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Blowup equations for 6d SCFTs. Part I

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ABSTRACT: We propose novel functional equations for the BPS partition functions of 6d (1,0) SCFTs, which can be regarded as an elliptic version of Göttsche-Nakajima-Yoshioka's K-theoretic blowup equations. From the viewpoint of geometric engineering, these are the generalized blowup equations for refined topological strings on certain local elliptic Calabi-Yau threefolds. We derive recursion formulas for elliptic genera of self-dual strings on the tensor branch from these functional equations and in this way obtain a universal approach for determining refined BPS invariants. As examples, we study in detail the minimal 6d SCFTs with SU(3) and SO(8) gauge symmetry. In companion papers, we will study the elliptic blowup equations for all other non-Higgsable clusters.

KEYWORDS: Solitons Monopoles and Instantons, Topological Strings, Field Theories in Higher Dimensions, Conformal Field Models in String Theory

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1 Introduction

Quantum field theories with the highest amount of symmetries, namely supersymmetry as well as conformal symmetry, in the highest possible dimension are 6d superconformal field theories. The classification of such theories subdivides between two main classes of theories, namely the $\mathcal{N} = (2, 0)$ theories and the $\mathcal{N} = (1, 0)$ theories. The former have been studied intensively for a long time now and there are powerful techniques available for constructing their partition functions on various manifolds. The subject of $\mathcal{N} = (1, 0)$ theories, which preserve half of the supercharges of the $(2, 0)$ theories, has recently enjoyed a resurgence due to a proposed classification of such theories in terms of F-theory compactifications on non-compact elliptic Calabi-Yau three-folds [1, 2].

In this classification, the geometry of the base B of the Calabi-Yau manifold directly translates into the tensor multiplet sector of the 6d SCFTs where the number of tensor multiplets is given by the dimension of $H^{1,1}(B, \mathbb{Z})$ and the intersection form on B gives the couplings of these tensor multiplets to each other. Note that an action is not available as field strengths of tensor multiplets are constrained to be self-dual. Nevertheless, it is useful to write down a “formal” action on the tensor branch from which many properties of the theory and its compactifications can be deduced, see for example [3, 4]. Furthermore, the base B of the Calabi-Yau is non-compact and all curve classes inside it are required to be simultaneously shrinkable to zero volume in order to restore conformal invariance of the resulting 6d theory at its tension-less limit. This gives strong constraints on the geometry and in particular forces all curves to be \mathbb{P}^1 's which have negative self-intersection number. Moreover, for self-intersection numbers $-n$ lower than -2 the elliptic fiber above the corresponding curve Σ becomes singular with a singularity type determined by Kodaira’s classification of elliptic fibers [5, 6]. The singularity becomes worse when n increases such that beyond $n = 12$ it becomes too bad for a smooth description of the Calabi-Yau three-fold. The physical interpretation of these singularities is the emergence of a bulk gauge group (whose Lie Algebra \mathfrak{g}_Σ is determined by the intersection form of the resolved singularity) in the 6d SCFT on its tensor branch. If we have two curves Σ_1 and Σ_2 with non-trivial intersection number and gauge groups, then the corresponding 6d theory will also have bi-fundamental matter in suitable representations of the arising gauge groups. In the current paper we want to focus on the cases where the base B contains only one curve with self-intersection $-n$, i.e. it is a certain decompactification limit of a Hirzebruch surface \mathbb{F}_n . Then the possible gauge groups which arise as a function of n are as follows:¹

n	3	4	5	6	7	8	12
G_Σ	SU(3)	SO(8)	F_4	E_6	E_7	E_7	E_8

These theories are known as the Non-Higgsable Clusters [6].² Note that the E_7 Lie algebra appears twice, namely for the self-intersections -7 and -8 . The difference is that in the -7

¹The cases of $n = 9, 10, 11$ involve points of enhanced singularities on the base curve which must be resolved giving rise to additional curves in the base.

²The Non-Higgsable Clusters also include three examples of intersecting chains of two or three curves, which we do not cover here.

case there is also fundamental matter which is non-Higgsable. These theories are the subject of the current paper where we focus on the cases $n = 3$ and $n = 4$ and aim at testing a novel method for computing BPS partition functions of the corresponding minimal 6d SCFTs. Let us describe our procedure for computing such partition functions in the following.

In order to be able to compute a partition function for our 6d theories we first need to make a choice for a background geometry. As it turns out, an appropriate choice for 6d SCFTs is the Omega background $\mathbb{R}^4 \times_{\epsilon_1, \epsilon_2} T^2$ [7]. This background not only regularizes the infinities arising from non-compactness of \mathbb{R}^4 but also serves as a building block for computing partition functions for other backgrounds like superconformal indices and $T^2 \times S^4$ partition functions [8, 9]. As was observed in [7], instantons on this background arise from self-dual strings wrapping the T^2 and localized at a point on \mathbb{R}^4 . From the F-theory point of view, such strings arise from D3 branes wrapping a curve Σ in the base which in our case is a $(-n)$ -curve. The bulk gauge group will descend to a flavor symmetry on the worldvolume theory of these strings which is a 2d $\mathcal{N} = (0, 4)$ supersymmetric theory with R -symmetry $SU(2)_{\epsilon_+} \times SU(2)_R$ where $SU(2)_R$ is the R -symmetry of the 6d SCFT. The partition function on the tensor branch Z^{6d} , i.e. when the volume t_b of the $(-n)$ -curve in the base is non-zero, is the generating function of the elliptic genera \mathbb{E}_k of k strings up to a prefactor [7, 10, 11].

For 6d theories corresponding to the particular choices $n = 3$ and $n = 4$ the worldvolume theory of the strings is known in terms of a quiver gauge theory whose single gauge node is of rank k [11, 12]. For the other cases, references [13, 14] give some descriptions for the $k = 1$ subsector of a single string but a complete description including a computation scheme for all the \mathbb{E}_k is still lacking. In this paper we want to remedy this gap by providing a novel computation scheme for Z^{6d} which allows us to derive expressions for the \mathbb{E}_k recursively. This is done by using the so-called *blow-up equations*.

The blowup equations have their origin in the studies of Donaldson invariants [15–18]. But the version we are most interested in is the generalized version proposed and later proved by Göttsche, Nakajima, and Yoshioka. Nakajima and Yoshioka [19] first considered the 4d $\mathcal{N} = 2$ $SU(N)$ supersymmetric Yang-Mills theory on the Omega background. The idea is to view $\mathbb{R}^4 \cong \mathbb{C}^2$ as the limit of its blow-up at the origin [19], denoted by $\widehat{\mathbb{C}^2}$, when one sends the size of the exceptional divisor \mathbb{P}^1 to zero. Then $U(1)_{\epsilon_1} \times U(1)_{\epsilon_2}$ has a natural action on $\widehat{\mathbb{C}^2}$ with two fixed points, one at the north pole and one at the south pole of the exceptional \mathbb{P}^1 . Computing the partition function on the background $\widehat{\mathbb{C}^2} \times_{\epsilon_1, \epsilon_2} T^2$ through localization then contributes a product of two copies of the partition function on our original background while one has to sum now over non-trivial fluxes of the B -field through the exceptional divisor. This idea can be put into functional equations for the Nekrasov partition function [19] (see also [20]), and they were instrumental for Nakajima and Yoshioka to prove Nekrasov’s conjecture [21]. Later together with Göttsche they generalized and then proved the blowup equations for 5d $\mathcal{N} = 1$ $SU(N)$ super-Yang-Mills theories on the Omega background $\mathbb{R}^4 \times_{\epsilon_1, \epsilon_2} S^1$ with a possible Chern-Simons term of level m [22–24]. On the other hand, the Nekrasov partition function of such a 5d theory can be computed by the refined topological string theory with target space the local toric Calabi-Yau threefold $X_{N,m}$, which is the resolution of the cone over the $Y^{N,m}$ singularity [25, 26].

This is an example of the geometric engineering [27]. Inspired by the consistency study of the exact quantization program of mirror curves of local Calabi-Yau threefolds [28–32], the Göttsche-Nakajima-Yoshioka blowup equations were reformulated completely in terms of the geometric data of the Calabi-Yau threefold $X_{N,m}$ [33]. The reformulation, however, was not complete, and the complete set of equations were provided in [34].

These geometrically reformulated or generalized blowup equations prove to be very powerful. First of all, just as in the case of the Göttsche-Nakajima-Yoshioka blowup equations [35], they can be used to compute the Nekrasov instanton partition functions [34]. Second, the generalized blowup equations open up possibilities for various directions of generalization. As we will see in the next subsection, the form of the generalized blowup equations is simple and universal, and it does not put any constraints on the target space of the topological string except that it has to be non-compact to allow for $U(1)$ isometry crucial for the preservation of supersymmetry in the presence of the Omega background. This naturally poses the question of the validity of the generalized blowup equations beyond 5d $SU(N)$ SYM engineered by the $X_{N,m}$ geometries. Indeed it was checked in [34] that the generalized blowup equations are satisfied by some toric Calabi-Yau threefolds which engineer 5d $SU(N)$ gauge theories with matter. Moreover, what is fascinating is that the generalized blowup equations may even be valid for 6d SCFTs as the topological string theory on non-compact elliptic Calabi-Yau threefolds used in F-theory compactifications computes precisely Z^{6d} of these 6d SCFTs on the Omega background. As a first step it was checked in [33, 34] that the simplest 6d SCFT, the E-string theory, respects half of the generalized blowup equations. The verification of the other half is a bit trickier, and it will be discussed in our upcoming work. In this paper, we demonstrate the validity of the generalized blowup equations through the already well-studied cases of $n = 3$ and $n = 4$ minimal SCFTs in the present paper, and illustrate their power by computing the elliptic genera as well as the BPS invariants with them. Furthermore, by reducing the $n = 4$ model down to the 5d $SO(8)$ SYM, we verify the validity of the generalized blowup equations for the latter theory as well, which is also new. We will cover all the remaining minimal SCFTs in companion papers. In the next subsection we give a quick overview of the generalized blowup equations.

1.1 Overview of geometric blowup equations

Consider putting the refined topological string theory on a non-compact Calabi-Yau threefold X . Let $H_{2i}(X, \mathbb{Z})$ be the homology groups of compact $2i$ -cycles. In particular $H_2(X, \mathbb{Z})$ includes compact curve classes $\{\Sigma_i\}$, and $H_4(X, \mathbb{Z})$ compact divisor classes $\{D_j\}$. We denote the complexified Kähler moduli of the compact curve classes by t_i with $\text{Vol}(\Sigma_i) = -\text{Re}(t_i)$, and the dimensions of the two homology groups by

$$b := \dim H_2(X, \mathbb{Z}), \quad g := \dim H_4(X, \mathbb{Z}). \tag{1.1}$$

Since X is not compact, these two numbers are not necessarily identical. We encode the intersection numbers of curve classes and divisor classes in a matrix

$$\mathbf{C} = (C_{ij}), \quad \text{with } C_{ij} = \Sigma_i \cdot D_j, \quad \Sigma_i \in H_2(X, \mathbb{Z}), \quad D_j \in H_4(X, \mathbb{Z}). \tag{1.2}$$

It is always possible to find $b-g$ independent linear combinations of the curve classes so that they have zero intersection number with any compact divisor. We call the corresponding Kähler moduli mass parameters and sometimes denote them by t_{m_i} , as they are interpreted as masses of hypermultiplets or instanton fugacity in 5d or 4d theories.

It was first observed in [36] and later confirmed in many examples that the non-vanishing BPS invariants $N_{j_L, j_R}^{\mathbf{d}}$ on a noncompact Calabi-Yau threefold respect a checkerboard pattern: there exists a b dimensional vector \mathbf{B} with entries in \mathbb{Z}_2 such that³

$$2j_L + 2j_R + 1 \equiv \mathbf{B} \cdot \mathbf{d} \pmod{2} \tag{1.3}$$

holds for any non-vanishing $N_{j_L, j_R}^{\mathbf{d}}$. Then we could define the twisted refined free energy via

$$\widehat{F}(\mathbf{t}, \epsilon_1, \epsilon_2) = F^{\text{pert}}(\mathbf{t}, \epsilon_1, \epsilon_2) + F^{\text{inst}}(\mathbf{t} + \pi i \mathbf{B}, \epsilon_1, \epsilon_2), \tag{1.4}$$

where only in the worldsheet instanton contributions are the Kähler moduli \mathbf{t} shifted by the vector \mathbf{B} . We call \mathbf{B} the \mathbf{B} field as it combines into the Kalb-Ramond part of the complexified \mathbf{t} . The twisted free energy appears prominently in the geometric engineering of Nekrasov partition functions [25, 26, 32] as well as in the program of exact quantum mirror curves [28, 29]. We also define the twisted partition function

$$\widehat{Z}(\mathbf{t}, \epsilon_1, \epsilon_2) = \exp(\widehat{F}(\mathbf{t}, \epsilon_1, \epsilon_2)) = Z^{\text{pert}}(\mathbf{t}, \epsilon_1, \epsilon_2) \widehat{Z}^{\text{inst}}(\mathbf{t}, \epsilon_1, \epsilon_2). \tag{1.5}$$

Note we do not put hat on Z^{pert} since the Kähler moduli are not shifted there.

In terms of these quantities, the blowup equations can be reformulated in the following way

$$\sum_{\mathbf{n} \in \mathbb{Z}^g} (-1)^{|\mathbf{n}|} \widehat{Z}(\mathbf{t} + \epsilon_1 \mathbf{R}, \epsilon_1, \epsilon_2 - \epsilon_1) \widehat{Z}(\mathbf{t} + \epsilon_2 \mathbf{R}, \epsilon_1 - \epsilon_2, \epsilon_2) = \Lambda(\mathbf{t}_m, \epsilon_1, \epsilon_2, \mathbf{r}) \widehat{Z}(\mathbf{t}, \epsilon_1, \epsilon_2), \tag{1.6}$$

with $|\mathbf{n}| = \sum_i n_i$ and

$$\mathbf{R} = \mathbf{C} \cdot \mathbf{n} + \mathbf{r}/2. \tag{1.7}$$

Here $\mathbf{r} = (r_i)$, which we call a \mathbf{r} field, is a b dimensional vector with entries in \mathbb{Z} satisfying

$$r_i \equiv B_i \pmod{2}. \tag{1.8}$$

Two \mathbf{r} fields \mathbf{r}, \mathbf{r}' are equivalent if

$$\mathbf{r} - \mathbf{r}' = 2\mathbf{C} \cdot \mathbf{n}', \quad \mathbf{n}' \in \mathbb{Z}^g \tag{1.9}$$

as the corresponding blowup equations can be identified by the shift $\mathbf{n} \rightarrow \mathbf{n} + \mathbf{n}'$. The prefactor Λ is trivial in the sense that it only depends on the mass parameters \mathbf{t}_m but not on the true moduli [34]. It also depends on the choice of the \mathbf{r} field, thus it gives rise to different blowup equations with different choices of the \mathbf{r} field. For some choices of the \mathbf{r} field, Λ vanishes all together, and we call the corresponding equations the *vanishing* blowup equations, while the other equations with non-vanishing Λ are called the *unity*

³The entries of the \mathbf{B} field may be fractional if it is not expanded in integral curve classes.

blowup equations. Note that since the row vector of the \mathbf{C} -matrix corresponding to a mass parameter is null, a multiplicative factor of \widehat{Z} which depends only on mass parameters but no other Kähler parameters decouples as its contributions to the blowup equations can be factored out of the summation in \mathbf{n} and be absorbed in Λ . We will thus discard this type of components in \widehat{Z} .

It was conjectured in [33, 34] that for any non-compact Calabi-Yau threefold there is a finite but non-empty set of \mathbf{r} fields so that the blowup equations (1.6) hold. It was further conjectured and verified for some toric Calabi-Yaus in [34] that the BPS invariants could be computed from the blowup equations using classical geometric data of X as input. Furthermore, Λ is modular invariant with respect to the monodromy group of the topological string moduli space. In this paper we demonstrate the validity of these statements for the partition functions of the minimal 6d $n = 3, 4$ SCFTs.

The partition function of a 6d SCFT on the Omega background $\mathbb{R}^4 \times_{\epsilon_1, \epsilon_2} T^2$ can be split to three components

$$Z(t_b, \tau, \mathbf{a}, \epsilon_1, \epsilon_2) = Z^{\text{pert}}(t_b, \tau, \mathbf{a}, \epsilon_1, \epsilon_2) Z^{1\text{-loop}}(\tau, \mathbf{a}, \epsilon_1, \epsilon_2) Z^{\text{ell}}(t_b, \tau, \mathbf{a}, \epsilon_1, \epsilon_2). \quad (1.10)$$

Here t_b, τ and \mathbf{a} are tensor modulus, complex structure of T^2 , and gauge fugacities (Wilson lines on T^2) respectively. Z^{pert} contains perturbative contributions. $Z^{1\text{-loop}}$ comes from Kaluza-Klein modes of 6d particle multiplets on T^2 . We denote it by the superscript 1-loop because it descends to 1-loop contributions in 4d when we shrink T^2 to a point. Finally $Z^{\text{ell}}(t_b, \tau, \mathbf{a}, \epsilon_1, \epsilon_2)$ splits by

$$Z^{\text{ell}}(t_b, \tau, \mathbf{a}, \epsilon_1, \epsilon_2) = 1 + \sum_{k=1}^{\infty} Q_{\text{ell}}^k \mathbb{E}_k(\tau, \mathbf{a}, \epsilon_1, \epsilon_2) \quad (1.11)$$

with $Q_{\text{ell}} = e^{t_{\text{ell}}}$ the counting parameter, and $\mathbb{E}_k(\tau, \mathbf{a}, \epsilon_1, \epsilon_2)$ the k string elliptic genus.

By the F-/M-theory duality and the relation of the BPS sector of the M-theory with the refined topological string theory, the partition function of a 6d SCFT on the tensor branch is computed by the partition function of the refined topological string theory on the same Calabi-Yau threefold X encoding the BPS invariants on X , and the moduli $t_{\text{ell}}, \tau, \mathbf{a}$ are identified with linear combinations of the Kähler moduli of compact curve classes in X . In particular, Z^{ell} includes the BPS states of M2 branes wrapping the base curve, while $Z^{1\text{-loop}}$ the BPS states of M2 branes not wrapping the base curve at all. They combine into the component Z^{inst} that encodes all the BPS invariants. Z^{pert} basically encodes the intersection numbers of divisors in X . If we further decompactify X along the direction of the elliptic fiber in the M-theory picture, the 6d (1, 0) SCFT reduces to a 5d $N = 1$ SYM with the same gauge group on the Omega background, where the tensor modulus t_b becomes the gauge coupling.

The organization of the rest of the paper is as follows. In section 2 we compute the initial data for blowup equations. These include the curve-divisor intersection \mathbf{C} -matrix, the B-field for the checkerboard pattern, Z^{pert} and $Z^{1\text{-loop}}$. We give explicit expressions for these initial data for the cases of 6d SCFTs with $SU(3)$ and $SO(8)$ bulk gauge groups. In section 3 we put everything together and first demonstrate the validity of the blowup

equations order by order in terms of Q_{ell} expansion and then proceed to recursively compute the elliptic genera of multiple strings as well as the corresponding BPS invariants. In section 4 we study reductions of the blowup equations in the 5d limit, that is when one of the circles of the T^2 which is wrapped by the strings shrinks to zero radius. Finally, in section 5 we present our conclusions and give an overview of applications and open problems.

2 Initial data for blowup equations

We explain here how to compute the initial data for the blowup equations: the curve-divisor intersection \mathbf{C} -matrix, the \mathbf{B} fields, as well as the perturbative and 1-loop partition functions Z^{pert} , $Z^{1\text{-loop}}$, for 6d minimal SCFTs with no matter. As we note in subsequent subsections, these data are necessary if we wish to derive compact formulas of elliptic genera from the blowup equations, while the piece $Z^{1\text{-loop}}$ is not needed if we wish to directly compute BPS invariants from the blowup equations.

2.1 Curve-divisor intersection matrix

The structures of the elliptic non-compact Calabi-Yau threefolds underlying the 6d minimal SCFTs are for instance discussed in [37]. Let the gauge group G be of rank r . There are $g = r + 1$ compact divisors. They result from the resolution of the singular elliptic fiber and they intersect with each other like the nodes of the Dynkin diagram of $\widehat{\mathfrak{g}}$. One of the divisors is special as it intersects with the base B and it corresponds to the affine node in the Dynkin diagram. We label the special divisor by D_{r+1} and the subsequent divisors D_r, D_{r-1}, \dots . All these divisors are Hirzebruch surfaces \mathbb{F}_{n_i} . The indices n_i of these divisors in different 6d $(1, 0)$ minimal SCFTs can be found in [37], and we give some examples in figure 1. The number of irreducible compact curves is $b = r + 2$. Of these $r + 1$ curves are the \mathbb{P}^1 fibers of the Hirzebruch surfaces Σ_i , $i = 1, \dots, r + 1$ and they stretch in the vertical direction. Their labelling follows the labelling of the underlying divisors. These curves satisfy

$$\sum_i a_i [\Sigma_i] = [\delta], \tag{2.1}$$

where a_i are marks of the affine Lie algebra $\widehat{\mathfrak{g}}$, and δ is the elliptic fiber. The last compact curve Σ_b is in the horizontal direction, and it projects down to the compact $-n$ curve Σ_B in the base. In accord with topological string calculations we choose it to be a Mori cone generator. It is always the \mathbb{P}^1 base of the Hirzebruch surface in the center of the chain of \mathbb{F}_{n_i} with the lowest index. It is therefore related to the base curve Σ_B by

$$[\Sigma_b] = [\Sigma_B] - \sum_{i=0}^{\lfloor (n-3)/2 \rfloor} (n - 2 - 2i) [\Sigma_{r+1-i}]. \tag{2.2}$$

In the case of $n = 3, 4$, we have

$$[\Sigma_b] = [\Sigma_B] - [\Sigma_{r+1}]. \tag{2.3}$$

We denote the volumes of these irreducible curves by t_i ($i = 1, \dots, r + 1$) and $t_{r+2} = t_b$.

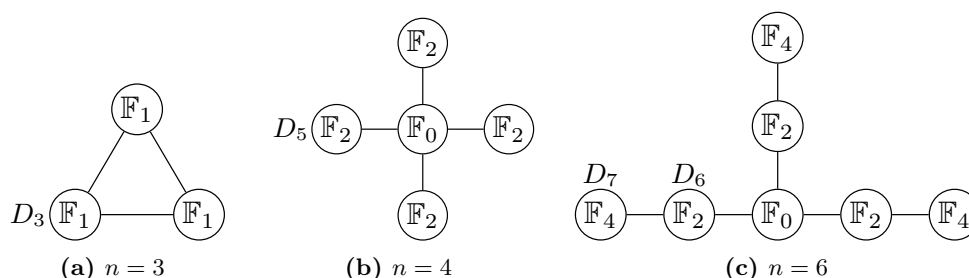


Figure 1. Schematic structure of compact divisors in elliptic fibrations over $\mathcal{O}(-n) \rightarrow \mathbb{P}^1$ as affine Dynkin diagrams for $n = 3, 4, 6$. The divisor D_{r+1} corresponds to the affine node.

This geometric picture allows us to write down the intersection **C**-matrix

$$\mathbf{C} = (\Sigma_i \cdot D_j) = \begin{pmatrix} & & -\mathbf{A} \\ * & \dots & * \end{pmatrix} \quad (2.4)$$

where the $(r+1) \times (r+1)$ submatrix $-\mathbf{A}$ is minus the Cartan matrix of the affine Lie algebra $\widehat{\mathfrak{g}}$, and the last row depends on the indices of the Hirzebruch surfaces $D_i = \mathbb{F}_{n_i}$. It is then easy to see that the only mass parameter is

$$\tau = \sum_{i=1}^{r+1} a_i t_i, \quad (2.5)$$

which is the complexified volume of the elliptic fiber δ . We will give the concrete expressions of the **C**-matrix of the $SU(3)$ and the $SO(8)$ theories in the example subsections.

Note that in most of the paper we expand Z^{ell} in terms of $Q_b = e^{t_b}$

$$Z^{\text{ell}} = 1 + \sum_{k=1}^{\infty} Q_b^k Z_k, \quad (2.6)$$

instead of $Q_{\text{ell}} = e^{t_{\text{ell}}}$ as the curve associated to t_{ell} may not be in an integral class. These parameters are related by⁴

$$t_{\text{ell}} = t_B - \frac{n-2}{2}\tau = t_b - \frac{n-2}{2}\tau + \sum_{i=0}^{\lfloor (n-3)/2 \rfloor} (n-2-2i)t_{r+1-i}. \quad (2.7)$$

2.2 The **B** field

We would like to compute the b dimensional \mathbb{Z}_2 **B** field which characterizes the checkerboard pattern of non-vanishing BPS invariants $N_{j_L, j_R}^{\mathbf{d}}$ with identity

$$2j_L + 2j_R + 1 \equiv \mathbf{B} \cdot \mathbf{d} \pmod{2}. \quad (2.8)$$

Since the r.h.s. is linear in the curve class \mathbf{d} , we only need to know the entries of the **B** field corresponding to each individual irreducible curves. Each irreducible curve can be

⁴The Kähler modulus t_b coincides with the volume of the curve class l_b in [38], but only coincides with the t_b defined in [38] for $n = 3, 4$.

embedded in an algebraic surface in the Calabi-Yau threefold X . Let us denote the curve and the surface where it is embedded by C and S respectively. The non-vanishing BPS invariants associated to this curve must have [33, 39]

$$2j_L^{\max} = \frac{C^2 + K_S \cdot C}{2} + 1, \quad 2j_R^{\max} = \frac{C^2 - K_S \cdot C}{2}, \quad (2.9)$$

where C^2 is the self-intersection number of the curve in the surface S , and K_S the canonical class of S . We have then

$$2j_L + 2j_R + 1 = C^2 \pmod{2}. \quad (2.10)$$

Thus the entry of the \mathbf{B} field corresponding to C is its self-intersection number in the surface S modulo two. On the other hand, the self-intersection number C^2 is identified with the degree of the normal bundle of C perpendicular to S . Recall that the normal bundle of a curve in a Calabi-Yau threefold has the form

$$\mathcal{O}(n) \oplus \mathcal{O}(-2+n) \rightarrow C, \quad n \in \mathbb{Z}. \quad (2.11)$$

Since the two degrees n and $-2+n$ are equivalent modulo two, we can take either of them to be the entry of the \mathbf{B} field corresponding to the curve C . Since we have a good understanding of irreducible curves and surfaces in the Calabi-Yau threefold underlying the 6d minimal SCFTs as we discussed in the beginning of the section, these numbers can be easily computed for each irreducible compact curve.

2.3 Perturbative partition function

The perturbative contribution $Z^{\text{pert}}(\mathbf{t}, \epsilon_1, \epsilon_2) = \exp(F^{\text{pert}}(\mathbf{t}, \epsilon_1, \epsilon_2))$ has the following form

$$\begin{aligned} F^{\text{pert}}(\epsilon_1, \epsilon_2; \mathbf{t}) &= \frac{1}{\epsilon_1 \epsilon_2} F_{(0,0)}^{\text{pert}}(\mathbf{t}, \epsilon_1, \epsilon_2) + F_{(1,0)}^{\text{pert}}(\mathbf{t}, \epsilon_1, \epsilon_2) - \frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2} F_{(0,1)}^{\text{pert}}(\mathbf{t}, \epsilon_1, \epsilon_2) \\ &= \frac{1}{\epsilon_1 \epsilon_2} \left(\frac{1}{6} \sum_{i,j,k=1}^b \kappa_{ijk} t_i t_j t_k \right) + \sum_{i=1}^b b_i^{\text{GV}} t_i - \frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2} \sum_{i=1}^b b_i^{\text{NS}} t_i. \end{aligned} \quad (2.12)$$

The perturbative prepotential $F^{\text{pert},(0,0)}$ is decided⁵ by the intersection numbers of divisors Poincaré dual to the curve classes Σ_i associated to t_i . Since the Poincaré duality is only rigorously defined in a compact manifold, we should compute $F_{\text{cmp}}^{\text{pert},(0,0)}$ in a compact Calabi-Yau threefold where the non-compact Calabi-Yau threefold X is embedded and take an appropriate decompactification limit. Fortunately the compactification of the Calabi-Yau threefolds underlying the 6d $SU(3)$ and $SO(8)$ gauge theories have been constructed in [11], and we use the compact models there to compute $F_{\text{cmp}}^{\text{pert},(0,0)}$ which is subsequently reduced to $F^{\text{pert},(0,0)}$ in the decompactification limit. On the other hand, once $F^{\text{pert},(0,0)}$ is computed for the 6d gauge theory, we could obtain $F_{5d}^{\text{pert},(0,0)}$ for the 5d gauge theory by further decompactifying the Calabi-Yau threefold X along the direction of the elliptic

⁵The perturbative prepotential can also include terms linear in \mathbf{t} . But they decouple from the blowup equations.

fiber while keeping the volumes of Σ_i ($i = 1, \dots, r$) finite.⁶ The latter is also given by the perturbative Nekrasov partition function [22]

$$F^{\text{Nek,pert}}(\mathbf{a}, \mathbf{q}, \epsilon_1, \epsilon_2) = -\frac{1}{\epsilon_1 \epsilon_2} \sum_{\alpha \in \Delta_+} \left(\frac{\langle \alpha, \mathbf{a} \rangle^3}{6} - \frac{\log(e^{-h_G^\vee \pi i \mathbf{q}})}{2h_G^\vee} \langle \alpha, \mathbf{a} \rangle^2 \right) - \frac{(\epsilon_1 + \epsilon_2)^2 + \epsilon_1 \epsilon_2}{\epsilon_1 \epsilon_2} \sum_{\alpha \in \Delta_+} \left(\frac{\langle \alpha, \mathbf{a} \rangle}{12} - \frac{\log(e^{-h_G^\vee \pi i \mathbf{q}})}{24h_G^\vee} \right) \quad (2.13)$$

where \mathbf{a} is the vector of Coulomb moduli, \mathbf{q} is the instanton counting parameter, Δ_+ is the set of positive roots of the Lie algebra \mathfrak{g} , and h_G^\vee the dual Coxeter number of G . $\langle \bullet, \bullet \rangle$ is the invariant bilinear form⁷ in the Lie algebra \mathfrak{g} . The dictionary between field theory parameters and geometric Kähler moduli is (see for instance [32])

$$\begin{cases} t_i = \langle \alpha_i, \mathbf{a} \rangle \\ t_m = -\log(e^{h_G^\vee \pi i \mathbf{q}}) \end{cases}, \quad (2.14)$$

where α_i are simple roots, and t_m the mass parameter whose associated curve, we recall, that does not intersect with compact divisors. Once we could identify the first line of (2.13) with $F_{5d}^{\text{pert,(0,0)}}$, we could uplift the second line of (2.13) to obtain the perturbative genus one free energies $F^{(1,0)}$, $F^{(0,1)}$ for the 6d theories, as we will do in example subsections. In particular, we find in the examples of the $n = 3, 4$ theories

$$b_i^{\text{GV}} + b_i^{\text{NS}} = 0, \quad i = 1, \dots, b. \quad (2.15)$$

2.4 One-loop partition function

$Z^{1\text{-loop}}$ has the contribution of the Kaluza-Klein modes on the 6d S^1 of the 6d particle multiplets. The contribution of a single supermultiplet of various types reads as follows [40]⁸

$$\begin{aligned} Z_{\text{tensor}} &= \text{PE} \left[-\frac{q_L + q_L}{(q_1^{1/2} - q_1^{-1/2})(q_2^{1/2} - q_2^{-1/2})} \left(\frac{Q_\tau}{1 - Q_\tau} \right) \right], \\ Z_{\text{vector}} &= \text{PE} \left[-\frac{q_R + q_R}{(q_1^{1/2} - q_1^{-1/2})(q_2^{1/2} - q_2^{-1/2})} Q_G^* \left(\frac{Q_\tau}{1 - Q_\tau} \right) \right], \\ Z_{\text{hyper}} &= \text{PE} \left[+\frac{1}{(q_1^{1/2} - q_1^{-1/2})(q_2^{1/2} - q_2^{-1/2})} Q_G^* Q_F^* \left(\frac{Q_\tau}{1 - Q_\tau} \right) \right], \end{aligned} \quad (2.16)$$

where $Q_\tau = e^\tau$,⁹ while $Q_G = e^{\mathbf{a}}$, $Q_F = e^{\mathbf{m}_G}$ are gauge and flavor symmetry fugacities, with powers $*$ appropriate charges of the supermultiplets. The plethystic exponential is defined as

$$\text{PE} [f(\cdot)] = \exp \left[\sum_{n=1}^{\infty} \frac{1}{n} f(\cdot^n) \right]. \quad (2.17)$$

⁶We send the volume of the curve class Σ_{r+1} which intersects with the base to infinity.

⁷Here we normalize it so that the longest root has norm square 2.

⁸To be in line with the refined Gopakumar-Vafa formula of topological string theory [41, 42], we suppress a term of 1/2 in [40].

⁹Note our convention here differs from the usual convention in the mathematics literature by a factor of $2\pi i$.

In the case of 6d minimal SCFTs, there is no contributions from charged hypermultiplets, while the contributions of tensor multiplets only depend on the mass parameter τ and no other Kähler moduli and can thus be factored out of the blowup equations. Therefore in this paper we only consider contributions of 6d vector multiplets to $Z^{1\text{-loop}}$. The spectrum of vector multiplets and thus their total contribution to $Z^{1\text{-loop}}$ can be computed by the refined topological string theory

$$\begin{aligned}
 & Z^{1\text{-loop}}(\mathbf{t}, \epsilon_1, \epsilon_2) \\
 &= \prod_{\Sigma \in H_2^{\text{vert}}(X, \mathbb{Z})} \prod_{k_L = -j_L}^{j_L} \prod_{k_R = -j_R}^{j_R} \prod_{m_1, m_2 = 1}^{\infty} \left(1 - t^{k_L + k_R + m_1 - \frac{1}{2}} q^{k_L - k_R + m_2 - \frac{1}{2}} \mathbf{Q}^\Sigma \right)^{M_\Sigma^{(j_L, j_R)}} \quad (2.18)
 \end{aligned}$$

where¹⁰

$$q = e^{\epsilon_1}, \quad t = e^{-\epsilon_2}, \quad M_\Sigma^{(j_L, j_R)} = (-1)^{2(j_L + j_R)} N_{j_L, j_R}^\Sigma, \quad (2.19)$$

with $(j_L, j_R) = (0, 1/2)$ for vector multiplets. Here $H_2^{\text{vert}}(X, \mathbb{Z})$ is the homology group of compact curves in the vertical direction, and it is generated by Σ_i ($i = 1, \dots, r + 1$). Using $N_{0, 1/2}^\Sigma = 1$, we obtain

$$\begin{aligned}
 Z^{1\text{-loop}}(\mathbf{t}, \epsilon_1, \epsilon_2) &= \prod_{\alpha \in \widehat{\Delta}_+} \prod_{i, j=0}^{\infty} (1 - t^i q^{j+1} Q^\alpha)^{-1} (1 - t^{i+1} q^j Q^\alpha)^{-1} \\
 &= \text{PE} \left[- \frac{q_R + q_R^{-1}}{(q_1^{1/2} - q_1^{-1/2})(q_2^{1/2} - q_2^{-1/2})} \sum_{\alpha \in \widehat{\Delta}_+} e^{\langle \alpha, \mathbf{a} \rangle} \right] \quad (2.20)
 \end{aligned}$$

where $\widehat{\Delta}_+$ is the set of positive roots of the affine Lie algebra. By the identification of the imaginary root with the elliptic fiber and (2.5), the expression for $Z^{1\text{-loop}}$ is equivalent to

$$Z^{1\text{-loop}}(\mathbf{t}, \epsilon_1, \epsilon_2) = \text{PE} \left[- \frac{q_R + q_R^{-1}}{(q_1^{1/2} - q_1^{-1/2})(q_2^{1/2} - q_2^{-1/2})} \sum_{\alpha \in \Delta_+} \left(e^{\langle \alpha, \mathbf{a} \rangle} + Q_\tau e^{-\langle \alpha, \mathbf{a} \rangle} \right) \frac{1}{1 - Q_\tau} \right]. \quad (2.21)$$

2.5 Examples

2.5.1 6d SU(3) theory

The non-compact Calabi-Yau threefold X underlying the 6d SU(3) model on the Omega background is the elliptic fibration over $\mathcal{O}(-3) \rightarrow \mathbb{P}^1$ with the singular fiber resolved. As explained in [37], there are $b = 4$ compact irreducible curves and $g = 3$ compact irreducible divisors. The latter D_1, D_2, D_3 are three \mathbb{F}_1 surfaces in the vertical direction intersecting at a common (-1) -curve $\Sigma_4 = \Sigma_b$, which projects to the (-3) -curve in the base. The other three curves Σ_i ($i = 1, 2, 3$) are the \mathbb{P}^1 fibers of the Hirzebruch surfaces. This geometry

¹⁰Here $M_\Sigma^{(j_L, j_R)}$ differs from that in [42] by 1 in order for the contributions of vector multiplets to be in the denominator, as they should.

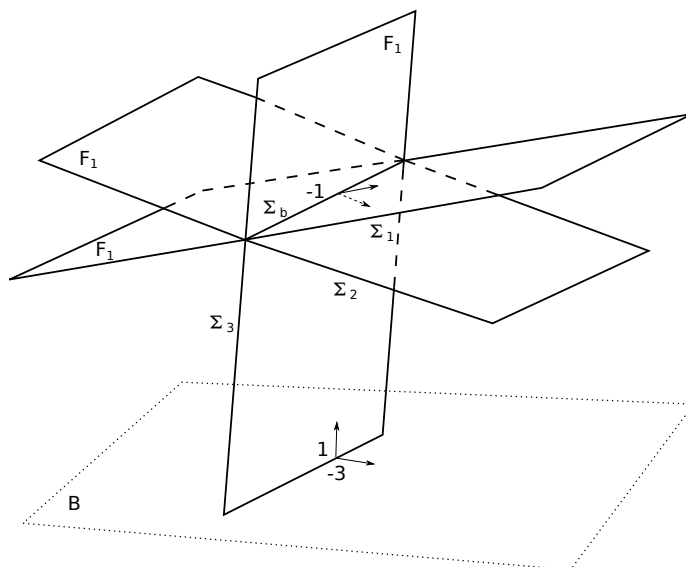


Figure 2. Compact curves and compact divisors in the elliptic fibration over $\mathcal{O}(-3) \rightarrow \mathbb{P}^1$.

is illustrated in figure 2. The intersection matrix of the curves Σ_i ($i = 1, \dots, 4$) and the divisors D_j ($j = 1, \dots, 3$) is

$$C = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \\ -1 & -1 & -1 \end{pmatrix}, \tag{2.22}$$

in accord with the general structure (2.4).

The understanding of the embedding of the curves Σ_i in surfaces in X allows us to write down the \mathbf{B} field

$$\mathbf{B} = (0, 0, 0, 1), \tag{2.23}$$

following the discussion in section 2.2.

To compute the perturbative prepotential $F^{\text{pert},(0,0)}$, we follow [11] and take X as the decompactification limit of the compact Calabi-Yau threefold \widehat{X} , the elliptic fibration over \mathbb{F}_3 , along the horizontal direction perpendicular to the (-3) -curve in the base. The compact model \widehat{X} can be realized as a hypersurface in a toric variety. Therefore its triple intersection numbers and thus the perturbative prepotential can be computed with the usual techniques in toric geometry (see for instance [43]). Then $F^{\text{pert},(0,0)}$ of the non-compact model is obtained by integrating over the periods which remain finite in the decompactification limit [44]. In this way, we find

$$F_{6d, \text{SU}(3)}^{\text{pert},(0,0)}(\mathbf{t}, \epsilon_1, \epsilon_2) = -\frac{1}{18}(t_1^3 + t_2^3 + t_3^3) - \frac{1}{6}t_b(t_1^2 + t_2^2 + t_3^2) - \frac{1}{6}t_b^2(t_1 + t_2 + t_3). \tag{2.24}$$

Keep in mind we use the convention $t_b = t_4$.

We can further decompactify X along the direction of the elliptic fiber by sending the volume of one of the curves in the vertical direction to infinity. Let us take the limit¹¹ $t_3 \rightarrow \infty$ and after following the same procedure of integrating over finite periods, we obtain

$$F_{5d, \text{SU}(3)}^{\text{pert}, (0,0)}(\mathbf{t}, \epsilon_1, \epsilon_2) = -\frac{1}{18}(t_1^3 + t_2^3) - \frac{1}{6}t_b(t_1^2 + t_2^2) - \frac{1}{6}t_b^2(t_1 + t_2) + \frac{1}{18}t_b^3. \quad (2.25)$$

This should coincide with the perturbative Nekrasov partition function for 5d $N = 1$ pure SYM with $G = \text{SU}(3)$. Combining (2.13) and (2.14), the latter reads

$$F_{5d, \text{SU}(3)}^{\text{pert}, (0,0), \text{Nek}}(\mathbf{t}, \epsilon_1, \epsilon_2) = -\frac{t_1^3}{3} - \frac{t_1^2 t_2}{2} - \frac{t_1 t_2^2}{2} - \frac{t_2^3}{3} - t_m \left(\frac{t_1^2}{3} + \frac{t_1 t_2}{3} + \frac{t_2^2}{3} \right). \quad (2.26)$$

To identify the mass parameter t_m in (2.25), we first write down the curve-divisor intersection \mathbf{C} -matrix of the 5d theory

$$C = \begin{pmatrix} -2 & 1 \\ 1 & -2 \\ -1 & -1 \end{pmatrix}, \quad (2.27)$$

which can be obtained by removing in the 6d \mathbf{C} -matrix (2.22) the third row corresponding to Σ_3 and the third column corresponding to the divisor D_3 containing Σ_3 . We find the curve $\Sigma_b - \Sigma_1 - \Sigma_2$ does not intersect with any compact divisor. The corresponding mass parameter for the 5d theory should be

$$t_m = t_b - t_1 - t_2. \quad (2.28)$$

With this identification, it is easy to see that $F_{5d, \text{SU}(3)}^{\text{pert}, (0,0), \text{Nek}}$ indeed coincides with $F_{5d, \text{SU}(3)}^{\text{pert}, (0,0)}$ from decompactification up to a pure mass parameter term

$$F_{5d, \text{SU}(3)}^{\text{pert}, (0,0)}(\mathbf{t}, \epsilon_1, \epsilon_2) - F_{5d, \text{SU}(3)}^{\text{pert}, (0,0), \text{Nek}}(\mathbf{t}, \epsilon_1, \epsilon_2) = \frac{1}{18}t_m^3. \quad (2.29)$$

We also notice that the 6d and the 5d perturbative prepotentials only differ by

$$F_{6d, \text{SU}(3)}^{(0,0)}(\mathbf{t}, \epsilon_1, \epsilon_2) - F_{5d, \text{SU}(3)}^{(0,0)}(\mathbf{t}, \epsilon_1, \epsilon_2) = -\frac{(t_3 + t_b)^3}{18} = -\frac{(\tau + t_m)^3}{18}. \quad (2.30)$$

This implies that the decompactification limit is really obtained by

$$t_3 + t_b = \tau + t_m \rightarrow -\infty, \quad Q_\tau Q_m \rightarrow 0, \quad t_b, t_m \text{ finite}. \quad (2.31)$$

This observation allow us to write down the perturbative contributions to genus one free energies of the 6d model. By writing down a generic linear ansatz for $F^{\text{pert}, (1,0)}$, $F^{\text{pert}, (0,1)}$, separating out $t_3 + t_b$, and demanding the remaining piece coincides with the second line

¹¹Since we wish to obtain the corresponding 5d gauge theory, we should decompactify the vertical curve which intersects with the base [37]. Nevertheless since the Kähler moduli of the three vertical curves appear to be on the equal footing in $F_{6d, \text{SU}(3)}^{\text{pert}, (0,0)}$, we can choose any of them to decompactify, keeping the others intact.

of (2.13), as well as imposing symmetry between t_1, t_2, t_3 , we fix $F^{\text{pert},(1,0)}, F^{\text{pert},(0,1)}$ for the 6d SU(3) model uniquely to be

$$\begin{aligned}
 F_{6d, \text{SU}(3)}^{\text{pert},(1,0)}(\mathbf{t}, \epsilon_1, \epsilon_2) &= -F_{6d, \text{SU}(3)}^{\text{pert},(0,1)}(\mathbf{t}, \epsilon_1, \epsilon_2) \\
 &= -\frac{t_1}{8} - \frac{t_2}{8} - \frac{t_3}{8} - \frac{t_b}{6}.
 \end{aligned}
 \tag{2.32}$$

The difference from the 5d free energy is

$$F_{6d, \text{SU}(3)}^{(1,0)} - F_{5d, \text{SU}(3)}^{(1,0), \text{Nek}} = -\left(F_{6d, \text{SU}(3)}^{(0,1)} - F_{5d, \text{SU}(3)}^{(0,1), \text{Nek}}\right) = -\frac{1}{8}(t_3 + t_b) = -\frac{1}{8}(\tau + t_m). \tag{2.33}$$

By specializing (2.21), we get the one-loop partition function

$$\begin{aligned}
 &Z_{6d, \text{SU}(3)}^{1\text{-loop}} \\
 &= \text{PE} \left[-\frac{q_R^{1/2} + q_R^{-1/2}}{(q_1^{1/2} - q_1^{-1/2})(q_2^{1/2} - q_2^{-1/2})} \frac{1}{1 - Q_\tau} (Q_1 + Q_2 + Q_3 + Q_1 Q_2 + Q_1 Q_3 + Q_2 Q_3) \right].
 \end{aligned}
 \tag{2.34}$$

2.5.2 6d SO(8) theory

The non-compact Calabi-Yau threefold X underlying the 6d SO(8) model on the Omega background is the elliptic fibration over $\mathcal{O}(-4) \rightarrow \mathbb{P}^1$ with the singular fiber resolved. As explained in [37] there are $b = 6$ compact irreducible curves and $g = 5$ compact irreducible divisors. The divisors $D_1, D_2, D_3, D_4, D_5 = D_c$ are Hirzebruch surfaces $\mathbb{F}_2, \mathbb{F}_2, \mathbb{F}_2, \mathbb{F}_2, \mathbb{F}_0$ linking up with each other like the Dynkin diagram of $\widehat{\mathfrak{so}}(8)$, where D_c plays the role of the central node, while D_4 lays the role of the affine node and intersects with the base. D_1, D_2, D_3, D_4 intersect with D_c by the (-2) curves which are all homologously equivalent on D_c . We take this curve to be $\Sigma_6 = \Sigma_b$. The remaining irreducible curves Σ_i ($i = 1, \dots, 4$) and $\Sigma_5 = \Sigma_c$ are the \mathbb{P}^1 fibers of D_i ($i = 1, \dots, 4$) and D_c . This geometry is illustrated in figure 3. The intersection matrix of the curves Σ_i ($i = 1, \dots, 6$) and the divisors D_j ($j = 1, \dots, 5$) is

$$C = \begin{pmatrix} -2 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & -2 & 1 \\ 1 & 1 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}, \tag{2.35}$$

in accord with the general structure (2.4).

Next, with the picture in figure 3 we can write down the \mathbf{B} field

$$\mathbf{B} = (0, 0, 0, 0, 0, 0), \tag{2.36}$$

following the discussion in section 2.2. As in the case of SU(3) theory, we compute the perturbative prepotential $F^{\text{pert},(0,0)}$ by following [11] and taking X as the decompactification limit of the compact Calabi-Yau threefold \widehat{X} , the elliptic fibration over \mathbb{F}_4 , along the horizontal direction perpendicular to the (-4) curve in the base. The compact model \widehat{X} can

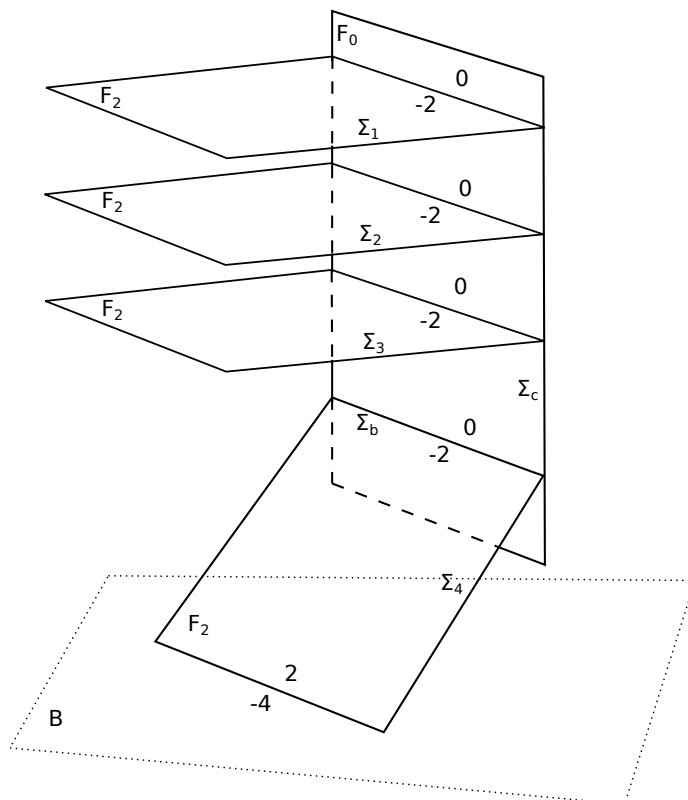


Figure 3. Compact curves and compact divisors in the elliptic fibration over $\mathcal{O}(-4) \rightarrow \mathbb{P}^1$.

also be realized as a hypersurface in a toric variety. Therefore its triple intersection numbers and thus the perturbative prepotential are readily computable using usual techniques in toric geometry. We then obtain $F^{\text{pert},(0,0)}$ of the non-compact model by integrating over finite periods in the decompactification limit [44] and we get

$$F_{6d,SO(8)}^{(0,0)} = -\frac{1}{6} (t_1^3 + t_2^3 + t_3^3 + t_4^3) - \frac{1}{4} t_b (t_1^2 + t_2^2 + t_3^2 + t_4^2) - \frac{1}{8} t_b^2 (t_1 + t_2 + t_3 + t_4 + 2t_c). \quad (2.37)$$

Keep in mind we use the convention $t_b = t_6, t_c = t_5$.

We further decompactify X along the direction of the elliptic fiber by sending $t_4 \rightarrow -\infty$. After following the same procedure of integrating over finite periods, we obtain

$$F_{5d,SO(8)}^{(0,0)}(\mathbf{t}, \epsilon_1, \epsilon_2) = -\frac{1}{6} (t_1^3 + t_2^3 + t_3^3) - \frac{t_b}{4} (t_1^2 + t_2^2 + t_3^2) - \frac{t_b^2}{8} (t_1 + t_2 + t_3 + 2t_c) + \frac{t_b^3}{48}. \quad (2.38)$$

This should coincide with the perturbative Nekrasov partition function for 5d $N = 1$ pure SYM with $G = SO(8)$. Combining (2.13) and (2.14), the latter reads

$$\begin{aligned} & F_{5d,SO(8)}^{(0,0),\text{Nek}}(\mathbf{t}, \epsilon_1, \epsilon_2) \\ &= -t_1^3 - t_2^3 - t_3^3 - \frac{8t_c^3}{3} - 4t_c^2(t_1 + t_2 + t_3) - 3t_c(t_1^2 + t_2^2 + t_3^2) - 4t_c(t_1t_2 + t_1t_3 + t_2t_3) \\ &\quad - \frac{3}{2}(t_1^2t_2 + t_1t_2^2 + t_1^2t_3 + t_1t_3^2 + t_1t_3^2 + t_2^2t_3) - 2t_1t_2t_3 \\ &\quad - \frac{t_m}{2}(t_1^2 + t_2^2 + t_3^2 + 2t_c^2 + 2t_c(t_1 + t_2 + t_3) + t_1t_2 + t_1t_3 + t_2t_3). \end{aligned} \quad (2.39)$$

To identify the mass parameter t_m in (2.38), we first obtain the curve-divisor intersection \mathbf{C} -matrix of the 5d theory

$$C = \begin{pmatrix} -2 & 0 & 0 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & -2 & 1 \\ 1 & 1 & 1 & -2 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad (2.40)$$

which is done by removing in the 6d \mathbf{C} -matrix (2.35) the fourth row corresponding to Σ_4 and the fourth column corresponding to the divisor D_4 containing Σ_4 . We find the curve $\Sigma_b - 2\Sigma_1 - 2\Sigma_2 - 2\Sigma_3 - 4\Sigma_c$ does not intersect with any compact divisor. The corresponding mass parameter for the 5d theory should be

$$t_m = t_b - 2t_1 - 2t_2 - 2t_3 - 4t_c. \quad (2.41)$$

With this identification, it is easy to see that $F_{5d,SO(8)}^{\text{pert,(0,0),Nek}}$ indeed coincides with $F_{5d,SO(8)}^{\text{pert,(0,0)}}$ from decompactification up to a pure mass parameter term

$$F_{5d,SO(8)}^{(0,0)} - F_{5d,SO(8)}^{(0,0),\text{Nek}} = \frac{t_m^3}{48}. \quad (2.42)$$

We also notice that the 6d and the 5d perturbative prepotentials only differ by

$$F_{6d,SO(8)}^{(0,0)}(\mathbf{t}, \epsilon_1, \epsilon_2) - F_{5d,SO(8)}^{(0,0)}(\mathbf{t}, \epsilon_1, \epsilon_2) = -\frac{1}{6} \left(t_4 + \frac{1}{2}t_b \right)^3 = -\frac{1}{6} \left(\tau + \frac{1}{2}t_m \right)^3. \quad (2.43)$$

This implies that the decompactification limit is really obtained by

$$t_4 + \frac{1}{2}t_b = \tau + \frac{1}{2}t_m \rightarrow -\infty, \quad Q_\tau Q_m^{1/2} \rightarrow 0, \quad t_b, t_m \text{ finite}. \quad (2.44)$$

This observation allow us to write down the perturbative contributions to genus one free energies of the 6d model. By writing down a generic linear ansatz for $F^{\text{pert,(1,0)}}$, $F^{\text{pert,(0,1)}}$, separating out $t_4 + 1/2t_b$, and demanding the remaining piece coincides with the second line of (2.13), as well as imposing symmetry between t_1, t_2, t_3, t_4 , we fix $F^{\text{pert,(1,0)}}$, $F^{\text{pert,(0,1)}}$ for the 6d $SO(8)$ model uniquely to be

$$\begin{aligned} F_{6d,SO(8)}^{(1,0)}(\mathbf{t}, \epsilon_1, \epsilon_2) &= -F_{6d,SO(8)}^{(0,1)}(\mathbf{t}, \epsilon_1, \epsilon_2) \\ &= -\frac{t_1}{3} - \frac{t_2}{3} - \frac{t_3}{3} - \frac{t_4}{3} - \frac{t_c}{2} - \frac{t_b}{4}. \end{aligned} \quad (2.45)$$

The difference from the 5d theory is

$$F_{6d,SO(8)}^{(1,0)} - F_{5d,SO(8)}^{(1,0),\text{Nek}} = -\left(F_{6d,SO(8)}^{(0,1)} - F_{5d,SO(8)}^{(0,1),\text{Nek}} \right) = -\frac{1}{3} \left(t_4 + \frac{1}{2}t_b \right) = -\frac{1}{3} \left(\tau + \frac{1}{2}t_m \right). \quad (2.46)$$

Finally, by specializing (2.21), we get the 1-loop partition function

$$\begin{aligned} Z_{6d,SO(8)}^{1\text{-loop}} = \text{PE} &\left[-\frac{q_R^{1/2} + q_R^{-1/2}}{(q_1^{1/2} - q_1^{-1/2})(q_2^{1/2} - q_2^{-1/2})} \frac{1}{1 - Q_\tau} \left(Q_c^2 \sum_{1 \leq i < j < k \leq 4} Q_i Q_j Q_k \right. \right. \\ &\left. \left. + Q_c \left(1 + \sum_{i=1}^4 Q_i + \sum_{1 \leq i < j \leq 4} Q_i Q_j + \sum_{1 \leq i < j < k \leq 4} Q_i Q_j Q_k + Q_1 Q_2 Q_3 Q_4 \right) + \sum_{i=1}^4 Q_i \right) \right]. \end{aligned} \quad (2.47)$$

3 Elliptic genera from blowup equations

In this section we put everything together, and demonstrate for the 6d SU(3) and SO(8) SCFTs the validity of blowup equations order by order in an expansion in Q_b with the help of the well-known results of the elliptic genera of these two theories [12, 38]. Then we reverse the logic and show that the blowup equations can be used to solve the elliptic genera and BPS invariants, illustrating the power of blowup equations in the studies of 6d SCFTs.

3.1 Constraint on \mathbf{r} fields

We first find a mild condition on the \mathbf{r} field and argue that the number of inequivalent and admissible \mathbf{r} fields satisfying this condition is finite.

We first rewrite the blowup equation (1.6) by moving the unshifted partition function $\widehat{Z}(\mathbf{t}, \epsilon_1, \epsilon_2)$ to the other side of the equation

$$\widehat{Z}(\mathbf{t}, \epsilon_1, \epsilon_2)^{-1} \sum_{\mathbf{n} \in \mathbb{Z}^g} (-1)^{|\mathbf{n}|} \widehat{Z}(\mathbf{t} + \epsilon_1 \mathbf{R}, \epsilon_1, \epsilon_2 - \epsilon_1) \widehat{Z}(\mathbf{t} + \epsilon_2 \mathbf{R}, \epsilon_1 - \epsilon_2, \epsilon_2) = \Lambda(\tau, \epsilon_1, \epsilon_2), \quad (3.1)$$

where the dependence on \mathbf{r} is always understood, and

$$\mathbf{R} = \mathbf{C} \cdot \mathbf{n} + \mathbf{r}/2. \quad (3.2)$$

We have also used the fact that τ is the only mass parameter. When the l.h.s. of the blowup equations are expanded in terms of the Kähler moduli $Q_i = e^{t_i}$ ($i = 1, \dots, r+1$), $Q_{r+2} = Q_b = e^{t_b}$, the perturbative partition function Z^{pert} determines the leading order terms. The contributions of Z^{pert} to the l.h.s of (3.1) reads

$$\begin{aligned} & \log \left(Z^{\text{pert}}(\epsilon_1, \epsilon_2 - \epsilon_1) Z^{\text{pert}}(\epsilon_1 - \epsilon_2, \epsilon_2) / Z^{\text{pert}}(\epsilon_1, \epsilon_2) \right) \\ &= (\epsilon_1 + \epsilon_2) \left(-\frac{1}{6} \sum_{i,j,k=1}^{r+2} \kappa_{ijk} R_i R_j R_k + \sum_{i=1}^{r+2} (b_i^{\text{GV}} - b_i^{\text{NS}}) R_i \right) + \sum_{k=1}^{r+2} \left(-\frac{1}{2} \sum_{i,j=1}^{r+2} \kappa_{ijk} R_i R_j \right) t_k \\ &=: f_0(\mathbf{n}) + \sum_{k=1}^{r+2} f_k(\mathbf{n}) t_k, \end{aligned} \quad (3.3)$$

where we have used (2.15). For the blowup equations to hold at the leading order of Q_k , we must have

$$\sum_{\mathbf{n} \in \cap_{k=1}^g I_k} (-1)^{|\mathbf{n}|} e^{f_0(\mathbf{n})} e^{f_k(\mathbf{n}) t_k} = \Lambda(\tau, \epsilon_1, \epsilon_2). \quad (3.4)$$

Here I_k is the set of integral vectors \mathbf{n} that minimize $f_k(\mathbf{n})$ for true Kähler moduli (not mass parameters). The latter can be written as

$$\begin{aligned} f_k(\mathbf{n}) &= -\frac{1}{2} \sum_{i,j} \kappa_{ijk} \left(\sum_{\ell} C_{i\ell} n_{\ell} + \frac{1}{2} r_i \right) \left(\sum_m C_{jm} n_m + \frac{1}{2} r_j \right) \\ &= -\frac{1}{2} \sum_{\ell,m} \left(\sum_{i,j} \kappa_{ijk} C_{i\ell} C_{jm} \right) n_{\ell} n_m - \frac{1}{2} \sum_{\ell} \left(\sum_{i,j} \kappa_{ijk} r_i C_{j\ell} \right) n_{\ell} - \frac{1}{8} \sum_{i,j} \kappa_{ijk} r_i r_j. \end{aligned} \quad (3.5)$$

Note that the functions $f_k(\mathbf{n})$ for $k = 1, \dots, r$ and $f_0(\mathbf{n})$ are dependent on the \mathbf{r} field as well.

In order for the blowup equation to make sense, $f_k(\mathbf{n})$ as functions of the integral vector \mathbf{n} must have a minimum for any $k = 1, \dots, r + 2$, which allows us to determine the valid \mathbf{r} fields [34]. We find that in the case of the $n = 3, 4$ theories, the valid \mathbf{r} fields satisfy

$$\sum_{i=1}^{r+1} a_i r_i = 0, \tag{3.6}$$

in other words, the entry of the \mathbf{r} field corresponding to τ is zero. With this feature we can write down the blowup equations as identities of Jacobi forms, as we will see in section 3.2, and proceed to prove the blowup equations order by order in terms of the Q_b expansion using the modularity argument. We will call the \mathbf{r} fields satisfying the constraint (3.6) *admissible*.

Interestingly, we notice that (3.6) is equivalent to a slightly stronger condition that $f_b(\mathbf{n}) = f_{r+2}(\mathbf{n})$ in particular has a minimum for a real vector \mathbf{n} .¹² To see this, let us recall that κ_{ijk} are intersection numbers of divisors K_i dual to the curve class Σ_i , satisfying

$$K_i \cdot \Sigma_j = \delta_{ij}. \tag{3.7}$$

In the elliptic Calabi-Yau threefold underlying a 6d minimal SCFT with a bulk pure gauge theory, the curves Σ_i ($i = 1, \dots, r + 2$) have mutual intersection numbers identical to minus the Cartan matrix of the affine lie algebra

$$(\Sigma_i \cdot \Sigma_j)_b = -A_{ij}, \quad i, j = 1, \dots, r + 1, \tag{3.8}$$

(the subscript b means restriction to K_b) when restricted to the vertical divisor K_b perpendicular to Σ_b , which is in fact the Poincaré dual¹³ of Σ_b . Let D_i ($i = 1, \dots, r + 1$) be the irreducible compact divisors coming from Σ_i fibered over the \mathbb{P}^1 in the base. We should thus have

$$D_i \cdot K_b = \Sigma_i, \quad i = 1, \dots, r + 1. \tag{3.9}$$

We note that the two sets of divisors D_i and K_j should be related by the \mathbf{C} -matrix

$$D_i = \sum_{\ell=1}^{r+2} K_\ell C_{\ell i}, \tag{3.10}$$

so that (1.2) still holds.

Let us come back to the discussion of the functions $f_k(\mathbf{n})$. Take the direction $k = r + 2 = b$. The coefficients can then be explicitly evaluated

$$\sum_{i,j=1}^{r+2} \kappa_{ijb} C_{il} C_{jm} = D_\ell \cdot D_m \cdot K_b = (\Sigma_\ell \cdot \Sigma_m)_b = -A_{\ell m}, \tag{3.11}$$

$$\sum_{i,j=1}^{r+2} \kappa_{ijb} r_i C_{j\ell} = \sum_{i=1}^{r+2} r_i K_i \cdot D_\ell \cdot K_b = \sum_{i=1}^{r+2} r_i K_i \cdot \Sigma_\ell = r_\ell, \tag{3.12}$$

¹²This condition is stronger because if a minimal real \mathbf{n} exists, a minimal integral \mathbf{n} must exist nearby; on the other hand, if a minimal integral \mathbf{n} exists, there can be a non-integral flat direction of $f_b(\mathbf{n})$.

¹³We understand that the Poincaré is only rigorously defined in a compact manifold, while the elliptic Calabi-Yau threefold here is non-compact. So we are presenting here an argument not a proof. We also checked the validity of (3.6) for the SU(3), SO(8) as well as some other 6d gauge theories.

and the function $f_b(\mathbf{n})$ reads¹⁴

$$f_b(\mathbf{n}) = \frac{1}{2} \sum_{\ell,m=1}^{r+1} A_{\ell m} n_\ell n_m - \frac{1}{2} \sum_{\ell=1}^{r+1} r_\ell n_\ell - \frac{1}{8} \sum_{i,j=1}^{r+2} \kappa_{ij,b} r_i r_j. \quad (3.13)$$

If $f_b(\mathbf{n})$ can be minimized for real values of \mathbf{n} , its derivatives with respect to components of \mathbf{n} should have a common zero. These equations are encapsulated in a single linear equation

$$\mathbf{A} \cdot \mathbf{n} = \frac{1}{2} \mathbf{r}. \quad (3.14)$$

For this linear equation to have a solution, the \mathbf{r} field must be annihilated by vectors in the (left) kernel of \mathbf{A} . Since \mathbf{A} is the Cartan of $\hat{\mathfrak{g}}$, there is only one vector,

$$\mathbf{l} = (a_i), \quad i = 1, \dots, r+1, \quad (3.15)$$

which annihilates \mathbf{A} when multiplied from the left. We thus get $\mathbf{l} \cdot \mathbf{r} = 0$, which is the condition (3.6).

Let us now give a counting of inequivalent and admissible \mathbf{r} fields. Given the condition (1.8), we can parameterize \mathbf{r} fields by

$$\mathbf{r} = \mathbf{B} + 2\mathbf{v}, \quad \mathbf{v} \in \mathbb{Z}^b, \quad (3.16)$$

and the equivalence condition (1.9) is translated to

$$\mathbf{v} - \mathbf{v}' = \mathbf{C} \cdot \mathbf{n}'. \quad (3.17)$$

The domain of inequivalent \mathbf{v} -vectors, defined to be the lattice \mathbb{Z}^b modulo the equivalence relation (3.17), has only a finite number of points along g directions, and extends freely along the remaining $b - g$ directions. In practise, we can always make linear combinations of curve classes so that the last $b - g$ rows of the intersection matrix \mathbf{C} are empty, i.e.

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}_{\text{sub}} \\ 0 \end{pmatrix}, \quad (3.18)$$

and the $g \times g$ submatrix \mathbf{C}_{sub} is of full rank. The Kähler moduli of the curve classes corresponding to the first g rows of \mathbf{C} are true moduli, while the Kähler moduli of those for the remaining rows are mass parameters. Inequivalent \mathbf{v} -vectors take the form

$$\mathbf{v} = (v_1, \dots, v_g, *, *, \dots) \quad (3.19)$$

where $v_i, i = 1, \dots, g$ can only take a finite number of integral values, while the remaining entries denoted by $*$ can take any value in \mathbb{Z} . The equivalence condition for the truncated \mathbf{v} -vectors defined by

$$\bar{\mathbf{v}} = (v_1, \dots, v_g), \quad (3.20)$$

¹⁴This equation and one below need slight modification if the bulk gauge group is not of the *ADE* type, as we will see in the companion paper that discusses more general cases.

reads

$$\bar{\mathbf{v}} - \bar{\mathbf{v}}' = \mathbf{C}_{\text{sub}} \cdot \mathbf{n}' . \tag{3.21}$$

Clearly the matrix \mathbf{C}_{sub} defines a lattice embedding $\mathbb{Z}^g \hookrightarrow \mathbb{Z}^g$. Therefore the number of inequivalent truncated \mathbf{v} -vectors is $\det \mathbf{C}_{\text{sub}}$.

In the case of 6d minimal SCFT with a pure bulk gauge theory, there is only one mass parameter τ corresponding to the volume of elliptic fiber, and we have seen that the corresponding entry of \mathbf{r} field must be zero. Therefore, the number of inequivalent and admissible \mathbf{r} fields must be identical with that of inequivalent truncated \mathbf{v} -vectors, which is $\det \mathbf{C}_{\text{sub}}$. We mentioned in the previous sections that \mathbf{C} barring the last row is identified with the opposite of the affine Cartan matrix. Hence in practice, we can construct the full rank square submatrix \mathbf{C}_{sub} of \mathbf{C} by throwing away the row corresponding to the affine node in the Dynkin diagram.¹⁵ Once all the inequivalent truncated \mathbf{v} -vectors are found, we can convert them to \mathbf{r} fields with the help of (3.16) and (3.6). In the 6d $n = 3, 4$ minimal SCFTs, we find that all admissible \mathbf{r} fields give rise to valid blowup equations.

3.2 Recursion relations

Here we derive recursion relations of elliptic genera from the blowup equations. Later when we discuss individual models in sections 3.3 and 3.4, we will demonstrate the validity of these recursion relations and then inverse the logic solving elliptic genera from these relations.

The blowup equations for the partition function of a 6d SCFT can be written as follows

$$\sum_{\mathbf{n} \in \mathbb{Z}^g} A(\mathbf{t}, \epsilon_1, \epsilon_2; \mathbf{n}) \widehat{Z}^{\text{ell}}(\mathbf{t} + \epsilon_1 \mathbf{R}, \epsilon_1, \epsilon_2 - \epsilon_1) \widehat{Z}^{\text{ell}}(\mathbf{t} + \epsilon_2 \mathbf{R}, \epsilon_1 - \epsilon_2, \epsilon_2) = \Lambda(\tau, \epsilon_1, \epsilon_2) \widehat{Z}^{\text{ell}}(\mathbf{t}, \epsilon_1, \epsilon_2) , \tag{3.22}$$

where it is understood that everything depends on the choice of \mathbf{r} field. Here Z^{ell} is the generating function of elliptic genera

$$Z^{\text{ell}}(\mathbf{t}, \epsilon_1, \epsilon_2) = 1 + \sum_{k=1}^{\infty} Q_b^k Z_k(t_\ell, \epsilon_1, \epsilon_2) , \tag{3.23}$$

where Z_k is proportional to the k -string elliptic genus with a model-dependent prefactor, and it only depends on Kähler moduli of vertical curves t_ℓ , $\ell = 1, \dots, r + 1$. When Z^{ell} is twisted with \mathbf{t} shifted to $\mathbf{t} + \pi i \mathbf{B}$, Q_b is multiplied with a phase $(-1)^{B_b}$, while Z_k is unchanged, as t_ℓ are volumes of (-2) curves and thus the corresponding entries of \mathbf{B} vanish according to the discussion in section 2.2. The function A is given by

$$A(\mathbf{t}, \epsilon_1, \epsilon_2; \mathbf{n}) = (-1)^{|\mathbf{n}| + (k_1 + k_2 - k) B_b} D^{\text{pert}}(\mathbf{t}, \epsilon_1, \epsilon_2; \mathbf{n}) D^{1\text{-loop}}(\mathbf{t}, \epsilon_1, \epsilon_2; \mathbf{n}) \tag{3.24}$$

¹⁵Note the mark associated to the affine node is 1. We can construct another full rank \mathbf{C}'_{sub} by throwing away a different row corresponding to a different node. If the associated mark a_i is greater than 1, the number of truncated \mathbf{v} -vectors might be larger. But we cannot recover integral \mathbf{r} fields from all integral truncated \mathbf{v} -vectors because of (3.6). In the end the number of integral \mathbf{r} fields is still the determinant of \mathbf{C}_{sub} constructed from throwing away the affine node.

including the perturbative contribution

$$D^{\text{pert}}(\mathbf{t}, \epsilon_1, \epsilon_2; \mathbf{n}) = Q_b^{f_b(\mathbf{n})} \exp \left[f_0(\mathbf{n})(\epsilon_1 + \epsilon_2) + \sum_{\ell=1}^{r+1} f_\ell(\mathbf{n})t_\ell \right], \quad (3.25)$$

and the one-loop contribution

$$D^{1\text{-loop}}(t_\ell, \epsilon_1, \epsilon_2; \mathbf{n}) = \frac{Z^{1\text{-loop}}(t_\ell + \epsilon_1 R_\ell, \epsilon_1, \epsilon_2 - \epsilon_1) Z^{1\text{-loop}}(t_\ell + \epsilon_2 R_\ell, \epsilon_1 - \epsilon_2, \epsilon_2)}{Z^{1\text{-loop}}(t_\ell, \epsilon_1, \epsilon_2)}, \quad (3.26)$$

with f_0, f_ℓ defined in (3.3). We don't have to put hat over $Z^{1\text{-loop}}$ because entries of \mathbf{B} associated to t_ℓ ($\ell = 1, \dots, r+1$) are all zero. By comparing the coefficients of Q_b^k on both sides of (3.22), we find the recursion relation

$$\begin{aligned} & \Lambda(\tau, \epsilon_1, \epsilon_2) Z_k(t_\ell, \epsilon_1, \epsilon_2) \\ &= \sum_{f_b(\mathbf{n})+k_1+k_2=k} (-1)^{|\mathbf{n}|+(k_1+k_2-k)B_b} \exp \left[f_0(\mathbf{n})(\epsilon_1 + \epsilon_2) + \sum_{\ell=1}^{r+1} f_\ell(\mathbf{n})t_\ell + (k_1\epsilon_1 + k_2\epsilon_2)R_b(\mathbf{n}) \right] \\ & \times D^{1\text{-loop}}(t_\ell, \epsilon_1, \epsilon_2, \mathbf{n}) Z_{k_1}(t_\ell + \epsilon_1 R_\ell, \epsilon_1, \epsilon_2 - \epsilon_1) Z_{k_2}(t_\ell + \epsilon_2 R_\ell, \epsilon_1 - \epsilon_2, \epsilon_2). \end{aligned} \quad (3.27)$$

The expression above can be simplified due to the following observation. Given the expression (3.13) of $f_b(\mathbf{n})$ and the condition (3.6) on the \mathbf{r} field, it is clear that $f_b(\mathbf{n})$ is invariant under the shift

$$\mathbf{n} \rightarrow \mathbf{n} + m\mathbf{a}, \quad m \in \mathbb{Z}, \quad (3.28)$$

where $\mathbf{a} = (a_k)$ the vector of marks. Similarly

$$R_k = \sum_{\ell=1}^{r+1} C_{k,\ell} n_\ell + \frac{1}{2} r_k = -A_{k,\ell} n_\ell + \frac{1}{2} r_k \quad (3.29)$$

for $k = 1, \dots, r+1$ is also invariant under the integral shift (3.28). Therefore, we could define representatives $\hat{\mathbf{n}}$ of \mathbf{n} by

$$\mathbf{n} = \hat{\mathbf{n}} + m\mathbf{a}, \quad m \in \mathbb{Z} \quad (3.30)$$

so that no two representatives differ by $\mathbf{a}\mathbb{Z}$. Then the summation in (3.27) can be split into two steps

$$\begin{aligned} & \Lambda(\tau, \epsilon_1, \epsilon_2) Z_k(t_\ell, \epsilon_1, \epsilon_2) \\ &= \sum_{f_b(\hat{\mathbf{n}})+k_1+k_2=k} \left(\sum_{m \in \mathbb{Z}} (-1)^{|\mathbf{n}|+(k_1+k_2-k)B_b} \exp \left[f_0(\mathbf{n})(\epsilon_1 + \epsilon_2) + \sum_{\ell=1}^{r+1} f_\ell(\mathbf{n})t_\ell + (k_1\epsilon_1 + k_2\epsilon_2)R_b(\mathbf{n}) \right] \right) \\ & \times D^{1\text{-loop}}(t_\ell, \epsilon_1, \epsilon_2, \hat{\mathbf{n}}) Z_{k_1}(t_\ell + \epsilon_1 R_\ell(\hat{\mathbf{n}}), \epsilon_1, \epsilon_2 - \epsilon_1) Z_{k_2}(t_\ell + \epsilon_2 R_\ell(\hat{\mathbf{n}}), \epsilon_1 - \epsilon_2, \epsilon_2), \end{aligned} \quad (3.31)$$

where the summation of m gives a theta function. These are the equations whose validity we will demonstrate order by order through a modularity argument in the following example sections.

In the remainder of this section, we illustrate how to derive elliptic genera from the recursion relation (3.31). We start with $k = 0$. Given the expression (3.13) for $f_b(\mathbf{n})$, its value is already non-negative since the affine Cartan matrix $A_{\ell m}$ is positive semi-definite. If the minimal value of $f_b(\mathbf{n})$ is greater than zero, the identity (3.31) could not hold at $k = 0$ unless $\Lambda = 0$. Thus we should have a vanishing blowup equation. If the minimal value of $f_b(\mathbf{n})$ is zero, there is a chance that (3.31) is satisfied at $k = 0$ and we get a non-vanishing Λ , which should result in a unity blow up equation. In the latter case, using $Z_0 = 1$, we find the following expression for Λ

$$\Lambda(\tau, \epsilon_1, \epsilon_2) = \sum_{\hat{\mathbf{n}} \in \hat{I}_b} D^{1\text{-loop}}(t_\ell, \epsilon_1, \epsilon_2, \hat{\mathbf{n}}) \sum_{m \in \mathbb{Z}} (-1)^{|\mathbf{n}|} \exp \left[f_0(\mathbf{n})(\epsilon_1 + \epsilon_2) + \sum_{\ell=1}^{r+1} f_\ell(\mathbf{n}) t_\ell \right], \quad (3.32)$$

where \hat{I}_b is the set of representatives $\hat{\mathbf{n}}$ which minimize f_b to zero.

Let us now focus on unity blowup equations. As we will see in example sections, the associated \mathbf{r} fields have zero entries except for r_b : $\mathbf{r} = (0, \dots, 0, r_b)$; besides, one can always choose the representative $\hat{\mathbf{n}}$ in \hat{I}_b to be the null vector. As a result, t_ℓ are not shifted in Z_{k_1}, Z_{k_2} if either $k_1 = k$ or $k_2 = k$. We can thus put the unity recursion relations for $k \geq 1$ in a more explicit form

$$Z_k(t_\ell, \epsilon_1, \epsilon_2) = Z_k(t_\ell, \epsilon_1, \epsilon_2 - \epsilon_1) J_k^{(1)}(\tau, \epsilon_1, \epsilon_2) + Z_k(t_\ell, \epsilon_1 - \epsilon_2, \epsilon_2) J_k^{(2)}(\tau, \epsilon_1, \epsilon_2) + I_k(t_\ell, \epsilon_1, \epsilon_2). \quad (3.33)$$

The coefficients are

$$J_k^{(1)}(\tau, \epsilon_1, \epsilon_2) = \frac{\Lambda_k^{(1)}(\tau, \epsilon_1, \epsilon_2)}{\Lambda(\tau, \epsilon_1, \epsilon_2)}, \quad J_k^{(2)}(\tau, \epsilon_1, \epsilon_2) = \frac{\Lambda_k^{(2)}(\tau, \epsilon_1, \epsilon_2)}{\Lambda(\tau, \epsilon_1, \epsilon_2)}, \quad (3.34)$$

with

$$\Lambda_k^{(1)}(\tau, \epsilon_1, \epsilon_2) = \sum_{\hat{\mathbf{n}} \in \hat{I}_b} \sum_{m \in \mathbb{Z}} (-1)^{|\mathbf{n}|} Q_\tau^{f_\tau(\mathbf{n})} (q_1 q_2)^{f_0(\mathbf{n})} q_1^{k R_b(\mathbf{n})} \quad (3.35)$$

$$\Lambda_k^{(2)}(\tau, \epsilon_1, \epsilon_2) = \sum_{\hat{\mathbf{n}} \in \hat{I}_b} \sum_{m \in \mathbb{Z}} (-1)^{|\mathbf{n}|} Q_\tau^{f_\tau(\mathbf{n})} (q_1 q_2)^{f_0(\mathbf{n})} q_2^{k R_b(\mathbf{n})}. \quad (3.36)$$

I_k is the summation on the r.h.s. of (3.31) with $k_1, k_2 < k$, and thus are known data in a recursive calculation.

The relations (3.33) can be solved to give compact expressions of Z_k , following the procedure in [35] for a similar calculation for 5d gauge theories. The key observation is that $Z_k(t_\ell, \epsilon_1, \epsilon_2), Z_k(t_\ell, \epsilon_1, \epsilon_2 - \epsilon_1), Z_k(t_\ell, \epsilon_1 - \epsilon_2, \epsilon_2)$ do not depend on the choice of \mathbf{r} fields. If there are at least three unity \mathbf{r} fields, we can pick three of them, and write down three equations of the form (3.33), and combine them into the linear system

$$\begin{pmatrix} -\Lambda(r_1) & \Lambda_k^{(1)}(r_1) & \Lambda_k^{(2)}(r_1) \\ -\Lambda(r_2) & \Lambda_k^{(1)}(r_2) & \Lambda_k^{(2)}(r_2) \\ -\Lambda(r_3) & \Lambda_k^{(1)}(r_3) & \Lambda_k^{(2)}(r_3) \end{pmatrix} \cdot \begin{pmatrix} Z_k(\epsilon_1, \epsilon_2) \\ Z_k(\epsilon_1, \epsilon_2 - \epsilon_2) \\ Z_k(\epsilon_1 - \epsilon_2, \epsilon_2) \end{pmatrix} = - \begin{pmatrix} \Lambda(r_1) I_k(r_1) \\ \Lambda(r_2) I_k(r_2) \\ \Lambda(r_3) I_k(r_3) \end{pmatrix}. \quad (3.37)$$

unity	(0, 0, 0, -1)	(0, 0, 0, 1)	(0, 0, 0, 3)
vanishing	(2, -2, 0, 1)	(-2, 0, 2, 1)	(0, 2, -2, 1)
	(0, -2, 2, 1)	(2, 0, -2, 1)	(-2, 2, 0, 1)

Table 1. The list of all inequivalent and admissible \mathbf{r} fields for 6d SU(3) gauge theory.

Here we only write down the most important parameters each function depends on. If the matrix M_{Λ_k} of coefficients $\Lambda, \Lambda_k^{(1)}, \Lambda_k^{(2)}$ on the l.h.s. of the linear system is of full rank, it can be inverted to give us the answer of Z_k in terms of I_k and thus lower base degree partition functions. We will demonstrate that this method also works for 6d theories, except for the solution of Z_1 for the SU(3) model.

3.3 SU(3) theory

3.3.1 Base degree zero

By combining (2.22) and (2.23) and following section 3.1, we find all the inequivalent and admissible \mathbf{r} fields. By analysing the recursion relation (3.31) at $k = 0$ we can divide the resulting blowup equations into unity and vanishing equations, as listed in table 1.

For the unity blowup equations, at base degree $k = 0$ they reduce to the computation of Λ . For all the three \mathbf{r} fields $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ of unity blowup equations in the first row of table 1 there is only one $\hat{\mathbf{n}} = (0, 0, 0)$ which minimize $f_b(\mathbf{n})$. Then using (3.32) we find the following results for Λ

$$\Lambda(\mathbf{r}_1) = \sum_{n \in \mathbb{Z}} (-1)^n Q^{\frac{3}{2}n^2 + \frac{1}{2}n + \frac{1}{24}} (q_1 q_2)^{n + \frac{1}{6}} = e^{-\frac{\pi i}{6}} \theta_4^{\left[\frac{1}{6}\right]}(3\tau, \epsilon_1 + \epsilon_2), \quad (3.38)$$

$$\Lambda(\mathbf{r}_2) = \sum_{n \in \mathbb{Z}} (-1)^n Q^{\frac{3}{2}n^2 - \frac{1}{2}n + \frac{1}{24}} (q_1 q_2)^{n - \frac{1}{6}} = e^{\frac{\pi i}{6}} \theta_4^{\left[-\frac{1}{6}\right]}(3\tau, \epsilon_1 + \epsilon_2), \quad (3.39)$$

$$\Lambda(\mathbf{r}_3) = \sum_{n \in \mathbb{Z}} (-1)^n Q^{\frac{3}{2}n^2 - \frac{3}{2}n + \frac{3}{8}} (q_1 q_2)^{n - \frac{1}{2}} = e^{\frac{\pi i}{2}} \theta_4^{\left[-\frac{1}{2}\right]}(3\tau, \epsilon_1 + \epsilon_2). \quad (3.40)$$

Here and later in this paper we use the following notation of Jacobi theta functions with characteristics

$$\theta_3^{[\alpha]}(\tau, z) = \sum_{n \in \mathbb{Z}} e^{\frac{1}{2}\tau(n+\alpha)^2 + z(n+\alpha)}, \quad (3.41)$$

$$\theta_4^{[\alpha]}(\tau, z) = \sum_{n \in \mathbb{Z}} e^{\frac{1}{2}\tau(n+\alpha)^2 + (z+\pi i)(n+\alpha)}. \quad (3.42)$$

Indeed all the Λ only depend on τ and no other Kähler moduli.¹⁶

¹⁶We notice that Λ are all Jacobi forms of weight 1/2 with respect to the modular group acting on τ . This is a subgroup of the monodromy group Γ of the total modular space, and we thus seem to have a contradiction with the claim [34] that Λ is supposed to have weight zero with respect to Γ . To reconcile them, we recall that we have thrown away the contribution of the tensor multiplet to $Z^{1\text{-loop}}$. It is not difficult to verify with the help of the identities in appendix A that if included it contributes to an additional factor $\eta(\tau)^{-1}$ to Λ reducing the weight of the latter to zero.

As for the vanishing blowup equations, we only have to check one of them, as their \mathbf{r} fields are related to each other by S_3 symmetry acting on the first three entries. Consider the \mathbf{r} field $(-2, 2, 0, 1)$. There are three sets of \mathbf{n} which minimize $f_b(\mathbf{n})$ represented by

$$\hat{I}_b = \{(0, 0, 0), (-1, 0, 0), (0, 1, 0)\}. \quad (3.43)$$

At the lowest base degree with $k_1 = k_2 = 0$, the elliptic genera contribute trivially with $Z_0 = 1$. The recursion relation only takes contributions from perturbative and 1-loop partition functions from the first line of (3.31). Summing over $\hat{\mathbf{n}}$ in (3.43), the recursion relation (3.31) at lowest base degree reads

$$\Theta_{(0,0,0)}\theta_{(0,0,0)} + \Theta_{(-1,0,0)}\theta_{(-1,0,0)} + \Theta_{(0,1,0)}\theta_{(0,1,0)} = 0, \quad (3.44)$$

where Θ s encapsulate contributions from perturbative partition functions and have the form

$$\begin{aligned} \Theta_{(0,0,0)} &= \sum_{n=-\infty}^{\infty} (-1)^n Q_{\tau}^{\frac{3}{2}(n-\frac{1}{6})^2} Q_1^n Q_2^{-n+\frac{1}{3}} = e^{\frac{\pi i}{6}} Q_1^{\frac{1}{6}} Q_2^{\frac{1}{6}} \theta_4^{[-\frac{1}{6}]}(3\tau, t_1 - t_2), \\ \Theta_{(-1,0,0)} &= \sum_{n=-\infty}^{\infty} (-1)^n Q_{\tau}^{\frac{3}{2}(n-\frac{1}{6})^2} Q_1^{-2n+1} Q_2^{-n+\frac{1}{3}} = e^{\frac{\pi i}{6}} Q_1^{\frac{2}{3}} Q_2^{\frac{1}{6}} \theta_4^{[-\frac{1}{6}]}(3\tau, -2t_1 - t_2), \\ \Theta_{(0,1,0)} &= \sum_{n=-\infty}^{\infty} (-1)^n Q_{\tau}^{\frac{3}{2}(n-\frac{1}{6})^2} Q_1^n Q_2^{2n+\frac{1}{3}} = e^{\frac{\pi i}{6}} Q_1^{\frac{1}{6}} Q_2^{\frac{2}{3}} \theta_4^{[-\frac{1}{6}]}(3\tau, t_1 + 2t_2), \end{aligned} \quad (3.45)$$

while θ s encapsulate contributions from one-loop partition functions and have the form

$$\begin{aligned} \theta_{(0,0,0)} &= \text{PE} \left[(Q_1 + Q_2 + Q_1 Q_3 + Q_2 Q_3) \frac{1}{1 - Q_{\tau}} \right] = -\frac{Q_1^{-\frac{1}{2}} Q_2^{-\frac{1}{2}} Q_{\tau}^{\frac{1}{6}} \eta(\tau)^2}{\theta_1(\tau, t_1) \theta_1(\tau, t_2)}, \\ \theta_{(-1,0,0)} &= \text{PE} \left[(Q_1 + Q_3 + Q_1 Q_2 + Q_2 Q_3) \frac{1}{1 - Q_{\tau}} \right] = -\frac{Q_1^{-1} Q_2^{-\frac{1}{2}} Q_{\tau}^{\frac{1}{6}} \eta(\tau)^2}{\theta_1(\tau, t_1) \theta_1(\tau, t_1 + t_2)}, \\ \theta_{(0,1,0)} &= \text{PE} \left[(Q_2 + Q_3 + Q_1 Q_2 + Q_1 Q_3) \frac{1}{1 - Q_{\tau}} \right] = -\frac{Q_1^{-\frac{1}{2}} Q_2^{-1} Q_{\tau}^{\frac{1}{6}} \eta(\tau)^2}{\theta_1(\tau, t_2) \theta_1(\tau, t_1 + t_2)}. \end{aligned} \quad (3.46)$$

In the derivation of these expressions, the identities in appendix A are very useful. With the reparametrisation

$$t_1 = v_1 - v_2, \quad t_2 = v_2 - v_3 \quad (3.47)$$

subject to $v_1 + v_2 + v_3 = 0$, the identity (3.44) can be written as

$$\theta_4^{[-\frac{1}{6}]}(3\tau, -3v_1) \theta_1(\tau, v_2 - v_3) + \theta_4^{[-\frac{1}{6}]}(3\tau, -3v_2) \theta_1(\tau, v_3 - v_1) + \theta_4^{[-\frac{1}{6}]}(3\tau, -3v_3) \theta_1(\tau, v_1 - v_2) = 0. \quad (3.48)$$

This last identity can be proved by noticing that each term and therefore the total sum is a Jacobi form¹⁷ for $\Gamma(3)$ of weight 1 and index polynomial

$$\frac{1}{2}(v_1^2 + v_2^2 + v_3^2 - 2v_1 v_2 - 2v_2 v_3 - 2v_3 v_1), \quad (3.49)$$

and by verifying that the first few terms in the Q_{τ} expansion vanish, which we have checked up to very high orders.

¹⁷Strictly speaking, this is a component of a vector-valued Jacobi form.

3.3.2 Modularity at generic base degree

Here we give an argument for the validity of the recursion relation (3.31) by demonstrating that both sides of the equation (3.31) are multivariate meromorphic Jacobi forms of the same weight and index polynomial at any base degree k . Once this is achieved, after multiplying both sides of (3.31) with the common denominator, we get an identity of multivariate weak Jacobi forms of the same weight and index, which can then be proved by plugging in the expression of Z_k given in [12] and comparing the first few terms in the Q_τ expansion.

Consider a blowup equation with $\mathbf{r} = (r_1, r_2, r_3, r_b)$ subject to the condition $r_1 + r_2 + r_3 = 0$. The perturbative contribution to the recursion (3.31) is

$$\begin{aligned} D^{\text{pert},'} &:= \exp \left[f_0(\mathbf{n})(\epsilon_1 + \epsilon_2) + \sum_{\ell=1}^{r+1} f_\ell(\mathbf{n})t_\ell + (k_1\epsilon_1 + k_2\epsilon_2)R_b(\mathbf{n}) \right] \\ &= Q_1^{f_1(\mathbf{n})} Q_2^{f_2(\mathbf{n})} Q_3^{f_3(\mathbf{n})} (q_1 q_2)^{f_0(\mathbf{n})} \left(q_1^{k_1} q_2^{k_2} \right)^{R_b(\mathbf{n})} \end{aligned} \quad (3.50)$$

where

$$f_\ell(\mathbf{n}) = \frac{3}{2} \left(n_\ell - \frac{r_\ell + r_b}{6} \right)^2, \quad \ell = 1, 2, 3 \quad (3.51)$$

$$\begin{aligned} f_0(\mathbf{n}) &= \frac{\sum_{\ell=1}^3 r_\ell^3}{144} + r_b \left(-\frac{1}{6} + \frac{\sum_{\ell=1}^3 r_\ell^2}{48} \right) + \frac{n_1 + n_2 + n_3}{3} - \frac{\sum_{\ell=1}^3 r_\ell^2 n_\ell}{8} - \frac{r_b}{4} \sum_{\ell=1}^3 r_\ell n_\ell \\ &\quad + \frac{3}{4} \sum_{\ell=1}^3 r_\ell n_\ell^2 + \frac{r_b}{2} \left(\sum_{\ell=1}^3 n_\ell^2 - \sum_{1 \leq \ell < m \leq 3} n_\ell n_m \right) - \frac{4}{3} \sum_{\ell=1}^3 n_\ell^3 + \frac{1}{2} \sum_{\ell \neq m} n_\ell^2 n_m + n_1 n_2 n_3, \end{aligned} \quad (3.52)$$

$$R_b = -n_1 - n_2 - n_3 + \frac{r_b}{2}. \quad (3.53)$$

In addition

$$f_b(\mathbf{n}) = \frac{1}{24} \sum_{\ell=1}^3 r_\ell^2 - \frac{1}{2} \sum_{\ell=1}^3 r_\ell n_\ell - \sum_{1 \leq \ell < m \leq 3} n_\ell n_m + \sum_{\ell=1}^3 n_\ell^2. \quad (3.54)$$

Following the discussion in section 3.2, we can split \mathbf{n} to a representative $\hat{\mathbf{n}}$, which we uniquely fix by setting $n_3 = 0$, and (m, m, m) , i.e.

$$\mathbf{n} = (n_1, n_2, 0) + (m, m, m). \quad (3.55)$$

Then $D^{\text{pert},'}$ can be written as

$$\begin{aligned} &D^{\text{pert},'} \\ &= Q_\tau^{\frac{3}{2} \left(m - \frac{r_3 + r_b}{6} \right)^2} \left(Q_1^{n_1 + \frac{r_3 - r_1}{6}} Q_2^{n_2 + \frac{r_3 - r_2}{6}} \right)^3 \left(m - \frac{r_3 + r_b}{6} \right) (q_1 q_2)^{\left(m - \frac{r_3 + r_b}{6} \right) (1 - 3f_b(\hat{\mathbf{n}}))} \left(q_1^{k_1} q_2^{k_2} \right)^{-3 \left(m - \frac{r_3 + r_b}{6} \right)} \\ &\quad \times Q_1^{\frac{3}{2} \left(n_1 + \frac{r_3 - r_1}{6} \right)^2} Q_2^{\frac{3}{2} \left(n_2 + \frac{r_3 - r_2}{6} \right)^2} (q_1 q_2)^{f_0(\hat{\mathbf{n}}) + \frac{r_3 + r_b}{6} (1 - 3f_b(\hat{\mathbf{n}}))} \left(q_1^{k_1} q_2^{k_2} \right)^{-n_1 - n_2 - \frac{r_3}{2}}. \end{aligned} \quad (3.56)$$

For the contributions from vector multiplets, using the notation (A.7), we have

$$\begin{aligned}
 D^{1\text{-loop}} = \text{PE} \left[& - (Bl_{(0,1/2,-2n_1+n_2+\frac{r_1}{2})}(q_1, q_2)Q_1 + Bl_{(0,1/2,2n_1-n_2+\frac{r_2+r_3}{2})}(q_1, q_2)Q_2Q_3 \right. \\
 & + Bl_{(0,1/2,n_1-2n_2+\frac{r_2}{2})}(q_1, q_2)Q_2 + Bl_{(0,1/2,-n_1+2n_2+\frac{r_1+r_3}{2})}(q_1, q_2)Q_1Q_3 \\
 & \left. + Bl_{(0,1/2,n_1+n_2+\frac{r_3}{2})}(q_1, q_2)Q_3 + Bl_{(0,1/2,-n_1-n_2+\frac{r_1+r_2}{2})}(q_1, q_2)Q_1Q_2) / (1 - Q_\tau) \right], \tag{3.57}
 \end{aligned}$$

where we used the property that $D^{1\text{-loop}}(\mathbf{n}) = D^{1\text{-loop}}(\hat{\mathbf{n}})$ and set $n_3 = 0$. Then noticing $Q_\tau = Q_1Q_2Q_3$ and using the notation (A.14), we have

$$\begin{aligned}
 D^{1\text{-loop}} &= T_{-2n_1+n_2+\frac{r_1}{2}}(t_1) T_{n_1-2n_2+\frac{r_2}{2}}(t_2) T_{-n_1-n_2+\frac{r_1+r_2}{2}}(t_1+t_2) \\
 &= (-1)^{f_b(\hat{\mathbf{n}})} Q_\tau^{\frac{f_b(\hat{\mathbf{n}})}{2}} Q_1^{-f_b(\hat{\mathbf{n}})-\frac{3}{2}\left(n_1+\frac{r_3-r_1}{6}\right)^2} Q_2^{-f_b(\hat{\mathbf{n}})-\frac{3}{2}\left(n_2+\frac{r_3-r_2}{6}\right)^2} (q_1q_2)^{d_R(\mathbf{a}_2)} \\
 &\quad \times \check{\theta}_{-2n_1+n_2+\frac{r_1}{2}}(t_1) \check{\theta}_{n_1-2n_2+\frac{r_2}{2}}(t_2) \check{\theta}_{-n_1-n_2+\frac{r_1+r_2}{2}}(t_1+t_2), \tag{3.58}
 \end{aligned}$$

where

$$\begin{aligned}
 d_R(\mathbf{a}_2) &= \frac{4n_1^3+4n_2^3}{3} - \frac{n_1n_2(n_1+n_2)}{2} - r_1 \frac{5n_1^2-2n_1n_2+2n_2^2}{4} - r_2 \frac{2n_1^2-2n_1n_2+5n_2^2}{4} \\
 &\quad - \frac{n_1+n_2}{3} + \frac{3r_1^2n_1+3r_2^2n_2}{8} + \frac{r_1r_2(n_1+n_2)}{4} + \frac{r_1+r_2}{6} - \frac{r_1^3+r_2^3}{24} - \frac{r_1r_2(r_1+r_2)}{16}. \tag{3.59}
 \end{aligned}$$

In Appendix A, we show $\check{\theta}_R(t)$ is a meromorphic Jacobi form of weight 0, and thus so is $D^{1\text{-loop}}$ up to the prefactor.

Finally, given the relation between Z_k and the k -string elliptic genus \mathbb{E}_k for the SU(3) theory

$$Z_k(t_\ell, \epsilon_1, \epsilon_2) = \left(\frac{Q_\tau^{1/2}}{Q_1Q_2} \right)^k \mathbb{E}_k(t_\ell, \epsilon_1, \epsilon_2), \tag{3.60}$$

we also have

$$\begin{aligned}
 Z_{k_1}(t_\ell + \epsilon_1 R_\ell, \epsilon_1, \epsilon_2 - \epsilon_1) &= q_1^{k_1\left(n_1+n_2-\frac{r_1+r_2}{2}\right)} \left(\frac{Q_\tau^{1/2}}{Q_1Q_2} \right)^{k_1} \mathbb{E}_{k_1}(t_\ell + \epsilon_1 R_\ell, \epsilon_1, \epsilon_2 - \epsilon_1), \\
 Z_{k_2}(t_\ell + \epsilon_2 R_\ell, \epsilon_1 - \epsilon_2, \epsilon_2) &= q_2^{k_2\left(n_1+n_2-\frac{r_1+r_2}{2}\right)} \left(\frac{Q_\tau^{1/2}}{Q_1Q_2} \right)^{k_2} \mathbb{E}_{k_2}(t_\ell + \epsilon_2 R_\ell, \epsilon_1 - \epsilon_2, \epsilon_2), \tag{3.61}
 \end{aligned}$$

for the last two factors.

Combining (3.56), (3.56) and (3.56) all together, we find the following expression for the r.h.s. of the recursion relation (3.31) in terms of (meromorphic) Jacobi forms

$$\begin{aligned}
 \text{r.h.s.} &= e^{\frac{\pi i(r_3+r_b)}{6}} \left(\frac{Q_\tau^{1/2}}{Q_1 Q_2} \right)^k \sum_{f_b(\mathbf{n})+k_1+k_2=k} (-1)^{n_1+n_2} \\
 &\times \theta_4^{[-(r_3+r_b)/6]} \left(3\tau, 3 \left(\left(n_1 + \frac{r_3-r_1}{6} \right) t_1 + \left(n_2 + \frac{r_3-r_2}{6} \right) t_2 \right) \right. \\
 &\quad \left. + (1-3(k-k_2))\epsilon_1 + (1-3(k-k_1))\epsilon_2 \right) \\
 &\times \check{\theta}_{-2n_1+n_2+\frac{r_1}{2}}(t_1) \check{\theta}_{n_1-2n_2+\frac{r_2}{2}}(t_2) \check{\theta}_{-n_1-n_2+\frac{r_1+r_2}{2}}(t_1+t_2) \\
 &\times \mathbb{E}_{k_1}(t_\ell + \epsilon_1 R_\ell, \epsilon_1, \epsilon_2 - \epsilon_1) \mathbb{E}_{k_2}(t_\ell + \epsilon_2 R_\ell, \epsilon_1 - \epsilon_2, \epsilon_2). \tag{3.62}
 \end{aligned}$$

Using the fact that the elliptic genus $\mathbb{E}_k(t_\ell, \epsilon_1, \epsilon_2)$ is a meromorphic Jacobi form of weight zero and index polynomial [13, 38]

$$\text{Ind}(\mathbb{E}_k) = -\frac{k}{2}(\epsilon_1^2 + \epsilon_2^2) + \frac{3k(k-1)}{2}\epsilon_1\epsilon_2 - \frac{3k}{2}(\mathbf{a}, \mathbf{a})_{\mathbf{a}_2}, \tag{3.63}$$

where

$$(\mathbf{a}, \mathbf{a})_{\mathbf{a}_2} = \frac{2}{3}(t_1^2 + t_1 t_2 + t_2^2) = v_1^2 + v_2^2 + v_3^2. \tag{3.64}$$

It is not difficult to show that up to the common prefactor every term in (3.62) is a meromorphic Jacobi form¹⁸ for $\Gamma(3)$ of weight 1/2 and index polynomial

$$\text{Ind}(\text{r.h.s.}) = \frac{3k-1}{6}(\epsilon_1^2 + \epsilon_2^2) + \frac{(3k-2)(3k-1)}{6}\epsilon_1\epsilon_2 - \frac{3k}{2}(\mathbf{a}, \mathbf{a})_{\mathbf{a}_2}, \tag{3.65}$$

which is independent of the summation indices n_1, n_2, k_1, k_2 , and thus so is the total sum.

On the other hand, if the blowup equation is of vanishing type, the l.h.s. of (3.31) vanish; if the blowup equation is of unity type, we have $\mathbf{r} = (0, 0, 0, r_b)$, and after plugging in the expression of Λ , we find the l.h.s. of (3.31) to be

$$\text{l.h.s.} = e^{\frac{\pi i r_b}{6}} \left(\frac{Q_\tau^{1/2}}{Q_1 Q_2} \right)^k \theta_4^{[-\frac{r_b}{6}]}(3\tau, \epsilon_1 + \epsilon_2) \mathbb{E}_k(t_\ell, \epsilon_1, \epsilon_2), \tag{3.66}$$

which is also a meromorphic Jacobi form of the same weight and the same index (3.65) up to the same prefactor. In both cases, after multiplied with a common denominator, the recursion relations (3.31) can be cast as identities of (weak) Weyl invariant Jacobi forms of identical weights and indices. As the ring of Jacobi forms is finitely generated, these identities can be proved by checking that when the *correct* \mathbf{r} (table 1) are plugged in the first few terms in Q_τ expansion are correct. For instance when $k = 0$ we find (3.62) indeed reduces to the computation of Λ in the unity cases and the identity (3.48) in the vanishing cases. When $k = 1$ and with a unity \mathbf{r} plugged in, the recursion relations reduce to

$$\begin{aligned}
 &\theta_4^{[-\frac{r_b}{6}]}(3\tau, -2\epsilon_1 + \epsilon_2) \mathbb{E}_1(\mathbf{v}, \epsilon_1, \epsilon_2 - \epsilon_1) + \theta_4^{[-\frac{r_b}{6}]}(3\tau, \epsilon_1 - 2\epsilon_2) \mathbb{E}_1(\mathbf{v}, \epsilon_1 - \epsilon_2, \epsilon_2) \\
 &- \theta_4^{[-\frac{r_b}{6}]}(3\tau, \epsilon_1 + \epsilon_2) \mathbb{E}_1(\mathbf{v}, \epsilon_1, \epsilon_2) + I_1^{[-\frac{r_b}{6}]}(\epsilon_1, \epsilon_2) = 0, \tag{3.67}
 \end{aligned}$$

¹⁸The same as footnote 17.

in which

$$I_1^{[-\frac{r_b}{6}]}(\epsilon_1, \epsilon_2) = - \sum_{i \neq j \neq k} \frac{\theta_4^{[-\frac{r_b}{6}]}(3\tau, 3v_{ij} - 2\epsilon_1 - 2\epsilon_2) \eta^6}{\theta_1(v_{ij}) \theta_1(v_{ij} - \epsilon_1) \theta_1(v_{ij} - \epsilon_2) \theta_1(v_{ij} - \epsilon_1 - \epsilon_2) \theta_1(v_{ik}) \theta_1(v_{jk})}. \quad (3.68)$$

We have used the variables $\mathbf{v} = (v_1, v_2, v_3)$ for Kähler moduli defined in (3.47) to make the Weyl symmetry of $SU(3)$ more transparent, and we define $v_{ij} = v_i - v_j$. Here and in the following we suppress the modular parameter of a theta function if it is simply τ . One can readily verify by using the expression (3.51) of \mathbb{E}_1 in [12] and by the first few terms in the Q_τ expansion that (3.67) is valid if r_b is odd but not if r_b is even. Similarly when the vanishing $\mathbf{r} = (-2, 2, 0, 1)$ is plugged in, we find the identity

$$\begin{aligned} & \sum_{i=1}^3 \frac{\theta_4^{[-\frac{1}{6}]}(3\tau, -3v_i - 3\epsilon_1) \eta^2}{\prod_{j \neq i} \theta_1(v_{ij})} \mathbb{E}_1(\mathbf{v} + \epsilon_1 \mathbf{R}_v^{\{i\}}, \epsilon_1, \epsilon_2 - \epsilon_1) \\ & + \sum_{i=1}^3 \frac{\theta_4^{[-\frac{1}{6}]}(3\tau, -3v_i - 3\epsilon_2) \eta^2}{\prod_{j \neq i} \theta_1(v_{ij})} \mathbb{E}_1(\mathbf{v} + \epsilon_2 \mathbf{R}_v^{\{i\}}, \epsilon_1 - \epsilon_2, \epsilon_2) \\ & - \sum_{i=1}^3 \frac{\theta_4^{[-\frac{1}{6}]}(3\tau, 6v_i - 3\epsilon_1 - 3\epsilon_2) \eta^8}{\prod_{j \neq i} \theta_1(v_{ij}) \theta_1(v_{ij} - \epsilon_1) \theta_1(v_{ij} - \epsilon_2) \theta_1(v_{ij} - \epsilon_1 - \epsilon_2)} = 0, \end{aligned} \quad (3.69)$$

with the \mathbf{R} fields

$$\begin{aligned} \mathbf{R}_v^{\{1\}} &= \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3} \right), \\ \mathbf{R}_v^{\{2\}} &= \left(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3} \right), \\ \mathbf{R}_v^{\{3\}} &= \left(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3} \right), \end{aligned} \quad (3.70)$$

which can also be explicitly verified. The cases of the other vanishing \mathbf{r} fields can be obtained by permutations of the components r_1, r_2, r_3 of the \mathbf{r} field. We have also checked the cases of $k = 2$. Identities for $k > 2$ can be checked in a similar manner.

3.3.3 Recursion formula for elliptic genera

We would like to inverse the logic and illustrate here that it is possible to solve \mathbb{E}_k ($k \geq 2$) for the $SU(3)$ theory from the recursion relations (3.33) following the argument at the end of section 3.2. In the case of 6d $SU(3)$ theory, there are three unity \mathbf{r} fields. The corresponding $\Lambda_k^{(1)}, \Lambda_k^{(2)}$ are

- $\mathbf{r}_1 = (0, 0, 0, -1)$

$$\begin{aligned} \Lambda_k^{(1)}(\mathbf{r}_1) &= \sum_{n \in \mathbb{Z}} (-1)^n Q_\tau^{\frac{3}{2}n^2 + \frac{1}{2}n + \frac{1}{24}} (q_1^{-3k+1} q_2)^{n + \frac{1}{6}} = e^{-\frac{\pi i}{6}} \theta_4^{[\frac{1}{6}]}(3\tau, -(3k-1)\epsilon_1 + \epsilon_2), \\ \Lambda_k^{(2)}(\mathbf{r}_1) &= \sum_{n \in \mathbb{Z}} (-1)^n Q_\tau^{\frac{3}{2}n^2 + \frac{1}{2}n + \frac{1}{24}} (q_1 q_2^{-3k+1})^{n + \frac{1}{6}} = e^{-\frac{\pi i}{6}} \theta_4^{[\frac{1}{6}]}(3\tau, \epsilon_1 - (3k-1)\epsilon_2). \end{aligned} \quad (3.71)$$

- $\mathbf{r}_2 = (0, 0, 0, 1)$

$$\begin{aligned}\Lambda_k^{(1)}(\mathbf{r}_2) &= \sum_{n \in \mathbb{Z}} (-1)^n Q_\tau^{\frac{3}{2}n^2 - \frac{1}{2}n + \frac{1}{24}} (q_1^{-3k+1} q_2)^{n - \frac{1}{6}} = e^{\frac{\pi i}{6}} \theta_4^{[-\frac{1}{6}]}(3\tau, -(3k-1)\epsilon_1 + \epsilon_2), \\ \Lambda_k^{(2)}(\mathbf{r}_2) &= \sum_{n \in \mathbb{Z}} (-1)^n Q_\tau^{\frac{3}{2}n^2 - \frac{1}{2}n + \frac{1}{24}} (q_1 q_2^{-3k+1})^{n - \frac{1}{6}} = e^{\frac{\pi i}{6}} \theta_4^{[-\frac{1}{6}]}(3\tau, \epsilon_1 - (3k-1)\epsilon_2).\end{aligned}\quad (3.72)$$

- $\mathbf{r}_3 = (0, 0, 0, 3)$

$$\begin{aligned}\Lambda_k^{(1)}(\mathbf{r}_3) &= \sum_{n \in \mathbb{Z}} (-1)^n Q_\tau^{\frac{3}{2}n^2 - \frac{3}{2}n + \frac{3}{8}} (q_1^{-3k+1} q_2)^{n - \frac{1}{2}} = e^{\frac{\pi i}{2}} \theta_4^{[-\frac{1}{2}]}(3\tau, -(3k-1)\epsilon_1 + \epsilon_2), \\ \Lambda_k^{(2)}(\mathbf{r}_3) &= \sum_{n \in \mathbb{Z}} (-1)^n Q_\tau^{\frac{3}{2}n^2 - \frac{3}{2}n + \frac{3}{8}} (q_1 q_2^{-3k+1})^{n - \frac{1}{2}} = e^{\frac{\pi i}{2}} \theta_4^{[-\frac{1}{2}]}(3\tau, \epsilon_1 - (3k-1)\epsilon_2).\end{aligned}\quad (3.73)$$

Surprisingly, we find that at base degree one the matrix M_{Λ_1} is actually not of full-rank. Therefore one cannot invert M_{Λ_1} to solve Z_1 from the recursion relation.

That $\det M_{\Lambda_1} = 0$ may have something to do with the curious coincidence that while the characteristics of the theta functions enjoy a \mathbb{Z}_3 symmetry connected to the gauge group $SU(3)$, the elliptic parameters of theta functions enjoy some S_3 symmetry. Note that the three types of theta functions

$$\theta_4^{[\frac{1}{6}]}, \quad \theta_4^{[-\frac{1}{6}]}, \quad \theta_4^{[-\frac{1}{2}]}, \quad (3.74)$$

are invariant under the shift of the upper characteristic $\alpha \rightarrow \alpha - 1/3$ because $\theta_4^{[\alpha]} = \theta_4^{[\alpha+1]}$. On the other hand, the three elliptic parameters for each theta function

$$\epsilon_1 + \epsilon_2, \quad -2\epsilon_1 + \epsilon_2, \quad \epsilon_1 - 2\epsilon_2 \quad (3.75)$$

sum up to zero, and enjoy a S_3 symmetry.

On the other hand, $\det M_{\Lambda_k}$ does not vanish at base degrees $k > 1$. For instance the leading order contribution in the Q_τ expansion is

$$\begin{aligned}\det M_{\Lambda_k} &= -(q_1 q_2)^{-\frac{3k+1}{2}} \left((q_1 q_2)^{\frac{k}{2}} - (q_1 q_2)^{\frac{1}{2}} \right) \left((q_1 q_2)^{\frac{k}{2}} + (q_1 q_2)^{\frac{1}{2}} \right) \\ &\quad \times \left(q_1^k + q_2^k + q_1^{2k} + q_2^{2k} + q_1^{2k} q_2^k + q_1^k q_2^{2k} \right) Q_\tau^{11/24} + \mathcal{O}(Q_\tau^{11/24+1}).\end{aligned}\quad (3.76)$$

We can thus obtain compact expressions of Z_k from the linear equation (3.37) by inverting M_{Λ_k} . We do not give explicit formulas for Z_k here as they are quite lengthy, and the results of Z_k are already well known in the literature [12, 33]. Instead we will compute and list the BPS invariants in the next subsection, which also serves as another check on the blowup equations.

unity	$(0, 0, 0, 0, 0, -2)$	$(0, 0, 0, 0, 0, 0)$	$(0, 0, 0, 0, 0, 2)$	$(0, 0, 0, 0, 0, 4)$
vanishing	$(-2, 0, 0, 2, 0, 0)$	$(0, -2, 0, 2, 0, 0)$	$(0, 0, -2, 2, 0, 0)$	
	$(-2, 2, 0, 0, 0, 0)$	$(-2, 0, 2, 0, 0, 0)$	$(0, -2, 2, 0, 0, 0)$	
	$(-2, -2, 0, 0, 2, 2)$	$(-2, 0, -2, 0, 2, 2)$	$(0, -2, -2, 0, 2, 2)$	
	$(-2, 0, 0, -2, 2, 2)$	$(0, -2, 0, -2, 2, 2)$	$(0, 0, -2, -2, 2, 2)$	

Table 2. The list of all inequivalent and admissible \mathbf{r} fields for 6d SO(8) gauge theory.

3.3.4 Solving refined BPS invariants

Among the recursion relations those from unity blowup equations (3.33) are most useful as it is rather easy to solve them and obtain compact formulas of elliptic genera; recursion relations from vanishing blowup equations are rather complicated and it is difficult to get headway with them.

Another way to solve the blowup equations (1.6) is to expand them in terms of all Kähler moduli Q_i and extract equations of refined BPS invariants. There are two advantages to this method: one can equally easily extract equations from vanishing blowup equations and thus increase the number of available constraint equations; one can in fact start without the input of $Z^{1\text{-loop}}$ but with only the truly perturbative data: the \mathbf{C} -matrix, the \mathbf{B} -field, and Z^{pert} .

We have succeeded to exploit this method to great effect. We have used the equations extracted from the blowup equations associated to the following \mathbf{r} fields

$$(0, 0, 0, 1), (0, 0, 0, 3), (-2, 2, 0, 1) + \text{permutations of } r_1, r_2, r_3, \tag{3.77}$$

and computed the BPS invariants up to total degree $d_1 + d_2 + d_3 + d_b = 7$. They are listed in table 5. These BPS invariants respect the permutation symmetry of the degrees d_1, d_2, d_3 . Therefore we only list the non-vanishing invariants with $d_1 \leq d_2 \leq d_3$ and omit the invariants which can be obtained by permuting d_1, d_2, d_3 . All the other curve classes which are not listed have vanishing BPS invariants. These invariants agree with the results in the existing literature. In this way we have not only demonstrated the validity of the generalized blowup equations but also shown the power of the blowup equations as a computational tool. We expect that BPS invariants of higher degrees can also be computed with enough time.

3.4 SO(8) theory

3.4.1 Base degree zero

Following the same analysis as in the SU(3) theory, we can find all the inequivalent and admissible \mathbf{r} fields and divide the resulting blowup equations into unity and vanishing equations. The results are listed in table 2.

For the unity blowup equations, at base degree $k = 0$ they reduce to the computation of Λ . We find the following results for the four \mathbf{r} fields in the first row of table 2

$$\Lambda(\mathbf{r}_1) = \sum_{n \in \mathbb{Z}} Q_\tau^{2n^2+n+\frac{1}{8}} (q_1 q_2)^{2n+\frac{1}{2}} = \theta_3^{[\frac{1}{4}]}(4\tau, 2\epsilon_1 + 2\epsilon_2), \quad (3.78)$$

$$\Lambda(\mathbf{r}_2) = \sum_{n \in \mathbb{Z}} Q_\tau^{2n^2} (q_1 q_2)^{2n} = \theta_3^{[0]}(4\tau, 2\epsilon_1 + 2\epsilon_2), \quad (3.79)$$

$$\Lambda(\mathbf{r}_3) = \sum_{n \in \mathbb{Z}} Q_\tau^{2n^2-n+\frac{1}{8}} (q_1 q_2)^{2n-\frac{1}{2}} = \theta_3^{[-\frac{1}{4}]}(4\tau, 2\epsilon_1 + 2\epsilon_2), \quad (3.80)$$

$$\Lambda(\mathbf{r}_4) = \sum_{n \in \mathbb{Z}} Q_\tau^{2n^2-2n+\frac{1}{2}} (q_1 q_2)^{2n-1} = \theta_3^{[-\frac{1}{2}]}(4\tau, 2\epsilon_1 + 2\epsilon_2). \quad (3.81)$$

Notice that indeed all the Λ only depend on τ and no other Kähler moduli.

As for the vanishing blowup equations, we check two of them with \mathbf{r} fields

$$(-2, 2, 0, 0, 0, 0), \quad (-2, -2, 0, 0, 2, 2) \quad (3.82)$$

while the other \mathbf{r} fields could be obtained by acting S_4 symmetry on the first four entries.

In the case of $\mathbf{r} = (-2, 2, 0, 0, 0, 0)$, there are eight sets of \mathbf{n} which minimize $f_b(\mathbf{n})$ and they are represented by

$$\begin{aligned} \hat{I}_b = \{ & (-1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (-1, 1, 0, 0, 0), (0, 0, 0, 0, 0), \\ & (0, 1, 0, 0, 1), (0, 1, 1, 1, 1), (0, 1, 0, 1, 1), (0, 1, 1, 0, 1) \}. \end{aligned} \quad (3.83)$$

Summing over them, the lowest order blowup equation is

$$\sum_{\mathbf{n} \in \hat{I}_b} (-1)^{|\mathbf{n}|} \Theta_{\mathbf{n}} \theta_{\mathbf{n}} = 0 \quad (3.84)$$

where

$$\begin{aligned} \Theta_{(-1,0,0,0,0)} &= \Theta_{(0,1,0,0,0)} = \sum_{n \in \mathbb{Z}} Q_\tau^{2n^2} Q_1^{2n+\frac{1}{2}} Q_2^{2n+\frac{1}{2}} = Q_1^{\frac{1}{2}} Q_2^{\frac{1}{2}} \theta_3(4\tau, 2t_1 + 2t_2), \\ \Theta_{(-1,1,0,0,0)} &= \Theta_{(0,0,0,0,0)} = \sum_{n \in \mathbb{Z}} Q_\tau^{2n^2} Q_1^{2n+\frac{1}{2}} Q_2^{-2n+\frac{1}{2}} = Q_1^{\frac{1}{2}} Q_2^{\frac{1}{2}} \theta_3(4\tau, 2t_1 - 2t_2), \\ \Theta_{(0,1,0,0,1)} &= \Theta_{(0,1,1,1,1)} = \sum_{n \in \mathbb{Z}} Q_\tau^{2(n+\frac{1}{2})^2} Q_3^{2n+\frac{3}{2}} Q_4^{2n+\frac{3}{2}} = Q_3^{\frac{1}{2}} Q_4^{\frac{1}{2}} \theta_3^{[\frac{1}{2}]}(4\tau, 2t_3 + 2t_4), \\ \Theta_{(0,1,0,1,1)} &= \Theta_{(0,1,1,0,1)} = \sum_{n \in \mathbb{Z}} Q_\tau^{2(n+\frac{1}{2})^2} Q_3^{2n+\frac{3}{2}} Q_4^{-2n-\frac{1}{2}} = Q_3^{\frac{1}{2}} Q_4^{\frac{1}{2}} \theta_3^{[\frac{1}{2}]}(4\tau, 2t_3 - 2t_4). \end{aligned} \quad (3.85)$$

and

$$\begin{aligned}
 \theta_{(-1,0,0,0,0)} &= \theta_{(0,1,0,0,0)} \\
 &= -\frac{Q_1^{-\frac{1}{2}} Q_2^{-\frac{1}{2}} \eta^6}{\theta_1(t_1)\theta_1(t_2)\theta_4\left(\frac{-t_1-t_2-t_3-t_4}{2}\right)\theta_4\left(\frac{-t_1-t_2+t_3-t_4}{2}\right)\theta_4\left(\frac{-t_1-t_2-t_3+t_4}{2}\right)\theta_4\left(\frac{+t_1+t_2-t_3-t_4}{2}\right)}, \\
 \theta_{(-1,1,0,0,0)} &= \theta_{(0,0,0,0,0)} \\
 &= -\frac{Q_1^{-\frac{1}{2}} Q_2^{-\frac{1}{2}} \eta^6}{\theta_1(t_1)\theta_1(t_2)\theta_4\left(\frac{+t_1-t_2-t_3-t_4}{2}\right)\theta_4\left(\frac{-t_1+t_2-t_3-t_4}{2}\right)\theta_4\left(\frac{+t_1-t_2+t_3-t_4}{2}\right)\theta_4\left(\frac{+t_1-t_2-t_3+t_4}{2}\right)}, \\
 \theta_{(0,1,0,0,1)} &= \theta_{(0,1,1,1,1)} \\
 &= -\frac{Q_3^{-\frac{1}{2}} Q_4^{-\frac{1}{2}} \eta^6}{\theta_1(t_3)\theta_1(t_4)\theta_4\left(\frac{-t_1-t_2-t_3-t_4}{2}\right)\theta_4\left(\frac{+t_1-t_2-t_3-t_4}{2}\right)\theta_4\left(\frac{-t_1+t_2-t_3-t_4}{2}\right)\theta_4\left(\frac{+t_1+t_2-t_3-t_4}{2}\right)}, \\
 \theta_{(0,1,0,1,1)} &= \theta_{(0,1,1,0,1)} \\
 &= -\frac{Q_3^{-\frac{1}{2}} Q_4^{-\frac{1}{2}} \eta^6}{\theta_1(t_3)\theta_1(t_4)\theta_4\left(\frac{-t_1-t_2+t_3-t_4}{2}\right)\theta_4\left(\frac{-t_1-t_2-t_3+t_4}{2}\right)\theta_4\left(\frac{+t_1-t_2+t_3-t_4}{2}\right)\theta_4\left(\frac{+t_1-t_2-t_3+t_4}{2}\right)}. \tag{3.86}
 \end{aligned}$$

It is equivalent to the identity

$$\begin{aligned}
 0 &= -\theta_3(4\tau, 2t_1 + 2t_2)\theta_1(t_3)\theta_1(t_4) \\
 &\quad \times \theta_4\left(\frac{+t_1-t_2-t_3-t_4}{2}\right)\theta_4\left(\frac{-t_1+t_2-t_3-t_4}{2}\right)\theta_4\left(\frac{+t_1-t_2+t_3-t_4}{2}\right)\theta_4\left(\frac{+t_1-t_2-t_3+t_4}{2}\right) \\
 &\quad + \theta_3(4\tau, 2t_1 - 2t_2)\theta_1(t_3)\theta_1(t_4) \\
 &\quad \times \theta_4\left(\frac{-t_1-t_2-t_3-t_4}{2}\right)\theta_4\left(\frac{-t_1-t_2+t_3-t_4}{2}\right)\theta_4\left(\frac{-t_1-t_2-t_3+t_4}{2}\right)\theta_4\left(\frac{+t_1+t_2-t_3-t_4}{2}\right) \\
 &\quad + \theta_3^{[1/2]}(4\tau, 2t_3 + 2t_4)\theta_1(t_1)\theta_1(t_2) \\
 &\quad \times \theta_4\left(\frac{-t_1-t_2+t_3-t_4}{2}\right)\theta_4\left(\frac{-t_1-t_2-t_3+t_4}{2}\right)\theta_4\left(\frac{+t_1-t_2+t_3-t_4}{2}\right)\theta_4\left(\frac{+t_1-t_2-t_3+t_4}{2}\right) \\
 &\quad - \theta_3^{[1/2]}(4\tau, 2t_3 - 2t_4)\theta_1(t_1)\theta_1(t_2) \\
 &\quad \times \theta_4\left(\frac{-t_1-t_2-t_3-t_4}{2}\right)\theta_4\left(\frac{+t_1-t_2-t_3-t_4}{2}\right)\theta_4\left(\frac{-t_1+t_2-t_3-t_4}{2}\right)\theta_4\left(\frac{+t_1+t_2-t_3-t_4}{2}\right). \tag{3.87}
 \end{aligned}$$

It can be proved by noticing that each summand and thus the total sum is a Jacobi form for $\Gamma(4)$ of weight $7/2$ and index polynomial

$$t_1^2 + t_2^2 + t_3^2 + t_4^2, \tag{3.88}$$

and that the first few terms in Q_τ expansion vanish, which we checked up to very high orders.

In the case of $\mathbf{r} = (-2, -2, 0, 0, 2, 2)$, there are eight sets of \mathbf{n} which minimize $f_b(\mathbf{n})$ and they are represented by

$$\begin{aligned}
 \hat{I}_b &= \{(-1, -1, 0, 0, 0), (0, 0, 0, 0, 0), (-1, 0, 0, 0, 0), (0, -1, 0, 0, 0), \\
 &\quad (0, 0, 1, 1, 1), (0, 0, 0, 0, 1), (0, 0, 1, 0, 1), (0, 0, 0, 1, 1)\}. \tag{3.89}
 \end{aligned}$$

Summing over them, we get for the lowest order blowup equation

$$\sum_{\hat{\mathbf{n}} \in \hat{I}_b} (-1)^{|\mathbf{n}|} \Theta_{\mathbf{n}} \theta_{\mathbf{n}} = 0 \tag{3.90}$$

where

$$\begin{aligned}
 \Theta_{(-1,-1,0,0,0)} &= Q_1^{\frac{1}{2}} Q_2^{\frac{1}{2}} \theta_3^{[\frac{1}{4}]}(4\tau, 2t_1 + 2t_2), & \Theta_{(0,0,0,0,0)} &= Q_1^{\frac{1}{2}} Q_2^{\frac{1}{2}} \theta_3^{[-\frac{1}{4}]}(4\tau, 2t_1 + 2t_2), \\
 \Theta_{(-1,0,0,0,0)} &= Q_1^{\frac{1}{2}} Q_2^{\frac{1}{2}} \theta_3^{[\frac{1}{4}]}(4\tau, 2t_1 - 2t_2), & \Theta_{(0,-1,0,0,0)} &= Q_1^{\frac{1}{2}} Q_2^{\frac{1}{2}} \theta_3^{[-\frac{1}{4}]}(4\tau, 2t_1 - 2t_2), \\
 \Theta_{(0,0,1,1,1)} &= Q_3^{\frac{1}{2}} Q_4^{\frac{1}{2}} \theta_3^{[\frac{1}{4}]}(4\tau, 2t_3 + 2t_4), & \Theta_{(0,0,0,0,1)} &= Q_3^{\frac{1}{2}} Q_4^{\frac{1}{2}} \theta_3^{[-\frac{1}{4}]}(4\tau, 2t_3 + 2t_4), \\
 \Theta_{(0,0,1,0,1)} &= Q_3^{\frac{1}{2}} Q_4^{\frac{1}{2}} \theta_3^{[\frac{1}{4}]}(4\tau, 2t_3 - 2t_4), & \Theta_{(0,0,0,1,1)} &= Q_3^{\frac{1}{2}} Q_4^{\frac{1}{2}} \theta_3^{[-\frac{1}{4}]}(4\tau, 2t_3 - 2t_4).
 \end{aligned} \tag{3.91}$$

and

$$\begin{aligned}
 \theta_{(-1,-1,0,0,0)} &= \theta_{(0,0,0,0,0)} \\
 &= -\frac{Q_1^{-\frac{1}{2}} Q_2^{-\frac{1}{2}} \eta^6}{\theta_1(t_1) \theta_1(t_2) \theta_4\left(\frac{-t_1-t_2-t_3-t_4}{2}\right) \theta_4\left(\frac{-t_1-t_2+t_3-t_4}{2}\right) \theta_4\left(\frac{-t_1-t_2-t_3+t_4}{2}\right) \theta_4\left(\frac{+t_1+t_2-t_3-t_4}{2}\right)}, \\
 \theta_{(-1,0,0,0,0)} &= \theta_{(0,-1,0,0,0)} \\
 &= -\frac{Q_1^{-\frac{1}{2}} Q_2^{-\frac{1}{2}} \eta^6}{\theta_1(t_1) \theta_1(t_2) \theta_4\left(\frac{+t_1-t_2-t_3-t_4}{2}\right) \theta_4\left(\frac{-t_1+t_2-t_3-t_4}{2}\right) \theta_4\left(\frac{+t_1-t_2-t_3+t_4}{2}\right) \theta_4\left(\frac{+t_1-t_2+t_3-t_4}{2}\right)}, \\
 \theta_{(0,0,1,1,1)} &= \theta_{(0,0,0,0,1)} \\
 &= -\frac{Q_3^{-\frac{1}{2}} Q_4^{-\frac{1}{2}} \eta^6}{\theta_1(t_3) \theta_1(t_4) \theta_4\left(\frac{-t_1-t_2-t_3-t_4}{2}\right) \theta_4\left(\frac{+t_1-t_2-t_3-t_4}{2}\right) \theta_4\left(\frac{-t_1+t_2-t_3-t_4}{2}\right) \theta_4\left(\frac{+t_1+t_2-t_3-t_4}{2}\right)}, \\
 \theta_{(0,0,1,0,1)} &= \theta_{(0,0,0,1,1)} \\
 &= -\frac{Q_3^{-\frac{1}{2}} Q_4^{-\frac{1}{2}} \eta^6}{\theta_1(t_3) \theta_1(t_4) \theta_4\left(\frac{-t_1-t_2+t_3-t_4}{2}\right) \theta_4\left(\frac{-t_1-t_2-t_3+t_4}{2}\right) \theta_4\left(\frac{+t_1-t_2-t_3+t_4}{2}\right) \theta_4\left(\frac{+t_1-t_2+t_3-t_4}{2}\right)}.
 \end{aligned} \tag{3.92}$$

It is equivalent to

$$\begin{aligned}
 0 &= -\left(\theta_3^{[\frac{1}{4}]}(4\tau, 2t_1 + 2t_2) + \theta_3^{[-\frac{1}{4}]}(4\tau, 2t_1 + 2t_2)\right) \theta_1(t_3) \theta_1(t_4) \\
 &\quad \times \theta_4\left(\frac{+t_1-t_2-t_3-t_4}{2}\right) \theta_4\left(\frac{-t_1+t_2-t_3-t_4}{2}\right) \theta_4\left(\frac{+t_1-t_2+t_3-t_4}{2}\right) \theta_4\left(\frac{+t_1-t_2-t_3+t_4}{2}\right) \\
 &\quad + \left(\theta_3^{[\frac{1}{4}]}(4\tau, 2t_1 - 2t_2) + \theta_3^{[-\frac{1}{4}]}(4\tau, 2t_1 - 2t_2)\right) \theta_1(t_3) \theta_1(t_4) \\
 &\quad \times \theta_4\left(\frac{-t_1-t_2-t_3-t_4}{2}\right) \theta_4\left(\frac{-t_1-t_2+t_3-t_4}{2}\right) \theta_4\left(\frac{-t_1-t_2-t_3+t_4}{2}\right) \theta_4\left(\frac{+t_1+t_2-t_3-t_4}{2}\right) \\
 &\quad + \left(\theta_3^{[\frac{1}{4}]}(4\tau, 2t_3 + 2t_4) + \theta_3^{[-\frac{1}{4}]}(4\tau, 2t_3 + 2t_4)\right) \theta_1(t_1) \theta_1(t_2) \\
 &\quad \times \theta_4\left(\frac{-t_1-t_2+t_3-t_4}{2}\right) \theta_4\left(\frac{-t_1-t_2-t_3+t_4}{2}\right) \theta_4\left(\frac{+t_1-t_2+t_3-t_4}{2}\right) \theta_4\left(\frac{+t_1-t_2-t_3+t_4}{2}\right) \\
 &\quad - \left(\theta_3^{[\frac{1}{4}]}(4\tau, 2t_3 - 2t_4) + \theta_3^{[-\frac{1}{4}]}(4\tau, 2t_3 - 2t_4)\right) \theta_1(t_1) \theta_1(t_2) \\
 &\quad \times \theta_4\left(\frac{-t_1-t_2-t_3-t_4}{2}\right) \theta_4\left(\frac{+t_1-t_2-t_3-t_4}{2}\right) \theta_4\left(\frac{-t_1+t_2-t_3-t_4}{2}\right) \theta_4\left(\frac{+t_1+t_2-t_3-t_4}{2}\right), \tag{3.93}
 \end{aligned}$$

which can be similarly proved.

Note that in order to write recursion relations in terms of appropriate Jacobi forms, we need to absorb one of the five Kähler moduli t_1, t_2, t_3, t_4, t_c completely into τ . It is canonical to absorb t_4 associated to the affine node as one goes down from the affine Lie algebra to the simple Lie algebra. Here we choose to absorb t_c so that the symmetry between t_1, t_2, t_3, t_4 still survives.

3.4.2 Modularity at generic base degree

Following the example of the SU(3) theory, we show here that both sides of the recursion relation (3.31) for the SO(8) theory are meromorphic Jacobi forms of the same weight and index polynomial at any base degree k . When this is established, one can multiply both sides with the common denominator and obtain an identity of weak Jacobi forms and proceeds to prove it by comparing the first few terms in the Q_τ expansion.

Consider a blowup equation with $\mathbf{r} = (r_1, r_2, r_3, r_4, r_c, r_b)$ subject to the condition $r_1 + r_2 + r_3 + r_4 + 2r_c = 0$. The perturbative contribution is

$$\begin{aligned} D^{\text{pert},'} &:= \exp \left[f_0(\mathbf{n})(\epsilon_1 + \epsilon_2) + \sum_{\ell=1}^{r+1} f_\ell(\mathbf{n})t_\ell + (k_1\epsilon_1 + k_2\epsilon_2)R_b(\mathbf{n}) \right] \\ &= Q_1^{f_1(\mathbf{n})} Q_2^{f_2(\mathbf{n})} Q_3^{f_3(\mathbf{n})} Q_4^{f_4(\mathbf{n})} Q_c^{f_c(\mathbf{n})} (q_1 q_2)^{f_0(\mathbf{n})} (q_1^{k_1} q_2^{k_2})^{R_b(\mathbf{n})}, \end{aligned} \quad (3.94)$$

where

$$\begin{aligned} f_\ell(\mathbf{n}) &= 2 \left(n_\ell - \frac{r_\ell + r_b/2}{4} \right)^2, \quad \ell = 1, 2, 3, 4 \\ f_c(\mathbf{n}) &= \left(n_c - \frac{r_b}{4} \right)^2, \\ f_0(\mathbf{n}) &= -\frac{4}{3} \sum_{i=1}^5 n_i^3 + n_c^2 \sum_{\ell=1}^4 n_\ell + \sum_{\ell=1}^4 \left(r_\ell + \frac{r_b}{2} \right) n_\ell^2 - \frac{r_b n_c}{2} \sum_{\ell=1}^4 n_\ell + \frac{r_b + r_c}{2} n_c^2 \\ &\quad + \frac{\sum_{i=1}^5 n_i}{3} - \frac{\sum_{\ell=1}^4 r_\ell^2 n_\ell}{4} - \frac{r_b \sum_{\ell=1}^4 r_\ell n_\ell}{4} - \frac{r_b r_c n_c}{4} \\ &\quad + \frac{\sum_{\ell=1}^4 r_\ell^3}{48} + \frac{r_b \sum_{\ell=1}^4 r_\ell^2}{32} + \frac{r_b^2 \sum_{\ell=1}^4 r_\ell}{64} + \frac{r_b^2 r_c}{32} + \frac{r_c}{6} - \frac{r_b}{4}, \\ R_b(\mathbf{n}) &= -2n_c + \frac{r_b}{2}, \end{aligned} \quad (3.95)$$

with the notation $r_5 = r_c$. In addition

$$f_b(\mathbf{n}) = \frac{\sum_{\ell=1}^4 r_\ell^2}{16} - \frac{\sum_{\ell=1}^5 r_\ell n_\ell}{2} + \sum_{i=1}^5 n_i^2 - n_c \sum_{\ell=1}^4 n_\ell. \quad (3.96)$$

We split \mathbf{n} to a representative $\hat{\mathbf{n}}$, which we fix uniquely by setting $n_4 = 0$, and $(m, m, m, m, 2m)$, i.e.

$$\mathbf{n} = (n_1, n_2, n_3, 0, n_c) + (m, m, m, m, 2m). \quad (3.97)$$

Then separating m -dependent and $-$ -independent parts, $D^{\text{pert},'}$ can be written as

$$\begin{aligned}
 D^{\text{pert},'} &= Q_\tau^2 \left(m - \frac{r_4 + r_b/2}{4} \right)^2 \prod_{\ell=1}^3 Q_\ell^{4(m - \frac{r_4 + r_b/2}{4})(n_\ell + \frac{r_4 - r_\ell}{4})} Q_c^{4(m - \frac{r_4 + r_b/2}{4})(n_c + \frac{r_4}{2})} \\
 &\quad \times (q_1 q_2)^{(m - \frac{r_4 + r_b/2}{4})(2 - 4f_b(\hat{\mathbf{n}}))} (q_1^{k_1} q_2^{k_2})^{-4(m - \frac{r_4 + r_b/2}{4})} \\
 &\quad \times \prod_{\ell=1}^3 Q_\ell^{2(n_\ell + \frac{r_4 - r_\ell}{4})^2} Q_c^{(n_c + \frac{r_4}{2})^2} (q_1 q_2)^{f_0(\hat{\mathbf{n}}) + \frac{1}{4}(r_4 + r_b/2)(2 - 4f_b(\hat{\mathbf{n}}))} (q_1^{k_1} q_2^{k_2})^{-2n_c - r_4}.
 \end{aligned} \tag{3.98}$$

The contribution of 1-loop partition function is

$$\begin{aligned}
 &D^{\text{1-loop}} \\
 &= T_{-2n_1 + n_c + \frac{r_1}{2}}(t_1) T_{-2n_2 + n_c + \frac{r_2}{2}}(t_2) T_{-2n_3 + n_c + \frac{r_3}{2}}(t_3) T_{n_1 + n_2 + n_3 - 2n_c + \frac{r_c}{2}}(t_c) \\
 &\quad \times T_{-n_1 + n_2 + n_3 - n_c + \frac{r_1 + r_c}{2}}(t_1 + t_c) T_{n_1 - n_2 + n_3 - n_c + \frac{r_2 + r_c}{2}}(t_2 + t_c) T_{n_1 + n_2 - n_3 - n_c + \frac{r_3 + r_c}{2}}(t_3 + t_c) \\
 &\quad \times T_{-n_1 - n_2 + n_3 + \frac{r_1 + r_2 + r_c}{2}}(t_1 + t_2 + t_c) T_{-n_1 + n_2 - n_3 + \frac{r_1 + r_3 + r_c}{2}}(t_1 + t_3 + t_c) \\
 &\quad \times T_{n_1 - n_2 - n_3 + \frac{r_2 + r_3 + r_c}{2}}(t_2 + t_3 + t_c) T_{-n_1 - n_2 - n_3 + n_c + \frac{r_1 + r_2 + r_3 + r_c}{2}}(t_1 + t_2 + t_3 + t_c) \\
 &\quad \times T_{-n_c + \frac{r_1 + r_2 + r_3 + 2r_c}{2}}(t_1 + t_2 + t_3 + 2t_c) \\
 &= Q_\tau^{f_b(\hat{\mathbf{n}})} \prod_{\ell=1}^3 Q_\ell^{-2f_b(\hat{\mathbf{n}}) - 2(n_\ell + \frac{r_4 - r_\ell}{4})^2} Q_c^{-4f_b(\hat{\mathbf{n}}) - (n_c + \frac{r_4}{2})^2} (q_1 q_2)^{d_R(\mathfrak{d}_4)} \\
 &\quad \times \check{\theta}_{-2n_1 + n_c + \frac{r_1}{2}}(t_1) \check{\theta}_{-2n_2 + n_c + \frac{r_2}{2}}(t_2) \check{\theta}_{-2n_3 + n_c + \frac{r_3}{2}}(t_3) \check{\theta}_{n_1 + n_2 + n_3 - 2n_c + \frac{r_c}{2}}(t_c) \\
 &\quad \times \check{\theta}_{-n_1 + n_2 + n_3 - n_c + \frac{r_1 + r_c}{2}}(t_1 + t_c) \check{\theta}_{n_1 - n_2 + n_3 - n_c + \frac{r_2 + r_c}{2}}(t_2 + t_c) \check{\theta}_{n_1 + n_2 - n_3 - n_c + \frac{r_3 + r_c}{2}}(t_3 + t_c) \\
 &\quad \times \check{\theta}_{-n_1 - n_2 + n_3 + \frac{r_1 + r_2 + r_c}{2}}(t_1 + t_2 + t_c) \check{\theta}_{-n_1 + n_2 - n_3 + \frac{r_1 + r_3 + r_c}{2}}(t_1 + t_3 + t_c) \\
 &\quad \times \check{\theta}_{n_1 - n_2 - n_3 + \frac{r_2 + r_3 + r_c}{2}}(t_2 + t_3 + t_c) \check{\theta}_{-n_1 - n_2 - n_3 + n_c + \frac{r_1 + r_2 + r_3 + r_c}{2}}(t_1 + t_2 + t_3 + t_c) \\
 &\quad \times \check{\theta}_{-n_c + \frac{r_1 + r_2 + r_3 + 2r_c}{2}}(t_1 + t_2 + t_3 + 2t_c),
 \end{aligned} \tag{3.99}$$

where

$$\begin{aligned}
 d_R(\mathfrak{d}_4) &= \frac{4}{3} \sum_{\ell=1}^3 n_\ell^3 - n_c^2 \sum_{\ell=1}^3 n_\ell + \frac{4}{3} n_c^3 + \sum_{\ell=1}^3 (r_4 - r_\ell) n_\ell^2 - r_4 n_c \sum_{\ell=1}^3 n_\ell + \left(r_4 - \frac{r_c}{2} \right) n_c^2 \\
 &\quad + \sum_{\ell=1}^3 \left(-\frac{1}{3} + \frac{r_\ell^2}{4} - \frac{r_4 r_\ell}{2} \right) n_\ell + \left(-\frac{1}{3} - \frac{r_4 r_c}{2} \right) n_c \\
 &\quad - \frac{r_c}{6} - \frac{r_4}{2} - \frac{\sum_{\ell=1}^3 r_\ell^3 - r_4^3}{16} - \frac{\sum_{\ell=1}^3 r_\ell^2 - r_4^2}{8} r_c - \frac{r_c^2 \sum_{\ell=1}^3 r_\ell}{4} - \frac{r_c^3}{3} + \frac{r_1 r_2 r_3}{8}.
 \end{aligned} \tag{3.100}$$

Here $T_R(t)$ and $\check{\theta}_R(t)$ are defined in appendix A; in particular, $\check{\theta}_R(t)$ is a Jacobi form of weight 0 and index given by (A.17).

Finally, Z_k is related to the k -string elliptic genus \mathbb{E}_k for the $\text{SO}(8)$ theory by

$$Z_k(t_\ell, \epsilon_1, \epsilon_2) = \left(\frac{Q_\tau^{1/2}}{Q_1 Q_2 Q_3 Q_c^2} \right)^{2k} \mathbb{E}_k(t_\ell, \epsilon_1, \epsilon_2), \quad (3.101)$$

and the latter is a meromorphic Jacobi form of weight 0 and index [13, 38]

$$\text{Ind}(\mathbb{E}_k) = -k(\epsilon_1^2 + \epsilon_2^2) + k(2k - 3)\epsilon_1\epsilon_2 - 2k(\mathbf{a}, \mathbf{a})_{\mathfrak{d}_4}, \quad (3.102)$$

with the invariant bilinear form normalized as¹⁹

$$(\mathbf{a}, \mathbf{a})_{\mathfrak{d}_4} = t_1^2 + t_2^2 + t_3^2 + t_1 t_2 + t_2 t_3 + t_3 t_1 + 2t_c(t_1 + t_2 + t_3) + 2t_c^2. \quad (3.103)$$

We also have

$$\begin{aligned} Z_{k_1}(t_\ell + \epsilon_1 R_\ell, \epsilon_1, \epsilon_2 - \epsilon_1) &= q_1^{k_1(2n_c+r_4)} \left(\frac{Q_\tau^{1/2}}{Q_1 Q_2 Q_3 Q_c^2} \right)^{2k_1} \mathbb{E}_{k_1}(t_\ell + \epsilon_1 R_\ell, \epsilon_1, \epsilon_2 - \epsilon_1), \\ Z_{k_2}(; t_\ell + \epsilon_2 R_\ell, \epsilon_1 - \epsilon_2, \epsilon_2) &= q_2^{k_2(2n_c+r_4)} \left(\frac{Q_\tau^{1/2}}{Q_1 Q_2 Q_3 Q_c^2} \right)^{2k_2} \mathbb{E}_{k_2}(t_\ell + \epsilon_2 R_\ell, \epsilon_1 - \epsilon_2, \epsilon_2), \end{aligned} \quad (3.104)$$

Combining (3.98), (3.99) and (3.104) all together, we get for the r.h.s. of the recursion relation (3.31)

$$\begin{aligned} \text{r.h.s.} &= \left(\frac{Q_\tau^{1/2}}{Q_1 Q_2 Q_3 Q_c^2} \right)^{2k} \sum_{f_b(\hat{\mathbf{n}})+k_1+k_2=k} (-1)^{n_1+n_2+n_3+n_5} \\ &\times \theta_3^{\left[-\frac{2r_4+r_b}{8} \right]} \left(4\tau, 4 \sum_{\ell=1}^3 \left(n_\ell + \frac{r_4 - r_\ell}{4} \right) t_\ell + 4 \left(n_c + \frac{r_4}{2} \right) t_c + (2-4(k-k_2))\epsilon_1 + (2-4(k-k_1))\epsilon_2 \right) \\ &\times \check{\theta}_{-2n_1+n_c+\frac{r_1}{2}}(t_1) \check{\theta}_{-2n_2+n_c+\frac{r_2}{2}}(t_2) \check{\theta}_{-2n_3+n_c+\frac{r_3}{2}}(t_3) \check{\theta}_{n_1+n_2+n_3-2n_c+\frac{r_c}{2}}(t_c) \\ &\times \check{\theta}_{-n_1+n_2+n_3-n_c+\frac{r_1+r_c}{2}}(t_1+t_c) \check{\theta}_{n_1-n_2+n_3-n_c+\frac{r_2+r_c}{2}}(t_2+t_c) \check{\theta}_{n_1+n_2-n_3-n_c+\frac{r_3+r_c}{2}}(t_3+t_c) \\ &\times \check{\theta}_{-n_1-n_2+n_3+\frac{r_1+r_2+r_c}{2}}(t_1+t_2+t_c) \\ &\times \check{\theta}_{-n_1+n_2-n_3+\frac{r_1+r_3+r_c}{2}}(t_1+t_3+t_c) \check{\theta}_{n_1-n_2-n_3+\frac{r_2+r_3+r_c}{2}}(t_2+t_3+t_c) \\ &\times \check{\theta}_{-n_1-n_2-n_3+n_c+\frac{r_1+r_2+r_3+r_c}{2}}(t_1+t_2+t_3+t_c) \check{\theta}_{-n_c+\frac{r_1+r_2+r_3+2r_c}{2}}(t_1+t_2+t_3+2t_c) \\ &\times \mathbb{E}_{k_1}(t_\ell + \epsilon_1 R_\ell, \epsilon_1, \epsilon_2 - \epsilon_1) \mathbb{E}_{k_2}(t_\ell + \epsilon_2 R_\ell, \epsilon_1 - \epsilon_2, \epsilon_2). \end{aligned} \quad (3.105)$$

Up to the common prefactor, each summand happens to be a meromorphic Jacobi form²⁰ for Γ_4 of the same weight 1/2 and index polynomial

$$\text{Ind}(\text{r.h.s.}) = \frac{-2k+1}{2}(\epsilon_1^2 + \epsilon_2^2) + (k-1)(2k-1)\epsilon_1\epsilon_2 - 2k(\mathbf{a}, \mathbf{a})_{\mathfrak{d}_4}, \quad (3.106)$$

which is independent of the summation indices n_i, k_1, k_2 and thus so is the total sum.

¹⁹Recall $t_i = \langle \alpha_i, \mathbf{a} \rangle$.

²⁰The same as footnote 17.

On the other hand, if the blowup equation is of vanishing type, the l.h.s. of (3.31) vanish; if the blowup equation is of unity type, we have $\mathbf{r} = (0, 0, 0, 0, 0, r_b)$, and after plugging in the expression of Λ , we find the l.h.s. of (3.31) to be

$$\text{l.h.s.} = \left(\frac{Q_\tau^{1/2}}{Q_1 Q_2 Q_3 Q_c^2} \right)^{2k} \theta_3^{\left[-\frac{r_b}{8}\right]}(4\tau, 2\epsilon_1 + 2\epsilon_2) \mathbb{E}(t_\ell, \epsilon_1, \epsilon_2), \quad (3.107)$$

which is a meromorphic Jacobi form of the same weight and the same index (3.106). In both cases, after multiplied with a common denominator, the recursion relations (3.31) become identities of (weak) Weyl invariant Jacobi forms of identical weights and indices. As in the case of SU(3) theory, these identities can be proved by checking that when the *correct* \mathbf{r} (table 2) are plugged in the first few terms in Q_τ expansion are correct. For instance, when $k = 0$ we find (3.105) indeed reduces to the computation of Λ in the unity cases, and the identities (3.87), (3.93) in the vanishing cases. When $k = 1$, let us first reparametrise the Kähler moduli by

$$m_i = \langle \mathbf{e}_i, \mathbf{a} \rangle \quad (3.108)$$

with the standard basis $\{\mathbf{e}_i\}$ of \mathbb{R}^4 , in which the root lattice of SO(8) is embedded, so that the Weyl symmetry of SO(8) is more transparent. The variables m_i are related to t_i by

$$\begin{cases} t_1 = m_1 - m_2 \\ t_2 = m_3 - m_4 \\ t_3 = m_3 + m_4 \\ t_c = m_2 - m_3 \\ t_4 = \tau - m_1 - m_2 \end{cases} \quad (3.109)$$

In the case of unity equations, $f_b(\hat{\mathbf{n}})$ can only be 0 or 1. It has one set of solution $\hat{\mathbf{n}} = (0, 0, 0, 0, 0)$ in the former case, and 24 sets of solutions in the latter case, which are

$$\begin{aligned} \hat{I}_b^{(1)} = \{ & (-1, 0, 0, 0, 0), (1, 0, 0, 0, 0), (0, 0, 0, 0, 1), (1, 0, 0, 0, 1), (1, 1, 0, 0, 1), \\ & (1, 1, 1, 0, 1), (1, 1, 1, 1, 1) + \text{permutations of first four entries} \}. \end{aligned} \quad (3.110)$$

Then the recursion relations (3.105), (3.107) become

$$\begin{aligned} & \theta_3^{\left[-\frac{r_b}{8}\right]}(4\tau, -2\epsilon_1 + 2\epsilon_2) \mathbb{E}_1(\mathbf{v}, \epsilon_1, \epsilon_2 - \epsilon_1) + \theta_3^{\left[-\frac{r_b}{8}\right]}(4\tau, 2\epsilon_1 - 2\epsilon_2) \mathbb{E}_1(\mathbf{v}, \epsilon_1 - \epsilon_2, \epsilon_2) + I^{\left[-\frac{r_b}{8}\right]} \\ & = \theta_3^{\left[-\frac{r_b}{8}\right]}(4\tau, 2\epsilon_1 + 2\epsilon_2) \mathbb{E}_1(\mathbf{v}, \epsilon_1, \epsilon_2), \end{aligned} \quad (3.111)$$

where

$$\begin{aligned} I_1^{\left[-\frac{r_b}{8}\right]} = & - \sum_{i < j} \sum_{\substack{r = \pm 1 \\ s = \pm 1}} \frac{\theta_3^{\left[-\frac{r_b}{8}\right]}(4\tau, 4(rm_i + sm_j) - 2\epsilon_1 - 2\epsilon_2) \eta^4}{\theta_1(rm_i + sm_j) \theta_1(rm_i + sm_j - \epsilon_1) \theta_1(rm_i + sm_j - \epsilon_2) \theta_1(rm_i + sm_j - \epsilon_1 - \epsilon_2)} \\ & \times \prod_{\substack{k \neq i \\ k \neq j}} \frac{\eta^4}{\theta_1(m_i \pm m_k) \theta_1(m_j \pm m_k)}. \end{aligned} \quad (3.112)$$

Here we define $\theta_1(m_i \pm m_k)$ as $\theta_1(m_i + m_k)\theta_1(m_i - m_k)$. One can readily verify by the first terms in the Q_τ expansion and using the expression of \mathbb{E}_1 in (3.24) of [11]²¹ that the identity (3.111) holds only when r_b is even. The vanishing equations at $k = 1$ and the cases of $k \geq 2$ can be checked in a similar manner.

3.4.3 Recursion formula for elliptic genera

In the case of 6d SO(8) theory, there are four unity \mathbf{r} fields. The corresponding $\Lambda_k^{(1)}, \Lambda_k^{(2)}$ are

- $\mathbf{r}_1 = (0, 0, 0, 0, 0, -2)$

$$\Lambda_k^{(1)}(\mathbf{r}_1) = \sum_{n \in \mathbb{Z}} Q_\tau^{2n^2 + n + \frac{1}{8}} (q_1^{-2k+1} q_2)^{2n + \frac{1}{2}} = \theta_3^{\left[\frac{1}{4}\right]}(4\tau, -(4k-2)\epsilon_1 + 2\epsilon_2), \quad (3.113)$$

$$\Lambda_k^{(2)}(\mathbf{r}_1) = \sum_{n \in \mathbb{Z}} Q_\tau^{2n^2 + n + \frac{1}{8}} (q_1 q_2^{-2k+1})^{2n + \frac{1}{2}} = \theta_3^{\left[\frac{1}{4}\right]}(4\tau, 2\epsilon_1 - (4k-2)\epsilon_2). \quad (3.114)$$

- $\mathbf{r}_2 = (0, 0, 0, 0, 0, 0)$

$$\Lambda_k^{(1)}(\mathbf{r}_2) = \sum_{n \in \mathbb{Z}} Q_\tau^{2n^2} (q_1^{-2k+1} q_2)^{2n} = \theta_3(4\tau, -(4k-2)\epsilon_1 + 2\epsilon_2), \quad (3.115)$$

$$\Lambda_k^{(2)}(\mathbf{r}_2) = \sum_{n \in \mathbb{Z}} Q_\tau^{2n^2} (q_1 q_2^{-2k+1})^{2n} = \theta_3(4\tau, 2\epsilon_1 - (4k-2)\epsilon_2). \quad (3.116)$$

- $\mathbf{r}_3 = (0, 0, 0, 0, 0, 2)$

$$\Lambda_k^{(1)}(\mathbf{r}_3) = \sum_{n \in \mathbb{Z}} Q_\tau^{2n^2 - n + \frac{1}{8}} (q_1^{-2k+1} q_2)^{2n - \frac{1}{2}} = \theta_3^{\left[-\frac{1}{4}\right]}(4\tau, -(4k-2)\epsilon_1 + 2\epsilon_2), \quad (3.117)$$

$$\Lambda_k^{(2)}(\mathbf{r}_3) = \sum_{n \in \mathbb{Z}} Q_\tau^{2n^2 - n + \frac{1}{8}} (q_1 q_2^{-2k+1})^{2n - \frac{1}{2}} = \theta_3^{\left[-\frac{1}{4}\right]}(4\tau, 2\epsilon_1 - (4k-2)\epsilon_2). \quad (3.118)$$

- $\mathbf{r}_4 = (0, 0, 0, 0, 0, 4)$

$$\Lambda_k^{(1)}(\mathbf{r}_4) = \sum_{n \in \mathbb{Z}} Q_\tau^{2n^2 - 2n + \frac{1}{2}} (q_1^{-2k+1} q_2)^{2n-1} = \theta_3^{\left[-\frac{1}{2}\right]}(4\tau, -(4k-2)\epsilon_1 + 2\epsilon_2), \quad (3.119)$$

$$\Lambda_k^{(2)}(\mathbf{r}_4) = \sum_{n \in \mathbb{Z}} Q_\tau^{2n^2 - 2n + \frac{1}{2}} (q_1 q_2^{-2k+1})^{2n-1} = \theta_3^{\left[-\frac{1}{2}\right]}(4\tau, 2\epsilon_1 - (4k-2)\epsilon_2). \quad (3.120)$$

In this case, the matrix M_{Λ_k} constructed out of any three of the four unity \mathbf{r} fields at any base degree have non-vanishing determinant and is thus of full rank. For instance when $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ are used, the leading order contribution to $\det M_{\Lambda_k}$ is

$$\det M_{\Lambda_k} = (q_1 q_2)^{-k} (q_1^k + q_2^k + q_1^{2k} + q_2^{2k} + q_1^{2k} q_2^k + q_1^k q_2^{2k}) Q_\tau^{1/4} + \mathcal{O}(Q_\tau^{5/4}). \quad (3.121)$$

One can therefore invert M_{Λ_k} to solve for Z_k or \mathbb{E}_k from the recursion relation.

²¹We need to multiply the expression in [11] by a factor of two.

For instance, using the identity (3.111) with the unity \mathbf{r} fields $(0, 0, 0, 0, 0, 2)$, $(0, 0, 0, 0, 0, 0)$, and $(0, 0, 0, 0, 0, -2)$, we obtain the following expression for the one string elliptic genus:

$$\mathbb{E}_1 = \frac{\Delta^{[-\frac{1}{4}]} I_1^{[-\frac{1}{4}]} + \Delta^{[0]} I_1^{[0]} + \Delta^{[\frac{1}{4}]} I_1^{[\frac{1}{4}]}}{\Delta}, \quad (3.122)$$

where Δ is the determinant of the matrix

$$\begin{pmatrix} \theta_3^{[-\frac{1}{4}]}(4\tau, -2\epsilon_1 + 2\epsilon_2) & \theta_3^{[-\frac{1}{4}]}(4\tau, 2\epsilon_1 - 2\epsilon_2) & \theta_3^{[-\frac{1}{4}]}(4\tau, 2\epsilon_1 + 2\epsilon_2) \\ \theta_3^{[0]}(4\tau, -2\epsilon_1 + 2\epsilon_2) & \theta_3^{[0]}(4\tau, 2\epsilon_1 - 2\epsilon_2) & \theta_3^{[0]}(4\tau, 2\epsilon_1 + 2\epsilon_2) \\ \theta_3^{[\frac{1}{4}]}(4\tau, -2\epsilon_1 + 2\epsilon_2) & \theta_3^{[\frac{1}{4}]}(4\tau, 2\epsilon_1 - 2\epsilon_2) & \theta_3^{[\frac{1}{4}]}(4\tau, 2\epsilon_1 + 2\epsilon_2) \end{pmatrix}, \quad (3.123)$$

and $\Delta^{[\alpha]}$ is the minor of $\theta_3^{[\alpha]}(4\tau, 2\epsilon_1 + 2\epsilon_2)$. Here Δ only has poles at $\epsilon_1 = 0$, $\epsilon_2 = 0$ and $\epsilon_1 - \epsilon_2 = 0$. It is a Jacobi form of weight $3/2$ and index $(3\epsilon_1^2 - 2\epsilon_1\epsilon_2 + 3\epsilon_2^2)/2$. The leading order is $Q_\tau^{9/4}$. The expressions of the elliptic genera for higher numbers of strings can be similarly written down, although they are much more lengthy.

Before ending this subsection, let us mention an interesting phenomenon. In the case of $SO(8)$, the four theta functions

$$\theta_3^{[\frac{1}{4}]}, \quad \theta_3^{[0]}, \quad \theta_3^{[-\frac{1}{4}]}, \quad \theta_3^{[-\frac{1}{2}]}, \quad (3.124)$$

enjoy a cyclic \mathbb{Z}_4 symmetry, as they are invariant under the shift of the upper characteristic $\alpha \rightarrow \alpha - 1/4$. The matrix

$$\begin{pmatrix} \theta_3^{[\frac{1}{4}]}(\tau, z_1) & \theta_3^{[\frac{1}{4}]}(\tau, z_2) & \theta_3^{[\frac{1}{4}]}(\tau, z_3) & \theta_3^{[\frac{1}{4}]}(\tau, z_4) \\ \theta_3(\tau, z_1) & \theta_3(\tau, z_2) & \theta_3(\tau, z_3) & \theta_3(\tau, z_4) \\ \theta_3^{[-\frac{1}{4}]}(\tau, z_1) & \theta_3^{[-\frac{1}{4}]}(\tau, z_2) & \theta_3^{[-\frac{1}{4}]}(\tau, z_3) & \theta_3^{[-\frac{1}{4}]}(\tau, z_4) \\ \theta_3^{[-\frac{1}{2}]}(\tau, z_1) & \theta_3^{[-\frac{1}{2}]}(\tau, z_2) & \theta_3^{[-\frac{1}{2}]}(\tau, z_3) & \theta_3^{[-\frac{1}{2}]}(\tau, z_4) \end{pmatrix}, \quad \text{with } z_1 + z_2 + z_3 + z_4 = 0, \quad (3.125)$$

which has S_4 symmetry amongst the elliptic parameters and thus is an analogue of the matrix M_{Λ_1} of $SU(3)$, has a vanishing determinant.

3.4.4 Solving refined BPS invariants

Here we compute the BPS invariants from the equations extracted from the expansion of the blowup equations with respect to all Kähler moduli. We used the blowup equations associated with the following \mathbf{r} fields

$$(0, 0, 0, 0, 0, 2), (-2, -2, 0, 0, 2, 2), (-2, 2, 0, 0, 0, 0) + \text{permutations of } r_1, r_2, r_3, r_4. \quad (3.126)$$

We managed to compute all the BPS invariants up to total degree of $d_1 + d_2 + d_3 + d_4 + d_c + d_b = 5$ and list them in table 6. They satisfy the obvious permutation symmetry of d_1, d_2, d_3, d_4 . Therefore we only list the non-vanishing invariants with $d_1 \leq d_2 \leq d_3 \leq d_4$ and omit those which can be obtained by permuting these degrees. The other curve classes that are not listed in the table all have vanishing BPS invariants. These invariants agree with the results in the literature.

Note that in addition there seems to be a curious symmetry between d_c, d_b if both are nonzero

$$N_{j_L, j_R}^{d_1, d_2, d_3, d_4, d_c, d_b} = N_{j_L, j_R}^{d_1, d_2, d_3, d_4, d_b, d_c}, \quad d_c, d_b \neq 0. \quad (3.127)$$

We trace this symmetry to fiber-base duality of D-type theories [45]. This can be understood as follows. Starting with an affine D_4 base, the central fiber (i.e. the fiber over the central node in \widehat{D}_4) is of affine $SU(2)$ type. Switching the role of fiber and base, one obtains an affine $SU(2)$ base consisting of a (-4) and a (-1) -curve with the fiber over the (-4) curve of affine D_4 type. Decompactifying the -1 one arrives exactly at our present setup. Thus what we have effectively done, when switching off all non-central nodes of \widehat{D}_4 is to swap the central node with the base curve. This is a remnant of the actual exact duality where the (-1) curve has finite size.

We believe that BPS invariants of higher degree can be computed similarly given more computing time and that the same symmetry will hold.

4 Reduction to blowup equations for 5d theories

We demonstrate here that the blowup equations for 6d gauge theories could be dimensionally reduced to the blowup equations for 5d gauge theories.

We have seen in sections 2.5.1, 2.5.2 that the perturbative free energy of the 6d theory is reduced to that of the 5d theory through the limit

$$\lim' : \tau + c t_m \rightarrow -\infty, \quad t_m \text{ finite}, \quad (4.1)$$

with a model-dependent constant c . When applied to Z^{inst} this limit is equivalent to keeping finite terms in the limit $Q_\tau = e^\tau \rightarrow 0$. The 6d one-loop partition function given by (2.21) becomes

$$\begin{aligned} Z^{\text{1-loop}} &\rightarrow \text{PE} \left[- \frac{q_R + q_R^{-1}}{(q_1^{1/2} - q_1^{-1/2})(q_2^{1/2} - q_2^{-1/2})} \sum_{\alpha \in \Delta_+} e^{-\alpha \cdot a} \right] \\ &= \prod_{i,j=1}^{\infty} \prod_{\alpha \in \Delta_+} \left(1 - q_1^{i+1} q_2^{j+1} e^{\langle \alpha, \mathbf{a} \rangle} \right) \left(1 - q_1^i q_2^j e^{\langle \alpha, \mathbf{a} \rangle} \right). \end{aligned} \quad (4.2)$$

Using the formal identity

$$\prod_{n=0}^{\infty} (1 - x q^n) = \prod_{n=1}^{\infty} (1 - x q^{-n})^{-1}. \quad (4.3)$$

The last line of (4.2) could be written as

$$\prod_{i,j=0}^{\infty} \prod_{\alpha \in \Delta_+} \left(1 - q^i t^{j+1} e^{\langle \alpha, \mathbf{a} \rangle} \right)^{-1} \left(1 - q^{i+1} t^j e^{\langle \alpha, \mathbf{a} \rangle} \right)^{-1} \quad (4.4)$$

with

$$q = e^{-\epsilon_1}, \quad t = e^{\epsilon_2} \quad (4.5)$$

which is precisely the 1-loop partition function of a 5d pure SYM theory [46].

Furthermore, the partition function component Z_k of the 6d gauge theory is identified with the k -string elliptic genus by²²

$$Z_k(\mathbf{t}, \epsilon_1, \epsilon_2) = \left(\frac{\prod_{i=0}^{\lfloor \frac{n-3}{2} \rfloor} Q_{r+1-i}^{n-2-2i}}{Q_\tau^{\frac{n-2}{2}}} \right)^k \mathbb{E}_k(\tau, \mathbf{m}, \epsilon_1, \epsilon_2). \quad (4.6)$$

When reduced to 5d gauge theory by sending Q_τ to 0, we recover 5d gauge instanton partition functions Z_k^{5d} by [12, 38]²³

$$\begin{cases} \frac{\prod_{i=0}^{\lfloor \frac{n-3}{2} \rfloor} Q_{r+1-i}^{n-2-2i}}{Q_\tau^{n-2}} Q_b = Q_\tau^{-(n-2)/2} Q_{\text{ell}} \rightarrow \mathfrak{q} \\ Q_\tau^{k(n-2)/2} \mathbb{E}_k \rightarrow Z_k^{\text{inst}} \end{cases} \quad (4.7)$$

such that

$$1 + \sum_{k=1}^{\infty} Q_b^k Z_k \rightarrow 1 + \sum_{k=1}^{\infty} \mathfrak{q}^k Z_k^{\text{inst}}. \quad (4.8)$$

Here \mathfrak{q} is the gauge instanton fugacity related to the 5d mass parameter t_m by $\mathfrak{q} = e^{t_m}$. The first line in the dictionary (4.7) is then consistent with the observation (2.28), (2.41).

These observations allow us to conclude that we can obtain the full partition function of the 5d pure SYM theory from the partition function of the 6d gauge theory through the operation

$$Z_{5d}(\mathbf{t}, \epsilon_1, \epsilon_2) = \lim' Z^{\text{dec}}(\tau, t_m, \epsilon_1, \epsilon_2)^{-1} Z_{6d}(\mathbf{T}, \epsilon_1, \epsilon_2), \quad (4.9)$$

where $Z^{\text{dec}}(\tau, t_m, \epsilon_1, \epsilon_2)$ is the component that runs off in the limit (4.1), which is the exponential of the extra piece in the perturbative free energy given by (2.30), (2.33) combined for SU(3) and (2.43), (2.46) combined for SO(8) theories respectively. Here we use \mathbf{T} for Kähler moduli in 6d instead of \mathbf{t} to stress that there is one more Kähler modulus in 6d theories. We make it explicit that Z^{dec} only depends on τ, t_m and no other Kähler moduli. Besides, we are free to twist the partition functions in the sense of (1.4) and put hats over Z_{5d}, Z_{6d} .

Then by multiplying both sides of the blowup equations for the 6d theory with an inverse power of Z^{dec} and taking the limit \lim' , we get

$$\begin{aligned} & \lim' \Lambda_{6d}(\tau, \epsilon_1, \epsilon_2) \widehat{Z}_{5d}(\mathbf{t}, \epsilon_1, \epsilon_2) \\ &= \lim' \sum_{\mathbf{n} \in \mathbb{Z}^{r+1}} (-1)^{|\mathbf{n}|} B^{\text{dec}}(\tau, t_m, \epsilon_{1,2}; \mathbf{n}) \widehat{Z}_{5d}(\mathbf{t} + \epsilon_1 \mathbf{R}, \epsilon_1, \epsilon_2 - \epsilon_1) \widehat{Z}_{5d}(\mathbf{t} + \epsilon_2 \mathbf{R}, \epsilon_1 - \epsilon_2, \epsilon_2) \end{aligned} \quad (4.10)$$

where we have defined

$$B^{\text{dec}}(\tau, t_m, \epsilon_{1,2}; \mathbf{n}) = \frac{Z^{\text{dec}}(\tau, t_m + \epsilon_1 R_{t_m}, \epsilon_1, \epsilon_2 - \epsilon_1) Z^{\text{dec}}(\tau, t_m + \epsilon_2 R_{t_m}, \epsilon_1 - \epsilon_2, \epsilon_2)}{Z^{\text{dec}}(\tau, t_m, \epsilon_1, \epsilon_2)}. \quad (4.11)$$

²²This relation coincides with that in [38] when $n = 3, 4$. The discrepancy for $n > 4$ is due to that Q_b defined in [38] is no longer the volume of a Mori cone generator.

²³The discussion in section 5.4 of [38] is slightly inaccurate.

As we will illustrate by the examples of SU(3) and SO(8) theories, if we expand Λ_{6d} and B^{dec} in terms of Q_τ , and keep only the coefficients of the lowest power on both sides of (4.10), we get the blowup equations for the 5d gauge theory

$$\Lambda_{5d}(t_m, \epsilon_{1,2}) \widehat{Z}_{5d}(\mathbf{t}, \epsilon_{1,2}) = \sum_{\mathbf{n} \in \mathbb{Z}^r} (-1)^{|\mathbf{n}|} \widehat{Z}_{5d}(\mathbf{t} + \epsilon_1 \mathbf{R}, \epsilon_1, \epsilon_2 - \epsilon_1) \widehat{Z}_{5d}(\mathbf{t} + \epsilon_2 \mathbf{R}, \epsilon_1 - \epsilon_2, \epsilon_2). \quad (4.12)$$

Note that the dimension of \mathbf{n} is reduced by 1 in the 5d blowup equations.

4.1 SU(3) model

In the case of 6d SU(3) model, we have concretely

$$\lim'_{\text{SU}(3)} = \lim_{\tau+t_m \rightarrow -\infty} = \lim_{t_3+t_b \rightarrow -\infty}, \quad t_m, t_b \text{ finite}. \quad (4.13)$$

We take the Nekrasov partition function to be the partition function of the 5d gauge theory. Z^{dec} should include an extra piece from (2.29), and it reads

$$Z_{\text{SU}(3)}^{\text{dec}}(\mathbf{t}, \epsilon_1, \epsilon_2) = \exp \left(-\frac{(\tau + t_m)^3 - t_m^3}{18\epsilon_1\epsilon_2} - \frac{(\epsilon_1^2 + \epsilon_2^2 + 3\epsilon_1\epsilon_2)(\tau + t_m)}{8\epsilon_1\epsilon_2} \right) \quad (4.14)$$

Then

$$B_{\text{SU}(3)}^{\text{dec}} = \exp \left[\tau \left(\frac{3}{2}n_3^2 - \frac{1}{2}n_3(r_3 + r_4) + \frac{1}{24}(r_3 + r_4)^2 \right) + \dots \right] \quad (4.15)$$

where the remaining pieces in the ellipses are linear in t_m and ϵ_R and depend only on n_3 but not on n_1, n_2 . Clearing, depending on the value of $r_3 + r_4$, there are only one or two integral values of n_3 which minimize the power of Q_τ . If we only keep the minimal power of Q_τ in (4.10), although we still sum n_1, n_2 over all integers, we only sum n_3 over one or two values. This is the reason the dimension of the summation index vector \mathbf{n} is reduced by 1 in the 5d blowup equations. In the case where n_3 can take two values (this happens when $r_3 + r_4 = 6k + 3, k \in \mathbb{Z}$), one 6d blowup equation splits to two 5d blowup equations.

Note that given the definition of \mathbf{R} in (3.2), when we sum $\mathbf{n} \in \mathbb{Z}$ in the blowup equations, we are effectively summing $2\mathbf{R}$ over all the \mathbf{r} fields in the same equivalence class. Therefore when dimensionally reducing blowup equations, we can get the equivalence classes of \mathbf{r} fields for the 5d theory simply by fixing the value of n_3 , as we prescribed above, and deleting the entry r_3 associated to t_3 .

This procedure also gives us immediately a way to compute Λ_{5d} from Λ_{6d} , which are given in section 3.3.1

$$\Lambda_{5d}^{(n_3^{\text{min}})}(t_m) = \lim_{\tau+t_m \rightarrow -\infty} \Lambda_{6d}(\tau) B^{\text{dec}}(\tau, t_m, \epsilon_{1,2}; n_3^{\text{min}})^{-1}, \quad (4.16)$$

where n_3^{min} is a value of n_3 that minimizes the power of Q_τ .

We list the 5d \mathbf{r} fields reduced from 6d \mathbf{r} fields as well as the corresponding Λ_{5d} in table 3. They are consistent with a direct compute with 5d blowup equations. Note that Λ_{5d} for the 5d unity \mathbf{r} fields $(2, -2, -3), (2, -2, 3), (-2, 2, -3), (-2, 2, 3)$ cannot be derived by (4.16) though, and they are instead computed from 5d blowup equations. They are also notably pairwise identical as they should since they descend from the same 6d vanishing blowup equations pairwise. Note that the last entry of 5d \mathbf{r} field is r_m related to r_b by

$$r_m = r_b - r_1 - r_2. \quad (4.17)$$

(r_1, r_2, r_3, r_b)	(r_1, r_2, r_m)	Λ_{5d}
$(0, 0, 0, -1)$	$(0, 0, -1)$	$q_R^{1/12}$
$(0, 0, 0, 1)$	$(0, 0, 1)$	$q_R^{-1/12}$
$(0, 0, 0, 3)$	$(0, 0, -3), (0, 0, 3)$	$q_R^{1/4}, q_R^{-1/4}$
$(-2, 0, 2, 1)$	$(2, -2, -3), (2, -2, 3)$	$q_R^{-3/4} Q_m^{1/3}, q_R^{-3/4} Q_m^{1/3}$
$(0, -2, 2, 1)$	$(-2, 2, -3), (-2, 2, 3)$	$q_R^{3/4} Q_m^{1/3}, q_R^{-3/4} Q_m^{1/3}$
$(-2, 2, 0, 1)$	$(-2, 2, 1)$	0
$(2, -2, 0, 1)$	$(2, -2, 1)$	0
$(2, 0, -2, 1)$	$(-2, 2, -1)$	0
$(0, 2, -2, 1)$	$(2, -2, -1)$	0

Table 3. Reduction of 6d \mathbf{r} fields to 5d \mathbf{r} fields and corresponding Λ_{5d} for the SU(3) model. Unity and vanishing \mathbf{r} fields are colored in blue and green respectively.

4.2 SO(8) model

In the case of SO(8) model, we have

$$\lim'_{\text{SO}(8)} = \lim_{\tau + \frac{1}{2}t_m \rightarrow -\infty} = \lim_{t_4 + \frac{1}{2}t_b \rightarrow -\infty}, \quad t_m, t_b \text{ finite}, \quad (4.18)$$

as well as

$$Z_{\text{SO}(8)}^{\text{dec}}(\mathbf{t}, \epsilon_1, \epsilon_2) = \exp\left(-\frac{(2\tau + t_m)^3 - t_m^3}{48\epsilon_1\epsilon_2} - \frac{(\epsilon_1^2 + \epsilon_2^2 + 3\epsilon_1\epsilon_2)(2\tau + t_m)}{6\epsilon_1\epsilon_2}\right), \quad (4.19)$$

which leads to

$$B_{\text{SO}(8)}^{\text{dec}} = \exp\left[\tau\left(2n_4^2 - \frac{2r_4 + r_6}{2}n_4 + \frac{4r_4^2 + r_6^2 + 4r_4r_6}{32}\right) + \dots\right]. \quad (4.20)$$

The analysis is completely analogous as in the case of the SU(3) theory. The splitting of 6d \mathbf{r} fields to pairs of 5d \mathbf{r} fields happens when $2r_4 + r_6 = 8k + 4$, $k \in \mathbb{Z}$. We list the resulting 5d \mathbf{r} fields and the corresponding Λ_{5d} in table 4. Note that the last entry of 5d \mathbf{r} field is r_m related to r_b by

$$r_m = r_b - 2r_1 - 2r_2 - 2r_3 - 4r_c. \quad (4.21)$$

5 Conclusion and discussion

In this paper, we consider the $n = 3, 4$ minimal 6d SCFTs in the tensor branch. These theories are obtained by F-theory compactification on non-compact elliptic Calabi-Yau threefolds. We demonstrate that the elliptic genera of these theories, which encode the refined BPS invariants of the underlying Calabi-Yau threefolds, satisfy the generalized blowup equations. Furthermore, we illustrate that the generalized blowup equations can be used to solve the elliptic genera as well as the refined BPS invariants.

$(r_1, r_2, r_3, r_4, r_c, r_b)$	$(r_1, r_2, r_3, r_c, r_m)$	Λ_{5d}
$(0, 0, 0, 0, 0, -2)$	$(0, 0, 0, 0, -2)$	$q_R^{1/3}$
$(0, 0, 0, 0, 0, 0)$	$(0, 0, 0, 0, 0)$	1
$(0, 0, 0, 0, 0, 2)$	$(0, 0, 0, 0, 2)$	$q_R^{-1/3}$
$(0, 0, 0, 0, 0, 4)$	$(0, 0, 0, 0, 4), (0, 0, 0, 2, -4)$	$q_R^{-2/3}, q_R^{2/3}$
$(-2, 0, 0, 2, 0, 0)$	$(-2, 0, 0, 0, 4), (-2, 0, 0, 2, -4)$	0, 0
$(0, -2, 0, 2, 0, 0)$	$(0, -2, 0, 0, 4), (0, -2, 0, 2, -4)$	0, 0
$(0, 0, -2, 2, 0, 0)$	$(0, 0, -2, 0, 4), (0, 0, -2, 2, -4)$	0, 0
$(-2, 2, 0, 0, 0, 0)$	$(-2, 2, 0, 0, 0)$	0
$(-2, 0, 2, 0, 0, 0)$	$(-2, 0, 2, 0, 0)$	0
$(0, -2, 2, 0, 0, 0)$	$(0, -2, 2, 0, 0)$	0
$(-2, -2, 0, 0, 2, 2)$	$(-2, -2, 0, 2, 2)$	0
$(-2, 0, -2, 0, 2, 2)$	$(-2, 0, -2, 2, 2)$	0
$(0, -2, -2, 0, 2, 2)$	$(0, -2, -2, 2, 2)$	0
$(-2, 0, 0, -2, 2, 2)$	$(-2, 0, 0, 2, -2)$	0
$(0, -2, 0, -2, 2, 2)$	$(0, -2, 0, 2, -2)$	0
$(0, 0, -2, -2, 2, 2)$	$(0, 0, -2, 2, -2)$	0

Table 4. Reduction of 6d \mathbf{r} fields to 5d \mathbf{r} fields and the corresponding Λ_{5d} for the SO(8) model. Unity and vanishing \mathbf{r} fields are colored in blue and green respectively.

We emphasize here that the generalized blowup equations is an extremely powerful tool for computing the BPS invariants of non-compact Calabi-Yau threefolds. All the currently existing techniques for computing BPS invariants in local geometry, being well established and very powerful, have their limitations in terms of accessible geometries. The topological vertex [42, 47, 48] is only applicable for toric geometries or generalizations thereof. The holomorphic anomaly equations [49, 50] and the topological recursion [51–53] are useful only if the mirror geometry is known, and in particular if the number of compact divisors g ,²⁴ in the original geometry is low ($g \leq 2$). The modular bootstrap [13, 33, 38, 54, 55] only works if the Calabi-Yau is elliptic and is most efficient if there is no singular elliptic fiber (see also [56]). On the other hand, the generalized blowup equations are more versatile than any of these individual methods. They have been applied in toric geometries [34], elliptic geometries (which are non-toric), and the cases where the number of compact divisors is greater than two (this paper). Up to this moment, there does not seem to be any restriction on the type of non-compact Calabi-Yau threefolds for which the generalized blowup equations are applicable.

²⁴If the mirror geometry can be reduced to a curve, this number is equal to the genus of the curve.

Nevertheless, we have to point out that why the generalized blowup equations work still remains a mystery. The only case where the blowup equations have a rigorous mathematical proof is when applied on the $X_{N,m}$ geometries [24]. A better mathematical or physical understanding of the generalized blowup equations would be extremely desirable. For example, what is the relation between blowup equations and refined holomorphic anomaly equations? To answer this question requires a non-holomorphic version of blowup equations. And what is the relation between blowup equations and refined topological vertex? For 6d SCFTs, this may involve the recently proposed elliptic topological vertex [57]. Furthermore the moduli space of the topological string theory usually contains both geometric and non-geometric phases. In this paper we only work deep in the geometric phase around the large volume limit. It is an interesting problem to study the blowup equations in the other phases of the moduli space as in [34]. Finally one could certainly push the computation of the BPS invariants for an almost infinite range of non-compact Calabi-Yau threefolds. The easiest targets and the most similar to what are studied here are those for the remaining cases of minimal 6d SCFTs, the results of which we will report in companion papers in the near future.

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A Useful identities

We collect some identities which are useful in the main text of the paper.

Using the triple product formula of θ_1

$$\theta_1(\tau, z) = iQ^{\frac{1}{2}}Q_z^{-\frac{1}{2}}\eta(\tau) \prod_{n=1}^{\infty} (1 - Q_z Q_\tau^{n-1}) \left(1 - \frac{Q_\tau^n}{Q_z}\right), \quad (\text{A.1})$$

we can simplify the following plethystic exponentials which often appear in the evaluation of vector multiplet contributions to the one-loop partition function

$$\text{PE} \left[\frac{Q}{1 - Q_\tau} \right] = \prod_{n=0}^{\infty} \frac{1}{1 - QQ_\tau^n}, \quad (\text{A.2})$$

and

$$\text{PE} \left[\left(Q_z + \frac{Q_\tau}{Q_z} \right) \left(\frac{1}{1 - Q_\tau} \right) \right] = \frac{iQ^{\frac{1}{2}}Q_z^{-\frac{1}{2}}\eta(\tau)}{\theta_1(\tau, z)}. \quad (\text{A.3})$$

In the following, we would like to present some elementary but useful formulas when dealing with blowup equations. Denote

$$f_{(j_L, j_R)}(q_1, q_2) = \frac{\chi_{j_L}(q_L)\chi_{j_R}(q_R)}{\left(q_1^{1/2} - q_1^{-1/2}\right)\left(q_2^{1/2} - q_2^{-1/2}\right)} \quad (\text{A.4})$$

which is the spin-related prefactor in the contribution to the one-loop partition function of a multiplet with spin (j_L, j_R) (see for instance (2.16)). It satisfies the relations

$$f_{(j_L, j_R)}(q_1^{-1}, q_2^{-1}) = f_{(j_L, j_R)}(q_1, q_2) = f_{(j_L, j_R)}(q_2, q_1), \quad (\text{A.5})$$

$$f_{(j_L, j_R)}(q_1^{-1}, q_2) = f_{(j_L, j_R)}(q_1, q_2^{-1}) = f_{(j_R, j_L)}(q_1, q_2). \quad (\text{A.6})$$

In the blowup equation this prefactor contributes by

$$Bl_{(j_L, j_R, R)}(q_1, q_2) = f_{(j_L, j_R)}(q_1, q_2/q_1)q_1^R + f_{(j_L, j_R)}(q_1/q_2, q_2)q_2^R - f_{(j_L, j_R)}(q_1, q_2), \quad (\text{A.7})$$

where $R = \mathbf{R} \cdot \mathbf{d} \in \frac{1}{2}\mathbb{Z}$ is the entry of \mathbf{R} associated to the Kähler modulus $\mathbf{Q}^{\mathbf{d}}$ multiplying this prefactor. The checkerboard pattern (1.3) translates to the condition

$$2j_L + 2j_R + 1 \equiv 2R \pmod{2}. \quad (\text{A.8})$$

It has been argued from the ϵ_1, ϵ_2 expansion of refined free energy and blowup equations [34] that under this condition the apparent denominator of $Bl_{(j_L, j_R, R)}(q_1, q_2)$ can always be factored out so that

$$Bl_{(j_L, j_R, R)}(q_1, q_2) = \text{finite series in } q_1, q_2. \quad (\text{A.9})$$

We call (A.9) *fundamental identities*. Note that since

$$Bl_{(j_L, j_R, -R)}(q_1, q_2) = Bl_{(j_L, j_R, R)}(q_1^{-1}, q_2^{-1}), \quad (\text{A.10})$$

we only need to consider the cases with $R \geq 0$.

In the following, we present some frequently used instances of the fundamental identities for small spins.

- For $(j_L, j_R) = (0, 0)$, R should be half integers. Then

$$Bl_{(0,0,R)}(q_1, q_2) = - \sum_{\substack{m,n \geq 0 \\ m+n \leq R-3/2}} q_1^{m+1/2} q_2^{n+1/2}, \quad R \geq 1/2. \quad (\text{A.11})$$

- For $(j_L, j_R) = (1/2, 0)$, R should be integers. Then

$$Bl_{(1/2,0,R)}(q_1, q_2) = \begin{cases} - \sum_{\substack{m,n \geq 0 \\ 1 \leq m+n \leq R}} q_1^m q_2^n - \sum_{\substack{m,n \geq 0 \\ m+n \leq R-3}} q_1^{m+1} q_2^{n+1}, & R \geq 1, \\ -1, & R = 0. \end{cases} \quad (\text{A.12})$$

- For $(j_L, j_R) = (0, 1/2)$, R should be integers. Then

$$Bl_{(0,1/2,R)}(q_1, q_2) = - \sum_{\substack{m,n \geq 0 \\ m+n \leq R-1}} q_1^m q_2^n - \sum_{\substack{m,n \geq 0 \\ m+n \leq R-2}} q_1^{m+1} q_2^{n+1}, \quad R \geq 0. \quad (\text{A.13})$$

As we have seen in the main text, the contribution of vector multiplets can always be factorized as products of

$$T_R(z) = \text{PE} \left[- \left(Bl_{(0,1/2,R)}(q_1, q_2) Q_z + Bl_{(0,1/2,-R)}(q_1, q_2) \frac{Q_\tau}{Q_z} \right) \left(\frac{1}{1 - Q_\tau} \right) \right]. \quad (\text{A.14})$$

Using (A.13) and (A.3) and assuming $R \geq 0$, it can be written as

$$\begin{aligned} T_R(z) &= \prod_{\substack{m,n \geq 0 \\ m+n \leq R-1}} \frac{i Q_\tau^{1/12} \eta(Q_z q_1^m q_2^n)^{-1/2}}{\theta_1(z + m\epsilon_1 + n\epsilon_2)} \prod_{\substack{m,n \geq 0 \\ m+n \leq R-2}} \frac{i Q_\tau^{1/12} \eta(Q_z q_1^{m+1} q_2^{n+1})^{-1/2}}{\theta_1(z + (m+1)\epsilon_1 + (n+1)\epsilon_2)} \\ &= \left(i Q_\tau^{1/12} Q_z^{-1/2} \right)^{R^2} (q_1 q_2)^{-\frac{(R-1)R(R+1)}{6}} \check{\theta}_R(z), \end{aligned} \quad (\text{A.15})$$

where

$$\check{\theta}_R(z) = \prod_{\substack{m,n \geq 0 \\ m+n \leq R-1}} \frac{\eta}{\theta_1(z + m\epsilon_1 + n\epsilon_2)} \prod_{\substack{m,n \geq 0 \\ m+n \leq R-2}} \frac{\eta}{\theta_1(z + (m+1)\epsilon_1 + (n+1)\epsilon_2)}. \quad (\text{A.16})$$

In the case of $R < 0$ we can use the above expression for $-R$ with $\epsilon_{1,2}$ replaced by $-\epsilon_{1,2}$ or equivalently $q_{1,2}$ replaced by $1/q_{1,2}$. In both cases, $\check{\theta}_R(z)$ is a multivariate Jacobi form of weight zero and index quadratic form

$$\text{Ind}_{\check{\theta}}^R(z) = -\frac{R^2 z^2}{2} - \frac{(R-1)R(R+1)}{3} z(\epsilon_1 + \epsilon_2) - \frac{(R-1)R^2(R+1)}{12} (\epsilon_1^2 + \epsilon_1 \epsilon_2 + \epsilon_2^2). \quad (\text{A.17})$$

B Refined BPS invariants

$\mathbf{d} = (d_1, d_2, d_3, d_b) \oplus N_{j_L, j_R}^{\mathbf{d}}(j_L, j_R)$	
(0, 0, 1, 0)	(0, 1/2)
(0, 1, 1, 0)	(0, 1/2)
(1, 1, 2, 0)	(0, 1/2)
(1, 2, 2, 0)	(0, 1/2)
(2, 2, 3, 0)	(0, 1/2)
(0, 0, 0, 1)	(0, 0)
(0, 0, 1, 1)	(0, 1)
(0, 1, 1, 1)	(0, 0) ⊕ (0, 1)
(0, 0, 2, 1)	(0, 2)
(1, 1, 1, 1)	3(0, 0) ⊕ 3(0, 1) ⊕ (1/2, 1/2)
(0, 1, 2, 1)	(0, 1) ⊕ (0, 2)
(0, 0, 3, 1)	(0, 3)
(1, 1, 2, 1)	2(0, 0) ⊕ 4(0, 1) ⊕ 2(0, 2) ⊕ (1/2, 1/2) ⊕ (1/2, 3/2)
(0, 2, 2, 1)	(0, 0) ⊕ (0, 1) ⊕ (0, 2)
(0, 1, 3, 1)	(0, 2) ⊕ (0, 3)
(0, 0, 4, 1)	(0, 4)
(1, 2, 2, 1)	4(0, 0) ⊕ 7(0, 1) ⊕ 3(0, 2) ⊕ 2(1/2, 1/2) ⊕ (1/2, 3/2)
(1, 1, 3, 1)	2(0, 1) ⊕ 4(0, 2) ⊕ 2(0, 3) ⊕ (1/2, 3/2) ⊕ (1/2, 5/2)
(0, 2, 3, 1)	(0, 1) ⊕ (0, 2) ⊕ (0, 3)
(0, 1, 4, 1)	(0, 3) ⊕ (0, 4)
(0, 0, 5, 1)	(0, 5)
(2, 2, 2, 1)	13(0, 0) ⊕ 15(0, 1) ⊕ 6(0, 2) ⊕ 7(1/2, 1/2) ⊕ 3(1/2, 3/2) ⊕ (1, 1)
(1, 2, 3, 1)	2(0, 0) ⊕ 6(0, 1) ⊕ 6(0, 2) ⊕ 2(0, 3) ⊕ (1/2, 1/2) ⊕ 2(1/2, 3/2) ⊕ (1/2, 5/2)
(1, 1, 4, 1)	2(0, 2) ⊕ 4(0, 3) ⊕ 2(0, 4) ⊕ (1/2, 5/2) ⊕ (1/2, 7/2)
(0, 3, 3, 1)	(0, 0) ⊕ (0, 1) ⊕ (0, 2) ⊕ (0, 3)
(0, 2, 4, 1)	(0, 2) ⊕ (0, 3) ⊕ (0, 4)
(0, 1, 5, 1)	(0, 4) ⊕ (0, 5)
(0, 0, 6, 1)	(0, 6)
(0, 0, 2, 2)	(0, 5/2)
(1, 1, 1, 2)	2(0, 1/2) ⊕ (0, 3/2)
(0, 1, 2, 2)	(0, 3/2) ⊕ (0, 5/2)
(0, 0, 3, 2)	(0, 5/2) ⊕ (0, 7/2) ⊕ (1/2, 4)
(1, 1, 2, 2)	3(0, 1/2) ⊕ 5(0, 3/2) ⊕ 3(0, 5/2) ⊕ (1/2, 1) ⊕ (1/2, 2)
(0, 2, 2, 2)	2(0, 1/2) ⊕ 2(0, 3/2) ⊕ 2(0, 5/2) ⊕ (0, 7/2)
(0, 1, 3, 2)	(0, 3/2) ⊕ 3(0, 5/2) ⊕ 2(0, 7/2) ⊕ (1/2, 3) ⊕ (1/2, 4)
(0, 0, 4, 2)	(0, 5/2) ⊕ (0, 7/2) ⊕ 2(0, 9/2) ⊕ (1/2, 4) ⊕ (1/2, 5) ⊕ (1, 11/2)
(1, 2, 2, 2)	12(0, 1/2) ⊕ 14(0, 3/2) ⊕ 8(0, 5/2) ⊕ 2(0, 7/2) ⊕ 2(1/2, 0) ⊕ 4(1/2, 1) ⊕ 3(1/2, 2) ⊕ (1/2, 3)
(1, 1, 3, 2)	2(0, 1/2) ⊕ 9(0, 3/2) ⊕ 13(0, 5/2) ⊕ 6(0, 7/2) ⊕ (1/2, 1) ⊕ 4(1/2, 2) ⊕ 5(1/2, 3) ⊕ 2(1/2, 4) ⊕ (1, 5/2) ⊕ (1, 7/2)
(0, 2, 3, 2)	2(0, 1/2) ⊕ 4(0, 3/2) ⊕ 5(0, 5/2) ⊕ 3(0, 7/2) ⊕ (0, 9/2) ⊕ (1/2, 2) ⊕ (1/2, 3) ⊕ (1/2, 4)
(0, 1, 4, 2)	(0, 3/2) ⊕ 3(0, 5/2) ⊕ 5(0, 7/2) ⊕ 3(0, 9/2) ⊕ (1/2, 3) ⊕ 3(1/2, 4) ⊕ 2(1/2, 5) ⊕ (1, 9/2) ⊕ (1, 11/2)
(0, 0, 5, 2)	(0, 5/2) ⊕ (0, 7/2) ⊕ 2(0, 9/2) ⊕ 2(0, 11/2) ⊕ (1/2, 4) ⊕ (1/2, 5) ⊕ 2(1/2, 6) ⊕ (1, 11/2) ⊕ (1, 13/2) ⊕ (3/2, 7)
(0, 0, 3, 3)	(0, 3) ⊕ (1/2, 9/2)
(1, 1, 2, 3)	(0, 1) ⊕ 2(0, 2) ⊕ (0, 3)
(0, 2, 2, 3)	(0, 0) ⊕ (0, 1) ⊕ (0, 2) ⊕ (0, 3) ⊕ (0, 4)
(0, 1, 3, 3)	(0, 2) ⊕ 2(0, 3) ⊕ (0, 4) ⊕ (1/2, 7/2) ⊕ (1/2, 9/2)
(0, 0, 4, 3)	(0, 2) ⊕ (0, 3) ⊕ 2(0, 4) ⊕ (0, 5) ⊕ (0, 6) ⊕ (1/2, 7/2) ⊕ 2(1/2, 9/2) ⊕ 2(1/2, 11/2) ⊕ (1, 5) ⊕ (1, 6) ⊕ (3/2, 13/2)

Table 5. BPS invariants of 6d $n = 3$ minimal model.

$\mathbf{d} = (d_1, d_2, d_3, d_4, d_c, d_b)$	$\oplus N_{j_L, j_R}^{\mathbf{d}}(j_L, j_R)$
(0, 0, 0, 1, 0, 0)	(0, 1/2)
(0, 0, 0, 0, 1, 0)	(0, 1/2)
(0, 0, 0, 1, 1, 0)	(0, 1/2)
(0, 0, 1, 1, 1, 0)	(0, 1/2)
(0, 1, 1, 1, 1, 0)	(0, 1/2)
(1, 1, 1, 1, 1, 0)	(0, 1/2)
(0, 1, 1, 1, 2, 0)	(0, 1/2)
(0, 0, 0, 0, 0, 1)	(0, 1/2)
(0, 0, 0, 1, 0, 1)	(0, 1/2)
(0, 0, 0, 2, 0, 1)	(0, 3/2)
(0, 0, 0, 3, 0, 1)	(0, 5/2)
(0, 0, 0, 4, 0, 1)	(0, 7/2)
(0, 0, 0, 0, 1, 1)	(0, 3/2)
(0, 0, 0, 1, 1, 1)	(0, 1/2) \oplus (0, 3/2)
(0, 0, 1, 1, 1, 1)	2(0, 1/2) \oplus (0, 3/2)
(0, 0, 0, 2, 1, 1)	(0, 1/2) \oplus (0, 3/2)
(0, 1, 1, 1, 1, 1)	3(0, 1/2) \oplus (0, 3/2)
(0, 0, 1, 2, 1, 1)	(0, 1/2) \oplus (0, 3/2)
(0, 0, 0, 3, 1, 1)	(0, 3/2) \oplus (0, 5/2)
(0, 0, 0, 0, 2, 1)	(0, 5/2)
(0, 0, 0, 1, 2, 1)	(0, 3/2) \oplus (0, 5/2)
(0, 0, 1, 1, 2, 1)	(0, 1/2) \oplus 2(0, 3/2) \oplus (0, 5/2)
(0, 0, 0, 2, 2, 1)	(0, 1/2) \oplus (0, 3/2) \oplus (0, 5/2)
(0, 0, 0, 0, 3, 1)	(0, 7/2)
(0, 0, 0, 1, 3, 1)	(0, 5/2) \oplus (0, 7/2)
(0, 0, 0, 0, 4, 1)	(0, 9/2)
(0, 0, 0, 3, 0, 2)	(0, 5/2)
(0, 0, 0, 0, 1, 2)	(0, 5/2)
(0, 0, 0, 1, 1, 2)	(0, 3/2) \oplus (0, 5/2)
(0, 0, 1, 1, 1, 2)	(0, 1/2) \oplus 2(0, 3/2) \oplus (0, 5/2)
(0, 0, 0, 2, 1, 2)	(0, 1/2) \oplus (0, 3/2) \oplus (0, 5/2)
(0, 0, 0, 0, 2, 2)	(0, 5/2) \oplus (0, 7/2) \oplus (1/2, 4)
(0, 0, 0, 1, 2, 2)	(0, 3/2) \oplus 3(0, 5/2) \oplus 2(0, 7/2) \oplus (1/2, 3) \oplus (1/2, 4)
(0, 0, 0, 0, 3, 2)	(0, 5/2) \oplus (0, 7/2) \oplus 2(0, 9/2) \oplus (1/2, 4) \oplus (1/2, 5) \oplus (1, 11/2)
(0, 0, 0, 0, 1, 3)	(0, 7/2)
(0, 0, 0, 1, 1, 3)	(0, 5/2) \oplus (0, 7/2)
(0, 0, 0, 0, 2, 3)	(0, 5/2) \oplus (0, 7/2) \oplus 2(0, 9/2) \oplus (1/2, 4) \oplus (1/2, 5) \oplus (1, 11/2)
(0, 0, 0, 0, 1, 4)	(0, 9/2)

Table 6. BPS invariants of 6d $n = 4$ minimal model.

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