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Taub-NUT from the Dirac monopole

Hadi Godazgar [1], **Mahdi Godazgar** [2] **and C.N. Pope** [3,4]

[1] *Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut), Mühlenberg 1, D-14476 Potsdam, Germany.*

[2] *School of Mathematical Sciences, Queen Mary University of London, Mile End Road, E1 4NS, United Kingdom.*

[3] *George P. & Cynthia Woods Mitchell Institute for Fundamental Physics and Astronomy, Texas A&M University, College Station, TX 77843, USA.*

[4] *DAMTP, Centre for Mathematical Sciences, Cambridge University, Wilberforce Road, Cambridge CB3 OWA, UK.*

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ABSTRACT

Writing the metric of an asymptotically flat spacetime in Bondi coordinates provides an elegant way of formulating the Einstein equation as a characteristic value problem. In this setting, we find that a specific class of asymptotically flat spacetimes, including stationary solutions, contains a Maxwell gauge field as free data. Choosing this gauge field to correspond to the Dirac monopole, we derive the Taub-NUT solution in Bondi coordinates.

E-mails: hadi.godazgar@aei.mpg.de, m.godazgar@qmul.ac.uk, pope@physics.tamu.edu

1 Introduction

Asymptotically flat spacetimes have been extensively studied over the last 50 years, particularly, recently, in the context of gravitational wave detections [1] and asymptotic symmetry groups [2, 3]. In this paper, motivated by recent work on dual gravitational charges [4–7], we show how treating asymptotically flat spacetimes in terms of a characteristic value problem [8] provides an intriguing way of viewing the Dirac magnetic monopole as a progenitor of the Taub-NUT spacetime.[1]

Our starting point is to consider a general class of asymptotically flat metrics, written in a Bondi coordinate system $(u, r, x^I = \{\theta, \phi\})$, such that the metric takes the form[2]

$$ds^2 = -Fe^{2\beta}du^2 - 2e^{2\beta}dudr + r^2 h_{IJ}\,(dx^I - C^I du)(dx^J - C^J du), \tag{1.1}$$

with the metric functions satisfying the following fall-off conditions at large r:

$$\begin{aligned} F(u,r,x^I) &= 1 + \frac{F_0(u,x^I)}{r} + \frac{F_1(u,x^I)}{r^2} + \frac{F_2(u,x^I)}{r^3} + \frac{F_3(u,x^I)}{r^4} + o(r^{-4}), \\ \beta(u,r,x^I) &= \frac{\beta_0(u,x^I)}{r^2} + \frac{\beta_1(u,x^I)}{r^3} + \frac{\beta_2(u,x^I)}{r^4} + o(r^{-4}), \\ C^I(u,r,x^I) &= \frac{C_0^I(u,x^I)}{r^2} + \frac{C_1^I(u,x^I)}{r^3} + \frac{C_2^I(u,x^I)}{r^4} + \frac{C_3^I(u,x^I)}{r^5} + o(r^{-5}), \\ h_{IJ}(u,r,x^I) &= \omega_{IJ} + \frac{C_{IJ}(u,x^I)}{r} + \frac{C^2\omega_{IJ}}{4r^2} + \frac{D_{IJ}(u,x^I)}{r^3} + \frac{E_{IJ}(u,x^I)}{r^4} + o(r^{-4}), \end{aligned} \tag{1.2}$$

where ω_{IJ} is the standard metric on the round 2-sphere with coordinates $x^I = \{\theta, \phi\}$ and $C^2 \equiv C_{IJ}C^{IJ}$. Moreover, a residual gauge freedom allows us to require that

$$h = \omega, \tag{1.3}$$

where $h \equiv \det(h_{IJ})$, and $\omega \equiv \det(\omega_{IJ}) = \sin\theta$. Following Ref. [7], we do not impose on the fields defined above any regularity conditions on the 2-sphere.

One may introduce a complex null frame of vector fields $e_\mu{}^a = (\ell^a, n^a, m^a, \bar{m}^a)$ with

[1] The relation between the Dirac monopole and Taub-NUT solutions is also encountered in a different setting: that of the double copy [9, 10], where, the focus of the investigation is on the (double) Kerr-Schild ansatz and the key insight is that the Kerr-Schild null vector(s) may be related to a Maxwell field.

[2] See section 2 of Ref. [11] for a more in-depth discussion of asymptotically-flat spacetimes and the notation we will use in this paper.

inverse $E^\mu{}_a$,

$$g_{ab} = E^\mu{}_a E^\nu{}_b \, \eta_{\mu\nu}, \qquad \eta_{\mu\nu} = \begin{pmatrix} \begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{pmatrix}, \tag{1.4}$$

where

$$\begin{aligned} \ell &= \frac{\partial}{\partial r}, \qquad n = e^{-2\beta}\left[\frac{\partial}{\partial u} - \tfrac{1}{2}F\frac{\partial}{\partial r} + C^I\frac{\partial}{\partial x^I}\right], \qquad m = \frac{\hat{m}^I}{r}\frac{\partial}{\partial x^I}, \\ \ell^\flat &= -e^{2\beta}du, \qquad n^\flat = -\Big(dr + \frac{1}{2}F du\Big), \qquad m^\flat = r\,\hat{m}_I\,(dx^I - C^I du), \end{aligned} \tag{1.5}$$

and

$$2\hat{m}^{(I}\bar{\hat{m}}^{J)} = h^{IJ}, \tag{1.6}$$

with h^{IJ} the matrix inverse of h_{IJ}.

Given the choice of Bondi coordinates, the formalism most adapted to the problem of constructing solutions from initial data is the characteristic approach [8]. In the characteristic value problem, such as that defined by Bondi coordinates, the spacetime is viewed as a foliation of null hypersurfaces, called the characteristic surfaces, corresponding to the level sets of u (see figure 1).

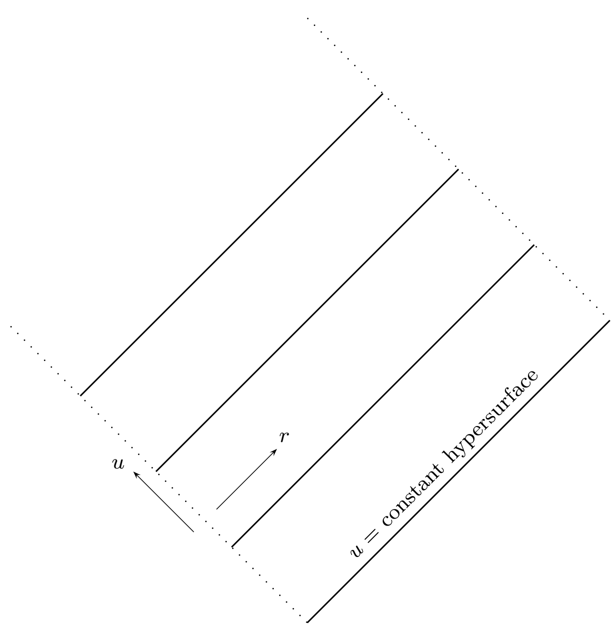

Figure 1: Hypersurfaces defining a characteristic value problem.

The vacuum[3] Einstein equation may then be divided into three types of equations [8]:

[3]One could, of course, include matter with corresponding energy-momentum tensor satisfying appropriate

1. **Hypersurface equations**: these are equations that hold in each $u =$ constant hypersurface and are of the form

$$\partial_r \mathcal{F} = H_{\mathcal{F}}(\mathcal{F}, \mathcal{G}), \tag{1.7}$$

where $\mathcal{F}$ denotes the set of hypersurface variables, which in Bondi coordinates correspond to $\{\beta, F, C^I\}$, while $\mathcal{G}$ denotes evolution variables, which in Bondi coordinates corresponds to h_{IJ}. The operator $H_{\mathcal{F}}$, as well as $H_{\mathcal{G}}$ defined below, is non-linear in derivatives with respect to the three hypersurface coordinates. The structure of the hypersurface equations is such that the r-dependence of the right hand side is always explicit, and therefore the hypersurface variables may simply be determined by integrating with respect to r.

2. **Evolution equations**: these are of the form

$$\partial_u \partial_r \mathcal{G} = H_{\mathcal{G}}(\mathcal{F}, \mathcal{G}, \partial_u \mathcal{G}). \tag{1.8}$$

Note that there are no second order derivatives in time.

3. **Conservation equations**: these are satisfied on $r =$ constant hypersurfaces transverse to the characteristics, and are of the form

$$\partial_u \mathcal{F} = h_{\mathcal{F}}(\mathcal{F}, \mathcal{G}, \partial_u \mathcal{G}), \tag{1.9}$$

where $h_{\mathcal{F}}$ is some non-linear operator in the two angular coordinates in the $u =$ constant hypersurfaces. These are to be thought of as conservation equations rather than evolution equations because their structure is such that once they are satisfied on a particular $r =$ constant hypersurface, they are guaranteed to hold for all values of r.

In the complex null frame introduced above, these groups correspond to the following components of the Einstein equation:

$$\textbf{Hypersurface equations}: \quad \ell^\mu \ell^\nu G_{\mu\nu} = 0\,, \quad \ell^\mu n^\nu G_{\mu\nu} = 0\,, \quad \ell^\mu m^\nu G_{\mu\nu} = 0\,. \tag{1.10}$$

$$\textbf{Evolution equations}: \quad m^\mu m^\nu G_{\mu\nu} = 0\,. \tag{1.11}$$

$$\textbf{Conservation equations}: \quad n^\mu n^\nu G_{\mu\nu} = 0\,, \quad n^\mu m^\nu G_{\mu\nu} = 0\,. \tag{1.12}$$

fall-off conditions. However, in this paper, we simply consider the vacuum Einstein equation.

Note that the $m^\mu\,\bar{m}^\nu\,G_{\mu\nu}=0$ component is automatically satisfied if we assume the other components to hold [12]. Since we are assuming specific expansions in inverse powers of r for the metric components,[4] at least for the first few orders, the corresponding hypersurface equations simplify to algebraic equations at fixed u for all scalar and vector fields on the right hand side of the fall-off conditions for $\{\beta, F, C^I\}$, except for the fields F_0 and C_1^I, which as we will explain later constitute initial data, whose evolutions are determined by the conservation equations. The evolution equations then determine the evolution of D_{IJ} and E_{IJ} (as well as the other terms at higher powers in the $1/r$ expansion of h_{IJ}), given prescribed initial data. Rather unusually, $C_{IJ}(u, x^I)$ constitutes free data. Furthermore, note that the initial data are unconstrained, in contrast to a Cauchy formulation where they would satisfy elliptic constraints. The explicit Einstein equations for the metric components are listed in section 2.2 of Ref. [11].

In summary, the Einstein equation is solved by prescribing initial data

$$\{F_0(u_0, x^I), C_1^I(u_0, x^I), D_{IJ}(u_0, x^I), E_{IJ}(u_0, x^I), \ldots\},$$

where the ellipses denote terms at higher powers in the $1/r$ expansion of h_{IJ}, and an arbitrary trace-free symmetric tensor $C_{IJ}(u, x^I)$. The hypersurface equations then determine all scalar, vector and tensor fields in the expansions (1.2) on the initial hypersurface. Now, the evolution and conservation equations can be integrated to determine $\{F_0, C_1^I, D_{IJ}, E_{IJ}, \ldots\}$ at a time step $u_0 + \Delta u$, before iterating the above procedure to find the full solution at this time step and so on.

One of the hypersurface equations, arising at order $1/r^3$ of the $\ell^\mu\, m^\nu\, G_{\mu\nu} = 0$ equation (see equation (2.16) of Ref. [11]), gives[5]

$$C_0{}^I = -\tfrac{1}{2} D_J\, C^{IJ}\,, \tag{1.13}$$

which determines C_0^I given the data $C_{IJ}(u, x^I)$. Alternatively, we may view this equation as determining C_{IJ} given a choice of C_0^I, up to functions of integration which may themselves be arbitrarily chosen. That is to say, we may exchange the freedom to choose a trace-free tensor $C_{IJ}(u, x^I)$ with the freedom to choose a vector C_0^I. It will be useful in what follows to take this perspective. Therefore, the data for the characteristic value problem are given

[4] This assumption is formally consistent [12, 13] in the sense that assuming such a fall off for the free initial data $h_{IJ}(u_0, r, x^I)$ implies a solution of the form defined in equation (1.2).

[5] $I, J, \ldots$ indices are raised or lowered using ω^{IJ} or ω_{IJ}.

by

$$\{F_0(u_0, x^I), C_0^I(u, x^I),\, C_1^I(u_0, x^I), D_{IJ}(u_0, x^I), E_{IJ}(u_0, x^I), \dots\},$$

where we emphasise that the choice for C_0^I is not only an initial value choice as with the other fields, but that we have the freedom to choose the vector completely, as a function also of u.

Let us consider, briefly, the physical significance of these quantities. The Bondi mass and angular momentum are given by [12, 13], [7][6]

$$M_B = -\frac{1}{8\pi}\int_S d\Omega\left(F_0 + \frac{1}{4}D_I D_J C^{IJ}\right), \qquad J_B = \frac{3}{8\pi}\int_S d\Omega\, \sin^2\theta\, L^\phi, \tag{1.14}$$

where

$$L^I = \frac{1}{2}C_1^I + \frac{1}{3}C^{IJ}C_{0\,J} \tag{1.15}$$

is the angular momentum aspect, so that if L^ϕ is constant then $J_B = L^\phi$. Moreover, we have that the dual Bondi mass or Bondi NUT charge is given by [7]

$$\widetilde{M}_B = \frac{1}{16\pi}\int_S dC_0, \tag{1.16}$$

where $C_0 = C_{0\,I}\, dx^I$. Thus F_0, C_0^I and C_1^I constitute the data determining the Bondi mass, NUT charge and angular momentum. Furthermore, the other data given by terms at higher powers in the $1/r$ expansion of h_{IJ} correspond to subleading BMS charges [11], [5]. For example, D_{IJ} determines the BMS charge at order $1/r^2$.

In this paper, our focus will be on constructing stationary solutions, meaning that all the scalar, vector and tensor fields in the expansions (1.2) will be independent of the retarded time u. This means that the evolution and conservation equations (1.11) and (1.12) will now give rise to genuine constraints on the initial data. Thus, all stationary asymptotically flat solutions are given by prescribing the following data

$$\{F_0(x^I),\, C_0^I(x^I),\, C_1^I(x^I),\, D_{IJ}(x^I),\, E_{IJ}(x^I), \dots\}$$

subject to the constraints implied by equations (1.11) and (1.12).

[6]We work in units where $G = 1$. Note that the total derivative term $D_I D_J C^{IJ}$ in the expression for the Bondi mass arises because the tensor C^{IJ} is not necessarily regular on the sphere [7].

2 $C_{0\,I}$ as a Maxwell gauge field

As can be verified from the form of the BMS transformations in the notation of Ref. [11] that we are using here, under a supertranslation with diffeomorphism parameter $\xi = s(\theta,\phi)\,\partial u + \cdots$, we have

$$\delta C_{0\,I} = \partial_u C_0^I + D_I\left(\frac{1}{2}\Box s + s\right). \tag{2.1}$$

Thus, if

$$\partial_u C_0^I = 0, \tag{2.2}$$

which is the case for stationary solutions, but also more generally, then

$$\delta C_{0\,I} = D_I \Lambda, \qquad \Lambda = \frac{1}{2}\Box s + s \tag{2.3}$$

Thus, in a very real sense, one can think of a u-independent 1-form C_0 as being analogous to a Maxwell gauge potential.

This interesting observation is one reason why we chose to view $C_{0\,I}$ rather than C_{IJ} as characteristic data in the previous section.

3 Dirac monopole

Given the interpretation of a u-independent C_0 as a Maxwell gauge field, it is natural to consider the case where this 1-form describes a Dirac magnetic monopole on the 2-sphere, with C_0 of the form

$$C_0 \equiv C_{0\,I}\, dx^I = 2p\cos\theta\, d\phi, \tag{3.1}$$

where p is a constant. This 1-form is singular at the north and the south poles of the sphere. We shall look for solutions that are stationary and axisymmetric,[7] so all the metric functions will be assumed to be independent of u and of ϕ. Of course, in order to specify a particular solution we must also prescribe initial data for the other fields. However, since we will have constraint equations implied by the requirement of stationarity, we choose for now to keep the other initial data arbitrary and choose them with the constraint equations in mind.

The singularities in the Dirac monopole gauge potential $A = 2p\cos\theta\, d\phi$ are purely gauge artefacts, with the field strength $F = dA = -2p\sin\theta\, d\theta\wedge d\phi$ being perfectly regular on the

[7] The most general such solutions have been classified in Weyl coordinates, and are given by four functions that satisfy simple coupled partial differential equations in two variables. For details, see chapter 20 of Ref. [14].

sphere. The Dirac string or wire singularities in the gauge potential can be moved around by means of large gauge transformations $A \to A + d\Lambda$. For example, taking $\Lambda = -2p\,\phi$ gives $A = -4p\sin^2\frac{\theta}{2}\,d\phi$, which is singular only at the south pole, whilst taking $\Lambda = 2p\,\phi$ gives $A = 4p\cos^2\frac{\theta}{2}\,d\phi$, which is singular only at the north pole.

By comparing with equation (2.3), and taking s to be a constant multiple of the azimuthal coordinate ϕ, we can perform precisely the kinds of gauge transformation we described above for the Dirac monopole.[8] We may take as our starting point the slight generalisation of (3.1) where

$$C_0 \equiv C_{0\,I}\,dx^I = 2p\cos\theta\,d\phi + 2k\,d\phi, \tag{3.2}$$

with k being a gauge-adjustable constant, which fixes the supertranslation gauge that we are working in. Already, it is clear that we are dealing with spacetimes that have a non-trivial NUT charge (1.16):

$$\widetilde{M}_B = -\frac{p}{2}. \tag{3.3}$$

Using the fact that C_{IJ} is symmetric, trace-free and depends only on θ, by the axisymmetry assumption, we can substitute (3.2) into (1.13) and solve to find

$$C_{\theta\theta} = \frac{c_1}{\sin^2\theta}\,, \qquad C_{\phi\phi} = -c_1\,, \qquad C_{\theta\phi} = \frac{c_2 + 4k\cos\theta + 2p\cos^2\theta}{\sin\theta}\,, \tag{3.4}$$

where c_1 and c_2 are constants of integration. These constants will contribute to subleading BMS charges, and as is evident from the definition (1.15) of the angular momentum aspect L^I and the form of (3.2), c_1 will also contribute to the angular momentum. Note that $D_I D_J C^{IJ} = 0$, and therefore c_1 and c_2 do not contribute to the Bondi mass given in (1.14).

It is interesting to note that although in general this solution for C_{IJ} is singular at one or both of the poles of the sphere, we can find a non-singular solution in the special case where we choose $c_1 = 0$, $c_2 = -2p$ and $k = 0$, for which the only non-vanishing components of the symmetric C_{IJ} are specified just by

$$C_{\theta\phi} = -2p\sin\theta. \tag{3.5}$$

Of course, since our starting point was $C_{0\,I}$ having the form of a Dirac monopole, which is necessarily singular somewhere on the sphere, there is no particular reason why we should

[8] The function ϕ is of course singular on the sphere. However, since we are already entertaining the idea of using a monopole configuration for $C_{0\,I}$ that is singular on the sphere, there is no longer any reason to restrict ourselves to non-singular supertranslation parameters.

expect or require C_{IJ} to be non-singular. Indeed, as we shall see below, there are reasons to prefer different assignments for (c_1, c_2, k) for which C_{IJ} does have singularities.

As explained before, the other hypersurface equations are algebraic equations that determine the form of other fields given initial data. Therefore, we need not concern ourselves with those just yet. However, the evolution and conservation equations are now non-trivial equations, constraining the data. Therefore, we focus on these equations.

The $n^\mu\, n^\nu\, G_{\mu\nu} = 0$ projection of the Einstein equation at order $1/r^2$ implies the conservation equation [11]

$$\partial_u F_0 = -\frac{1}{2} D_I D_J \partial_u C^{IJ} + \frac{1}{4} \partial_u C^{IJ} \partial_u C_{IJ}, \tag{3.6}$$

which is trivially satisfied in the stationary case. The other conservation equation, which comes from the $n^\mu\, m^\nu\, G_{\mu\nu} = 0$ projection of the Einstein equation at order $1/r^3$, then gives (see equation (2.26) of [11])

$$\begin{aligned} 0 = 3\partial_u C_1^I &= D^I F_0 - D_J (D^J C_0^I - D^I C_0^J) \\ &= D^I F_0 - D_J F^{JI}, \end{aligned} \tag{3.7}$$

where $F_{IJ} = 2\partial_{[I} C_{0\, J]}$ is the field strength associated with the gauge field C_0. In the first line we have used equation (1.13), and the fact that $\partial_u C_{IJ} = 0$. Since the only non-zero component of F^{IJ} is $F^{\theta\phi} = -2p/\sin\theta$, and

$$D_J F^{JI} = \frac{1}{\sin\theta} \partial_J (\sin\theta F^{JI}) = 0, \tag{3.8}$$

equation (3.7) then implies that F_0 is independent of the coordinates on the sphere. Hence, we choose

$$F_0 = -2m, \tag{3.9}$$

with constant m parameterising the mass of the asymptotically-flat spacetime.

Moving on to the evolution equations given by the $m^\mu\, m^\nu\, G_{\mu\nu} = 0$ projection of the Einstein equation, at order $1/r^4$, this gives [11]

$$\begin{aligned} 0 = \partial_u D_{IJ} = &-\frac{1}{4} F_0 C_{IJ} - \frac{1}{2} D_{(I} C_{1\, J)} + \frac{1}{4} C_{IJ} D_K C_0^K + \frac{1}{32} D_I D_J C^2 \\ &- D_{(I} (C_{J)K} C_0^K) - \frac{1}{8} D_I C^{KL} D_J C_{KL} \\ &+ \frac{1}{4} \omega_{IJ} \Big[D_K C_1^K - \frac{1}{16} \Box C^2 + 2D^K (C_{KL} C_0^L) + \frac{1}{4} D^M C^{KL} D_M C_{KL} \Big]. \end{aligned} \tag{3.10}$$

With C_0 and C_{IJ} given by (3.2) and (3.4), respectively, we may now seek to solve for $C_{1\, I}(\theta)$.

Since the solutions are a little complicated in general we shall not present them here, but just remark that in general they involve terms that have power-law singularities at $\theta = 0$ and $\theta = \pi$, and also terms proportional to $\log\cot\frac{\theta}{2}$ with logarithmic singularities at the poles of the sphere. The solution for $C_{1\,I}$ will be free of logarithmic singularities if and only if we choose the c_1 and c_2 integration constants in (3.4) to be

$$c_1 = 0\,, \qquad c_2 = 2p. \tag{3.11}$$

The solution is then given by

$$\begin{aligned} C_{1\,\theta} &= c_3\,\sin\theta + \frac{3k\cos 4\theta + 4(k^2+p^2)\cos 3\theta - 12kp\cos 2\theta - 36(k^2+p^2)\cos\theta - 55kp}{4\sin^5\theta}\,, \\ C_{1\,\phi} &= c_4\sin^2\theta - 4m(p\cos\theta + k), \end{aligned} \tag{3.12}$$

where c_3 and c_4 are constants of integration.

This is the most general solution for $C_{1\,I}$ without logarithmic singularities, starting from the Dirac monopole connection (3.2). We may now evaluate the angular momentum given in (1.14), obtaining

$$J_B = \frac{1}{2}(c_4 - 6km). \tag{3.13}$$

However, the Komar angular momentum

$$J_K = \frac{1}{16\pi}\int_S \star dj^\flat \tag{3.14}$$

with $j = \partial/\partial\phi$ is divergent. This is easy to see, since

$$(\star dj^\flat)_{\theta\phi} = 2[2k\sin\theta + p\sin 2\theta]\,r + [2(c_4 - 6km)\sin\theta - 6pm\sin 2\theta] + \mathcal{O}(r^{-1}). \tag{3.15}$$

Therefore, before sending r to infinity, the right hand side of the expression for J_K gives

$$kr + \frac{1}{2}(c_4 - 6km) + \mathcal{O}(r^{-1}). \tag{3.16}$$

A simple explanation for why the divergence above vanishes for $k = 0$ is that the wire singularity provides a divergent contribution, however, when $k = 0$ the divergent contributions from the wire singularities at the north and south poles cancel. There has been a proposal to resolve this divergence issue for general k in Ref. [15], by redefining the Komar integral. However, given that k parameterises a large gauge transformation, we may simply avoid these difficulties by choosing to set $k = 0$. (Recall that k is the gauge parameter appearing in (3.2).) Now, we obtain

$$J_B = \frac{1}{2} c_4. \tag{3.17}$$

Thus, c_4 is a Kerr-like angular momentum parameter, which, for simplicity, we shall for now set to zero. The integration constant c_3 will contribute to subleading BMS charges [11], [5], and we shall also set this to zero for simplicity.

With $c_3 = c_4 = 0$, and making the gauge choice $k = 0$ as discussed above, the expressions for $C_{1\,I}$ in (3.12) become

$$C_{1\,\theta} = -\frac{4p^2\,(2+\sin^2\theta)\cos\theta}{\sin^5\theta}\,, \qquad C_{1\,\phi} = -4mp\,\cos\theta. \tag{3.18}$$

The previous expressions (3.2) and (3.4) for $C_{0\,I}$ and C_{IJ} then give

$$\begin{aligned} &C_{0\,\theta} = 0\,, \qquad C_{0\,\phi} = 2p\cos\theta, \\ &C_{\theta\theta} = 0\,, \qquad C_{\phi\phi} = 0\,, \qquad C_{\theta\phi} = \frac{2p\,(1+\cos^2\theta)}{\sin\theta}. \end{aligned} \tag{3.19}$$

Substituting these expressions into the $\ell^\mu\, n^\nu\, G_{\mu\nu} = 0$ component of the Einstein equation at order $1/r^4$ then gives

$$F_1 = \frac{p^2\,(4 + 4\sin^2\theta - 11\sin^4\theta)}{2\sin^4\theta}. \tag{3.20}$$

Comparing with the expressions in appendix A, we see that the above fields are in exact agreement with the corresponding components in the expansion of the Taub-NUT metric in Bondi coordinates, with $p = \ell$.

Clearly we could in principle continue to arbitrarily high orders in the $1/r$ expansions of the Einstein equation. In particular, the structure of the constraint equations coming from the evolution equations should be clear now. For example, at the next order (see equation (2.21) of [11]), the equation implied by the u-independence of E_{IJ} implies an ordinary differential equation for $D_{IJ}(\theta)$. Following this iterative process, and making appropriate choices for constants of integration as they arise, reproduces the Taub-NUT metric in Bondi coordinates to any desired order. Note that had we not chosen to set $k = 0$, we could have obtained the Bondi metric corresponding to the Taub-NUT metric with the string singularity correspondingly shifted.

Other choices of the integration constants will give more general solutions, generically presumably with more severe singular behaviour on the sphere, such as the logarithmic behaviour in $C_{1\,I}$ that we avoided above by the judicious choice (3.11) for the constants of integration c_1 and c_2.

We have already seen, (3.17), that the constant of integration c_4 in the expression (3.12) for $C_{1\,\phi}$ is related to angular momentum. In fact, if we turn off the Dirac monopole altogether by choosing $p = 0$ (and $k = 0$), i.e. setting $C_{0\,I} = 0$, and letting $c_3 = 0$ and $c_4 = 2ma$, we find that our solution (3.12) becomes

$$C_{1\,\theta} = 0\,, \qquad C_{1\,\phi} = 2ma\sin^2\theta. \tag{3.21}$$

As can be seen from the expressions in appendix A of [7], the $C_{1\,I}$ above is precisely that of the Kerr metric in Bondi coordinates, where a is the Kerr rotation parameter. In fact the Kerr metric could be derived in Bondi coordinates by using the same iterative technique we have illustrated in this paper for the Taub-NUT metric, by starting with $F_0 = -2m$, $C_{0\,I} = 0$ and $C_{1\,I}$ given by (3.21), and then sequentially solving the Einstein equation order by order in powers of $1/r$ with the evolution equations determining lower order powers of h_{IJ}, under the assumption of stationarity. Of course one could also derive the Kerr-Taub-NUT metric in Bondi coordinates, by starting out as we did above for Taub-NUT, but now at the stage where we obtained the expressions (3.12) for $C_{1\,I}$ we would take c_4 to be non-zero.

4 Discussion

We showed that, in a stationary setting, starting from a choice of the gauge field C_0 as the Dirac monopole, we can integrate the vacuum Einstein equation to recover the Taub-NUT solution in Bondi coordinates. Assuming the solution to be stationary, the evolution and conservation equations transformed into constraint equations on the other characteristic data. For example, we found that F_0 must be constant, giving a Bondi mass. Solving another of the constraint equations, we found C_1^I up to two constants of integration. One of these constants corresponded to a Kerr-like angular momentum parameter. Working through the other constraint equations would also generate other constants, which contribute to subleading BMS charges. Therefore, what the Dirac monopole actually generates is a *family* of Taub-NUT-like solutions, of which the usual Taub-NUT solution is one member. In general, the other members of the family would correspond to other stationary axisymmetric Weyl solutions with a non-trivial NUT charge.

The 1-form C_0 can in fact be regarded as a gauge connection more generally, namely in any spacetime, even a time-dependent one, provided that C_0 itself is still time-independent. Therefore the iterative procedure that we carried out, starting with a Dirac monopole configuration, can be repeated in a more general non-stationary setting. It would be interesting to see if dynamical solutions with Taub-NUT charge can be constructed this way.

Moreover, in this work, for simplicity, we have assumed a $1/r$-expansion of the Bondi metric, at least for the first few orders. However, this is not necessary and only simplifies the hypersurface, (1.10), and evolution, (1.11), equations. Therefore, one could consider the more general system of equations with weaker fall-off conditions, which would presumably be necessary when considering time-dependent solutions.

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A Taub-NUT Metric in Bondi Coordinates

Here, we construct the Taub-NUT metric in Bondi coordinates, working in the "symmetric" coordinate gauge where there are string singularities at both the north and south poles. The starting metric in this case is the standard one,

$$ds^2 = -f(\bar{r})\,(d\bar{t} + 2\ell\,\cos\bar{\theta}\,d\bar{\phi})^2 + f(\bar{r})^{-1}\,d\bar{r}^2 + (\bar{r}^2\ell^2)\,(d\bar{\theta}^2 + \sin^2\,\bar{\theta}\,d\bar{\phi}^2)\,. \tag{A.1}$$

We have placed bars on the coordinates, because we now make an expansion of the form described in [7], imposing the Bondi metric conditions $g_{rr} = g_{u\theta} = g_{u\phi} = 0$, and $\det(h_{IJ}) = \det(\omega_{IJ})$, order by order in the expansion in $1/r$. Proceeding to the first few orders, we find

$$\bar{t} = u + r + 2m\log r - \frac{4m^2 - 2\ell^2\,(\csc^2\theta + \csc^4\theta - \frac{11}{4})}{r} - \frac{2m\,[2m^2 + \ell^2\,(3 - 2\csc^2\theta)]}{r^2} + \cdots\,,$$

$$\bar{\phi} = \phi - \frac{2\ell\cos\theta}{r\,\sin^2\theta} - \frac{\ell^3\,(\sin^4\theta + 4\sin^2\theta - \frac{20}{3})}{r^3\,\sin^6\theta} - \frac{2m\,\ell^3\,\cos^3\theta}{r^4\,\sin^4\theta} + \cdots\,,$$

$$\bar{r} = r + \frac{\ell^2\,(4\csc^4\theta - 3)}{2r} - \frac{2m\ell^2\,\cot^2\theta}{r^2}$$
$$+ \frac{\ell^4\,(-6 + 16\sin^2\theta + 11\sin^4\theta - 25\sin^6\theta + \frac{45}{8}\sin^8\theta)}{3r^3\,\sin^8\theta} + \frac{m\ell^4\,(7\cos 2\theta + 1)\cos^2\theta}{r^4\,\sin^6\theta} + \cdots\,,$$

$$\bar{\theta} = \theta + \frac{2\ell^2 \cos\theta}{r^2 \sin^3\theta} + \frac{2\ell^4 (\sin^4\theta + 10\sin^2\theta - 15) \cos\theta}{3r^4 \sin^7\theta} + \frac{4m\ell^4 \cos^3\theta}{r^5 \sin^5\theta} + \cdots \,. \tag{A.2}$$

We have actually worked to a higher order than the terms presented here, sufficient for our later purposes. Using these expansions, we then obtain the Taub-NUT metric in Bondi form, finding

$$\begin{aligned}
g_{uu} &= -1 + \frac{2m}{r} + \frac{2\ell^2}{r^2} + \frac{m\ell^2 (1 - 4\csc^4\theta)}{r^3} - \frac{2\ell^2 [2\ell^2 (\sin^4\theta - 2) + m^2 \sin 2\theta]}{r^4 \sin^4\theta} + \mathcal{O}(r^{-5})\,,\\
g_{ur} &= -1 + \frac{\ell^2 (1+\cos^2\theta)^2}{2r^2 \sin^4\theta} + \frac{\ell^4 [6 - 28\sin^2\theta + 15\sin^4\theta + 8\sin^6\theta - \frac{21}{8}\sin^8\theta]}{r^4 \sin^8\theta} + \mathcal{O}(r^{-5})\,,\\
g_{u\theta} &= \frac{4\ell^2 (2 - \sin^2\theta + \sin^4\theta) \cos\theta}{r \sin^5\theta} + \frac{4m\ell^2 (\cot^2\theta - 1)}{r^2}\\
&\quad - \frac{2\ell^4 (8 - 28\sin^2\theta + \frac{50}{3}\sin^4\theta - 4\sin^6\theta + 3\sin^8\theta) \cos\theta}{r^3 \sin^9\theta} + \mathcal{O}(r^{-4})\,,\\
g_{u\phi} &= -2\ell \cos\theta + \frac{4m\ell\cos\theta}{r} + \frac{4\ell^3 \cos\theta}{r^2} + \frac{2m\ell^3 (1 - 4\csc^4\theta)\cos\theta}{r^3} + \mathcal{O}(r^{-4})\,,\\
g_{rr} &= \mathcal{O}(r^{-6})\,, \qquad g_{r\theta} = \mathcal{O}(r^{-5})\,, \qquad g_{r\phi} = \mathcal{O}(r^{-5})\,,\\
g_{\theta\theta} &= r^2 + \frac{2\ell^2 (1+\cos^2\theta)^2}{\sin^4\theta} - \frac{4m\ell^2 \cot^2\theta}{r}\\
&\quad - \frac{2\ell^4 (40 - 16\sin^2\theta - 45\sin^4\theta + 27\sin^6\theta)}{3r^2 \sin^6\theta} + \mathcal{O}(r^{-3})\,,\\
g_{\theta\phi} &= \frac{2\ell r (1+\cos^2\theta)}{\sin\theta} + \frac{\ell^3 (8 - \frac{20}{3}\sin^2\theta + 6\sin^4\theta - 5\sin^6\theta)}{r \sin^5\theta} - \frac{10m\ell^3 \cos^2\theta}{r^2 \sin\theta} + \mathcal{O}(r^{-3})\,,\\
g_{\phi\phi} &= r^2 \sin^2\theta + \frac{2\ell^2 (1+\cos^2\theta)^2}{\sin^2\theta} + \frac{4m\ell^2 \cos^2\theta}{r}\\
&\quad + \frac{2\ell^4 (8 - \frac{16}{3}\sin^2\theta - \sin^4\theta - \sin^6\theta)}{r^2 \sin^4\theta} + \mathcal{O}(r^{-3})\,.
\end{aligned} \tag{A.3}$$

Comparing with the expansions for the Bondi metric as defined in (1.2), we have, for example,

$$\begin{aligned}
C_{IJ}: &\quad C_{\theta\theta} = 0\,, \qquad C_{\phi\phi} = 0\,, \qquad C_{\theta\phi} = 2\ell\,(1+\cos^2\theta)\,\csc\theta\,,\\
D_{IJ}: &\quad D_{\theta\theta} = -4m\ell^2 \cot^2\theta\,, \qquad D_{\phi\phi} = 4m\ell^2 \cos^2\theta\,,\\
&\quad D_{\theta\phi} = \ell^3\,(8 - \tfrac{20}{3}\sin^2\theta + 6\sin^4\theta - 5\sin^6\theta)\,\csc^5\theta\,,\\
C_{0\,I}: &\quad C_{0\,\theta} = 0\,, \qquad C_{0\,\phi} = 2\ell\cos\theta\,,\\
C_{1\,I}: &\quad C_{1\,\theta} = -4\ell^2\,(2+\sin^2\theta)\cot\theta\csc^4\theta\,, \qquad C_{1\,\phi} = -4m\ell\,\cos\theta\,,
\end{aligned}$$

$$F_0 = -2m\,, \qquad F_1 = \frac{\ell^2\,(4+4\sin^2\theta - 11\sin^4\theta)}{2\sin^4\theta}\,. \tag{A.4}$$

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