\{l \in \mathbb{Z} : 0 < l < n, l \equiv 0, 1, 2 \text{ or } 4 \mod 8\}$.

Proof. Write $n-2 = d = 2m+1$, and let $r := m+d+1$. It is enough to show that $\mathrm{Im}(\mathrm{tr}) \cong \ker(p^*) \cong 2^{\delta(n)} W^0(k)$. By Lemma 6.8 and Definition 6.9, the trace map $\mathrm{tr} : W(C_0(q)_\sigma) \to W(k)$ takes a symmetric space $b : M \times M^{\mathrm{op}} \to C_0^{0,n}$ in $W^0((C_0^{0,n})_\sigma)$ to the symmetric space

$$M_r \times M_r \to C_0^{0,n} \xrightarrow{\mathrm{tr}} k : (m \otimes e) \otimes (n \otimes e') \mapsto \mathrm{tr}(b(m,n) \cdot \bar{e}e')$$

in $W^0(k)$ where $M_r := C_r^{0,n} \otimes_{C_0^{0,n}} M$. Note that $C_0^{0,n}$ is isomorphic to $C_1^{0,n}$ as k-modules by multipling an anisotropic vector e with $e^2 = 1$ (We can do this because $q = n\langle 1 \rangle$ in this case). Therefore, the symmetric form

$$M \times M \to C_0^{0,n} \xrightarrow{\mathrm{tr}} k : m \otimes n \mapsto \mathrm{tr}(b(m,n))$$

over k is isometric to $M_r \times M_r \to C_0^{0,n} \xrightarrow{\mathrm{tr}} k$ above.

Note that there is a surjective group homomorphism $\mathbb{Z}[k^\times/k^{2\times}] \to W^0((C_0^{0,n})_\sigma)$. The reason is that, via the Morita equivalence, $W^0((C_0^{0,n})_\sigma)$ is isomorphic to $W^0(Y, \sigma_s)$ or $W^0(k)$. Any Hermitian form q over a division algebra with involution is diagonalizable, i.e. $q \simeq \langle a_1, \ldots, a_n \rangle$ where a_i are all in the involution center, cf. Chapter I.6 [24], and involution centers of the division algebras with involution in our case are all equal to k. This gives the surjective map $\mathbb{Z}[k^\times/k^{2\times}] \to W^0((C_0^{0,n})_\sigma)$.

By this observation, it is enough to investigate where a symmetric space represented by an irreducible $C_0^{0,n}$-module goes under the trace map. In this case, $C_0^{0,n}$ is simple. Let Σ_n be the naive representation of $C_0^{0,n}$, cf. Appendix B. By definition, Σ_n is a left ideal of $C_0^{0,n}$. As a k-vector space, Σ_n is of rank $2^{\delta(n)}$, cf. Appendix B.3. Consider an element $[b_n] \in W^0(C_0^{0,n})$ represented by the form $b_n : \Sigma_n \times \Sigma_n \hookrightarrow C_0^{0,n} \times C_0^{0,n} \xrightarrow{\beta} C_0^{0,n}$. [1] Conclude that $\mathrm{tr}([b_n]) = 2^{\delta(n)}\langle 1 \rangle$ by the property of the basis of Σ_n in Corollary B.11. [2] □

Case 3. $m = 0$ or $n = 0$, and $m+n$ is even. Let $X = k\langle\sqrt{-1}\rangle$. Again, we only consider $Q_{0,n}$ and copy parts of the table on [28, p. 125] relevant to this case.

Table 2	n=2	n=4	n=6	n= 8
$C^{n-1,0}$	X	$Y \times Y$	$M_4(X)$	$M_8(k) \times M_8(k)$
Type of σ	unitary	symplectic	unitary	orthogonal

The type of the canonical involution is determined in [25, Proposition 8.4]. The Clifford algebras are also 8-periodic. By Morita equivalence, we see that

$$W^0(C_0(q)_\sigma) \cong \begin{cases} W^0(X_{\sigma_u}) & \text{if } n \equiv 2, 6 \mod 8 \\ W^0(Y_{\sigma_s}) \oplus W^0(Y_{\sigma_s}) & \text{if } n \equiv 4 \mod 8 \\ W^0(k) \oplus W^0(k) & \text{if } n \equiv 0 \mod 8 \end{cases}$$

where σ_u is the unitary involution and σ_s is the involution which is identity on the center and -1 outside the center.

Theorem 8.10. *Let $n = d+2$ be an even number. Let k be a field in which -1 is not a sum of two squares. If $d \equiv 2 \mod 4$, then $W^0(Q_d) \cong W^0(k)/2^{\delta(n)}W^0(k)$ and if $d \equiv 0 \mod 4$, then there is an exact sequence*

$$0 \longrightarrow W^0(k)/2^{\delta(n)}W^0(k) \longrightarrow W^0(Q_d) \longrightarrow W^0(k) .$$

Proof. To obtain this result, we need to compute the cokernel of the trace map of Witt groups. The situation is similar as in the proof of Theorem 8.9. □

[1] The form b_n is necessarily non-degenerate, because $\mathrm{tr} \circ b_n$ is non-degenerate by Corollary B.11 and the map $[M, C_0]_{C_0} \xrightarrow{[1,\mathrm{tr}]} [M, k]_k$ is an isomorphism.

[2] Since 2 is invertible in k, we have $\langle \frac{1}{2^m} \rangle \simeq \langle 1 \rangle$ for $m \geq 2$.

8.5. **Witt groups of real anisotropic quadrics.** If $k=\mathbb{R}$, then X is the field of complex numbers $\mathbb{C}$ and Y is just the quaternion algebra $\mathbb{H}$.

Theorem 8.11. *Let $Q_{0,n}$ be the quadric defined by the anisotropic quadratic form $n\langle 1\rangle$ over $\mathbb{R}$. In other words, $Q_{0,n}=\operatorname{Proj}\left(\mathbb{R}[X_1,\dots,X_n]/(X_1^2+\cdots+X_n^2)\right)$. Then, we have*

	$W^i(Q_{0,n})$			
	$i=0$	$i=1$	$i=2$	$i=3$
$n\equiv 1 \mod 8$	$\mathbb{Z}/2^{\delta(n)}$	0	0	0
$n\equiv 2 \mod 8$	$\mathbb{Z}/2^{\delta(n)}$	$\mathbb{Z}/2^{\kappa(n)}$	0	0
$n\equiv 3 \mod 8$	$\mathbb{Z}/2^{\delta(n)}$	$\mathbb{Z}/2$	0	0
$n\equiv 4 \mod 8$	$\mathbb{Z}/2^{\delta(n)}$	$(\mathbb{Z}/2)^2$	0	$\mathbb{Z}/2^{\kappa(n)}$
$n\equiv 5 \mod 8$	$\mathbb{Z}/2^{\delta(n)}$	$\mathbb{Z}/2$	0	0
$n\equiv 6 \mod 8$	$\mathbb{Z}/2^{\delta(n)}$	$\mathbb{Z}/2^{\kappa(n)}$	0	0
$n\equiv 7 \mod 8$	$\mathbb{Z}/2^{\delta(n)}$	0	0	0
$n\equiv 8 \mod 8$	$\mathbb{Z}/2^{\delta(n)}$	0	0	$\mathbb{Z}/2^{\kappa(n)}$

where $\kappa(n)$ is an integer in $\mathbb{Z}_{\geq 0}$ depending on n.

Proof. Suppose that $n=d+2$ is odd. By [24, Chapter I (10.5)], we have $W^0(\mathbb{R})\cong W^0(\mathbb{H}_{\sigma_s})\cong\mathbb{Z}$, $W^2(\mathbb{R})=0$ and $W^2(\mathbb{H}_{\sigma_s})\cong\mathbb{Z}/2$. It follows that $W^0(Q_{0,n})\cong\mathbb{Z}/2^{\delta(n)}\mathbb{Z}$ by Theorem 8.9. To compute $W^1(Q_{0,n})$, we need to study $W^2((C_0^{0,n})_\sigma)$ by Corollary 8.8. By previous discussion, we conclude that

$$W^2((C_0^{0,n})_\sigma)=\begin{cases} W^2(\mathbb{H}_{\sigma_s})\cong\mathbb{Z}/2\mathbb{Z} & \text{if } n\equiv 3,5 \mod 8\\ W^2(\mathbb{R})=0 & \text{if } n\equiv 1,7 \mod 8\end{cases}$$

If $n=d+2$ is odd, then we have $W^0((C_0^{0,n})_\sigma)\cong\mathbb{Z}$ and a commutative diagram

$$\begin{array}{ccc} W^0((C_0^{0,n})_\sigma) & \xrightarrow{\mathrm{tr}} & W^0(\mathbb{R})\\ \downarrow\cong & & \downarrow\cong\\ \mathbb{Z} & \xrightarrow{2^{\delta(n)}} & \mathbb{Z}\end{array}$$

by the proof of Theorem 8.9, so that $\ker(\mathrm{tr})=0$. By Corollary 8.8, we conclude $W^2(Q_{0,n})=W^3(Q_{0,n})=0$.

Assume that $n=d+2$ is even. By [24, Chapter I (10.5)] again, we have that $W^0(\mathbb{R})\cong W^0(\mathbb{H}_{\sigma_s})\cong W^0(\mathbb{C}_{\sigma_u})\cong W^2(\mathbb{C}_{\sigma_u})\cong\mathbb{Z}$, $W^2(\mathbb{R})=0$ and $W^2(\mathbb{H}_{\sigma_s})\cong\mathbb{Z}/2$ where σ_u is just the complex conjugation. From previous discussions, we conclude that

$$W^0((C_0^{0,n})_\sigma)\cong\begin{cases} W^0(\mathbb{C}_{\sigma_u})\cong\mathbb{Z} & \text{if } n\equiv 2,6 \mod 8\\ W^0(\mathbb{H}_{\sigma_s})\oplus W^0(\mathbb{H}_{\sigma_s})\cong\mathbb{Z}\oplus\mathbb{Z} & \text{if } n\equiv 4 \mod 8\\ W^0(\mathbb{R})\oplus W^0(\mathbb{R})\cong\mathbb{Z}\oplus\mathbb{Z} & \text{if } n\equiv 0 \mod 8\end{cases}$$

and that

$$W^2((C_0^{0,n})_\sigma)\cong\begin{cases} W^2(\mathbb{C}_{\sigma_u})\cong\mathbb{Z} & \text{if } n\equiv 2,6 \mod 8\\ W^2(\mathbb{H}_{\sigma_s})\oplus W^2(\mathbb{H}_{\sigma_s})\cong\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & \text{if } n\equiv 4 \mod 8\\ W^2(\mathbb{R})\oplus W^2(\mathbb{R})=0 & \text{if } n\equiv 0 \mod 8.\end{cases}$$

If $n \equiv 0 \mod 4$, we have a commutative diagram

$$\begin{array}{ccc} W^0((C_0^{0,n})_\sigma) & \xrightarrow{\mathrm{tr}} & W^0(\mathbb{R}) \\ \downarrow\cong & & \downarrow\cong \\ \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{(2^{\delta(n)},2^{\delta(n)})} & \mathbb{Z} \end{array}$$

so that $\ker(\mathrm{tr}) \cong \mathbb{Z}$. If $n \equiv 2 \mod 4$, we get a commutative diagram

$$\begin{array}{ccc} W^0((C_0^{0,n})_\sigma) & \xrightarrow{\mathrm{tr}} & W^0(\mathbb{R}) \\ \downarrow\cong & & \downarrow\cong \\ \mathbb{Z} & \xrightarrow{2^{\delta(n)}} & \mathbb{Z} \end{array}$$

so that $\ker(\mathrm{tr}) = 0$. By [12, Proposition 3.1], we also note that $W^i(Q_{0,n})$ are all 2-primary torsion groups. Thus, if d is even, $W^0(Q_{0,n})$ is isomorphic to $\mathbb{Z}/2^{\delta(n)}\mathbb{Z}$ by Theorem 8.10 (because $\mathrm{Hom}(\mathbb{Z}/2^c\mathbb{Z},\mathbb{Z}) = 0$ for any integer $c > 0$). If $n \equiv 2 \mod 4$, then there is a commutative diagram of exact sequences

$$\begin{array}{ccccccccccc} 0 \to & \mathrm{coker}(\mathrm{tr}) & \to & W^0(Q_{0,n}) & \to & W^0(\mathbb{R}) & \to & W^2((C_0^{0,n})_\sigma) & \to & W^1(Q_{0,n}) & \to 0 \\ & \downarrow\cong & & \| & & \downarrow\cong & & \downarrow\cong & & \| & \\ 0 \to & \mathbb{Z}/2^{\delta(n)}\mathbb{Z} & \xrightarrow{\cong} & W^0(Q_{0,n}) & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & W^1(Q_{0,n}) & \to 0 \end{array}$$

Since $W^1(Q_{0,n})$ is a 2-primary torsion group, we see $W^1(Q_{0,n})$ is either zero or a cyclic group of order some powers of two. If $n \equiv 0 \mod 4$, then there is a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 \to & W^2(Q_{0,n}) & \to & W^0(\mathbb{R}) & \to & \ker(\mathrm{tr}) & \to & W^3(Q_{0,n}) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \| & \\ 0 \to & W^2(Q_{0,n}) & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & W^3(Q_{0,n}) & \to 0. \end{array}$$

which gives $W^2(Q_{0,n}) = 0$. Since $W^3(Q_{0,n})$ is a 2-primary torsion group, $W^3(Q_{0,n})$ is either zero or a cyclic group of order some powers of two. Other cases can be observed from Proposition 8.8. □

Appendix A. Witt groups of Clifford algebras

The purpose of this appendix is to give examples for which Proposition 8.8 can be applied. Let k be a commutative ring containing $\frac{1}{2}$. Let A be a k-algebra with an involution σ. Let $W^i(A_\sigma)$ be Balmer's Witt groups of finitely generated projective modules over A, which are written as $W^i(A_\sigma - \mathrm{proj})$ in Balmer and Preedi's work [5]. Note that $W^0(A_\sigma)$ is isomorphic to the classical Witt group of the algebra A with the involution σ. Let $W^i(A_\sigma - \mathrm{free})$ be Balmer's Witt groups of finite free modules over A (see [5] for this notation). The following result may be compared to [13, Theorem 5.2].

Theorem A.1. *Let k be a commutative ring with $\frac{1}{2}$. Let A be a semilocal k-algebra with an involution σ. Let $J(A)$ be the Jacobson radical of A and $\bar{A} := A/J(A)$. If the natural map $K_0(A) \to K_0(\bar{A})$ is an isomorphism, then $W^{2l+1}(A_\sigma) = 0$.*

Proof. Let $\bar{\sigma}$ be the induced involution on $\bar{A}$. Consider the following diagram consisting of Hornbostel-Schlichting exact sequences (cf. [18, Appendix A])

$$\begin{array}{ccccccc} \hat{H}^{i-1}(C_2, \widetilde{K}_0(A)) & \longrightarrow & W^i(A_\sigma - \mathrm{free}) & \longrightarrow & W^i(A_\sigma) & \longrightarrow & \hat{H}^i(C_2, \widetilde{K}_0(A)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \hat{H}^{i-1}(C_2, \widetilde{K}_0(\bar{A})) & \longrightarrow & W^i(\bar{A}_\sigma - \mathrm{free}) & \longrightarrow & W^i(\bar{A}_\sigma) & \longrightarrow & \hat{H}^i(C_2, \widetilde{K}_0(\bar{A})) \end{array}$$

where $\widetilde{K}_0(A) := \operatorname{coker}\big(K_0(A-\text{free}) \to K_0(A)\big)$. If $K_0(A) \to K_0(\bar{A})$ is an isomorphism, then both the left and right vertical maps are isomorphisms. The second vertical map is injective for i odd by [13, Theorem 5.2]. Since $\bar{A}$ is semisimple, we have $W^i(\bar{A}_{\bar{\sigma}}) = 0$ for i odd by [31](or [5, Theorem 4.2]). Finally, we conclude $W^i(A_\sigma) = 0$ by diagram chasing. □

Recall that a k-algebra A is called semiperfect if A is semilocal and idempotents of $A/J(A)$ can be lifted to A. For examples, see Lemma A.8.

Corollary A.2. *Let k be a commutative ring containing $\frac{1}{2}$. If A is a semiperfect k-algebra with an involution σ, then $W^{2l+1}(A_\sigma) = 0$.*

Proof. The proof of [7, Proposition 2.12 Chapter III on p. 90] can be applied to show that $K_0(A) \to K_0(\bar{A})$ is an isomorphism when A is semiperfect. □

Let (P, q) be a nondegenerate quadratic form of rank n over k. Let $C(q)$ be the Clifford algebra of (P, q) and let $C_0(q)$ be its even part. Let σ be an involution of the first kind on $C(q)$ (the one that induces the identity on the center). Certainly, $W^{2l+1}(C_0(q)_\sigma) = W^{2l+1}(C(q)_\sigma) = 0$ if the base k is a field of characteristic $\neq 2$ by [31], since Clifford algebras are semisimple over a field. In this appendix, we go a little further. If k is local of characteristic $\neq 2$, then we can assume P is free and we can find a basis $\{e_i\}_{1\le i\le n}$ of (P, q) and write $q \cong \langle \alpha_1, \ldots, \alpha_n\rangle$ with $\alpha_i = q(e_i) \in k^\times$.

Theorem A.3. *Let k be a commutative semilocal ring containing $\frac{1}{2}$. Then*

$$W^{2l+1}(C_0(q)_\sigma) = W^{2l+1}(C(q)_\sigma) = 0$$

if one of the following additional assumptions hold.

(i) $(k, J(k))$ is a Henselian pair,
(ii) k is a local ring containing a field F such that F is isomorphic to the residue field and $q(e_i) \in F^\times$,
(iii) k is a regular local ring containing a field F.

To prove this theorem, we need the following lemmas.

Lemma A.4. *Let k be a commutative ring. If A is a separable k-algebra, then $J(A) = J(k)A$.*

Proof. See [19, Lemma 1.1 (d)]. □

Lemma A.5. *Let k be a commutative semilocal ring. Then finite k-algebras are semilocal.*

Proof. See [27, Proposition 20.6 on p. 297] □

Lemma A.6. *Under the assumption (i) of Theorem A.3, separable k-algebras are semiperfect.*

Proof. By Lemma A.5, we learn that separable algebras over commutative semilocal rings are semilocal. Under the assumption (i) of Theorem A.3, the idempotent lifting property with respect to the Jacobson radical holds by [14, Theorem 2.1] and Lemma A.4. □

Lemma A.7. *Let k be a commutative ring. Then both $C(q)$ and $C_0(q)$ are separable k-algebras.*

Proof. See [17, Theorem 9.9 on p. 127]. □

Lemma A.8. *Under the assumption (ii) of Theorem A.3, both $C(q)$ and $C_0(q)$ are semiperfect.*

Proof. Let $\mathfrak{m}$ be the unique maximal ideal of k. Note that $J(C(q)) = \mathfrak{m}C(q)$ by Lemma A.4 and A.7. It is enough to show that idempotents of $C(q)/\mathfrak{m}C(q)$ can be lifted to $C(q)$. Let $\{e_i\}_{1\le i\le n}$ be an orthogonal basis of (P, q). If $\Delta = (c_1, \ldots, c_n) \in \mathbb{F}_2^n$, then we write $e^\Delta := e_1^{c_1} \cdots e_n^{c_n}$. It is well-known that $C(q)$ has a k-basis $\{e^\Delta : \Delta \in \mathbb{F}_2^n\}$. Let $\bar{e} = \sum_i \bar{a}_i \bar{e}^{\Delta_i}$ (with $\bar{a}_i \in k/\mathfrak{m}$) be a non-zero idempotent in $C(q)/\mathfrak{m}C(q)$. For each nonzero $\bar{a}_i \in k/\mathfrak{m}$, we can find an element $a_i \in F^\times$ lifting $\bar{a}_i$ via the isomorphism $F \hookrightarrow k \to k/\mathfrak{m}$. Define $e = \sum_i a_i e^{\Delta_i}$. I claim that e is an idempotent in $C(q)$. If not, then $e^2 - e = \sum_\lambda b_\lambda e^{\hat{\Delta}_\lambda}$ for some $b_\lambda \in F^\times$ (because a_i and $q(e_i)$ are all in $F^\times$). It follows that $\bar{e}^2 - \bar{e} \neq 0 \in C(q)/\mathfrak{m}C(q)$. This contradicts the assumption that $\bar{e}$ is an idempotent in $k/\mathfrak{m}$. Using the k-basis $\{e^\Delta : \Delta = (c_1, \ldots, c_n) \in \mathbb{F}_2^n$ and $\sum_i c_i = 0 \in \mathbb{F}_2\}$ for $C_0(q)$, we see that the same strategy works to show the idempotent lifting property of $C_0(q)$ with respect to $J(C_0(q))$. □

Proof of Theorem A.3. By Corollary A.2, Lemma A.6 and Lemma A.8, we obtain the result under the assumption (i) or (ii). The result under the assumption (iii) can be concluded by [15, Lemma in Section 7.6]. □

Remark A.9. Clifford algebras over local rings need not be semiperfect in general. For example, the quaternion algebra $(\frac{1-p,1-p}{\mathbb{Z}_{(p)}})$ is not semiperfect for any prime $p > 0$ where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at the multiplicative system $\mathbb{Z} - (p)$. However, it still might be the case for $W^{2l+1}(C_0(q)_\sigma) = W^{2l+1}(C(q)_\sigma) = 0$.

Appendix B. Irreducible representations of Clifford algebras

B.1. Some classical isomorphisms of algebras.

Lemma B.1. $\mathbb{C} \otimes \mathbb{H} \cong M_2(\mathbb{C})$.

Proof. Define a map $c : \mathbb{C} \otimes \mathbb{H} \to M_2(\mathbb{C})$ by

$$1 \otimes 1 \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad 1 \otimes i \mapsto \begin{pmatrix} & -i \\ -i & \end{pmatrix}, \quad 1 \otimes j \mapsto \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \quad 1 \otimes k \mapsto \begin{pmatrix} -i & \\ & i \end{pmatrix}$$
$$i \otimes 1 \mapsto \begin{pmatrix} i & \\ & i \end{pmatrix}, \quad i \otimes i \mapsto \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad i \otimes j \mapsto \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad i \otimes k \mapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

This map is an isomorphism of $\mathbb{R}$-algebras. □

Let A be a k-algebra and let $a \in A$. Denote $h_{s,t}(a)$ the matrix with $a \in A$ in the (s,t)-spot and 0 in all other entries. The first column $C_1(\mathbb{C})$ of $M_2(\mathbb{C})$ can be regarded as an irreducible left module of $M_2(\mathbb{C})$, and it has an $\mathbb{R}$-basis $\{h_{1,1}(1), h_{1,1}(i), h_{1,2}(1), h_{1,2}(i)\}$. Using the isomorphism $c : \mathbb{C} \otimes \mathbb{H} \cong M_2(\mathbb{C})$, we find an irreducible left module $c^{-1}C_1(\mathbb{C})$ of $\mathbb{C} \otimes \mathbb{H}$. In this case, we can write out an $\mathbb{R}$-basis for $c^{-1}C_1(\mathbb{C})$:

$$\begin{array}{lcl} h_{1,1}(1) & \mapsto & \frac{1}{2}(1 \otimes 1 + i \otimes k) \\ h_{1,1}(i) & \mapsto & \frac{1}{2}(i \otimes 1 - 1 \otimes k) \\ h_{1,2}(1) & \mapsto & \frac{1}{2}(i \otimes i - 1 \otimes j) \\ h_{1,2}(i) & \mapsto & \frac{1}{2}(-1 \otimes i - i \otimes j) \end{array} \tag{33}$$

Lemma B.2. $\mathbb{H} \otimes \mathbb{H} \cong M_4(\mathbb{R})$.

Proof. Define a map $d : \mathbb{H} \otimes \mathbb{H} \to M_4(\mathbb{R})$ by

$$1 \otimes 1 \mapsto \begin{pmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ &&& 1 \end{pmatrix}, \quad 1 \otimes i \mapsto \begin{pmatrix} & 1 && \\ -1 &&& \\ &&& -1 \\ && 1 & \end{pmatrix}, \quad 1 \otimes j \mapsto \begin{pmatrix} && 1 & \\ &&& 1 \\ -1 &&& \\ & -1 && \end{pmatrix}, \quad 1 \otimes k \mapsto \begin{pmatrix} &&& 1 \\ && -1 & \\ & 1 && \\ -1 &&& \end{pmatrix}$$
$$i \otimes 1 \mapsto \begin{pmatrix} & -1 && \\ 1 &&& \\ &&& -1 \\ && 1 & \end{pmatrix}, \quad i \otimes i \mapsto \begin{pmatrix} 1 &&& \\ & 1 && \\ && -1 & \\ &&& -1 \end{pmatrix}, \quad i \otimes j \mapsto \begin{pmatrix} &&& -1 \\ && 1 & \\ & 1 && \\ -1 &&& \end{pmatrix}, \quad i \otimes k \mapsto \begin{pmatrix} && 1 & \\ &&& 1 \\ 1 &&& \\ & 1 && \end{pmatrix}$$
$$j \otimes 1 \mapsto \begin{pmatrix} && -1 & \\ &&& 1 \\ 1 &&& \\ & -1 && \end{pmatrix}, \quad j \otimes i \mapsto \begin{pmatrix} &&& 1 \\ && 1 & \\ & 1 && \\ 1 &&& \end{pmatrix}, \quad j \otimes j \mapsto \begin{pmatrix} 1 &&& \\ & -1 && \\ && 1 & \\ &&& -1 \end{pmatrix}, \quad j \otimes k \mapsto \begin{pmatrix} & -1 && \\ -1 &&& \\ &&& 1 \\ && 1 & \end{pmatrix}$$
$$k \otimes 1 \mapsto \begin{pmatrix} &&& -1 \\ && -1 & \\ & 1 && \\ 1 &&& \end{pmatrix}, \quad k \otimes i \mapsto \begin{pmatrix} && -1 & \\ &&& 1 \\ -1 &&& \\ & 1 && \end{pmatrix}, \quad k \otimes j \mapsto \begin{pmatrix} & 1 && \\ 1 &&& \\ &&& 1 \\ && 1 & \end{pmatrix}, \quad k \otimes k \mapsto \begin{pmatrix} 1 &&& \\ & -1 && \\ && -1 & \\ &&& 1 \end{pmatrix}$$

This map is an isomorphism of $\mathbb{R}$-algebras. □

Let $C_1^4(\mathbb{R})$ be the first column of $M_4(\mathbb{R})$. Note that $C_1^4(\mathbb{R})$ is an irreducible left module of $M_4(\mathbb{R})$. Using the isomorphism $d : \mathbb{H} \otimes \mathbb{H} \cong M_4(\mathbb{R})$, we obtain an irreducible left module $d^{-1}C_1^4(\mathbb{R})$ of $\mathbb{C} \otimes \mathbb{H}$. There is an $\mathbb{R}$-basis for $d^{-1}C_1^4(\mathbb{R})$:

$$\begin{array}{lcl} h_{1,1}(1) & \mapsto & \frac{1}{4}(1 \otimes 1 + i \otimes i + j \otimes j + k \otimes k) \\ h_{1,2}(1) & \mapsto & \frac{1}{4}(i \otimes 1 - 1 \otimes i + k \otimes j - j \otimes k) \\ h_{1,3}(1) & \mapsto & \frac{1}{4}(j \otimes 1 - k \otimes i + i \otimes k - 1 \otimes j) \\ h_{1,4}(1) & \mapsto & \frac{1}{4}(k \otimes 1 + j \otimes i - i \otimes j - 1 \otimes k) \end{array} \tag{34}$$

Let $q_{m,n}$ be the quadratic form $m\langle -1\rangle \perp n\langle 1\rangle$. Following Karoubi (cf. [20, Chapter III.3] and [28, Chapter V]), we write $C^{m,n} := C(q_{m,n})$. Set $C_0^{m,n} := C_0(q_{m,n})$.

Lemma B.3. $\omega_n : C^{n-1,0} \xrightarrow{\cong} C_0^{0,n}$.

Proof. The map ω_n is extended from $V \to C_0^{0,n} : e_1, \cdots, e_{n-1} \mapsto f_n f_1, \cdots, f_n f_{n-1}$. □

Lemma B.4. $C^{1,0} \cong \mathbb{C}$ *and* $C^{0,1} \cong \mathbb{R} \times \mathbb{R}$.

Proof. We construct the maps $C^{1,0} \to \mathbb{C} : 1, e \mapsto 1, i$ and $C^{0,1} \to \mathbb{R} \times \mathbb{R} : 1, e \mapsto (1,0), (0,1)$. □

Lemma B.5. $C^{2,0} \cong \mathbb{H}$ *and* $C^{0,2} \cong M_2(\mathbb{R})$.

Proof. Construct the following isomorphisms of $\mathbb{R}$-algebras:

$$C^{2,0} \to \mathbb{H} : 1, e_1, e_2, e_1e_2 \mapsto 1, i, j, k$$

$$C^{0,2} \to M_2(\mathbb{R}) : 1, f_1, f_2, f_1f_2 \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

□

Let $C_1^2(\mathbb{R})$ be the first column of $M_2(\mathbb{R})$. Note that $C_1^2(\mathbb{R})$ is an irreducible left module of $M_2(\mathbb{R})$. Using the isomorphism $b : C^{0,2} \cong M_2(\mathbb{R})$, we obtain an irreducible left module $b^{-1}C_1^2(\mathbb{R})$ of $C^{0,2}$. There is an $\mathbb{R}$-basis for $b^{-1}C_1^2(\mathbb{R})$:

$$\begin{array}{ccc} h_{1,1}(1) & \mapsto & \frac{1}{2}(1+f_2) \\ h_{1,2}(1) & \mapsto & \frac{1}{2}(f_1+f_1f_2) \end{array} \tag{35}$$

Lemma B.6. $C^{0,k+2} \cong C^{k,0} \otimes C^{0,2}$ *and* $C^{k+2,0} \cong C^{0,k} \otimes C^{2,0}$.

Proof. The map $C^{0,k+2} \to C^{k,0} \otimes C^{0,2}$ is extended from

$$V \to C^{k,0} \otimes C^{0,2} : f_i \mapsto \begin{cases} 1 \otimes g_i & \text{if } i = 1,2 \\ e_{i-2} \otimes g_1g_2 & \text{if } k+2 \geq i \geq 3 \end{cases}$$

The map $C^{k+2,0} \to C^{0,k} \otimes C^{2,0}$ is extended from

$$V \to C^{0,k} \otimes C^{2,0} : e_i \mapsto \begin{cases} 1 \otimes h_i & \text{if } i = 1,2 \\ f_{i-2} \otimes h_1h_2 & \text{if } k+2 \geq i \geq 3 \end{cases}$$

See [2, Proposition 4.2] for details. □

Corollary B.7. *If* $n \geq 8$*, then* $C^{n,0} \cong C^{n-8,0} \otimes C^{8,0}$ *and* $C^{0,n} \cong C^{0,n-8} \otimes C^{0,8}$.

Proof. $C^{n,0} \cong C^{0,n-2} \otimes C^{2,0} \cong C^{n-4,0} \otimes C^{0,2} \otimes C^{2,0} \cong C^{0,n-6} \otimes C^{2,0} \otimes C^{0,2} \otimes C^{2,0} \cong C^{n-8,0} \otimes C^{0,2} \otimes C^{2,0} \otimes C^{0,2} \otimes C^{2,0} \cong C^{n-8,0} \otimes C^{8,0}$. The isomorphism $C^{0,n} \cong C^{0,n-8} \otimes C^{0,8}$ $\mathbb{R}$-algebras is obtained similarly. □

Proposition B.8.

$$\begin{array}{lclcl} \sigma_{8m} & : & C^{8m,0} & \xrightarrow{\cong} & M_{2^{4m}}(\mathbb{R}) \\ \sigma_{8m+1} & : & C^{8m+1,0} & \xrightarrow{\cong} & M_{2^{4m}}(\mathbb{C}) \\ \sigma_{8m+2} & : & C^{8m+2,0} & \xrightarrow{\cong} & M_{2^{4m}}(\mathbb{H}) \\ \sigma_{8m+3} & : & C^{8m+3,0} & \xrightarrow{\cong} & M_{2^{4m}}(\mathbb{H}) \times M_{2^{4m}}(\mathbb{H}) \\ \sigma_{8m+4} & : & C^{8m+4,0} & \xrightarrow{\cong} & M_{2^{4m+1}}(\mathbb{H}) \\ \sigma_{8m+5} & : & C^{8m+5,0} & \xrightarrow{\cong} & M_{2^{4m+2}}(\mathbb{C}) \\ \sigma_{8m+6} & : & C^{8m+6,0} & \xrightarrow{\cong} & M_{2^{4m+3}}(\mathbb{R}) \\ \sigma_{8m+7} & : & C^{8m+7,0} & \xrightarrow{\cong} & M_{2^{4m+3}}(\mathbb{R}) \times M_{2^{4m+3}}(\mathbb{R}) \end{array}$$

Proof. If $n = 1$, $C_0^{0,1} \cong \mathbb{R}$.

If $n = 2$, $C_0^{0,2} \cong C_0^{1,0} \cong \mathbb{C}$.

If $n = 3$, $C_0^{0,3} \cong C^{2,0} \cong \mathbb{H}$.

If $n = 4$, $C_0^{0,4} \cong C^{3,0} \cong C^{0,1} \otimes C^{2,0} \cong (\mathbb{R} \times \mathbb{R}) \otimes \mathbb{H} \cong \mathbb{H} \times \mathbb{H}$

If $n = 5$, $C_0^{0,5} \cong C^{4,0} \cong C^{0,2} \otimes C^{2,0} \cong M_2(\mathbb{R}) \otimes \mathbb{H} \cong M_2(\mathbb{H})$.

If $n = 6$, $C_0^{0,6} \cong C^{5,0} \cong C^{0,3} \otimes C^{2,0} \cong C^{1,0} \otimes C^{0,2} \otimes C^{2,0} \cong \mathbb{C} \otimes M_2(\mathbb{R}) \otimes \mathbb{H} \cong M_4(\mathbb{C})$.

If $n = 7$, $C_0^{0,7} \cong C^{6,0} \cong C^{0,4} \otimes C^{2,0} \cong C^{2,0} \otimes C^{0,2} \otimes C^{2,0} \cong \mathbb{H} \otimes M_2(\mathbb{R}) \otimes \mathbb{H} \cong M_8(\mathbb{R})$.

If $n = 8$, $C_0^{0,8} \cong C^{7,0} \cong C^{0,5} \otimes C^{2,0} \cong C^{3,0} \otimes C^{0,2} \otimes C^{2,0} \cong (\mathbb{H} \times \mathbb{H}) \otimes M_2(\mathbb{R}) \otimes \mathbb{H} \cong M_8(\mathbb{R}) \times M_8(\mathbb{R})$.

If $n = 9$, $C_0^{0,9} \cong C^{8,0} \cong C^{0,6} \otimes C^{2,0} \cong C^{4,0} \otimes C^{0,2} \otimes C^{2,0} \cong C^{0,2} \otimes C^{2,0} \otimes C^{0,2} \otimes C^{2,0} \cong M_2(\mathbb{R}) \otimes \mathbb{H} \otimes M_2(\mathbb{R}) \otimes \mathbb{H} \cong M_{16}(\mathbb{R})$.

For $n \geq 10$, we apply Corollary B.7 inductively. □

B.2. **Naive representations.** Let A be a k-algebra. The matrix algebra $M_n(A)$ (resp. $M_n(A) \times M_n(A)$) has a left module $\epsilon_n(A)$ (resp. two left modules $\epsilon_n^+(A)$ and $\epsilon_n^-(A)$) consisting of first column matrices. If $k = \mathbb{R}$ and A is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, then $\epsilon_n(A)$ (resp. $\epsilon_n^+(A)$ or $\epsilon_n^-(A)$) is an irreducible left module of $M_n(A)$ (resp. $M_n(A) \times M_n(A)$).

Definition B.9. We call $\epsilon_n(A)$ (resp. $\epsilon_n^+(A)$) the *naive representation* of $M_n(A)$ (resp. $M_n(A) \times M_n(A)$). We define Θ_n to be the naive representation of the matrix algebra on the target of the map σ_n in Proposition B.8. Define $\Sigma_n := \omega_n^{-1}\sigma_n^{-1}\Theta_n$. Then, Σ_n is an irreducible left module of $C_0^{n,0}$. Call Σ_n the naive representation of the even Clifford algebra $C_0^{n,0}$.

B.3. **A basis of the naive representation Σ_n.** In view of Lemma B.3 and Proposition B.8, we see that the rank of Σ_n as a k-vector space is $2^{\delta(n)}$, because $\delta(8m + t + 1) = 4m + s$ where $s = \#\{0 < \ell < t + 1 : \ell \equiv 0, 1, 2 \text{ or } 4 \mod 8\}$. Let $\{e_1, \ldots, e_n\}$ be an orthogonal basis of $(\mathbb{R}^n, q_{0,n})$ and let

$$\{e^\Delta = e_1^{a_1} \cdots e_2^{a_n} : \Delta = (a_1, \ldots, a_n) \in \mathbb{F}_2^n \text{ and } |\Delta| = \sum_i a_i \text{ is even}\}$$

be a basis of $C_0^{0,n}$ with $q(e_i) = \bar{e}_i e_i = 1$.

Lemma B.10. *Let $\varsigma(n) = \#\{0 < \ell < n : \ell \equiv 0, 4, 5, 6 \mod 8\}$. There exists a basis $\Psi_n = \{\zeta_i : 1 \leq i \leq 2^{\delta(n)}\}$ of $\Sigma_n \subset C_0^{0,n}$ such that $\zeta_i = \sum_{1 \leq j \leq 2^{\varsigma(n)}} (-1)^{t_{ij}} e^{\Delta_j^i}$ for some $\Delta_j^i \in \mathbb{F}_2^n$, and $e^{\Delta_j^i} \neq e^{\Delta_{j'}^{i'}}$ if $i \neq i'$ or $j \neq j'$.*

Proof. If $n = 1$, $\Sigma_1 = C_0^{0,1} \cong \mathbb{R}$. Set $\Psi_1 = \{1\}$.

If $n = 2$, $\Sigma_2 = C_0^{0,2} \cong \mathbb{C}$. Set $\Psi_2 = \{1, e_1e_2\}$

If $n = 3$, $\Sigma_3 = C_0^{0,3} \cong \mathbb{H}$. Set $\Psi_3 = \{1, e_3e_1, e_3e_2, e_1e_2\}$.

If $n = 4$, $C_0^{0,4} \cong \mathbb{H} \times \mathbb{H}$. Via this isomorphism, we define Ψ_4 to be the basis of Σ_4 corresponding to $\{(1, 0), (i, 0), (j, 0), (k, 0)\}$ in $\mathbb{H} \times \mathbb{H}$.

If $n = 5$, $C_0^{0,5} \cong M_2(\mathbb{R}) \otimes \mathbb{H} \cong M_2(\mathbb{H})$. Define Ψ_5 to be the basis of Σ_5 corresponding to $\{h_{1,s}(1) \otimes t : 1 \leq s \leq 2, t \in \{1, i, j, k\}\}$ in $M_2(\mathbb{R}) \otimes \mathbb{H}$.

If $n = 6$, $C_0^{0,6} \cong M_2(\mathbb{C}) \otimes M_2(\mathbb{R}) \cong M_4(\mathbb{C})$. Define Ψ_6 to be the basis of Σ_6 corresponding to $\{h_{1,s}(r) \otimes h_{1,t}(1) : 1 \leq s, t \leq 2, r \in \{1, i\}\}$ in $M_2(\mathbb{C}) \otimes M_2(\mathbb{R})$.

If $n = 7$, $C_0^{0,7} \cong M_8(\mathbb{R})$. Define Ψ_7 to be the basis of Σ_7 corresponding to $\{h_{1,s}(1) \otimes h_{1,t}(1) : 1 \leq s \leq 4, 1 \leq t \leq 2\}$ in $M_4(\mathbb{R}) \otimes M_2(\mathbb{R})$.

If $n = 8$, $C_0^{0,8} \cong M_8(\mathbb{R}) \times M_8(\mathbb{R})$. Define Ψ_8 to be the basis of Σ_8 corresponding to $\{(h_{1,s}(1), 0) : 1 \leq s \leq 8\}$ in $M_8(\mathbb{R}) \times M_8(\mathbb{R})$.

If $n = 9$, $C_0^{0,9} \cong M_{16}(\mathbb{R})$. Define Ψ_9 to be the basis of Σ_9 corresponding to $\{(h_{1,s}(1), 0) : 1 \leq s \leq 16\}$ in $M_{16}(\mathbb{R})$.

Note that every element in the basis of $\mathbb{C} \otimes \mathbb{H}$ occurs only once on the targets of the maps (33). This phenomenon also happens to $M_2(\mathbb{R})$ and $\mathbb{H} \otimes \mathbb{H}$ comparing to the maps (34) and (35). It follows that the lemma is true for $n \leq 9$. For $n \geq 10$, we apply Corollary B.7 inductively. □

Corollary B.11. *Let $\varsigma(n) = \#\{0 < \ell < n : \ell \equiv 0, 4, 5, 6 \mod 8\}$. There exists a basis $\Psi_n = \{\zeta_i : 1 \leq i \leq 2^{\delta(n)}\}$ of $\Sigma_n \subset C_0^{0,n}$ such that*

$$\mathrm{tr}(\bar{\zeta}_i \zeta_{i'}) = \begin{cases} \frac{1}{2^{n-\varsigma(n)}} & \text{if } i = i' \\ 0 & \text{if } i \neq i' \end{cases}$$

Remark B.12. The above results and proofs over $\mathbb{R}$ can be applied to the cases over a field k in which -1 is not a sum of two squares by replacing $\mathbb{H}$ (resp. $\mathbb{C}$) with $Y = (\frac{-1,-1}{k})$ (resp. $X = k\langle\sqrt{-1}\rangle$).

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