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Homotopy theory of cocomplete quasicategories

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We prove that the homotopy theory of cocomplete quasicategories is equivalent to the homotopy theory of cofibration categories. This is achieved by presenting both theories as fibration categories and constructing an explicit exact equivalence between them.

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Introduction

There are a few notions that formalize the concept of a cocomplete homotopy theory, but it is not clear how they compare to each other. We consider two of them: cofibration categories and cocomplete quasicategories and prove that they are indeed equivalent. More precisely, our main result (Theorems 1.10, 2.14 and 4.9) is as follows.

Theorem *Both the category of cofibration categories and the category of cocomplete quasicategories carry structures of fibration categories and these two fibration categories are equivalent.*

These two models of cocomplete homotopy theories exemplify two different approaches to abstract homotopy theory: *homotopical algebra* and *higher category theory*. Homotopical algebra refers broadly to the theory of categories with equivalences and some further structure which provides tools for constructing derived functors. It was started by Quillen when he introduced model categories [17], but there are other notions of a similar flavor, eg *(co)fibration categories*, first defined by K Brown [6], which are crucial in the present paper. Higher category theory refers, in this context, to various models of $(\infty, 1)$–categories which provide the language to express homotopy coherent universal properties. Examples of such models include *quasicategories* introduced by Boardman and Vogt [5] and studied in detail by Joyal [14] and Lurie [16], *Segal categories* introduced by Dwyer, Kan and Smith [9] and developed by Hirschowitz and Simpson [12], and *complete Segal spaces* introduced by Rezk [18].

These (and other) notions of an $(\infty, 1)$–category are known to be equivalent to each other by the results of Bergner [4] and Joyal and Tierney [15]. An abstract axiomatization was also developed by Toën [25] and Barwick and Schommer-Pries [3]. Moreover,

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Barwick and Kan [2; 1] established that these concepts are also equivalent to the notion of a *relative category*, ie a category equipped with a class of weak equivalences and no further structure.

Our main theorem can be seen as a structured version of the latter result that concerns cocomplete homotopy theories as opposed to arbitrary ones. In particular, the comparison between cofibration categories and cocomplete quasicategories includes a direct translation between homotopy colimits computed as derived functors of cofibration categories and colimits in quasicategories characterized by homotopy coherent universal properties. The result can be seen as an answer to a version of [13, Problem 8.2] which asks for a comparison between the theories of model categories and complete Segal spaces.

This paper is the last in the series of three that summarize the results of the author's thesis [21; 22] and relies heavily on the techniques of the previous two. The main result of the first one [24] was existence of a fibration category of cofibration categories. In the second one [23] we introduced the *quasicategory of frames* which is a new construction of the $(\infty, 1)$–category associated to a cofibration category. In the present paper we construct a fibration category of cocomplete quasicategories and prove that the quasicategory of frames functor is an equivalence of fibration categories.

Section 2 contains the basic theory of quasicategories, which is mostly cited from Joyal [14] and Dugger and Spivak [8]. In particular, we establish fibration categories of quasicategories and of cocomplete quasicategories. This section contains no new results, except possibly for the existence of the latter fibration category. (The completeness of the homotopy theory of cocomplete quasicategories is discussed in Lurie [16], but it is not stated in terms of fibration categories.)

In Section 4 we prove that $\mathrm{N_f}$ is a weak equivalence of fibration categories. To this end we associate with every cocomplete quasicategory $\mathcal{D}$ a cofibration category $\mathrm{Dg}\,\mathcal{D}$ called the *category of diagrams in* $\mathcal{D}$. This yields a functor Dg which is not exact but is an inverse to $\mathrm{N_f}$ up to weak equivalence. This suffices to conclude that $\mathrm{N_f}$ is an equivalence of homotopy theories.

The results are parametrized by a regular cardinal number κ and concern κ–cocomplete cofibration categories and κ–cocomplete quasicategories. In Section 4 the arguments split into two cases. First, we consider the easier case of $\kappa > \aleph_0$ and then point out the modifications necessary for the proof when $\kappa = \aleph_0$.

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1 Review of cofibration categories

Our results are based on the techniques of [24; 23] and we start by summarizing the contents of the first of these papers. The central notion is that of *cofibration categories* which are slightly modified duals of Brown's *categories of fibrant objects* [6].

Definition 1.1 [24, Definition 1.1] A *cofibration category* is a category $\mathcal{C}$ equipped with two subcategories: the subcategory of *weak equivalences* (denoted by $\xrightarrow{\sim}$) and the subcategory of *cofibrations* (denoted by $\rightarrowtail$) such that the following axioms are satisfied. (Here, an *acyclic cofibration* is a morphism that is both a weak equivalence and a cofibration.)

(C0) Weak equivalences satisfy the 2-out-of-6 property, ie if

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$$

are morphisms of $\mathcal{C}$ such that both gf and hg are weak equivalences, then so are f, g and h (and thus also hgf).

(C1) Every isomorphism of $\mathcal{C}$ is an acyclic cofibration.

(C2) An initial object exists in $\mathcal{C}$.

(C3) Every object X of $\mathcal{C}$ is cofibrant, ie if 0 is the initial object of $\mathcal{C}$, then the unique morphism $0 \to X$ is a cofibration.

(C4) Cofibrations are stable under pushouts along arbitrary morphisms of $\mathcal{C}$ (in particular these pushouts exist in $\mathcal{C}$). Acyclic cofibrations are stable under pushouts along arbitrary morphisms of $\mathcal{C}$.

(C5) Every morphism of $\mathcal{C}$ factors as a composite of a cofibration followed by a weak equivalence.

(C6) Cofibrations are stable under sequential colimits, ie given a sequence of cofibrations

$$\begin{tikzcd} A_0 \arrow[r, tail] & A_1 \arrow[r, tail] & A_2 \arrow[r, tail] & \cdots \end{tikzcd}$$

its colimit A_∞ exists and the induced morphism $A_0 \to A_\infty$ is a cofibration. Acyclic cofibrations are stable under sequential colimits.

(C7-κ) Coproducts of κ–small families of objects exist. Cofibrations and acyclic cofibrations are stable under κ–small coproducts.

The last two axioms are optional. If we drop them, then cofibration categories can be considered as models of finitely cocomplete homotopy theories. If we include (C6) and (C7-κ) for a fixed regular cardinal $\kappa > \aleph_0$, we obtain models of κ–cocomplete homotopy theories; we call them *(homotopy) κ–cocomplete cofibration categories*. For $\kappa = \aleph_0$ the name *(homotopy) $\aleph_0$–cocomplete cofibration category* will refer to a cofibration category satisfying the axioms (C0)–(C5). The definition readily dualizes to yield *fibration categories* which are models of finitely complete homotopy theories or κ–complete homotopy theories depending on the choice of axioms.

First, we recall some classical results about cofibration categories, mostly following [20]. We fix a cofibration category $\mathcal{C}$.

Definition 1.2 (1) A *cylinder* of an object X is a factorization of the codiagonal morphism $X \sqcup X \to X$ as $X \sqcup X \rightarrowtail IX \xrightarrow{\sim} X$.

(2) A *left homotopy* between morphisms $f, g\colon X \to Y$ via a cylinder $X \sqcup X \rightarrowtail IX \xrightarrow{\sim} X$ is a commutative square of the form

$$\begin{tikzcd} X \sqcup X \arrow[r, "{[f,g]}"] \arrow[d, tail] & Y \arrow[d, tail, "\sim"] \\ IX \arrow[r] & Z \end{tikzcd}$$

(3) Morphisms $f, g\colon X \to Y$ are *left homotopic* (notation: $f \simeq_l g$) if there exists a left homotopy between them via some cylinder on X.

The definition of left homotopies differs from the standard definition as usually given in the context of model categories where the morphism $Y \overset{\sim}{\rightarrowtail} Z$ is required to be the identity. This modification is dictated by the lack of fibrant objects in cofibration categories and makes the definition well-behaved for arbitrary Y while the standard definition in a model category is only well-behaved for a fibrant Y.

We denote the homotopy category of $\mathcal{C}$ (ie its localization with respect to weak equivalences) by $\operatorname{Ho}\mathcal{C}$ and for a morphism f of $\mathcal{C}$ we write $[f]$ for its image under the localization functor $\mathcal{C} \to \operatorname{Ho}\mathcal{C}$. The homotopy category can be constructed in two steps: first dividing out left homotopies and then applying the calculus of fractions.

Proposition 1.3 *The relation of left homotopy is a congruence on $\mathcal{C}$. Moreover, every morphism of $\mathcal{C}$ that becomes an isomorphism in $\mathcal{C} / \simeq_l$ is a weak equivalence. Thus left homotopic morphisms become equal in $\operatorname{Ho}\mathcal{C}$ and $\mathcal{C} / \simeq_l$ comes equipped with a canonical functor $\mathcal{C} / \simeq_l \to \operatorname{Ho}\mathcal{C}$.*

Proof The first statement is [20, Theorem 6.3.3(1)]. The remaining ones follow by straightforward 2-out-of-3 arguments. □

The next theorem is a crucial tool in the theory of cofibration categories and can be used to verify many of their fundamental properties. It says that up to left homotopy all cofibration categories satisfy the left calculus of fractions in the sense of Gabriel and Zisman [10, Chapter I]. This fact was first proven by Brown [6, Proposition I.2]. In general, constructing $\operatorname{Ho}\mathcal{C}$ may involve using arbitrarily long zig-zags of morphisms in $\operatorname{Ho}\mathcal{C}$ and identifying them via arbitrarily long chains of relations. However, the previous proposition implies that $\mathcal{C} / \simeq_l \to \operatorname{Ho}\mathcal{C}$ is also a localization functor and in that case Theorem 1.4 says that it suffices to consider two-step zig-zags (called *left fractions*) up to a much simplified equivalence relation.

Theorem 1.4 *A cofibration category $\mathcal{C}$ satisfies the left calculus of fractions up to left homotopy, ie*

(1) *Every morphism $\varphi \in \operatorname{Ho}\mathcal{C}(X, Y)$ can be written as a left fraction $[s]^{-1}[f]$, where $f\colon X \to \tilde{Y}$ and $s\colon Y \xrightarrow{\sim} \tilde{Y}$ are morphisms of $\mathcal{C}$.*

(2) *Two fractions $[s]^{-1}[f]$ and $[t]^{-1}[g]$ are equal in $\operatorname{Ho}\mathcal{C}(X, Y)$ if and only if there exist weak equivalences u and v such that*

$$us \simeq_l vt \quad \text{and} \quad uf \simeq_l vg.$$

(3) *If $\varphi \in \operatorname{Ho}\mathcal{C}(X, Y)$ and $\psi \in \operatorname{Ho}\mathcal{C}(Y, Z)$ can be written as $[s]^{-1}[f]$ and $[t]^{-1}[g]$ respectively and a square*

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "s"', "\sim"] & \widetilde{Z} \arrow[d, "u"', "\sim"] \\
\widetilde{Y} \arrow[r, "h"'] & \widehat{Z}
\end{tikzcd}$$

commutes up to homotopy, then $\psi\varphi$ *can be written as* $[ut]^{-1}[hf]$.

Proof Parts (1) and (2) follow from [20, Theorem 6.4.4(1)], and (3) follows from the proof of [20, Theorem 6.4.1]. □

We will need the following technical lemma. Even though cofibrations in a cofibration category do not necessarily satisfy any lifting property, they can still be shown to have a version of the "homotopy extension property" with respect to left homotopies.

Lemma 1.5 *Let* $i\colon A \rightarrowtail B$ *be a cofibration in* $\mathcal{C}$. *Let* $f\colon A \to X$ *and* $g\colon B \to X$ *be morphisms such that* gi *is left homotopic to* f. *Then there exist a weak equivalence* $s\colon X \to \widehat{X}$ *and a morphism* $\widetilde{g}\colon B \to \widehat{X}$ *such that* $\widetilde{g}$ *is left homotopic to* sg *and* $\widetilde{g}i = sf$.

Proof Pick compatible cylinders on A and B, ie a diagram

$$\begin{tikzcd}
A \amalg A \arrow[r, tail] \arrow[d, "i \amalg i"'] & IA \arrow[r, "\sim"] \arrow[d] & A \arrow[d, "i"] \\
B \amalg B \arrow[r, tail] & IB \arrow[r, "\sim"] & B
\end{tikzcd}$$

such that the induced morphism $IA \amalg_{(A \amalg A)} (B \amalg B) \to IB$ is a cofibration. Let δ_0 and δ_1 denote the two structure morphisms $A \rightarrowtail IA$.

Pick a left homotopy

$$\begin{tikzcd}
A \amalg A \arrow[r, "{[f, gi]}"] \arrow[d, tail, "{[\delta_0, \delta_1]}"'] & X \arrow[d, tail, "j"', "\sim"] \\
IA \arrow[r, "H"'] & \widetilde{X}
\end{tikzcd}$$

between f and gi. Then we have in particular $jgi = H\delta_1$ and thus there is an induced morphism $[H, jg]\colon IA \amalg_A B \to \widetilde{X}$ so we can take a pushout:

$$\begin{tikzcd} IA \amalg_A B \arrow[r, "{[H, jg]}"] \arrow[d, tail, "\sim"'] & \widetilde{X} \arrow[d, tail, "\widetilde{j}"', "\sim"] \\ IB \arrow[r, "\widetilde{H}"'] & \widehat{X} \end{tikzcd}$$

Set $s = \widetilde{j}j$ and $\widetilde{g} = \widetilde{H}$. We have $sf = \widetilde{g}i$, and $\widetilde{H}$ and $\mathrm{id}_{\widehat{X}}$ constitute a left homotopy between $\widetilde{g}$ and sg. □

The main result of [24] establishes the homotopy theory of cofibration categories in the form of a fibration category. We recall the prerequisite definitions before stating the theorem.

Definition 1.6 A functor $F\colon \mathcal{C} \to \mathcal{D}$ between cofibration categories is *exact* if it preserves cofibrations, acyclic cofibrations, initial objects and pushouts along cofibrations.

If $\mathcal{C}$ and $\mathcal{D}$ are κ–cocomplete, then F is *κ–cocontinuous* if, in addition, it preserves colimits of sequences of cofibrations and κ–small coproducts.

The category of (small) κ–cocomplete cofibration categories and κ–cocontinuous functors will be denoted by CofCat_κ. It is equipped with classes of weak equivalences and fibrations as defined below.

Definition 1.7 An exact functor $F\colon \mathcal{C} \to \mathcal{D}$ is a *weak equivalence* if it induces an equivalence $\mathrm{Ho}\,\mathcal{C} \to \mathrm{Ho}\,\mathcal{D}$.

A typical way of proving that an exact functor is a weak equivalence is by using the *approximation properties* of the following proposition. They were originally introduced by Waldhausen [27, Section 1.6] in his work on algebraic K–theory and later adapted to the context of cofibration categories by Cisinski.

Proposition 1.8 [7, Théorème 3.19] *An exact functor $F\colon \mathcal{C} \to \mathcal{D}$ is a weak equivalence if and only if it satisfies the following properties:*

(App1) *F reflects weak equivalences.*

(App2) *Given a morphism $f\colon FA \to Y$ in $\mathcal{D}$, there exists a morphism $i\colon A \to B$ in $\mathcal{C}$ and a commutative diagram*

$$\begin{tikzcd} FA \arrow[r, "f"] \arrow[d, "Fi"'] & Y \arrow[d, "\sim"] \\ FB \arrow[r, "\sim"'] & Z \end{tikzcd}$$

in $\mathcal{D}$. □

Definition 1.9 [24, Definition 2.3] Let $P\colon \mathcal{E} \to \mathcal{D}$ be an exact functor of cofibration categories.

(1) P is an *isofibration* if for every object $A \in \mathcal{E}$ and an isomorphism $g\colon PA \to Y$ there is an isomorphism $f\colon A \to B$ such that $Pf = g$.

(2) It is said to satisfy the *lifting property for factorizations* if for any morphism $f\colon A \to B$ of $\mathcal{E}$ and a factorization

$$\begin{tikzcd}
PA \arrow[rr, "Pf"] \arrow[dr, tail, "j"'] & & PB \\
& X \arrow[ur, "t"', "\sim"] &
\end{tikzcd}$$

there exists a factorization

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, tail, "i"'] & & B \\
& C \arrow[ur, "s"', "\sim"] &
\end{tikzcd}$$

such that $Pi = j$ and $Ps = t$ (in particular, $PC = X$).

(3) It has the *lifting property for pseudofactorizations* if for any morphism $f\colon A \to B$ of $\mathcal{E}$ and a diagram

$$\begin{tikzcd}
PA \arrow[r, "Pf"] \arrow[d, tail, "j"'] & PB \arrow[d, tail, "v", "\sim"'] \\
X \arrow[r, "\sim", "t"'] & Y
\end{tikzcd}$$

there exists a diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, tail, "i"'] & B \arrow[d, tail, "u", "\sim"'] \\
C \arrow[r, "\sim", "s"'] & D
\end{tikzcd}$$

such that $Pi = j$, $Ps = t$ and $Pu = v$ (in particular, $PC = X$ and $PD = Y$).

(4) We say that P is a *fibration* if it is an isofibration and satisfies the lifting properties for factorizations and pseudofactorizations.

Theorem 1.10 [24, Theorem 2.9] *The category* CofCat_κ *of small* κ*–cocomplete cofibration categories with weak equivalences and fibrations as above is a fibration category.* □

The goal of the paper is to prove that this fibration category is equivalent to the corresponding fibration category of κ–cocomplete quasicategories.

2 Cocomplete quasicategories

We will start with a concise summary of the theory of quasicategories. It is well covered in [14] and [16] so we do not go into much detail. Our main goal is to establish a fibration category of finitely cocomplete quasicategories in Theorem 2.14. We refer to [24] for background on fibration categories. We cite [14] for the proof that the fibration category of all quasicategories can be obtained without constructing the entire Joyal model structure (Theorem 2.4) which makes the proof rather elementary. (A more streamlined exposition of the same results can be found in the appendices to [8].) Then we briefly introduce colimits in quasicategories and state their basic properties used in the proof of Theorem 2.14.

We will denote the groupoid freely generated by an isomorphism $0 \to 1$ by $E(1)$ and its nerve by $E[1]$. Quasicategories are defined as certain special simplicial sets and are to be thought of as models of $(\infty, 1)$–categories where vertices are objects, edges are morphisms and higher simplices are higher morphisms (or higher homotopies). Functors between quasicategories are just simplicial maps. In particular, maps out of $E[1]$ are equivalences in quasicategories and $E[1]$–homotopies are natural equivalences between functors. The account of the homotopy theory of quasicategories below closely follows the classical approach to simplicial homotopy theory (see eg [11, Chapter I]) with Kan complexes replaced by quasicategories and usual simplicial homotopies replaced by $E[1]$–homotopies.

Definition 2.1 (1) Let $f, g\colon K \to L$ be simplicial maps. An $E[1]$*–homotopy* from f to g is a simplicial map $K \times E[1] \to L$ extending $[f, g]\colon K \times \partial\Delta[1] \to L$.

(2) Two simplicial maps $f, g\colon K \to L$ are $E[1]$*–homotopic* if there exists a zig-zag of $E[1]$–homotopies connecting f to g. (It suffices to consider sequences instead of zig-zags since $E[1]$ has an automorphism that exchanges the vertices.)

(3) A simplicial map $f\colon K \to L$ is an $E[1]$*–homotopy equivalence* if there is a simplicial map $g\colon L \to K$ such that fg is $E[1]$–homotopic to id_L and gf is $E[1]$–homotopic to id_K.

Definition 2.2 (1) A simplicial map is an *inner fibration* if it has the right lifting property with respect to the inner horn inclusions.

(2) A simplicial map is an *inner isofibration* if it is an inner fibration and has the right lifting property with respect to $\Delta[0] \hookrightarrow E[1]$.

(3) A simplicial map is an *acyclic Kan fibration* if it has the right lifting property with respect to $\partial\Delta[m] \hookrightarrow \Delta[m]$ for all m.

(4) A simplicial set $\mathcal{C}$ is a *quasicategory* if the unique map $\mathcal{C} \to \Delta[0]$ is an inner fibration.

We will refer to $E[1]$–equivalences between quasicategories as *categorical equivalences* and use them to introduce the homotopy theory of quasicategories. (It is also possible to extend this notion to maps of general simplicial sets, but we have no need to do it.) If K is any simplicial set and $\mathcal{C}$ is a quasicategory, then the relation of "being connected by a single $E[1]$–homotopy" is already an equivalence relation on the set of simplicial maps $K \to \mathcal{C}$ by [8, Proposition 2.3]. This simplifies the definition of categorical equivalences since it is always sufficient to consider one-step $E[1]$–homotopies. The following lemma provides a useful criterion for verifying that a functor between quasicategories is a categorical equivalence.

Lemma 2.3 [26] *A functor $F\colon \mathcal{C} \to \mathcal{D}$ between quasicategories is a categorical equivalence provided that for every commutative square of the form*

$$\begin{tikzcd}
\partial\Delta[m] \arrow[r, "u"] \arrow[d, hook] & \mathcal{C} \arrow[d, "F"] \\
\Delta[m] \arrow[r, "v"'] & \mathcal{D}
\end{tikzcd}$$

there exists a map $w\colon \Delta[m] \to \mathcal{C}$ such that $w|\partial\Delta[m] = u$ and Fw is $E[1]$–homotopic to v relative to $\partial\Delta[m]$. □

Theorem 2.4 *The category of small quasicategories with simplicial maps as morphisms, categorical equivalences as weak equivalences and inner isofibrations as fibrations is a fibration category.*

Proof Only two of the axioms require nontrivial proofs: stability of acyclic fibrations under pullbacks, which follows from the fact that acyclic (inner iso-) fibrations coincide with acyclic Kan fibrations by [14, Theorem 5.15], and the factorization axiom which is verified in [14, Proposition 5.16]. □

This fibration category is a part of the Joyal model structure on simplicial sets established in [14, Theorem 6.12]. Indeed, the theorem above is an intermediate step in the construction of this model category.

Quasicategories are models for homotopy theories and as such they have homotopy categories. Two morphisms $f, g\colon x \to y$ of a quasicategory $\mathcal{D}$ are *homotopic* if there exists a simplex $H\colon \Delta[2] \to \mathcal{D}$ such that $H\delta_0 = y\sigma_0$, $H\delta_1 = g$ and $H\delta_2 = f$. The *homotopy category* of $\mathcal{D}$ is the category $\operatorname{Ho}\mathcal{D}$ with the same objects as $\mathcal{D}$, homotopy classes of morphisms of $\mathcal{D}$ as morphisms and the composition induced by filling horns.

If f is a morphism of a quasicategory $\mathcal{C}$, then we say that f is an *equivalence* if the simplicial map $f\colon \Delta[1] \to \mathcal{C}$ extends to $E[1] \to \mathcal{C}$. (By [14, Proposition 4.22] a morphism is an equivalence if and only if it becomes an isomorphism in the homotopy category.) Two objects of $\mathcal{C}$ are *equivalent* if they are connected by an equivalence.

We proceed to the discussion of colimits in quasicategories. Such colimits are homotopy invariant by design and they serve as models for homotopy colimits. However, in quasicategories there is no corresponding notion of a "strict" colimit and thus it is customary to refer to "homotopy colimits" in quasicategories simply as colimits. The general theory of colimits is explored in depth in [16, Chapter 4]; here we only discuss its most basic aspects.

The quasicategorical notion of colimit is defined using the join construction for simplicial sets. As a functor $\star\colon \Delta \times \Delta \to \Delta$ it is defined by concatenation: $[m], [n] \mapsto [m+1+n]$. Then the general join is defined as the unique functor $\mathsf{sSet} \times \mathsf{sSet} \to \mathsf{sSet}$ which agrees with the above on the representable simplicial sets and such that for each K the resulting functor $K \star -\colon \mathsf{sSet} \to K \downarrow \mathsf{sSet}$ preserves colimits. As such, the functor $K \star -$ has a right adjoint which we will denote by $(X\colon K \to M) \mapsto X \setminus M$. ($X \setminus M$ is called the *slice* of M under X.)

Lemma 2.5 *Let $P\colon \mathcal{C} \twoheadrightarrow \mathcal{D}$ be a inner isofibration of quasicategories and $X\colon K \to \mathcal{C}$ a diagram. Then the induced map $X \setminus \mathcal{C} \to PX \setminus \mathcal{D}$ is an inner isofibration. In particular, $X \setminus \mathcal{C}$ is a quasicategory.*

Proof This follows from [14, Theorem 3.19(i) and Proposition 4.10]. □

For any simplicial set K we define the *under-cone* on K as $K^{\rhd} = K \star \Delta[0]$. We also fix a regular cardinal number κ.

Definition 2.6 Let $\mathcal{C}$ be a quasicategory and let $X\colon K \to \mathcal{C}$ be any simplicial map (which we consider as a K–indexed diagram in $\mathcal{C}$).

(1) A *cone under X* is a diagram $S\colon K^{\rhd} \to \mathcal{C}$ such that $S|K = X$.

(2) A cone S under X is *universal* or a *colimit of X* if for any $m > 0$ and any diagram of solid arrows

$$\begin{tikzcd} K \star \partial\Delta[m] \arrow[r, "U"] \arrow[d] & \mathcal{C} \\ K \star \Delta[m] \arrow[ur, dashed] & \end{tikzcd}$$

where $U|K^{\triangleright} = S$, there exists a dashed arrow making the diagram commute.

(3) An *initial object* of $\mathcal{C}$ is a colimit of the unique empty diagram in $\mathcal{C}$.

(4) A simplicial map $f\colon K \to L$ is *cofinal* if for every quasicategory $\mathcal{C}$ and every universal cone $S\colon L^{\triangleright} \to \mathcal{C}$ the induced cone $Sf^{\triangleright}$ is also universal.

(5) The quasicategory $\mathcal{C}$ is *κ–cocomplete* if for every κ–small simplicial set K every diagram $K \to \mathcal{C}$ has a colimit.

(6) A functor $F\colon \mathcal{C} \to \mathcal{D}$ between finitely cocomplete quasicategories is said to be *κ–cocontinuous* if for every κ–small simplicial set K and every universal cone $S\colon K^{\triangleright} \to \mathcal{C}$ the cone FS is also universal.

Lemma 2.7 *A cone S under X is universal if and only if it is an initial object of $X \setminus \mathcal{C}$.*

Proof This follows directly from the fact that the slice functor is a right adjoint of the join functor. □

We will now discuss the counterparts of a few classical statements of category theory saying that colimits are essentially unique and invariant under equivalences. For a quasicategory $\mathcal{C}$ and a diagram $X\colon K \to \mathcal{C}$ we let $(X \setminus \mathcal{C})^{\mathrm{univ}}$ denote the simplicial subset of $X \setminus \mathcal{C}$ consisting of those simplices whose vertices are all universal.

Lemma 2.8 *The simplicial set $(X \setminus \mathcal{C})^{\mathrm{univ}}$ is empty or a contractible Kan complex.*

Proof A simplicial set is empty or a contractible Kan complex if and only if it has the right lifting property with respect to the boundary inclusions $\partial\Delta[m] \hookrightarrow \Delta[m]$ for all $m > 0$. For $(X \setminus \mathcal{C})^{\mathrm{univ}}$ such lifting problems are equivalent to the lifting problems

$$\begin{tikzcd} K \star \partial\Delta[m] \arrow[r, "U"] \arrow[d] & \mathcal{C} \\ K \star \Delta[m] \arrow[ur, dashed] & \end{tikzcd}$$

with $U|(K \star \{i\})$ universal for each $i \in [m]$, which have solutions by the definition of universal cones. □

Corollary 2.9 *If $X\colon K \to \mathcal{C}$ is a diagram in a quasicategory and S and T are two universal cones under X, then they are equivalent under X, ie as objects of $X \setminus \mathcal{C}$.*

Proof The simplicial set $(X \setminus \mathcal{C})^{\mathrm{univ}}$ is nonempty and thus a contractible Kan complex by the previous lemma. Hence it has the right lifting property with respect to the inclusion $\partial\Delta[1] \hookrightarrow E[1]$, which translates to the lifting property

$$\begin{tikzcd}
K \star \partial\Delta[1] \arrow[r, "{[S,T]}"] \arrow[d] & \mathcal{C} \\
K \star E[1] \arrow[ur, dashed]
\end{tikzcd}$$

which yields an equivalence of S and T. □

Lemma 2.10 *If $\mathcal{C}$ is a quasicategory and X and Y are equivalent objects of $\mathcal{C}$, then X is initial if and only if Y is.*

Proof Assume that X is initial and let $U\colon \partial\Delta[m] \to \mathcal{C}$ be such that $U|\Delta[0] = Y$. We can consider an equivalence from X to Y as a diagram $f\colon \Delta[0] \star \Delta[0] \to \mathcal{C}$. Then by the universal property of X there is a diagram $\Delta[0] \star \partial\Delta[m]$ extending both f and U. (We can iteratively choose extensions over $\Delta[0] \star \Delta[k]$ for all faces $\Delta[k] \hookrightarrow \partial\Delta[m]$.) This diagram is a special outer horn (under the isomorphism $\Delta[0] \star \partial\Delta[m] \cong \Lambda^0[m+1]$) and thus has a filler by [14, Theorem 3.14]. Therefore U extends over $\Delta[m]$ and hence Y is initial. □

Our goal is to compare cofibration categories to quasicategories, but we expect κ–cocomplete cofibration categories to correspond to κ–cocomplete quasicategories, not to arbitrary ones. In the remainder of this section we will restrict the fibration structure of Theorem 2.4 to the subcategory of κ–cocomplete quasicategories and κ–cocontinuous functors.

First, we need two lemmas about lifting colimits along inner isofibrations.

Lemma 2.11 *Consider a pullback square of quasicategories*

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "G"] \arrow[d, "Q"'] & \mathcal{E} \arrow[d, two heads, "P"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{D}
\end{tikzcd}$$

where P is an inner isofibration, and let $S\colon K^{\triangleright} \to \mathcal{P}$ be a cone. If all GS, QS and $PGS = FQS$ are universal, then so is S.

Proof Under these assumptions the square

$$\begin{tikzcd}
X \setminus \mathcal{P} \arrow[r, "G"] \arrow[d, "Q"'] & GX \setminus \mathcal{E} \arrow[d, two heads, "P"] \\
QX \setminus \mathcal{C} \arrow[r, "F"'] & PGX \setminus \mathcal{D}
\end{tikzcd}$$

(where $X = S|K$) is also a pullback along an inner isofibration by Lemma 2.5. Hence it suffices to verify the conclusion for initial objects.

Thus assume that $K = \varnothing$ and let $m > 0$ and $U\colon \partial\Delta[m] \to \mathcal{P}$ be such that $U|\Delta[0] = S$. Then we have

$$GU|\Delta[0] = GS \quad \text{and} \quad QU|\Delta[0] = QS$$

and since both GS and QS are initial we can find $V_{\mathcal{E}} \in \mathcal{E}_m$ and $V_{\mathcal{C}} \in \mathcal{C}_m$ such that $V_{\mathcal{E}}|\partial\Delta[m] = GU$ and $V_{\mathcal{C}}|\partial\Delta[m] = QU$. Next, define $\widetilde{V}\colon \partial\Delta[m+1] \to \mathcal{D}$ by replacing the 1st face of $PV_{\mathcal{E}}\sigma_1|\partial\Delta[m+1]$ with $FV_{\mathcal{C}}$ and $\widetilde{W}\colon \Lambda^1[m+1] \to \mathcal{E}$ by setting it to $V_{\mathcal{E}}\sigma_1|\Lambda^1[m+1]$.

By the assumption PGS is initial and $\widetilde{V}|\Delta[0] = PGS$ so $\widetilde{V}$ extends to $V \in \mathcal{D}_{m+1}$. Then we have a commutative square

$$\begin{tikzcd}
\Lambda^1[m+1] \arrow[r, "\widetilde{W}"] \arrow[d] & \mathcal{E} \arrow[d, two heads, "P"] \\
\Delta[m+1] \arrow[r, "V"'] & \mathcal{D}
\end{tikzcd}$$

which admits a lift W since P is an inner isofibration and $0 < 1 < m+1$. We have $FV_{\mathcal{C}} = PW\delta_1$ and thus $(V_{\mathcal{C}}, W\delta_1)$ is an m–simplex of $\mathcal{P}$ whose boundary is U. Hence S is initial. □

Lemma 2.12 *Let $P\colon \mathcal{C} \twoheadrightarrow \mathcal{D}$ be an inner isofibration, $X\colon K \to \mathcal{C}$ a diagram and $T\colon K^{\triangleright} \to \mathcal{D}$ a colimit of PX. If X has a colimit in $\mathcal{C}$ which is preserved by P, then there exists a colimit $S\colon K^{\triangleright} \to \mathcal{C}$ of X such that $PS = T$.*

Proof Let $\widetilde{S}\colon K^{\triangleright} \to \mathcal{C}$ be some colimit of X. Since both T and $P\widetilde{S}$ are universal, we have a simplicial map $U\colon K \star E[1] \to \mathcal{D}$ such that $U|(K \star \partial\Delta[1]) = [T, P\widetilde{S}]$ by Corollary 2.9. The conclusion now follows from Lemmas 2.5 and 2.10. □

The homotopical content of the next proposition is the same as that of [16, Lemma 5.4.5.5]. However, we need a stricter point-set level statement. See also [19, Sections 3 and 4] for a systematic approach to results of this type.

Proposition 2.13 *Let $F\colon \mathcal{C} \to \mathcal{D}$ and $P\colon \mathcal{E} \twoheadrightarrow \mathcal{D}$ be κ–cocontinuous functors between κ–cocomplete quasicategories with P an inner isofibration. Then a pullback of P along F exists in the category of κ–cocomplete quasicategories and κ–cocontinuous functors.*

Proof Form a pullback of P along F in the category of quasicategories:

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "G"] \arrow[d, "Q"'] & \mathcal{E} \arrow[d, two heads, "P"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{D}
\end{tikzcd}$$

We will check that this square is also a pullback in the category of κ–cocomplete quasicategories and κ–cocontinuous functors.

First, we verify that $\mathcal{P}$ has κ–small colimits. Let $X\colon K \to \mathcal{P}$ be a diagram with K κ–small. Let $S\colon K^{\rhd} \to \mathcal{C}$ be a colimit of QX, then FS is a colimit of $FQX = PGX$ in $\mathcal{D}$. Lemma 2.12 implies that we can choose a colimit T of GX in $\mathcal{E}$ so that $PT = FS$. Then it follows by Lemma 2.11 that (S, T) is a colimit of $X = (QX, GX)$ in $\mathcal{P}$.

It remains to see that given a square

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "P"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{D}
\end{tikzcd}$$

of κ–cocomplete quasicategories and κ–cocontinuous functors, the induced functor $\mathcal{F} \to \mathcal{P}$ preserves κ–small colimits. Indeed, this follows directly from Lemma 2.11. □

Theorem 2.14 *The category QCat_κ of small κ–cocomplete quasicategories with κ–cocontinuous functors as morphisms, categorical equivalences as weak equivalences and (κ–cocontinuous) inner isofibrations as fibrations is a fibration category.*

Proof By Theorem 2.4 it suffices to observe:

(1) A terminal quasicategory is also a terminal κ–cocomplete quasicategory (which is clear).

(2) A pullback (in the category of all quasicategories) of κ–cocomplete quasicategories and κ–cocontinuous functors one of which is an inner isofibration is also a pullback in the category of κ–cocomplete quasicategories, which follows by (the proof of) Proposition 2.13.

(3) For a κ–cocomplete quasicategory $\mathcal{C}$, the functor $\mathcal{C}^{E[1]} \to \mathcal{C} \times \mathcal{C}$ is a κ–cocontinuous functor between κ–cocomplete quasicategories. Indeed, $\mathcal{C}^{E[1]}$ is κ–cocomplete since it is categorically equivalent to $\mathcal{C}$ (by Lemmas 2.7 and 2.10) and $\mathcal{C} \times \mathcal{C}$ is κ–cocomplete by (2). Finally, $\mathcal{C}^{E[1]} \to \mathcal{C} \times \mathcal{C}$ preserves κ–small colimits by (2) since both projections $\mathcal{C}^{E[1]} \to \mathcal{C}$ do. □

3 The quasicategory of frames

In [23] we introduced a functor $\mathrm{N_f}\colon \mathsf{CofCat}_\kappa \to \mathsf{QCat}_\kappa$. Let us briefly recall the construction. For each m let $D[m]$ be the category of elements of $\Delta[m]$ with the face operators as morphisms. It comes equipped with a functor $p_{[m]}\colon D[m] \to [m]$ that evaluates a map $[k] \to [m]$ at m. We consider $D[m]$ as a homotopical category with weak equivalences created by $p_{[m]}$ (from the isomorphisms of $[m]$). Then for a cofibration category $\mathcal{C}$ we define a simplicial set $\mathrm{N_f}\,\mathcal{C}$ (called the *quasicategory of frames* in $\mathcal{C}$) whose m–simplices are homotopical, Reedy cofibrant diagrams $D[m] \to \mathcal{C}$. See [23, Section 2] for full details.

Theorem 3.1 *For a κ–cocomplete cofibration category $\mathcal{C}$, the simplicial set $\mathrm{N_f}\,\mathcal{C}$ is a κ–cocomplete quasicategory and $\mathrm{N_f}\colon \mathsf{CofCat}_\kappa \to \mathsf{QCat}_\kappa$ is an exact functor of fibration categories.*

Proof By [23, Theorem 2.3] $\mathrm{N_f}\,\mathcal{C}$ is a κ–cocomplete quasicategory. Moreover, [23, Propositions 3.5, 3.8 and 3.9] imply that $\mathrm{N_f}$ is indeed exact. □

The results of the last section heavily depend on the methods of [24; 23] which in turn involve a lot of notation useful in expressing properties of $\mathrm{N_f}\,\mathcal{C}$ in terms of various diagrams in $\mathcal{C}$. In this section, we recall some of that notation and prove a few auxiliary lemmas.

First of all, the categories $D[m]$ introduced above generalize to homotopical categories DK for all simplicial sets K. The underlying category of DK has all simplices of K as objects and face operators between them as morphisms. The weak equivalences in DK are induced from degenerate simplices of K in a manner described in [23, Section 2]. The following fact is a fundamental tool for translating between properties of $\mathcal{C}$ and $\mathrm{N_f}\,\mathcal{C}$.

Proposition 3.2 [23, Proposition 2.6] *Let $\mathcal{C}$ be a cofibration category and K a simplicial set. There is a natural bijection between*

- *the set of homotopical Reedy cofibrant diagrams $DK \to \mathcal{C}$, and*
- *the set of simplicial maps $K \to \mathrm{N_f}\,\mathcal{C}$.* □

Moreover, this construction admits useful variations most conveniently described in terms of marked simplicial complexes. A *marked simplicial complex* is a simplicial set K equipped with an embedding $K \hookrightarrow NP$, where P is a homotopical poset. In this case DK stands for the same category as above but with (possibly) richer homotopical structure, ie one created by the composite $DK \hookrightarrow DP \to P$. Here, DP stands for DNP and the latter functor evaluates an object $[k] \to P$ at k. $\operatorname{Sd} K$ stands for the homotopical poset defined as the full homotopical subcategory of DK spanned by the nondegenerate simplices of K. Diagrams over $\operatorname{Sd} K$ have the same homotopical content as diagrams over DK, as made precise by the following lemma.

Lemma 3.3 [23, Lemma 3.12] *Let $K \hookrightarrow L$ be an injective map of finite marked simplicial complexes (which means that it covers an injective homotopical map of the underlying homotopical posets). Then for every cofibration category $\mathcal{C}$ the inclusion $DK \cup \operatorname{Sd} L \hookrightarrow DL$ induces an acyclic fibration $\mathcal{C}_{\mathrm{R}}^{DL} \to \mathcal{C}_{\mathrm{R}}^{DK \cup \operatorname{Sd} L}$.* □

This lemma will be useful in various ways, for example in constructing $E[1]$–homotopies between maps into $\mathrm{N_f}\,\mathcal{C}$. An $E[1]$–homotopy $K \times E[1] \to \mathrm{N_f}\,\mathcal{C}$ corresponds to a homotopical Reedy cofibrant diagram $D(K \times E[1]) \to \mathcal{C}$. Moreover, [23, Corollary 3.7] says that in order to specify such a homotopy it is enough to give a diagram $D(K \times \widehat{[1]}) \to \mathcal{C}$. (Here, $\widehat{[1]}$ stands for the poset $[1]$ with all morphisms as weak equivalences.) These observations allow us to state and prove the following lemma.

Lemma 3.4 *Let $K \hookrightarrow L$ be an inclusion of marked simplicial complexes, X and Y homotopical Reedy cofibrant diagrams $DL \to \mathcal{C}$, and $f\colon X|\operatorname{Sd} L \to Y|\operatorname{Sd} L$ a natural weak equivalence such that $f|\operatorname{Sd} K$ is an identity transformation. Then X and Y are $E[1]$–homotopic relative to K as diagrams in $\mathrm{N_f}\,\mathcal{C}$.*

Proof By [23, Corollary 3.7] it suffices to construct a homotopical Reedy cofibrant diagram $D(L \times \widehat{[1]}) \to \mathcal{C}$ that restricts to $[X, Y]$ on $D(L \times \partial\Delta[1])$ and to the identity on $D(K \times \widehat{[1]})$, ie to a degenerate edge of $(\mathrm{N_f}\,\mathcal{C})^K$.

First, observe that we have a homotopical diagram $[f, \mathrm{id}]\colon (\operatorname{Sd} L \cup DK) \times \widehat{[1]} \to \mathcal{C}$ which is Reedy cofibrant when seen as a diagram $\operatorname{Sd} L \cup DK \to \mathcal{C}^{\widehat{[1]}}$. Hence by Lemma 3.3 it extends to a Reedy cofibrant diagram $DL \to \mathcal{C}^{\widehat{[1]}}$. We consider it as a diagram $DL \times \widehat{[1]} \to \mathcal{C}$ and pull it back to $D(L \times \widehat{[1]}) \to \mathcal{C}$. It restricts to $[X, Y]$ on $D(L \times \partial\Delta[1])$ and to the identity on $D(K \times \widehat{[1]})$. Thus it can be replaced Reedy cofibrantly relative to $D(L \times \partial\Delta[1] \cup K \times \widehat{[1]})$ by [23, Lemma 1.9], which finishes the proof. □

Another lemma that we will need says that up to equivalence all frames are Reedy cofibrant replacements of constant diagrams.

Lemma 3.5 *Any object of $X \in \mathrm{N_f}\,\mathcal{C}$ is equivalent to a Reedy cofibrant replacement of $p_{[0]}^* X_0$.*

Proof By [23, Lemma 3.2] there are homotopical functors $f\colon [0] \to D[0]$ and $s\colon D[0] \to D[0]$ such that $p_{[0]} f = \mathrm{id}_{[0]}$ and there are weak equivalences

$$\mathrm{id} \xrightarrow{\sim} s \xleftarrow{\sim} f p_{[0]}.$$

These equivalences evaluated at X form a diagram $D[0] \times \mathrm{Sd}\,[\widehat{1}] \to \mathcal{C}$ which we can pull back along $D[\widehat{1}] \to D[0] \times \mathrm{Sd}\,[\widehat{1}]$ and then replace Reedy cofibrantly to obtain a homotopical Reedy cofibrant diagram $Y\colon D[\widehat{1}] \to \mathcal{C}$ such that $Y\delta_1 = X$ by [23, Lemma 1.9]. By [23, Corollary 3.7] Y is an equivalence and by the construction $Y\delta_0$ is a Reedy cofibrant replacement of $p_{[0]}^* X_0$. □

Perhaps the most useful result of [23] characterizes universal cones $K^{\rhd} \to \mathrm{N_f}\,\mathcal{C}$ in terms of the corresponding diagram $D(K^{\rhd}) \to \mathcal{C}$. It comes in two versions depending on whether $\kappa > \aleph_0$ or $\kappa = \aleph_0$. First, we state it in the case of $\kappa > \aleph_0$.

Theorem 3.6 [23, Theorem 4.6] *Let $\mathcal{C}$ be a κ–cocomplete cofibration category, K a κ–small simplicial set and $S\colon K^{\rhd} \to \mathrm{N_f}\,\mathcal{C}$. Then S is universal as a cone under $S|K$ if and only if the induced morphism*

$$\operatorname{colim}_{DK} S \to \operatorname{colim}_{D(K^{\rhd})} S$$

is a weak equivalence (with S seen, by Proposition 3.2, as a homotopical Reedy cofibrant diagram $D(K^{\rhd}) \to \mathcal{C}$). □

Observe that the assumption $\kappa > \aleph_0$ is necessary for the colimits in the statement of the theorem to exist. If $\kappa = \aleph_0$, then K is a finite simplicial set, but DK is still infinite (unless K is empty). This problem makes both the statement and the proof more technical in the case of $\kappa = \aleph_0$.

We filter the category DK by finite subcategories

$$D^{(0)}K \hookrightarrow D^{(1)}K \hookrightarrow D^{(2)}K \hookrightarrow \cdots$$

as described in detail in [23, Section 5]. Then given a homotopical Reedy cofibrant diagram $X\colon DK \to \mathcal{C}$ the colimits of its restrictions to all $D^{(k)}K$ exist. The homotopy type of these colimits stabilizes for k sufficiently large and this stable value is the homotopy colimit of X. This allows us to state the remaining case of the theorem.

Theorem 3.7 [23, Theorem 5.12] *Let $\mathcal{C}$ be a cofibration category and K a finite simplicial set. A cone $S\colon K^{\rhd} \to \mathrm{N_f}\,\mathcal{C}$ is universal if and only if the induced morphism*

$$\operatorname{colim}_{D^{(k)}K} S \to \operatorname{colim}_{D^{(k)}(K^{\rhd})} S$$

is a weak equivalence for k sufficiently large (where S is seen as a homotopical Reedy cofibrant diagram $D(K^{\rhd}) \to \mathcal{C}$ by Proposition 3.2). □

Both these theorems will be instrumental in the proof of our main result.

4 Cofibration categories of diagrams in quasicategories

In this section we will prove our main result, ie that $\mathrm{N_f}$ is a weak equivalence of fibration categories. This will be achieved by defining a functor Dg_κ from the category of κ–cocomplete quasicategories to the category of κ–cocomplete cofibration categories. The functor Dg_κ fails to be exact (eg it does not preserve the terminal object), but it will be verified to induce an inverse to $\mathrm{N_f}$ on the level of homotopy categories which is sufficient to complete the proof.

Definition 4.1 Let sSet_κ denote the category of κ–small simplicial sets. If $\mathscr{C}$ is a κ–cocomplete quasicategory we consider the slice category $\mathrm{sSet}_\kappa \downarrow \mathscr{C}$, we denote it by $\mathrm{Dg}_\kappa\,\mathscr{C}$ and call it the *category of κ–small diagrams in* $\mathscr{C}$. Then we define a morphism

$$\begin{tikzcd} K \arrow[rr, "f"] \arrow[dr, "X"'] & & L \arrow[dl, "Y"] \\ & \mathscr{C} & \end{tikzcd}$$

to be

- a *weak equivalence* if the induced morphism $\operatorname{colim}_K X \to \operatorname{colim}_L Y$ is an equivalence in $\mathscr{C}$ (more precisely, if for any universal cone $S\colon L^{\rhd} \to \mathscr{C}$ under Y the induced cone $Sf^{\rhd}$ is universal under X),
- a *cofibration* if f is injective.

In particular, a morphism of $\mathrm{Dg}_\kappa\,\mathscr{C}$ as above is a weak equivalence whenever f is cofinal, but there are of course many weak equivalences with f not cofinal. We will make use of the class of *right anodyne maps*, which is generated by the *right horn* inclusions $\Lambda^i[m] \hookrightarrow \Delta[m]$ (ie the ones with $0 < i \leq m$) under coproducts, pushouts along arbitrary maps, sequential colimits and retracts.

Proposition 4.2 *With weak equivalences and cofibrations as defined above* $\mathrm{Dg}_\kappa \mathcal{C}$ *is a* κ*–cocomplete cofibration category.*

Proof (C0) Weak equivalences satisfy 2-out-of-6 since equivalences in $\mathcal{C}$ do.

(C1) Isomorphisms are weak equivalences since isomorphisms of simplicial sets are cofinal.

(C2)–(C3) The empty diagram is an initial object and hence every object is cofibrant.

(C4) Pushouts are created by the forgetful functor $\mathrm{Dg}_\kappa \mathcal{C} \to \mathsf{sSet}_\kappa$, thus pushouts along cofibrations exist and cofibrations are stable under pushouts. By [20, Lemma 1.4.3(1)] it suffices to verify that the gluing lemma holds, which follows by [16, Proposition 4.4.2.2].

(C5) It will suffice to verify that in the usual mapping cylinder factorization

$$K \to Mf \to L$$

the second map is cofinal. Indeed, we have a diagram

$$\begin{tikzcd}
K \times \Delta[0] \arrow[r, "f"] \arrow[d, "K \times \delta_0"'] & L \arrow[d, "j"'] \arrow[dr, bend left, "\mathrm{id}_L"] & \\
K \times \Delta[1] \arrow[r] & Mf \arrow[r] & L
\end{tikzcd}$$

where the square is a pushout. The map $K \times \delta_0$ is right anodyne by [14, Theorem 2.17] and thus so is j. Hence it is cofinal by [16, Proposition 4.1.1.3(4)].

(C6)–(C7-κ) The proof is similar to that of (C4). (But there is no analogue of [16, Proposition 4.4.2.2] for sequential colimits explicitly stated in [16]. Instead, it follows from the more general [16, Proposition 4.2.3.10 and Remark 4.2.3.9].) □

Lemma 4.3 *A* κ*–cocontinuous functor* $F\colon \mathcal{C} \to \mathcal{D}$ *induces a* κ*–cocontinuous functor* $\mathrm{Dg}_\kappa F = \mathrm{Dg}_\kappa \mathcal{C} \to \mathrm{Dg}_\kappa \mathcal{D}$ *and thus we obtain a functor* $\mathrm{Dg}_\kappa\colon \mathsf{QCat}_\kappa \to \mathsf{CofCat}_\kappa$.

Proof Colimits in both $\mathrm{Dg}_\kappa \mathcal{C}$ and $\mathrm{Dg}_\kappa \mathcal{D}$ are created in sSet_κ and thus are preserved by $\mathrm{Dg}_\kappa F$. Cofibrations are clearly preserved and so are weak equivalences since F preserves κ–small colimits. □

For the moment, we focus on the case of $\kappa > \aleph_0$. The case of $\kappa = \aleph_0$ will be dealt with later.

Definition 4.4 For a κ–cocomplete cofibration category $\mathcal{C}$ we define a functor

$$\Phi_{\mathcal{C}}\colon \mathrm{Dg}_\kappa \mathrm{N_f}\,\mathcal{C} \to \mathcal{C}$$

by sending a diagram $X\colon K \to \mathrm{N_f}\,\mathcal{C}$ to $\mathrm{colim}_{DK} X$.

Observe that DK is κ–small since K is and $\kappa > \aleph_0$, so the colimit used in this definition exists in $\mathcal{C}$. It is clear that $\Phi_{\mathcal{C}}$ is a functor. While we may not be able to choose colimits so that $\Phi_{\mathcal{C}}$ is natural in $\mathcal{C}$, it is pseudonatural, ie natural up to coherent natural isomorphism.

Lemma 4.5 *The functor $\Phi_{\mathcal{C}}$ is κ–cocontinuous and a weak equivalence.*

Proof Preservation of cofibrations follows by [20, Theorem 9.4.1(1a)] since if $K \hookrightarrow L$ is an injective map of simplicial sets, then the induced functor $DK \hookrightarrow DL$ is a sieve. Colimits in $\mathcal{C}$ are compatible with colimits of indexing categories and thus $\Phi_{\mathcal{C}}$ is κ–cocontinuous. (Preservation of weak equivalences follows from the argument below.)

To see that it is a weak equivalence, it is enough to verify the approximation properties of Proposition 1.8. Lemma 4.1 of [23] and Theorem 3.6 imply that a morphism f in $\mathrm{Dg}_\kappa \mathrm{N_f}\,\mathcal{C}$ is a weak equivalence if and only if $\Phi_{\mathcal{C}} f$ is. Therefore $\Phi_{\mathcal{C}}$ preserves weak equivalences and satisfies (App1). It remains to check (App2), but it follows directly from [23, Lemma 4.2]. □

Next, we need a functor $\mathcal{D} \to \mathrm{N_f}\,\mathrm{Dg}_\kappa \mathcal{D}$ for every κ–cocomplete quasicategory $\mathcal{D}$. Let's start with unraveling the definition of $\mathrm{N_f}\,\mathrm{Dg}_\kappa \mathcal{D}$.

An m–simplex of $\mathrm{N_f}\,\mathrm{Dg}_\kappa \mathcal{D}$ consists of a Reedy cofibrant diagram $K\colon D[m] \to \mathsf{sSet}_\kappa$ and for each $\varphi \in D[m]$ a diagram $X_\varphi\colon K_\varphi \to \mathcal{D}$. These diagrams are compatible with each other in the sense that they form a cone under K with the vertex $\mathcal{D}$. Moreover, the entire structure is homotopical as a diagram in $\mathrm{Dg}_\kappa \mathcal{D}$, ie if $\varphi, \psi \in D[m]$ and $\chi\colon \varphi \to \psi$ is a weak equivalence, then the induced morphism $\mathrm{colim}_{K_\varphi} X_\varphi \to \mathrm{colim}_{K_\psi} X_\psi$ is an equivalence in $\mathcal{D}$.

If $\mu\colon [n] \to [m]$, then $(K, X)\mu = (K\mu, X\mu)$ is defined simply by $(K\mu)_\varphi = K_{\mu\varphi}$ and $(X\mu)_\varphi = X_{\mu\varphi}$.

We can now define a functor $\Psi_{\mathcal{D}}\colon \mathcal{D} \to \mathrm{N_f}\,\mathrm{Dg}_\kappa \mathcal{D}$ as follows.

Definition 4.6 For $x \in \mathcal{D}_m$ we set the underlying simplicial diagram of $\Psi_{\mathcal{D}} x$ to $\varphi \mapsto \Delta[k]$, where $\varphi\colon [k] \to [m]$, and the corresponding diagram in $\mathcal{D}$ to $x\varphi\colon \Delta[k] \to \mathcal{D}$. Then $\Psi_{\mathcal{D}} x$ is homotopical as a diagram $D[m] \to \mathrm{Dg}_\kappa \mathcal{D}$ since any weak equivalence in $D[m]$ induces a right anodyne (and hence cofinal by [16, Proposition 4.1.1.3(4)]) map of simplices.

Clearly, $\Psi_{\mathcal{D}}$ is a functor and is natural in $\mathcal{D}$. We check that it is also a categorical equivalence.

Proposition 4.7 *For every κ–cocomplete quasicategory $\mathcal{D}$ the functor $\Psi_{\mathcal{D}}$ is a categorical equivalence.*

Proof Consider a square as follows:

$$\begin{tikzcd}
\partial\Delta[m] \arrow[r, "x"] \arrow[d] & \mathcal{D} \arrow[d, "\Psi_{\mathcal{D}}"] \\
\Delta[m] \arrow[r, "Y"'] & \mathrm{N_f}\,\mathrm{Dg}_\kappa\,\mathcal{D}
\end{tikzcd}$$

By Lemma 2.3 it will be enough to extend x to a simplex $\hat{x}\colon \Delta[m] \to \mathcal{D}$ and construct an $E[1]$–homotopy from $\Psi_{\mathcal{D}}\hat{x}$ to Y relative to $\partial\Delta[m]$.

Let's start by finding $\hat{x}$. Consider $Y_{[m]}\colon A_{[m]} \to \mathcal{D}$. Since Y agrees with $\Psi_{\mathcal{D}}x$ over $\partial\Delta[m]$ the $[m]^{\text{th}}$ latching object of Y is $x\colon \partial\Delta[m] \to \mathcal{D}$, ie we have an induced injective map $\partial\Delta[m] \hookrightarrow A_{[m]}$ and $Y_{[m]}|\partial\Delta[m] = x$. Choose a universal cone

$$\tilde{Y}_{[m]}\colon A^{\rhd}_{[m]} \to \mathcal{D}$$

under $Y_{[m]}$ and consider $\tilde{Y}_{[m]}|\partial\Delta[m]^{\rhd}$. We have

$$\partial\Delta[m]^{\rhd} \cong \Lambda^{m+1}[m+1]$$

which is an outer horn. However, $\tilde{Y}_{[m]}|\partial\Delta[m]^{\rhd}$ is special since $\Psi_{\mathcal{D}}x$ is homotopical, and thus extends to $z\colon \Delta[m]^{\rhd} \to \mathcal{D}$ by [14, Theorem 4.13]. We set $\hat{x} = z|\Delta[m]$.

By Proposition 3.2, finding an $E[1]$–homotopy from $\Psi_{\mathcal{D}}\hat{x}$ to Y translates into constructing a homotopical Reedy cofibrant diagram $D([m] \times E(1)) \to \mathrm{Dg}_\kappa\,\mathcal{D}$ restricting to $[\Psi_{\mathcal{D}}\hat{x}, Y]$ on $D(\Delta[m] \times \partial\Delta[1])$. By [23, Corollary 3.7] it will be sufficient to construct such a diagram on $D([m] \times \widehat{[1]})$ and by Lemma 3.3 it will suffice to define it on $\mathrm{Sd}([m] \times \widehat{[1]})$.

We form a pushout on the left in $\mathrm{Dg}_\kappa\,\mathcal{D}$:

$$\begin{tikzcd}
\tilde{Y}|\partial\Delta[m]^{\rhd} \arrow[r, tail] \arrow[d, tail] & \tilde{Y} \arrow[d, tail] & \partial\Delta[m] \arrow[r, hook] \arrow[d, hook] & A_{[m]} \arrow[d, hook] \\
z \arrow[r, tail] & Z & \Delta[m] \arrow[r, hook] & B
\end{tikzcd}$$

Its underlying square of simplicial sets is $(-)^{\rhd}$ applied to the square on the right.

This yields the following sequence of morphisms of $\mathrm{Dg}_\kappa\,\mathcal{D}$ (with morphisms of the underlying simplicial sets displayed below):

$$\begin{tikzcd}
\hat{x} \arrow[r, tail] & z \arrow[r, tail] & Z & \widetilde{Y}_{[m]} \arrow[l, tail] & Y_{[m]} \arrow[l, tail] \\
\Delta[m] \arrow[r, hook] & \Delta[m]^{\triangleright} \arrow[r, hook] & B^{\triangleright} & A^{\triangleright}_{[m]} \arrow[l, hook'] & A_{[m]} \arrow[l, hook']
\end{tikzcd}$$

The first morphism is a weak equivalence since z is a filler of a special horn. So are the middle two since the underlying maps of simplicial sets preserve the cone points. The last one is also a weak equivalence since $\widetilde{Y}_{[m]}$ is universal. All these morphisms are maps of cones under $Y\,|\,\mathrm{Sd}\,\partial\Delta[m] = \Psi_{\mathcal{D}}x\,|\,\mathrm{Sd}\,\partial\Delta[m]$ and hence can be seen as transformations of diagrams over $\mathrm{Sd}[m]$ which restrict to identities over $\mathrm{Sd}\,\partial\Delta[m]$. The conclusion follows by Lemma 3.4. □

Before we can prove the main theorem we need to know the following:

Lemma 4.8 *The functor* Dg_κ *is homotopical.*

Proof We begin by constructing a natural equivalence $\Theta_{\mathcal{C}}\colon \mathrm{Ho}\,\mathrm{N_f}\,\mathcal{C} \to \mathrm{Ho}\,\mathcal{C}$ for every cofibration category $\mathcal{C}$. We send an object $X\colon D[0] \to \mathcal{C}$ to X_0 and a morphism $Y\colon D[1] \to \mathcal{C}$ to the composite $[v_1]^{-1}[v_0]$, where v_0 and v_1 are the structure morphisms

$$Y_0 \xrightarrow{v_0} Y_{01} \xleftarrow[\sim]{v_1} Y_1.$$

This assignment is well-defined and functorial since $\mathcal{C}$ has homotopy calculus of fractions, see Theorem 1.4.

We check that $\Theta_{\mathcal{C}}$ is an equivalence. It is surjective and full since both $\mathrm{Sd}[0] \hookrightarrow D[0]$ and $D\partial\Delta[1] \cup \mathrm{Sd}[1] \hookrightarrow D[1]$ have the Reedy left lifting property with respect to all cofibration categories by Lemma 3.3. For faithfulness, consider $X, \widetilde{X}\colon D[1] \to \mathcal{C}$ such that $X|D\partial\Delta[1] = \widetilde{X}|D\partial\Delta[1]$ and $\Theta_{\mathcal{C}}(X) = \Theta_{\mathcal{C}}(\widetilde{X})$. Since we have already verified that $\Theta_{\mathcal{C}}$ is essentially surjective, Lemma 3.5 allows us to assume that $X\delta_0$ is a Reedy cofibrant replacement of $p^*_{[0]}X_1$ so that the structure morphisms of X fit into a cylinder

$$X_1 \amalg X_1 \rightarrowtail X_{11} \xrightarrow{\sim} X_1.$$

By Theorem 1.4(2) we have a diagram

$$\begin{tikzcd}
 & X_{01} \arrow[d, "\varphi"', "\sim"] & \\
X_0 \arrow[ur, tail] \arrow[dr, tail] & Y \arrow[r, phantom, "\nu", shift left=2] \arrow[r, phantom, "\tilde{\nu}"', shift right=2] & X_1 \arrow[ul, tail, "\sim"'] \arrow[dl, tail, "\sim"] \\
 & \widetilde{X}_{01} \arrow[u, "\tilde{\varphi}", "\sim"'] &
\end{tikzcd}$$

where both squares commute up to left homotopy. By Lemma 1.5 we can assume that the left square commutes strictly. Let

$$\begin{tikzcd}
X_1 \amalg X_1 \arrow[r, "{[\varphi\nu, \widetilde{\varphi}\widetilde{\nu}]}"] \arrow[d, tail, "{[\delta_0, \delta_1]}"'] & Y \arrow[d, tail, "\psi"', "\sim"] \\
X_{11} \arrow[r, "\chi"'] & \widehat{Y}
\end{tikzcd}$$

be a left homotopy. Then we can form a diagram

$$\begin{tikzcd}
 & & X_1 \arrow[dl, "\nu", "\sim"'] \arrow[dr, "\sim"] & & \\
 & X_{01} \arrow[dr, "\psi\varphi", "\sim"'] & & X_{11} \arrow[dl, "\chi", "\sim"'] & \\
 & & \widehat{Y} & & \\
X_0 \arrow[uur] \arrow[rr] & & \widetilde{X}_{01} \arrow[u, "\psi\widetilde{\varphi}", "\sim"'] & & X_1 \arrow[ll, "\widetilde{\nu}"', "\sim"] \arrow[uul, "\sim"']
\end{tikzcd}$$

which is a homotopical diagram on Sd[2] and Reedy cofibrant over $\mathrm{Sd}\,\partial\Delta[2]$. Thus it can be replaced Reedy cofibrantly without modifying it over $\mathrm{Sd}\,\partial\Delta[2]$ by [23, Lemma 1.9]. Then X, $\widetilde{X}$ and $X\delta_0\sigma_0$ provide an extension over $D\partial\Delta[2]$. We know that the inclusion $D\partial\Delta[2] \cup \mathrm{Sd}[2] \hookrightarrow D[2]$ has the Reedy left lifting property with respect to all cofibration categories by Lemma 3.3, so we can find an extension to $D[2]$ which is a homotopy between X and $\widetilde{X}$ in $\mathrm{N_f}\,\mathcal{C}$.

Since equivalences of quasicategories induce equivalences of homotopy categories, it follows that $\mathrm{N_f}$ reflects equivalences. Thus Dg_κ is homotopical by Proposition 4.7. □

Finally, we are ready to prove the main theorem.

Theorem 4.9 *The functor* $\mathrm{N_f}\colon \mathsf{CofCat}_\kappa \to \mathsf{QCat}_\kappa$ *is a weak equivalence of fibration categories.*

Proof (for $\kappa > \aleph_0$) The functor Dg_κ is homotopical by Lemma 4.8 and thus induces a functor on the homotopy categories. Since Ψ is a natural categorical equivalence by Proposition 4.7 the induced transformation $\mathrm{Ho}\,\Psi$ is a natural isomorphism $\mathrm{id} \to (\mathrm{Ho}\,\mathrm{N_f})(\mathrm{Ho}\,\mathrm{Dg}_\kappa)$. The transformation Φ is merely pseudonatural, but natural isomorphisms of exact functors induce right homotopies in CofCat_κ (by the construction of path objects in the proof of [24, Theorem 2.8]). Therefore $\mathrm{Ho}\,\Phi$ is a natural transformation and by Lemma 4.5 it is an isomorphism $(\operatorname{Ho}\operatorname{Dg}_\kappa)(\operatorname{Ho}\mathrm{N_f}) \to \mathrm{id}$. Hence $\operatorname{Ho}\mathrm{N_f}$ is an equivalence. □

The only part of the argument above that does not work for $\kappa = \aleph_0$ is the construction of a natural weak equivalence $\Phi_{\mathcal{C}}\colon \operatorname{Dg}_\kappa \mathrm{N_f}\,\mathcal{C} \to \mathcal{C}$ for every cofibration category $\mathcal{C}$. Indeed, $\Phi_{\mathcal{C}}$ was defined using colimits over categories DK which are infinite even for finite simplicial sets K. Instead, we will define a zig-zag of (pseudonatural) weak equivalences connecting $\operatorname{Dg}_{\aleph_0} \mathrm{N_f}\,\mathcal{C}$ to $\mathcal{C}$, namely,

$$\begin{tikzcd}
\operatorname{Dg}_{\aleph_0} \mathrm{N_f}\,\mathcal{C} \arrow[r, "\Phi_{\mathcal{C}}^{(-)}"] & \mathcal{C}_{\mathrm{R}}^{\widetilde{\mathbb{N}}} & \mathcal{C}_{\mathrm{R}}^{\widehat{\mathbb{N}}} \arrow[l, hook'] \arrow[r, "\mathrm{ev}_0"] & \mathcal{C}.
\end{tikzcd}$$

Here, $\widehat{\mathbb{N}}$ is the homotopical poset of natural numbers with all morphisms as weak equivalences so that $\mathcal{C}_{\mathrm{R}}^{\widehat{\mathbb{N}}}$ is the cofibration category of Reedy cofibrant homotopically constant sequences. Similarly, $\mathcal{C}_{\mathrm{R}}^{\widetilde{\mathbb{N}}}$ stands for the cofibration category of Reedy cofibrant eventually homotopically constant sequences; see [23, Section 5] for details.

It was verified in [23, Lemma 5.9] that $\mathcal{C}_{\mathrm{R}}^{\widehat{\mathbb{N}}} \hookrightarrow \mathcal{C}_{\mathrm{R}}^{\widetilde{\mathbb{N}}}$ is a weak equivalence. Moreover, $\mathrm{ev}_0\colon \mathcal{C}_{\mathrm{R}}^{\widehat{\mathbb{N}}} \to \mathcal{C}$ is induced by a homotopy equivalence $[0] \to \widehat{\mathbb{N}}$ hence it is a weak equivalence, too.

It remains to define $\Phi_{\mathcal{C}}^{(-)}$ and prove that it is also a weak equivalence. For each k and an object $X\colon DK \to \mathrm{N_f}\,\mathcal{C}$ we set $\Phi_{\mathcal{C}}^{(k)} X = \operatorname{colim}_{D^{(k)}K} X$. This colimit exists since $D^{(k)}K$ is finite if K is finite.

Lemma 4.10 *For a cofibration category $\mathcal{C}$ the formula above defines an exact functor $\Phi_{\mathcal{C}}^{(-)}\colon \operatorname{Dg}_{\aleph_0} \mathrm{N_f}\,\mathcal{C} \to \mathcal{C}_{\mathrm{R}}^{\widetilde{\mathbb{N}}}$. Moreover, it is a weak equivalence.*

Proof First, we need to verify that $\Phi_{\mathcal{C}}^{(-)} X$ is an eventually constant sequence for all $(K, X) \in \operatorname{Dg}_{\aleph_0} \mathrm{N_f}\,\mathcal{C}$. Consider X as a diagram in $\mathrm{N_f}\,\mathcal{C}$ and choose a universal cone $S\colon K^{\rhd} \to \mathrm{N_f}\,\mathcal{C}$. Then [23, Lemma 4.8] implies that $\Phi_{\mathcal{C}}^{(-)} S$ is eventually constant and Theorem 3.7 implies that the induced morphism $\Phi_{\mathcal{C}}^{(-)} S \to \Phi_{\mathcal{C}}^{(-)} S$ is an eventual weak equivalence. Thus $\Phi_{\mathcal{C}}^{(-)} S$ is eventually constant.

Preservation of cofibrations follows by [20, Theorem 9.4.1(1a)] since if $K \hookrightarrow L$ is an injective map of simplicial sets, then the induced functors $D^{(k)}K \cup D^{(k-1)}L \to D^{(k)}L$ are sieves. Colimits in $\mathcal{C}$ are compatible with colimits of indexing categories and thus $\Phi_{\mathcal{C}}^{(-)}$ is exact. (Preservation of weak equivalences follows from the argument below.)

To see that it is a weak equivalence, it is enough to verify the approximation properties of Proposition 1.8. Theorem 3.7 and [23, Lemma 5.8] imply that a morphism f in $\operatorname{Dg}_{\aleph_0} \mathrm{N_f}\,\mathcal{C}$ is a weak equivalence if and only if $\Phi_{\mathcal{C}}^{(-)} f$ is an eventual weak equivalence. Therefore $\Phi_{\mathcal{C}}^{(-)}$ preserves weak equivalences and satisfies (App1). It remains to check (App2), but it follows directly from [23, Lemma 5.10]. □

This yields the proof of Theorem 4.9 in the remaining case of $\kappa = \aleph_0$ since the three weak equivalences described above induce a natural isomorphism

$$(\mathrm{Ho}\,\mathrm{Dg}_\kappa)(\mathrm{Ho}\,\mathrm{N}_\mathrm{f}) \to \mathrm{id}$$

and the rest of the argument applies verbatim.

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