

# AFFINE SURFACES AND THEIR VEECH GROUPS.

EDUARD DURYEV, CHARLES FOUGERON, AND SELIM GHAZOUANI

*To the memory of William Veech*

ABSTRACT. We introduce a class of objects which we call '*affine surfaces*'. These provide families of foliations on surfaces whose dynamics we are interested in. We present and analyze a couple of examples, and we define concepts related to these in order to motivate several questions and open problems. In particular we generalise the notion of Veech group to affine surfaces, and we prove a structure result about these Veech groups.

## NOTATIONS

- $\Sigma_g$  or only  $\Sigma$  is a compact surface of genus  $g \geq 2$ ;
- $\mathrm{Gl}_2^+(\mathbb{R})$  is the group of 2 by 2 matrices whose determinant is positive;
- $\mathrm{SL}_2(\mathbb{R})$  is the group of 2 by 2 matrices whose determinant is equal to 1;
- $\mathrm{Aff}(\mathbb{C})$  is the one dimensional affine complex group,  $\mathbb{C}^* \ltimes \mathbb{C}$ ;
- $\mathrm{Aff}_{\mathbb{R}_+^*}(\mathbb{C})$  is the subgroup of  $\mathrm{Aff}(\mathbb{C})$  of elements whose linear parts are real positive;
- $\mathbb{H}$  is the upper half plane of  $\mathbb{C}$ ;

## 1. INTRODUCTION.

A *translation* structure on a surface is a geometric structure modelled on the complex plane  $\mathbb{C}$  with structural group the set of translations. A large part of the interest that these structures have drawn lies in the directional foliations inherited from the standard directional foliations of  $\mathbb{C}$  (the latter being invariant under the action of translations). Examples of such structures are polygons whose sides are glued along parallel sides *of same length*, see Figure 1 below.

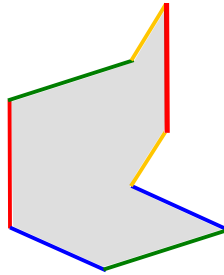


FIGURE 1. A translation surface of genus 2.

The directional foliation, say in the horizontal direction, can be drawn very explicitly: a leaf is a horizontal line until it meets a side, and continues as the horizontal line starting from the point on the other side to which it is identified. These foliations have been more than extensively studied over the past forty years. They are very closely linked to one dimensional dynamical systems called *interval exchange transformations*, and most of the basic features of these objects (as well as less basic ones !) have long been well understood, see [Zor06] for a broad and clear survey on the subject.

The starting point of this article is the following remark: to define the horizontal foliation discussed in the example above, there is no need to ask for the sides glued together to have same length, but only their being parallel in which case we can glue along affine identifications. In terms of geometric structures, it means that we extend the structural group to all the transformation of the form  $z \mapsto az + b$  with  $a \in \mathbb{R}_+^*$  and  $b \in \mathbb{C}$ . Formally, these corresponds to (branched) complex affine structures whose holonomy group lies in the subgroup of  $\text{Aff}(\mathbb{C})$  whose linear parts are real positive. A simple example of such an '*affine surface*' is given by the gluing below:

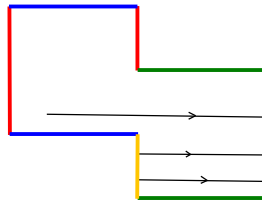


FIGURE 2. An '*affine surface*' of genus 2 and a leaf of its horizontal foliation.

A notable feature of these affine surfaces is that they present dynamical behaviours of *hyperbolic* type: the directional foliations sometimes have a closed leaf which 'attracts' all the nearby leaves. This is the case of the closed leaf drawn in black on Figure 3 below. This situation is in sharp contrast with the case of translation surfaces and promise a very different picture in the affine case.

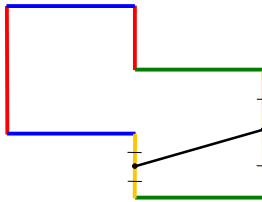


FIGURE 3. A '*hyperbolic*' closed leaf.

It is somewhat surprising to find no systematic study of these '*affine surfaces*' in the literature. However, related objects and concepts have kept popping up every

now and then, both of geometric and dynamical nature. To our knowledge, their earliest appearance is in the work of Prym on holomorphic 1-forms with values in a flat bundle, see [Pry69]. These provide an algebraico-geometric interpretation of these affine surfaces, see also Mandelbaum ([Man72, Man73]) and Gunning ([Gun81]). We would also like to mention Veech's remarkable papers [Vee93] and [Vee97] where he investigates moduli spaces of complex affine surfaces with singularities as well as Delaunay partitions for such surfaces. On the dynamical side, the first reference to related questions can be found in Levitt's paper on foliations on surfaces, [Lev82] where he builds an affine interval exchange (AIET) with a wandering interval (these AIETs must be thought of as the one-dimensional reduction of the foliations we are going to consider). It is followed by a series of works initiated by Camelier and Gutierrez [CG97] and pursued by Bressaud, Hubert and Maass [BHM10], and Marmi, Moussa, and Yoccoz [MMY10]. They generalize a well known construction of Denjoy to build out of a standard IET an AIET having a wandering interval, behaviour which is (conjecturally) highly non-generic. Very striking is that the question of the behavior of a typical AIET has been very little investigated. In this direction, we mention the nice article of Liousse [Lio95] where the author deals with the topological generic behaviour of transversely affine foliations on surfaces.

**Contents of the paper and results.** After introducing formal definitions as well as a couple of interesting examples of '*affine surfaces*', we prove a structure result about Veech groups of affine surfaces.

The Veech group of an affine surface  $\Sigma$  is the straightforward generalization of its translation analogue: it is the subgroup of  $SL_2(\mathbb{R})$  made of linear parts of locally affine transformation of  $\Sigma$ . It is a well-known fact that the Veech group of a translation surface is always discrete. This fails to be true in the more general case of affine surface, although the examples of surfaces whose Veech group is not discrete are fairly distinguishable. We completely describe the class of surfaces whose Veech group is not discrete. Roughly, those are the surface obtained starting from a ribbon graph and gluing to its edges a finite number of '*affine cylinders*'. We call such surfaces **Hopf surfaces**, because they must be thought of as higher genus analogues of Hopf tori, that are quotients  $\mathbb{C}^*/(z \sim \lambda z)$  with  $\lambda$  a positive constant different from 1. Precisely we prove

**Theorem 1.** Let  $\Sigma$  be an affine surface of genus  $\geq 2$ . There are two possible cases :

- (1)  $\mathbf{V}(\Sigma)$  is the subgroup of upper triangular elements of  $SL_2(\mathbb{R})$  and  $\Sigma$  is a Hopf surface.
- (2)  $\mathbf{V}(\Sigma)$  is discrete.

We also prove the following theorem on the existence of closed geodesics in genus 2 :

**Theorem 2.** Any affine surface of genus 2 has a closed regular geodesic.

The proof is elementary and relies on combinatoric arguments. Nonetheless it is a good motivation for a list of open problems we address in Section 6. We end the article with a short appendix reviewing Veech's results on the geometry of affine surfaces contained in the article [Vee97] and the unpublished material [Vee08] that W. Veech kindly shared.

**About Bill Veech's contribution.** Bill Veech's sudden passing away encouraged us to account for his important contribution to the genesis of the present article. About twenty years ago, he published a very nice paper called *Delaunay partitions* in the journal *Topology* (see [Vee97]), in which he investigated the geometry of complex affine surfaces (of which our 'affine surfaces' are particular cases). A remarkable result contained in it is that affine surfaces all have geodesic triangulations in the same way flat surfaces have. We used it extensively when we first started working on affine surfaces, overlooking the details of [Vee97]. But at some point, we discovered a family of affine surfaces that seemed to be a counterexample to Veech's result and which provides an obstruction for affine surfaces to have a geodesic triangulation. We then decided to contact Bill Veech, who replied to us almost instantly with the most certain kindness. He told us that he realized the existence of the mistake long ago, but since the journal *Topology* no longer existed and that the paper did not draw a lot of attention, he did not bother to write an erratum. However, he shared with us courses notes from 2008 in which he '*fixed the mistake*'. It was a pleasure for us to discover that in these long notes (more than 100p) he completely characterizes the obstruction for the slightly flawed theorem of *Delaunay partitions* to be valid, overcoming serious technical difficulties. We extracted from the notes the Proposition 5 which is somewhat the technical cornerstone of this paper.

A few weeks before his passing away, Bill Veech allowed us to reproduce some of the content of his notes in an appendix to this article. It is a pity he did not live to give his opinion and modify accordingly to his wishes this part of the paper.

**Acknowledgements.** We are very grateful to Vincent Delecroix, Bertrand Deroin, Pascal Hubert, Erwan Lanneau, Leonid Monid and William Veech for interesting discussions. The third author is grateful to Luc Pirio for introducing him to the paper [Vee97]. The third author acknowledges partial support of ANR Lambda (ANR-13-BS01-0002).

## 2. AFFINE SURFACES.

We give in this section formal definitions of *affine surfaces* and several concepts linked to both their geometry and dynamics.

**2.1. Basics.** An affine structure on a complex manifold  $M$  of dimension  $n$  is an atlas of chart  $(U_i, \varphi)$  with values in  $\mathbb{C}^n$  such that the transition maps belong to  $\text{Aff}_n(\mathbb{C}) = \text{Gl}_n(\mathbb{C}) \ltimes \mathbb{C}^n$ . It is well known that the only compact surfaces (thought of as 2-dimensional real manifolds) carrying an affine structure are tori. We make the definition of an affine structure less rigid, by allowing a finite number of

points where the structure is singular, in order to include the interesting examples mentioned in the introduction:

- Definition 1.** (1) An **affine surface**  $\mathcal{A}$  on  $\Sigma$  is a finite set  $\mathcal{S} = \{s_1, \dots, s_n\} \subset \Sigma$  together with an affine structure on  $\Sigma \setminus \mathcal{S}$  such that the latter extends to a euclidean cone structure of angle a multiple of  $2\pi$  at the  $s_i$ 's.
- (2) A **real affine surface** is an affine surface whose structural group has been restricted to  $\text{Aff}_{\mathbb{R}_+^*}(\mathbb{C})$ .

The type of singularities we allow results of what seems to be an arbitrary choice. We could have as well allowed singular points to look like affine cones, or the angles to be arbitrary. We will justify our choice very soon. A first important remark is that an affine surface satisfy a *discrete Gauss-Bonnet* equality. If  $\mathcal{S} = \{s_1, \dots, s_n\}$  is the set of singular points of an affine structure on  $\Sigma$ ; and  $k_i \geq 2$  is the integer such that the cone angle at  $s_i$  is  $2k_i\pi$ , then

$$\sum_{i=1}^n 1 - k_i = \chi(\Sigma) = 2 - 2g$$

From now on and until the end of the paper, we will consider only *real affine surfaces*, and therefore we will refer to those only as *affine surface*.

A general principle with geometric structures is that any object that is defined on the model and is invariant under the transformation group is well defined on the manifolds carrying such a structure. In our case the model is  $\mathbb{C}$  with structure group  $\text{Aff}_{\mathbb{R}_+^*}(\mathbb{C})$ . Among others, angles and (straight) lines are well defined on affine surfaces. More striking is the fact that it makes sense to say that the orientation of a line is well defined, and for each angle  $\theta \in S^1$  we can define a *foliation* oriented by  $\theta$  that we denote by  $\mathcal{F}_\theta$ , whose leaves are exactly the lines oriented by  $\theta$ .

Finally remark that although the speed of a path is only defined up to a fixed constant, it makes sense to say that a path has constant speed (speed which is not itself well defined), as well as to say that a path has finite or infinite length.

Formally:

- a **geodesic** is an affine immersion of a segment  $]a, b[$  ( $a$  or  $b$  can be  $\pm\infty$ );
- a **saddle connection** is a geodesic joining two singular points;
- a **leaf** of a directional foliation is a maximal geodesic in the direction of the foliation;
- a **closed geodesic** (or **closed leaf** if the direction of the foliation is unambiguous) is an affine embedding of  $\mathbb{R}/\mathbb{Z}$ ;
- the first return on a little segment orthogonal to such a closed geodesic is a map of the form  $x \mapsto \lambda x$  with  $\lambda \in \mathbb{R}_+^*$ . We say it is **flat** if  $\lambda = 1$  and that it is **hyperbolic** otherwise.

**2.2. Cylinders.** These definitions being set, we introduce a first fundamental example, the *Hopf torus*. Consider a real number  $\lambda \neq 1$  and identify every two

points on  $\mathbb{C}^*$  which differ by scalar multiplication by  $\lambda$ . The quotient surface  $\mathbb{C}^*/(z \sim \lambda z)$  through this identification is called a *Hopf torus* and we call  $\lambda$  its *affine factor*.

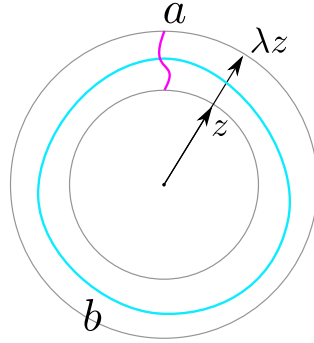


FIGURE 4. Hopf's torus

These provide a 1-parameter family of affine structures on the torus. These surfaces have a very specific kind of dynamics. Foliations in all directions have two closed leaves (the one corresponding to the ray from zero in this direction) one of which is attracting and the other repulsive.

Based on this torus, we can construct higher genus examples by gluing two of these tori along a slit in the same direction (see Figure 4). Take an embedded segment along the affine foliation in one direction on one torus, and another one in the same direction on the second torus. We cut the two surfaces along these segments and identify the upper part of one with the lower part of the other with the corresponding affine map.

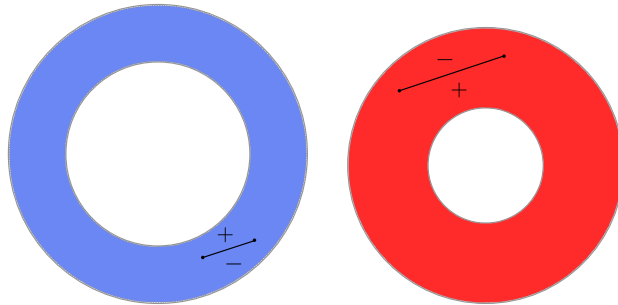


FIGURE 5. The franco-russian slit construction

Another construction based on the Hopf torus is given by considering a finite covers. Denote by  $a$  the closed curve of the Hopf torus in direction of the dilation and  $b$  the closed curve turning around zero once in the complex plane as in Figure 4. Take the  $k$  index subgroup of  $\pi_1(T^2)$  generated by  $a$  and  $b^k$  and consider the associated cover with the induced affine structure. It is also a torus which *makes*

$k$  turns around zero. We call it a  $k$ -Hopf torus. Similarly, a  $\infty$ -Hopf cylinder will be the cover associated to the subgroup generated by  $a$ .

**Remark.** We can also construct this structure as a slit construction along horizontal closed leaves of  $k$  different Hopf tori.

These structures have the remarkable property to be a disjoint union of closed geodesics in different directions. Each of these geodesics is an attractive leaf of the foliation in their direction. This is the property we want to keep track of, that is why we want to consider angular sectors of these tori embedded in an affine surfaces. This is the motivation for the following definition:

**Definition 2.** Consider  $\Sigma$  an affine surface. Let  $C_{\theta_2, \theta_1}$  be an angular domain of a  $\infty$ -Hopf cylinder between angles  $\theta_1 \in \mathbb{R}_+^*$  and  $\theta_1 < \theta_2$  such that  $\theta_2 - \theta_1 = \theta$ . A cylinder of angle  $\theta$  is the image of a maximal affine embedding of some  $C_{\theta_2, \theta_1}$  in  $\Sigma$ .

We call  $\lambda$  the *affine factor* of the cylinder and  $\theta$  its *angle*.

Remark the isomorphism class of an affine cylinder is completely determined by the two numbers  $\theta$  and  $\lambda$ .

**Proposition 1.** Let  $\Sigma$  be an affine surface and not a  $k$ -Hopf torus. Then the boundary of a maximal cylinder embedded in  $\Sigma$  is a union of saddle connections.

*Proof.* Suppose we embedded  $C_{0, \theta}$  in the surface  $\Sigma$ . If  $\theta = \infty$  we would have a half-infinite cylinder in the surface. But this half-cylinder would have an accumulation point at  $\infty$  in  $\Sigma$  which contradicts its being embedded.

When  $\theta < \infty$ , there are two reasons why  $C_{0, \theta'}$  cannot be embedded for  $\theta' > \theta$

- (1) The embedding  $C_{0, \theta} \rightarrow \Sigma$  extends continuously to the boundary of the cylinder in direction  $\theta$ , if the surface is not a  $k$ -Hopf torus the image contains a singular point. The image of the boundary is closed, and it is an union of saddle connections.
- (2) The embedding does not extend to the boundary, then there is a geodesic  $\gamma : [0, 1) \rightarrow C_{0, \theta}$  starting close to the boundary and ending orthogonally in the  $\theta$  boundary of  $C_{0, \theta}$  such that  $\gamma$  has no limit in  $\Sigma$  when approaching 1. Consider an open disk in  $C_{0, \theta}$  tangent to the boundary at the point to which  $\gamma$  is ending and centered on  $\gamma$  trajectory. Then Proposition 5 implies that  $\gamma$  starting from the center of the circle is a closed hyperbolic geodesic in  $\Sigma$ . This cannot happen since  $\gamma$  is embedded in  $\Sigma$

□

Again a cylinder will be the union of closed leaves. The dynamics of a geodesic entering such a cylinder is clear. If the cylinder is of angle less than  $\pi$  and the direction of the flow is not between  $\theta_1$  and  $\theta_2$  modulo  $2\pi$ , then it will leave the cylinder in finite time. Otherwise it will be attracted to the closest closed leaf corresponding to its direction, and be trapped in the cylinder.

As to enter a cylinder we have to cross its border, we see that for cylinder of angle larger than  $\pi$  every geodesic entering the cylinder is also trapped. These 'trap' cylinders can be ignored when studying dynamics. We can study instead the surface with boundaries where we remove all these cylinders. We will see in the following section that these cylinders are also responsible for degenerate behaviour when trying to triangulate the surface.

**Remark.** *A degenerate case of affine cylinder are flat cylinders. It is an embedding of the affine surface  $C_a = \{z \in \mathbb{C} \mid 0 < \Im(a) < a\} / (z \sim z + 1)$ . In this case, the length  $a$  of the domain of the strip we quotient by  $z \mapsto z + 1$  will be called the modulus of the cylinder.*

**2.3. Triangulations.** An efficient way to build affine surfaces is to glue the parallel sides of a (pseudo-)polygon. A surface obtained this way enjoys the property to have a *geodesic triangulation*. It is a triangulation whose edges are geodesic segments and whose set of vertices is exactly the set of singular points. It is natural to wonder if any affine surface has such triangulation from which we could easily deduce a polygonal presentation. Remark that the question only makes sense for surfaces of genus  $g \geq 2$ , for in genus 1 there are no singular points. Unfortunately, a simple example shows that it is not to be expected in general. The double Hopf torus constructed above cannot have a geodesic triangulation: any geodesic issued from the singular point accumulates on a closed regular geodesic, except for those coming from the slit. This obstruction can be extended to any affine surface containing an affine cylinder of angle  $\geq \pi$ : any geodesic entering such a cylinder never exits it which is incompatible with the fact that a triangulated surface deprived of its 1-skeleton is a union of triangles. A remarkable theorem of Veech proves that this obstruction is the only one:

**Theorem** (Veech, [Vee97, Vee08]). *Let  $\mathcal{A}$  be an affine structure which does not contain any affine cylinder of angle larger or equal to  $\pi$ . Then  $\mathcal{A}$  admits a geodesic triangulation.*

The exact theorem generalizes a classical construction known as *Delaunay partitions* to a more general class of affine surfaces. We review this construction and more of the material contained in [Vee97, Vee08] in Appendix A.

### 3. EXAMPLES.

**3.1. The two chambers surface.** By gluing the sides of same color of Figure 6 below, we get a genus 2 affine surface with a unique singular point of angle  $6\pi$ . For completely random reasons, we call it the two-chambers surface.

We believe that this example is particularly interesting because it is a first non trivial example for which we can describe the directional foliation in every direction.

**Proposition 2.** *Let  $\mathcal{F}_\theta$  be the directional foliation oriented by  $\theta$  on the two-chamber example.*



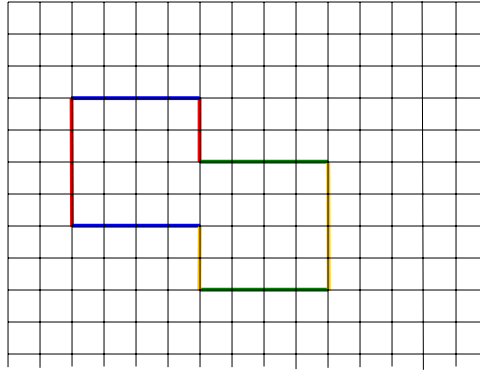


FIGURE 6. The two-chamber surface.

- (1) If  $\theta = \pm\frac{\pi}{2}$ , then the foliation is completely periodic; the surface decomposes into two euclidean cylinders.
- (2) If  $\theta = \arctan(n)$  or  $\arctan(n + \frac{1}{2})$  for  $n \in \mathbb{Z}$  and  $\theta \neq \pm\frac{\pi}{2}$ , the foliation accumulates on a closed saddle connection.
- (3) For any other  $\theta$ , the foliation accumulates on a hyperbolic closed leave.

*Proof.* Consider the case when  $\theta = \arctan(\frac{1}{4})$ . Here the segment linking the middle of the yellow sides projects to a closed leave of the directional foliation of angle  $\theta = \arctan(\frac{1}{4})$ . Take a little segment transverse to this leaf, the first return map is a dilation of factor  $\frac{1}{2}$  (resp 2). This closed leaf is therefore attractive in the sense that every leaf passing close by winds around and accumulate on it.

The other cases are similar, the reader can convince himself by looking for the two closed leaves in the given direction and remark that one will be attractive and the other repulsive.  $\square$

**3.2. Affine interval exchange transformations.** We mention in this subsection a construction of Camelier and Gutierrez ([CG97]), improved by Bressaud, Hubert and Maas ([BHM10]), and generalised by Marmi, Moussa and Yoccoz ([MMY10]).

An *affine interval exchange* is a piecewise affine bijective map from  $[0, 1]$  in itself. It can be thought of as a generalization of either standard interval exchanges or of piecewise affine homeomorphisms of the circle. To such an AIET (Affine Interval Exchange Transformation), we can associate an affine surface which is its *suspension*. It consists in taking a rectangle, identifying two vertical parallel sides in the standard way, and identifying the two horizontal according to the AIET, Figure 7b.

The dynamics of the vertical foliation of such an affine surface is exactly the same as the affine interval exchange we started from, for the orbits of the latter are in correspondence with the leaves of the foliation, singular leaves corresponding to discontinuity points of the AIET.

The construction mentioned above brings to light a surprising behaviour for certain affine interval exchanges.

**Theorem** (Marmi-Moussa-Yoccoz, [MMY10]). *For all combinatorics of genus at least 2, there exists a uniquely ergodic affine interval exchange whose invariant measure is supported by a Cantor set in  $[0, 1]$ .*

The implication of the theorem for the affine surfaces associated is that there are some leaves of the vertical foliation whose closure in the surface is union of leaves which intersects every transverse curve along a Cantor set. This in sharp contrast with both standard interval exchanges and piecewise affine homeomorphisms of the circle.

The construction is quite involved and we will not give any detail. Nonetheless it is worth noticing that in the example we have presented before, we have seen no trace of such a behaviour: the two-chambers surface has a very simple dynamics in every direction.

**3.3. The disco surface.** We build a first surface whose dynamics is *a priori* non-trivial. Choose  $a, b$  two positive real numbers, we consider the AIET associated to the permutation  $(1, 2)(3, 4)$ , and with top and bottom lengths  $a, b, b, a$ . Now take the suspension of this AIET with a rectangle of height 1, it defines an affine surface which two singularities of angle  $4\pi$  and genus 2, see Figure 7a. We call it the *disco surface* and denote it by  $D_{a,b}$ . Notice that the surface contains at hand several affine cylinders. We represent some of them on the following Figure 7b. These cylinders can overlap, and some zones can a priori be without cylinder coverage.

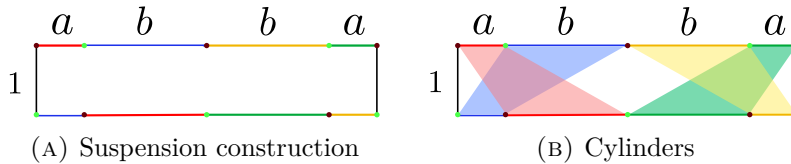


FIGURE 7. The disco surface with a horizontal cylinder

We give an alternative representation of the surface which makes a vertical flat cylinder decomposition appear. To do so we cut out the left part of the surface of width  $a$  along a vertical line. We now rescale it by a factor  $\frac{b}{a}$  and reglue it on the top  $b$  interval. Reproduce the same surgery with the right part of the surface and the new surface is the one drawn on Figure 8

#### 4. VEECH GROUPS.

Given a matrix  $M \in \text{GL}_2^+(\mathbb{R})$  and an affine structure  $\mathcal{A}$  on  $\Sigma$ , there is a way to create a new affine structure by replacing the atlas  $(U_i, \varphi_i)_{i \in I}$  by  $(U_i, M \cdot \varphi_i)_{i \in I}$ . This new affine structure is denoted by  $M \cdot \mathcal{A}$ . A way to put our hands on this operation is to describe it when  $\mathcal{A}$  is given by gluing sides of a polygon  $p$ . If one

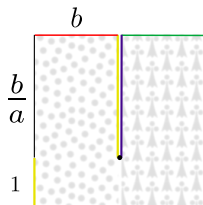


FIGURE 8. An alternative representation of the disco surface

sees  $P$  embedded in the complex plane  $\mathbb{C} \simeq \mathbb{R}^2$ ,  $M \cdot \mathcal{A}$  is the structure one gets after gluing sides of the polygon  $M \cdot P$  along the same pattern.

We have defined this way an action of  $\text{GL}_2^+(\mathbb{R})$  on the set of affine surfaces which factors through  $\text{SL}_2(\mathbb{R})$ , since the action of dilation is obviously trivial. If  $\mathcal{A}$  is an affine structure on  $\Sigma$ , we introduce its Veech group  $\mathbf{V}(\mathcal{A})$  which is the stabilizer in  $\text{SL}_2(\mathbb{R})$  of  $\mathcal{A}$ , namely

$$\mathbf{V}(\mathcal{A}) = \{M \in \text{SL}_2(\mathbb{R}) \mid M \cdot \mathcal{A} = \mathcal{A}\}$$

The Veech group is the set of real affine symmetries of the considered affine surface. It is the direct generalisation of the Veech group in the case of translation surfaces (see [HS06] for a nice introduction to the subject). For example, if  $\mathcal{T}$  is a Hopf torus,  $\mathbf{V}(\mathcal{T}) = \text{SL}_2(\mathbb{R})$ . It is a consequence of the fact that  $\mathcal{T} = \mathbb{C}^*/(z \sim \lambda z)$  for a certain  $\lambda > 1$ , and that the action of  $\text{SL}_2(\mathbb{R})$  commutes to  $z \mapsto \lambda z$ . This fact is in sharp contrast with the case of translation surfaces where the Veech group is known to always be discrete.

**4.1. The Veech group of the two-chambersurface.** We carry on the particular analysis of the examples introduced in the last section, beginning with the two-chambersurface. Our exhaustive understanding of the dynamics of every directional foliation will enable us to describe its Veech group completely.

**Proposition 3.** *The Veech group of the two-chambersurface is the group generated by the two following matrices*

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

*Proof.* There are only two directions on the two-chambersurface which are completely periodic which are  $\frac{\pi}{2}$  and  $-\frac{\pi}{2}$ . Any element of the Veech group must preserve this set of directions, this is why it must lie in the set of lower triangular matrices. The rotation of angle  $\pi$  belongs to the Veech group since both the polygon that defines the two-chambersurface and the identification are invariant under this rotation. Up to multiplying by the latter, we can always assume that an element of the Veech group fixes the direction  $\frac{\pi}{2}$ .

We prove now that if a matrix of the form  $\begin{pmatrix} \lambda & 0 \\ * & -\lambda^{-1} \end{pmatrix}$  belongs to the Veech group with  $\lambda > 0$ , then  $\lambda = 1$ . The two-chambersurface contains two flat cylinders of modulus 1. The image of these two cylinders by the action of such a matrix are two cylinders of modulus 1. So for  $\begin{pmatrix} \lambda & 0 \\ * & -\lambda^{-1} \end{pmatrix}$  to belong to the Veech of the two-chambersurface,  $\lambda$  must equal 1.

Now remark that  $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$  is in the Veech group. A simple cut and paste operation proves this fact, see Figure 9 below.

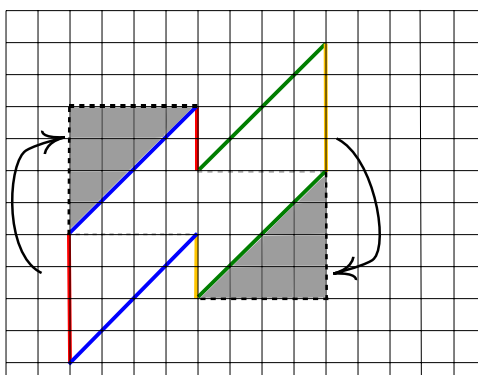


FIGURE 9

To complete the proof of the theorem, remark that the set of vectors of saddle connection must be preserved. This implies that  $t = 1$  is the smallest positive number such that  $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$  belongs to the Veech group of  $\Sigma$ .  $\square$

**4.2. The Veech group of the disco surface.** Let us describe some elements of the Veech group of the disco surface. First remark that when we act by the matrix

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

on a vertical cylinder of height 1 and width  $t$ , we can rearrange the surface and end up back to the same cylinder. It is exactly a Dehn twist on its core curve. This works also for any cylinder of modulus  $t$ . Hence if we have a surface which we can decompose in cylinders of same modulus  $t$  in the horizontal direction, the matrix above is in its Veech group.

As we remarked when introducing the disco surface  $D_{a,b}$ , they decompose into one cylinder of modulus  $2(a+b)$  in horizontal direction (Figure 7b) and into two cylinders of modulus  $\frac{1+b/a}{b} = \frac{1}{a} + \frac{1}{b}$  in vertical direction (Figure 8).

As a consequence,

$$\begin{pmatrix} 1 & 2(a+b) \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \frac{1}{a} + \frac{1}{b} & 1 \end{pmatrix} \in \mathbf{V}(D_{a,b})$$

Remark that these two matrices never generate a lattice in  $SL_2(\mathbb{R})$  since it would imply that  $2(a + b)(\frac{1}{a} + \frac{1}{b}) \leq 4$  which never happens.

**4.3. Hopf surfaces.** We present in this subsection a general construction of affine surfaces whose Veech group is conjugate to

$$\left\{ \begin{pmatrix} \lambda & t \\ 0 & \lambda^{-1} \end{pmatrix} \mid t \in \mathbb{R} \text{ and } \lambda \in \mathbb{R}_+^* \right\}$$

and we prove that these are the only surfaces whose Veech group is not discrete. In particular this construction includes the Hopf torus and their derivatives introduced in Section 2.2.

**Definition 3.** A *ribbon graph* is a finite graph with a cyclic ordering of its semi-edges at its vertices.

We can think of a ribbon graph as an embedding of a given graph in a surface, the manifold structure giving the ordering at the vertices. The structure of a tubular neighbourhood of the embedding of the graph completely determines the ribbon graph.

Given a ribbon graph, we can make the following construction: along the boundary components of the infinitesimal thickening of the ribbon graph, we can glue cylinders of angle  $k\pi$  respecting the orientation of the foliation to get an affine surface. We have to be careful to the factors of the cylinders we glue if we want the singular points to be Euclidean; we give an example from which it will be easy to deduce the general pattern.

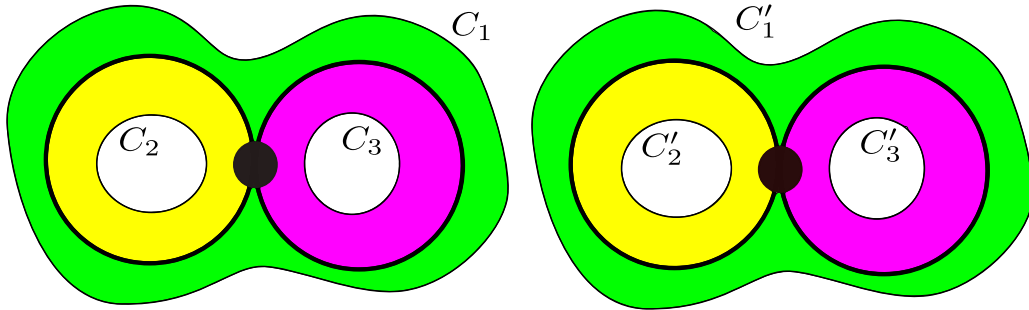


FIGURE 10. A ribbon graph with two vertices.

that we turn into a genus two surface by gluing three cylinders  $D_i$  to the boundary components, each joining  $C_i$  to  $C'_i$  for  $i = 1, 2, 3$ .

Let  $\lambda_i$  be the affine factor of  $D_i$ . For the singular points to be euclidean, it is necessary that the product of the factor of the cylinders adjacent at a singular point is trivial. In our case we get for an appropriate choice of orientation:

$$\lambda_1 = \lambda_2 \lambda_3$$

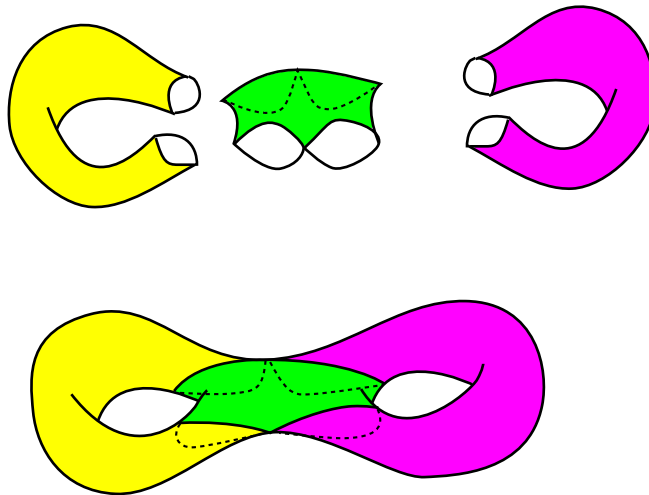


FIGURE 11. A cylinder decomposition of the surface of genus 2.

This constraint straightforwardly generalizes to the general case and is the only obstruction to complete the construction. We call an affine surface obtained by this construction a *Hopf surface*, because it generalizes the variations on the Hopf torus explained in Section 2.2.

We will see in the next section that this example corresponds exactly to the case when the Veech group is not discrete and is not  $\mathrm{SL}_2\mathbb{R}$

**4.4. Veech group dichotomy.** First we deal with the case of genus 1 with the following lemma,

**Lemma 1.** *An affine torus is the exponential of some flat torus  $\mathbb{C}/\alpha\mathbb{Z}\oplus(\beta + 2ik\pi)\mathbb{Z}$  where  $\alpha, \beta \in \mathbb{R}^+$  and  $k \in \mathbb{N}$ . Moreover its Veech group is  $\mathrm{SL}_2(\mathbb{R})$*

*Proof.* Consider an affine torus. Its developing map goes from  $\mathbb{C}$  to  $\mathbb{C}$  and its holonomy is commutative. Hence its holonomy is generated either by two translation or two affine maps with the same fixed point (which we assume to be zero). The former case is to be excluded since we are only considering strictly affine surface.

In the second case, we can choose a developing map  $f$  which avoids zero, and associate to it the 1-form  $d \log f = \frac{df}{f}$ . As the surface has no singularity, the derivative of  $f$  is never zero, and the 1-form is invariant with respect to the holonomy. Thus this *a priori* meromorphic form is defined on the torus, has no zeroes and by residue formula no poles, therefore it is holomorphic.

In conclusion, the logarithm form gives a flat structure on the torus that is isomorphic to  $\mathbb{C}/\alpha\mathbb{Z}\oplus\tau\mathbb{Z}$  where  $\alpha \in \mathbb{R}^+$  and  $\tau \in \mathbb{C}^*$ . The exponential of this flat torus is the initial affine structure, thus  $e^\tau \in \mathbb{R}_+^*$  and  $\Im(\tau) \equiv 0 \pmod{2\pi}$ .

Any matrix of  $\mathrm{SL}_2(\mathbb{R})$  commutes with the scalar multiplications, thus the Veech group of such a surface is the whole  $\mathrm{SL}_2(\mathbb{R})$ .  $\square$

This structure is like taking one  $\alpha$ -Hopf torus which we slit at one horizontal closed curve and glue  $k$  copies of it. When we glue back the  $k$ -th copy to the first one, we apply a  $\beta$  dilation.

Now that we have set aside the peculiar case of genus 1, we prove a classification theorem on affine surfaces of higher genus depending on the type of their Veech group.

**Theorem 1.** *Let  $\Sigma$  be an affine surface of genus  $\geq 2$ . There are two possible cases :*

- (1)  $\Sigma$  is a Hopf surface and  $\mathbf{V}(\Sigma)$  is the subgroup of upper triangular elements of  $\mathrm{SL}_2(\mathbb{R})$ ,

$$\left\{ \begin{pmatrix} \lambda & * \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{R}^* \right\}$$

- (2)  $\mathbf{V}(\Sigma)$  is discrete.

The end of the section is devoted to proving Theorem 1. To do so, we will distinguish between affine surfaces having saddle connections in at least two directions and those who do not. The former enjoy the property that their Veech group is automatically discrete thanks to a classical argument inspired by the case of translation surfaces (see Section 3.1 of [HS06]). The latter will turn out to be Hopf surfaces introduced in section 4.3.

**Lemma 2.** *Let  $\Sigma$  be an affine surface having two saddle connections in different directions. Then  $\mathbf{V}(\Sigma)$  is a discrete subgroup of  $\mathrm{SL}_2(\mathbb{R})$ .*

**Remark.** *As a direct corollary both the two-chambers surface and the disco surface have a discrete Veech group.*

*Proof.* Consider  $\mathbf{V}_0(\Sigma)$  the subgroup of  $\mathbf{V}(\Sigma)$  fixing point-wise the set of singular points of  $\Sigma$ . This subgroup has finite index in  $\mathbf{V}(\Sigma)$  and its being discrete implies discreteness for  $\mathbf{V}(\Sigma)$ .

Choose an arbitrary simply connected subset  $U \subset \Sigma$  containing all the singularities and an arbitrary developing map of the affine structure on  $U$ . Thanks to this developing map, we can associate to each oriented saddle connection a vector in  $\mathbb{R}^2$ . The set of such vectors enjoys two nice properties:

- it is discrete;
- it is invariant under the action of  $\mathbf{V}_0(\Sigma)$ .

We have made the hypothesis that a pair of those vectors  $(v_1, v_2)$  form a basis in  $\mathbb{R}^2$ . If  $a_n$  is a sequence of elements of  $\mathbf{V}_0(\Sigma)$  going to identity and  $A_n$  the matrices of their action in the normalization induced by  $U$  (notice that an element of  $\mathbf{V}(\Sigma)$  is an element of the quotient  $\mathrm{Gl}_2^+(\mathbb{R})/\mathbb{R}_+^*$  and therefore the matrix of its action depends on normalization both at the source and at the target).  $A_n \cdot v_1 \rightarrow v_1$  and  $A_n \cdot v_2 \rightarrow v_2$  but the discreteness of the set of saddle connection vectors implies that for  $n$  large enough  $A_n \cdot v_1 = v_1$  and  $A_n \cdot v_2 = v_2$ , and thus  $A_n = \mathrm{Id}$ . Which proves the discreteness of the Veech group.  $\square$

We will now characterize the affine surfaces having saddle connections in at most one direction. The two following lemmas will complete the classification.

**Lemma 3.** *If all the saddle connections of an affine surface  $\Sigma$  are in the same direction it is a Hopf surface.*

*Proof.* We make the assumption that  $\Sigma$  has at least a singular point (if not, Lemma 1 settles the question). We are going to prove that every singular point has at least a saddle connection in every angular sector of angle  $\pi$ . An affine surface whose all separatrices in one direction are saddle connection is easily seen to be a Hopf surface, see Subsection 4.3.

Consider the exponential map at a singular point  $p$  associated to a local affine normalization and let  $r > 0$  be smallest radius such that  $\Delta_r$  the (open) semi-disk of radius  $r$ , bounded below by two horizontal separatrices and containing a vertical separatrix, immerses in  $\Sigma$  by means of the exponential map. If  $r = \infty$ , we would have a maximal affine immersion of  $\mathcal{H}$ , whose boundary would project to a closed leaf containing  $p$  and there would be two saddle connections in the horizontal direction. Otherwise  $r < \infty$ . There can be two different reasons why  $\Delta_{r'}$  does not immerse for  $r' > r$ :

- (1) either the immersion  $\Delta_r \rightarrow \Sigma$  extends continuously to the boundary of  $\Delta_r$  and the image of this extension contains a singular point. In that case this singular point must be in the unique direction containing saddle connections (say the vertical one) and the vertical separatrix was actually a saddle connection;
- (2) or  $\Delta_r \rightarrow \Sigma$  does not extend to the boundary of  $\Delta_r$ .

We prove that the latter situation cannot occur. Otherwise there would be a geodesic  $\gamma$  issued at  $p$  affinely parametrized by  $[0, r)$  such that  $\gamma(t)$  does not have a limit in  $\Sigma$  when  $t$  tends to  $r$ . We make the confusion between  $\gamma$  as a subset of  $\Sigma$  and its pre-image in  $\Delta_r$ . Considering an open disk in  $\Delta_r$  which is tangent at the boundary to the point in  $\partial\Delta_r$  towards which  $\gamma$  is heading and such that its center belongs to  $\gamma$ . By means of the immersion of  $\Delta_r$  in  $\Sigma$ , it provides an affine immersion of  $\mathbb{D}$  satisfying the hypothesis of Proposition 5.  $\gamma$  would therefore project to a closed hyperbolic geodesic, which contradicts its being a separatrix.  $\square$

**4.5. The Veech group of an affine surface is probably not a lattice.** We start by showing the following lemma,

**Lemma 4.** *If  $\mathbf{V}(\Sigma)$  is discrete it cannot be cocompact.*

*Proof.* To show this we only need to find a continuous function on  $\mathbb{H}/\mathbf{V}(\Sigma)$  that is not bounded. For flat surface the systole does the trick, here we cannot define length of saddle connections but the ratio of the two lengths of saddle connections that intersect. Thus we can define the shortest and the second shortest simple saddle connections starting at a given singular point, and we denote by  $l$  and



$L$  their length in one arbitrary chart around the singularity. These two distinct saddle connections exist since we assume  $\mathbf{V}(\Sigma)$  discrete according to Theorem 1. Now the ratio  $L/l$  is independent from the previous choice of chart. And if we take the minimum value of this ratio on all the singularities, it will be a continuous function on the  $\mathbb{H}$  orbit of the surface invariant by the Veech group.

To finish the argument, apply the Teichmüller deformation of the surface with matrices of the form

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

where the horizontal direction will be the direction of the smallest saddle connection with the smallest ratio. This deformation will decrease the length  $l$  and increase  $L$  as  $t$  goes to infinity and thus make this function go to infinity.  $\square$

As we saw in Section 2.2, cylinders trap the linear flow in their corresponding angular sector. This behaviour restricts the potential directions for saddle connections around a singularity in the boundary of cylinders and prevent Veech groups from being lattices.

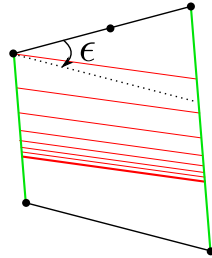


FIGURE 12. Angular section on which leaves are hyperbolic

Consider  $\Sigma$  an affine surface endowed with an affine cylinder, take any singular point at the border of this cylinder and flow a leaf heading inside the cylinder whose angular direction falls just in-between the two extreme angles of the cylinders. This leaf will be trapped inside the cylinder and accumulate to a close leaf (Figure 12). As a consequence none of these leaves will meet a singular point. There won't be any saddle connection starting from the chosen point in the angular sector define by the cylinder.

This implies the following proposition :

**Proposition 4.** *If  $\Sigma$  is an affine surface with an affine cylinder then  $\mathbf{V}(\Sigma)$  is not a lattice.*

*Proof.* Assume for a contradiction that  $\mathbf{V}(\Sigma)$  is a lattice. Take a finite index subgroup which stabilizes all the singularities of the surface. As an assumption on  $\Sigma$  there is an affine cylinder in the surface, Proposition 1 implies that there

is a singularity in the boundary of the cylinder, and a small angular section in which any separatrix from this singularity will accumulate to a closed leaf. As the subgroup is a lattice it contains a parabolic element and its limit set is the whole border of  $\mathbb{H}$ . Then we can conjugate this parabolic element to see that there is a dense set of direction in which there is a parabolic. In parabolic directions, the separatrices are all saddle connections. Indeed, if not there would be an accumulation point, and a neighborhood of this point would be crossed infinitely many times by the separatrix on which the parabolic acts as the identity. Hence the parabolic element would act as the identity on the whole neighborhood, which is a contradiction. This shall not be since we showed that there cannot be saddle connection in an open set of directions around the singularity.  $\square$

## 5. CYLINDERS ON GENUS 2 SURFACES.

The purpose of this section is to show the following result

**Theorem 2.** *Any affine surface of genus 2 has a cylinder.*

First, remark that Veech's theorem on Delaunay triangulation (see Veech's theorem in 2.3) tells us that if an affine surface does not have a triangulation, it must contain an affine cylinder (of angle at least  $\pi$ ). We can therefore forget about this case and assume that all the surfaces we are looking at have geodesic triangulations.

The theorem is also easy to prove when the surface has two singular points which both must be of angle  $4\pi$ . Consider a triangle of a Delaunay triangulation of the surface, at least two of its vertices are equal to the same singular point and the side corresponding to these two vertices is a simple closed curve. It must cut the angle of the associated singular point into two angular sectors of respective angle  $3\pi$  and  $\pi$ . Which implies that it bounds a cylinder on the side of the angle  $\pi$ .

For the remainder of the section,  $\Sigma$  is a surface of genus 2 together with a strictly affine structure whose unique singular point of angle  $6\pi$  is denoted by  $p$ . A geodesic triangulation of  $\Sigma$  must have exactly 9 edges and 6 triangles. In this particular case where the triangulation has a unique vertex, each edge defines a simple closed curve, geodesic away from  $p$  and cutting the latter into two angular sector. Because each directional foliation is oriented, such an edge must cut the angle  $6\pi$  into two angles of respective values either  $5\pi, \pi$  or  $3\pi, 3\pi$ .

The lemma below proves that any such geodesic triangulation has at least one edge cutting the singular point into two angles  $5\pi$  and  $\pi$ . The theorem is a direct consequence of this lemma.

**Lemma 5.** *A geodesic triangulation of  $\Sigma$  cannot have its 9 edges cutting  $p$  into two sectors of angles  $3\pi$ .*

*Proof.* The property on the edges implies that they all intersect with number  $\pm 1$  at the only vertex of the triangulation. Since every closed leaf separates the cylinder at  $p$  in two angular sectors of angle  $3\pi$  an oriented leaf can't have it's

in-going and outgoing parts in the same sector cut by any other leaf (the angle would have to be smaller).

We can always see topologically our surface as an octagon which boundary are edges of the triangulation. Take for example the maximal sub-graph of the 1-skeleton of the triangulation such as the complementary of it in the surface is connected and simply-connected, this complement will be the fundamental domain we are looking for. Now the intersection number property tells us that the configuration of the path at the order will be in the setting of Figure 13

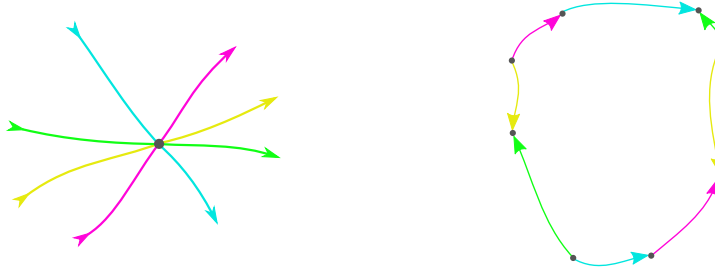


FIGURE 13. Topological setting of the separatrix diagram

Now consider yet another of the 9 edges, as it has to intersect all of the other path with  $\pm 1$  the only possibility is when the curves starts between two colors and end up between the two same colors.

Let's add one of these edges as in Figure 14. It is clear now that we won't be able to add any curve with the same property since they can't intersect a curve in other point than  $p$ .



FIGURE 14. Adding one curve

□

This completes the proof of Theorem 2. It is not completely satisfactory: we would like to prove that every affine surface contains *strictly affine cylinders*.

## 6. OPEN PROBLEMS.

We hope that at this point, most of the problems we are about to suggest seem natural to the reader.

We have so far put our hands on several dynamical behaviours for the directional foliations on our affine surfaces. Very often it happens that a finite number of *hyperbolic closed leaves* attract all the others, as it the case for all but one direction on the two-chambersurface. We say in that case that the dynamics is **hyperbolic**. The Camelier-Gutierrez construction discussed in Section 3.2 also proves the existence of directional foliations such that every leaf accumulates to a closed union of leaves which is transversely a *Cantor set*. Finally it is not to be excluded that some directional foliations are minimal (it is actually very easy to build examples of such affine surfaces). Because of the conjectural picture we are about to draw, we call both late cases **exceptional**.

We begin our list of open problems by very specific questions concerning the  $D_{1,2}$  example. It is a simple and very explicit one, but the dynamical questions that it raises are not straightforwardly answerable.

- (1) Does there exist a dense direction on the  $D_{1,2}$  example?
- (2) Does there exist a 'Cantor like' direction on the  $D_{1,2}$  example?
- (3) What does the set of hyperbolic directions on the  $D_{1,2}$  example look like? Is it dense? Has it full Lebesgue measure?

These might be a good starting point on the way to the general case. Although the combinatorial arguments used in its proof are unlikely to generalize to higher genus, Theorem 2 suggests that systematic dynamical behaviours are to be expected. Is it true that

- (4) every affine surface has a closed regular geodesic?
- (5) every affine surface has a closed, regular and hyperbolic geodesic?
- (6) the set of hyperbolic direction of every affine surface is dense? has full measure?

These questions have natural generalizations to the moduli space of affine surfaces together with a directional foliation. It is possible that in this setting some of the aforementioned questions are easier to answer, and that specific surfaces have a very different behaviour from the generic one. We conjecture that the answers to these three questions are positive, with a greater reserve about the full measure one. There are most probably very interesting things to say about the loci of exceptional directions, but making guesses without a clear picture of what happens for hyperbolic ones seems quite dairy. A very natural direction at this point is to investigate the geometrical properties of these affine surfaces:

- (7) Which are the affine surfaces having only finitely many saddle connections?
- (8) Is it true that trough a given point on an affine surface always passes either a closed geodesic or a saddle connection? It is the case in the two-chamberexample.
- (9) What does the set of vectors of saddle connections on a given surface look like?

Proposition 4 most probably prevents the Veech group of an affine surface to be a lattice in  $SL_2(\mathbb{R})$ . Nonetheless, it seems to be an interesting invariant of these affine surfaces.

(10) What kind of Fuchsian groups can appear as Veech groups ?

Finally we want to suggest that a nice source of questions is trying to describe the set of surfaces having specific interesting dynamical/geometric properties. For instance:

- (11) Is the set of affine surfaces having an exceptional direction dense? generic?
- (12) Is the set of affine surfaces having *no* exceptional direction dense? generic?
- (13) Does there exists a surface having infinitely many '*Cantor like*' directions?

APPENDIX A. VEECH'S RESULTS ON THE GEOMETRY OF AFFINE SURFACES.

We review in this appendix results of Veech on the geometry of affine surfaces appearing in [Vee97] and [Vee08]. Note that Veech works in the more general context of affine surfaces with singularities.

For the sake of clarity, we will restrict his results to the case under scrutiny in this paper namely branched affine surface with real positive linear holonomy. The notes [Vee08] remain unpublished and Veech kindly allowed us to reproduce here proofs that are contained in these notes.

**A.1. The property  $\mathcal{V}$ .** We say an affine surface  $\Sigma$  satisfies *the property  $\mathcal{V}$*  if there is no affine immersion of  $\mathbb{H}$  in  $\Sigma$ . It is equivalent to ask that  $\Sigma$  has no affine cylinder of angle larger than  $\pi$ .

**Theorem** (Veech, [Vee08]). *An equivalent formulation of the property  $\mathcal{V}$  is the following:*

( $\mathcal{V}'$ ) *Every affine immersion of the open unit disk  $\mathbb{D} \subset \mathbb{C}$  in  $\Sigma$  extends continuously to a map  $\overline{\mathbb{D}} \rightarrow \Sigma$ .*

We will only give the proof of one sense of the equivalence, namely that the property  $\mathcal{V}$  implies the property  $\mathcal{V}'$ , which will be sufficient for our purpose.

**Lemma 6.** *Let  $\varphi$  be an affine immersion of the open unit disk  $\mathbb{D} \subset \mathbb{C}$  in  $\Sigma$  that does not extend continuously to a map  $\overline{\mathbb{D}} \rightarrow \Sigma$ . Then  $\varphi$  extends to an affine immersion  $\mathbb{H} \rightarrow \Sigma$ .*

*Proof.* Since  $\varphi$  does not extend to  $\partial\mathbb{D}$ , there exists  $z \in \partial\mathbb{D}$  such that

$$\lim_{t \rightarrow 1} \varphi(tz)$$

does not exist. Let  $\gamma$  be the path  $t \mapsto \varphi(tz)$ . Let  $x$  be an accumulation point of  $\gamma$ . Since  $[0, 1]$  is connected, we can assume that  $x$  is not singular. Let  $t_k \rightarrow 1$  be an increasing sequence such that  $\gamma(t_k) \rightarrow x$  and  $f$  an affine chart at  $x$ ,  $f : U \mapsto \Delta$  where  $\Delta$  is the unit disk centered at 0 et  $U$  an open set containing  $x$  such that  $f(x) = 0$ . We denote by  $\frac{1}{2}U$  the pre-image by  $f$  in  $\Sigma$  of the disk centered at 0 or radius  $\frac{1}{2}$ .

For  $k$  large enough, we can assume that  $\gamma(t_k)$  belongs to  $\frac{1}{2}U$ . Denote by  $V_k$  the pre-image by  $\varphi$  in  $\Sigma$  of the disk of radius  $\frac{1}{2}$  centered at  $\varphi(\gamma(t_k))$ . In particular  $V_k$  contains  $x$ . We claim that the image under  $\varphi$  of  $D_k$  the open disk centered at  $t_k z \in \mathbb{U}$  tangent to  $\partial U$  at  $z$  contains  $V_k$ . Since  $\varphi$  does not extend at  $z$ , there is a closed disk centered at  $t_k z$  strictly contained in  $D_k$  whose image under  $\varphi$  is not contained in  $V_k$ . Since this image is a disk concentric at  $t_k z$  it must contain  $V_k$  and in particular  $x$ . Let  $w_k$  be a pre-image of  $x$  in  $D_k$ .

Let  $E_k$  be the largest disk with center  $w_k$  to which  $\varphi$  admits analytic continuation. Since  $w_k \rightarrow z$ , the radius of  $E_k$  converges to 0. Necessarily  $\varphi(E_k)$  contains  $U = f^{-1}(\Delta)$  because  $\varphi(E_k)$  is a maximal embedded disk of center  $x$ . Extending the map  $f^{-1} : \Delta \rightarrow U \subset \Sigma$  by means of  $\varphi$  defines  $F_k : \Delta_k \rightarrow \Sigma$ . The functions  $(F_k)_{k \in \mathbb{N}}$  have the following properties:

- $\Delta_k$  is a disk;
- $\Delta \subset \Delta_k$ ;
- $\forall \zeta \in \Delta$  we have that  $F_k(\zeta) = f^{-1}(\zeta)$ ;
- $\forall k, k' \in \mathbb{N}$ ,  $F_k = F_{k'}$  on  $\Delta_k \cap \Delta_{k'}$ ;
- the radius of  $\Delta_k$  tends to  $+\infty$ , because the radius of  $E_k$  tends to 0.

Since all the  $D_k$  are connected and  $\bigcap_{k \in \mathbb{N}} \Delta_k$  is non empty, the  $F_k$  define an affine immersion

$$F : \bigcup_{k \in \mathbb{N}} \Delta_k \longrightarrow \Sigma$$

and  $\bigcup_{k \in \mathbb{N}} \Delta_k$  must contain a half-plane since it is the union of disk whose radius tend to infinity whose intersection is not empty. This proves the lemma.  $\square$

**Lemma 7.** *Any affine immersion  $\varphi : \mathbb{H} \rightarrow \Sigma$  can be extended to  $\varphi' : \mathbb{H}' \rightarrow \Sigma$  such that the latter is invariant by the action by multiplication by a positive real number  $\lambda \neq 1$ .*

*Proof.* We show first that such an immersion cannot be one-to-one. Consider the geodesic  $\gamma[0, \infty[ \rightarrow [i, i\infty[ \subset \mathbb{H}$ . Its image by  $\varphi$  cannot have a limit in  $\Sigma$  for it has infinite (relative) length. Let  $x \in \Sigma$  be an accumulation point of this image, and  $U \subset \Sigma$  a closed embedded disk with center  $x$  such that  $\varphi(\gamma)$  is not completely contained in  $U$ . Since  $\varphi(\gamma)$  is a leaf of a directional foliation on  $\Sigma$ , it must cross  $U$  twice along parallel segment and therefore cannot be injective because the image of horizontal stripes at the first and second crossing will overlap.

There exists then  $v \neq w \in \mathbb{H}$  such that  $\varphi(z) = \varphi(w)$ . There exists then an affine map  $z \mapsto \lambda z + a$  with  $a \in \mathbb{R}_+^*$  and  $b \in \mathbb{C}$  such that  $w = \lambda v + a$  and for all  $z$  in a neighborhood of  $v$

$$\varphi(z) = \varphi(\lambda z + a).$$

The union of the iterated images of  $\mathbb{H}$  by  $z \mapsto \lambda z + a$  is a half-plane  $\mathbb{H}'$  to which  $\varphi$  extends by analytic continuation and on which the above invariance relation

holds for all  $z$ . If  $\lambda = 1$ , the image of  $\varphi'$  in  $\Sigma$  is an infinite flat cylinder, which is impossible since  $\Sigma$  is compact; therefore  $\lambda \neq 1$ . The fixed point of  $z \mapsto \lambda z + a$  must lie in  $\partial\mathbb{H}'$  and up to an affine transformation we can suppose that  $b = 0$ .  $\square$

**A.2. Delaunay decompositions and triangulations.** The main result of [Vee97] and [Vee08] combined is establishing the existence of geodesic triangulations for surfaces not verifying the only natural obstruction for this to happen. It is given by the following theorem:

**Theorem** (Veech). *An affine surface admits a geodesic triangulation if and only if it does not contain an embedded open affine cylinder of angle  $\pi$ . Equivalently, an affine surface admits a geodesic triangulation if and only if it satisfies the property  $\mathcal{V}$ .*

This theorem is a corollary of the existence of *Delaunay partitions* for affine surfaces that Veech deals with. We explain in the sequel the construction. Let  $\Sigma$  be an affine surface satisfying the property  $\mathcal{V}$  and let  $x \in \Sigma$  be a regular point. We are going to distinguish points depending on the number of singular points on the boundary of the largest immersed disk at  $x$ . We denote this number by  $\nu(x)$ . The set  $\{\nu(x) = 1\}$  is a open dense set in the surface. The special points are those such that  $\nu(x) \geq 3$ , who form a discrete and therefore finite set in the surface. At these points one can consider the largest embedded disk in the surface and consider in this disk the convex hull of the (at least) three singular point on the boundary. The crucial (but not completely obvious) facts are that:

- this convex hull projects onto an **embedded** convex polygon in the surface;
- the union of such polygons covers the whole surface;
- such polygons only intersect at their boundary;
- an intersection of two such polygons is union of some their (shared) sides;
- the set of vertices of such polygons is exactly the set of singular points;
- the union of the interior of these polygons is exactly the set  $\{\nu(x) = 1\}$ ;
- the union of the interior of their sides is exactly the set  $\{\nu(x) = 2\}$ .

This decomposition of the surface in convex polygons is called its **Delaunay polygonations**, is unique and only depends on its geometry. A triangulation of each polygon leads to a *geodesic triangulation* of the surface.

The crucial fact (we refer to [Vee97] for details) is that having the property  $\mathcal{V}$  implies that maximal affine embeddings of the disk extend to their boundary, which is the technical point that one needs to make sure the construction hinted above can be carried on.

**Remark.** *The converse of the triangulation theorem is quite easy. Any cylinder of angle at least  $\pi$  behave like a 'trap': any geodesic entering it never escapes. A surface containing such a cylinder therefore cannot be geodesically triangulated,*

because no edge of the triangulation could enter the cylinder and the complement of the 1-skeleton of such a triangulation would not be a union of cells.

**A.3. Closed geodesics and wild immersions of the disk.** We consider in this subsection a surface  $\Sigma$  which does not satisfy the property  $\mathcal{V}$ . This implies that there is an affine immersion  $\varphi : \mathbb{D} \rightarrow \Sigma$  which does not extend to  $\partial\mathbb{D}$  the boundary of  $\mathbb{D}$ . This also implies the existence of an affine cylinder and therefore closed hyperbolic geodesics. We give in this subsection a way to localize such a closed hyperbolic geodesic starting from  $\varphi$ . We believe that this result should be attributed to Veech. Even though not stated explicitly in [Vee08] (probably because Veech did not have in mind the dynamical questions we are interested in), the statement is obvious for anyone who has read carefully the sections 23 and 24 of [Vee08].

**Proposition 5** (Veech, [Vee08]). *Let  $\varphi : \mathbb{D} \rightarrow \Sigma$  be a wild immersion of the disk, i.e. that does not extend to  $\partial\mathbb{D}$ , and let  $z \in \partial\mathbb{D}$  such that the path  $\gamma : t \mapsto \varphi(tz)$  does not have a limit in  $\Sigma$  when  $t \rightarrow 1$ . Then  $\gamma([0,1))$  is a hyperbolic closed geodesic in  $\Sigma$ .*

*Proof.* According to both Lemma 6 and 7  $\varphi$  extends to a maximal  $\varphi' : \mathbb{H} \rightarrow \Sigma$  which is invariant by multiplication by a certain positive number  $\lambda \neq 1$ .  $\gamma$  in this extension must head toward  $0 \in \partial\mathbb{H}$  for otherwise it would have a limit in  $\Sigma$ . Its image is therefore a closed hyperbolic leaf because it is invariant by the action of  $\lambda$ .  $\square$

## REFERENCES

- [BHM10] Xavier Bressaud, Pascal Hubert, and Alejandro Maass. Persistence of wandering intervals in self-similar affine interval exchange transformations. *Ergodic Theory Dynam. Systems*, 30(3):665–686, 2010.
- [CG97] Ricardo Camelier and Carlos Gutierrez. Affine interval exchange transformations with wandering intervals. *Ergodic Theory Dynam. Systems*, 17(6):1315–1338, 1997.
- [Gun81] R. C. Gunning. Affine and projective structures on Riemann surfaces. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 225–244. Princeton Univ. Press, Princeton, N.J., 1981.
- [HS06] Pascal Hubert and Thomas A Schmidt. Chapter 6 - An Introduction to Veech Surfaces. In B Hasselblatt and A Katok, editors, *Handbook of Dynamical Systems*, volume 1, Part B of *Handbook of Dynamical Systems*, pages 501–526. Elsevier Science, 2006.
- [Lev82] Gilbert Levitt. Feuilletages des surfaces. *Ann. Inst. Fourier (Grenoble)*, 32(2):x, 179–217, 1982.
- [Lio95] Isabelle Liousse. Dynamique générique des feuilletages transversalement affines des surfaces. *Bull. Soc. Math. France*, 123(4):493–516, 1995.
- [Man72] Richard Mandelbaum. Branched structures on Riemann surfaces. *Trans. Amer. Math. Soc.*, 163:261–275, 1972.
- [Man73] Richard Mandelbaum. Branched structures and affine and projective bundles on Riemann surfaces. *Trans. Amer. Math. Soc.*, 183:37–58, 1973.
- [MMY10] S. Marmi, P. Moussa, and J.-C. Yoccoz. Affine interval exchange maps with a wandering interval. *Proc. Lond. Math. Soc. (3)*, 100(3):639–669, 2010.



- [Pry69] F. E. Prym. Zur Integration der gleichzeitigen Differentialgleichungen. *J. Reine Angew. Math.*, 70:354–362, 1869.
- [Vee93] William A. Veech. Flat surfaces. *Amer. J. Math.*, 115(3):589–689, 1993.
- [Vee97] W. A. Veech. Delaunay partitions. *Topology*, 36(1):1–28, 1997.
- [Vee08] W. A. Veech. Informal notes on flat surfaces. *Unpublished course notes*, 2008.
- [Zor06] Anton Zorich. Flat surfaces. In *Frontiers in number theory, physics, and geometry. I*, pages 437–583. Springer, Berlin, 2006.