

MAXIMUM LIKELIHOOD ESTIMATION IN HIDDEN MARKOV MODELS WITH INHOMOGENEOUS NOISE

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ABSTRACT. We consider parameter estimation in finite hidden state space Markov models with time-dependent inhomogeneous noise, where the inhomogeneity vanishes sufficiently fast. Based on the concept of asymptotic mean stationary processes we prove that the maximum likelihood and a quasi-maximum likelihood estimator (QMLE) are strongly consistent. The computation of the QMLE ignores the inhomogeneity, hence, is much simpler and robust. The theory is motivated by an example from biophysics and applied to a Poisson- and linear Gaussian model.

1. INTRODUCTION

Motivation. Hidden Markov models (HMMs) have a long history and are widely used in a plenitude of applications ranging from econometrics, chemistry, biology, speech recognition to neurophysiology. For example, transition rates between openings and closings of ion channels, see [1], are often assumed to be Markovian and the observed conductance levels from such experiments can be modeled with homogeneous HMMs. The HMM is typically justified if the underlying experimental conditions, such as the applied voltage in ion channel recordings, are kept constant over time, see [2, 3, 4, 5, 6].

However, if the conductance levels are measured in experiments with varying voltage over time, then the noise appears to be inhomogeneous, i.e., the noise has a voltage-dependent component. Such experiments play an important role in the understanding of the dependence of the gating behavior to the gradient of the applied voltage [7, 8]. To the best of our knowledge, there is a lack of a rigorous statistical methodology for analyzing such type of problems, for which we provide some first theoretical insights. More detailed, in this paper we are concerned with the consistency of the maximum likelihood estimator (MLE) in such models and with the question of how much maximum likelihood estimation in a homogeneous model is affected by inhomogeneity of the noise, a problem which appears to be relevant to many other situations, as well.

A *homogeneous hidden Markov model*, as considered in this paper, is given by a bivariate stochastic process $(X_n, Y_n)_{n \in \mathbb{N}}$, where $(X_n)_{n \in \mathbb{N}}$ is a Markov chain with

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finite state space S , and $(Y_n)_{n \in \mathbb{N}}$ is, conditioned on $(X_n)_{n \in \mathbb{N}}$, an independent sequence of random variables mapping to a Polish space G , such that the distribution of Y_n depends only on X_n . The Markov chain $(X_n)_{n \in \mathbb{N}}$ is not observable, but observations of $(Y_n)_{n \in \mathbb{N}}$ are available. A well known statistical method to estimate the unknown parameters is based on the maximum likelihood principle, see [9, 10]. The study of consistency and asymptotic normality of the MLE of such homogeneous HMMS has a long history and is nowadays well understood in quite general situations. We refer to the final paragraph of this section for a review but already mention that the approach of [11] is particularly useful for us.

In contrast to the classical setting, we consider an *inhomogeneous HMM*, namely a bivariate stochastic process $(X_n, Z_n)_{n \in \mathbb{N}}$, where conditioned on $(X_n)_{n \in \mathbb{N}}$ we assume that $(Z_n)_{n \in \mathbb{N}}$ is a sequence of independent random variables on space G , such that the distribution of Z_n depends not only on the value of X_n , but additionally on $n \in \mathbb{N}$. The dependence on n implies that the Markov chain $(X_n, Z_n)_{n \in \mathbb{N}}$ is inhomogeneous. In such generality a theory for maximum likelihood estimation in inhomogeneous hidden Markov models is, of course, a notoriously difficult task.

However, motivated by the example above (for details see below) we consider a specific situation where e.g. the inhomogeneity is caused by an exogenous quantity (e.g. the varying voltage) with decreasing influence as n increases. To this end, we introduce the concept of a *doubly hidden Markov model* (DHMM).

Definition 1 (DHMM). A doubly hidden Markov model is a trivariate stochastic process $(X_n, Y_n, Z_n)_{n \in \mathbb{N}}$ such that $(X_n, Y_n)_{n \in \mathbb{N}}$ is a non-observed homogeneous HMM and $(X_n, Z_n)_{n \in \mathbb{N}}$ is an inhomogeneous HMM with observations $(Z_n)_{n \in \mathbb{N}}$.

For such a DHMM we have in mind that the distribution of Z_n is getting “closer” to the distribution of Y_n for increasing n . A crucial point here is that $(Z_n)_{n \in \mathbb{N}}$ is observable whereas $(Y_n)_{n \in \mathbb{N}}$ is not. Because of the “proximity” of Z_n and Y_n one might hope to carry theoretical results from homogeneous HMMS to inhomogeneous ones.

We illustrate a setting of a DHMM by modeling the conductance level of ion channel data with varying voltage¹. In Figure 1 measurements of the current flow across the outer cell membrane of the porin PorB of *Neisseria meningitidis* are displayed in order to investigate the antibacterial resistance of the PorB channel. As the applied voltage $(u_n)_{n \in \mathbb{N}}$ increases linearly Ohm’s law suggests that the measured current increases also linearly, see Figure 1. A reasonable model for the observed current is to assume that it follows a Gaussian hidden Markov model, i.e., the dynamics can be described by

$$(1) \quad u_n(\mu^{(X_n)} + \sigma^{(X_n)}V_n) + \tilde{\varepsilon}_n.$$

Here the observation space $G = \mathbb{R}$ and the finite state space of the hidden Markov chain $(X_n)_{n \in \mathbb{N}}$ is assumed to be $S = \{1, 2\}$, which corresponds to an “open” and “closed” gate. For $i = 1, 2$, the expected slope is $\mu^{(i)} \in \mathbb{R}$, the noise level $\sigma^{(i)} \in (0, \infty)$ and $(V_n)_{n \in \mathbb{N}}$ is an i.i.d. standard normal sequence, i.e., $V_1 \sim \mathcal{N}(0, 1)$, where $\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. Further, $(\tilde{\varepsilon}_n)_{n \in \mathbb{N}}$ is another sequence of real-valued i.i.d. random variables, independent of $(V_n)_{n \in \mathbb{N}}$, with $\tilde{\varepsilon}_1 \sim \mathcal{N}(0, \kappa^2)$ and $\kappa^2 > 0$, which is necessary to model the background noise, even when $u_n = 0$.

¹Measurements are kindly provided by the lab of C. Steinem, Institute for Organic and Molecular Biochemistry, University of Göttingen

Dividing the dynamic (1) by u_n gives the conductivity of the channel, see Figure 2. This is now a sequence $(Z_n)_{n \in \mathbb{N}}$ of an inhomogeneous HMM. The state

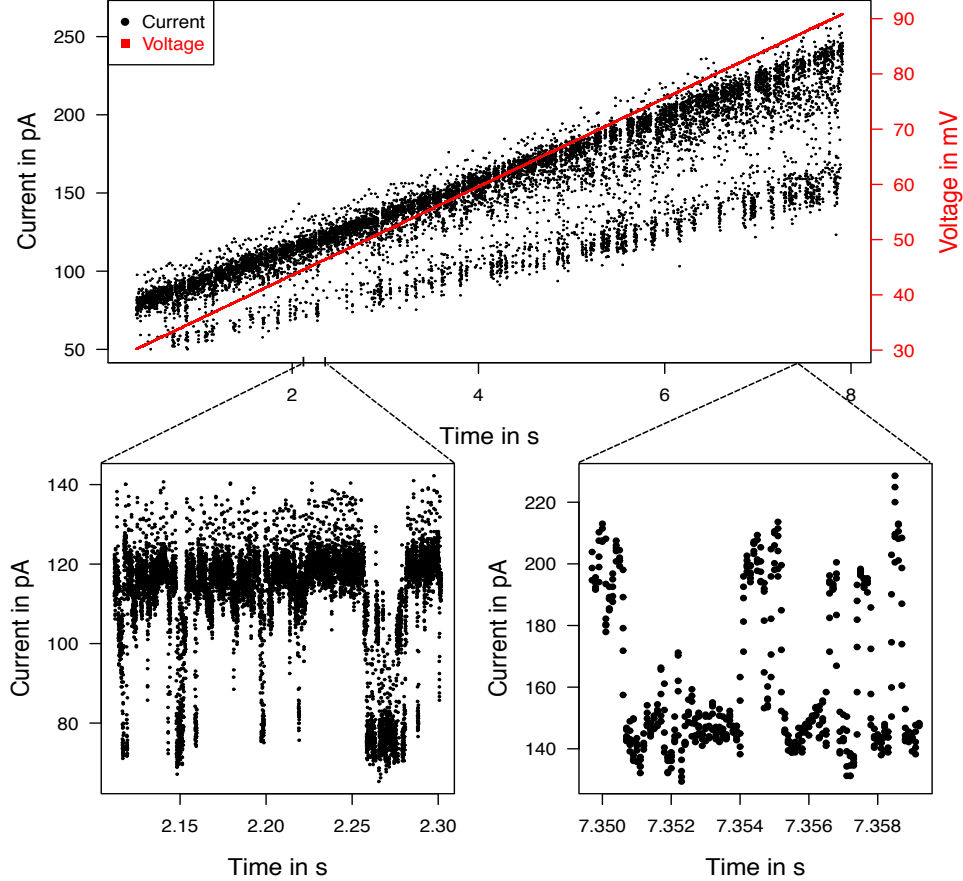


FIGURE 1. Above: Measurements at a large time scale (seconds) of the current flow of a PorB mutant protein driven by linear increasing voltage from $30mV$ - $120mV$. Below: Zoom into finer time scales (decisecond to millisecond).

of the Markov chain determines the parameter $(\mu^{(1)}, \sigma^{(1)})$ or $(\mu^{(2)}, \sigma^{(2)})$, both unknown. The non-observable sequence of random variables $(Y_n)_{n \in \mathbb{N}}$ of the homogeneous HMM is given by

$$(2) \quad Y_n := \mu^{(X_n)} + \sigma^{(X_n)} V_n.$$

The observation $(Z_n)_{n \in \mathbb{N}}$ of the inhomogeneous HMM is determined by

$$(3) \quad Z_n := Y_n + \varepsilon_n,$$

with $\varepsilon_n = \tilde{\varepsilon}_n / u_n$, such that $\varepsilon_n \sim \mathcal{N}(0, \beta_n^2)$ where $\beta_n = \kappa / u_n$ and $\lim_{n \rightarrow \infty} \beta_n^2 = 0$, as the voltage increases. Such a DHMM describes approximately the observed conductance level of ion channel recordings with linearly increasing voltage.

Intuitively, here one can already see that for sufficiently large n the influence of ε_n “washes out” as β_n decreases to zero and observations of Z_n are “close” to Y_n .

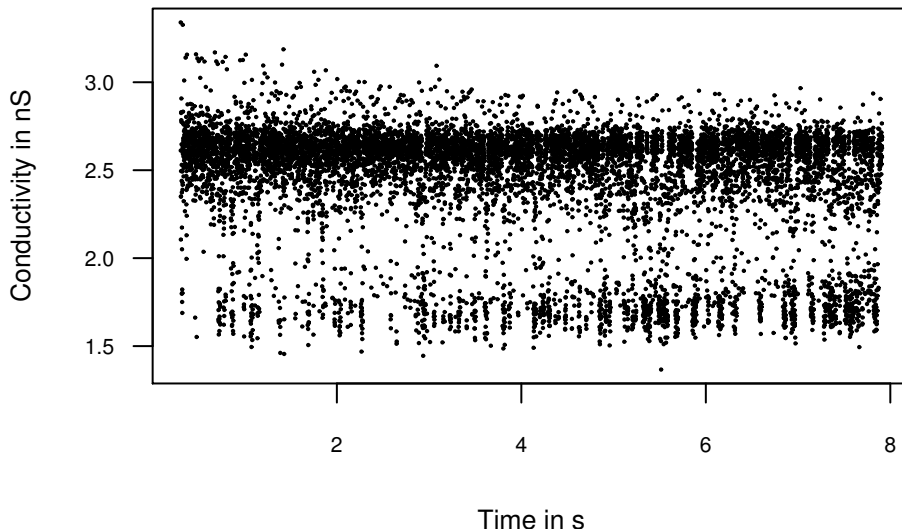


FIGURE 2. Conductivity of the protein PorB. The variance of the data decreases in time to a constant.

Main result. We explain now our main theoretical contribution for such a DHMM. Assume that we have a parametrized DHMM $(X_n, Y_n, Z_n)_{n \in \mathbb{N}}$ with compact parameter space $\Theta \subseteq \mathbb{R}^d$. For $\theta \in \Theta$ let q_θ^ν be the likelihood function of Y_1, \dots, Y_n and p_θ^ν be the likelihood function of Z_1, \dots, Z_n with $X_0 \sim \nu$. Both functions are assumed to be continuous in θ . Given observations z_1, \dots, z_n of Z_1, \dots, Z_n our goal is to estimate “the true” parameter $\theta^* \in \Theta$. The MLE $\theta_{\nu, n}^{\text{ML}}$, given by a parameter in the set of maximizers of the log-likelihood function, i.e.,

$$\theta_{\nu, n}^{\text{ML}} \in \operatorname{argmax}_{\theta \in \Theta} \log p_\theta^\nu(z_1, \dots, z_n),$$

is the canonical estimator for approaching this problem. Note that this set is non-empty due to the compactness of the parameter space and the continuity of $p_\theta^\nu(z_1, \dots, z_n)$ in θ . Unfortunately none of the strong consistency results of maximum likelihood parameter estimation provided for homogeneous HMMs are applicable, because of the inhomogeneity. Namely, all proofs for consistency in HMMs rely on the fact that the conditional distribution of Z_n given $X_n = x$ is constant for all $n \in \mathbb{N}$. In a DHMM this is usually not the case for $(Z_n)_{n \in \mathbb{N}}$, because of the time-dependent noise. This issue can be circumvented by proving that under suitable assumptions $(Z_n)_{n \in \mathbb{N}}$ is an *asymptotic mean stationary process*. This implies ergodicity and an ergodic theorem for $(Z_n)_{n \in \mathbb{N}}$, that can be used. However, for the computation of $\theta_{\nu, n}^{\text{ML}}$ explicit knowledge of the inhomogeneity is needed, i.e., of the time-dependent component of the noise which is hardly known in practice (recall our data example). That is the reason for us to introduce a quasi-maximum likelihood estimator (QMLE), given by a maximizer of the quasi-likelihood function, i.e.,

$$\theta_{\nu, n}^{\text{QML}} \in \operatorname{argmax}_{\theta \in \Theta} \log q_\theta^\nu(z_1, \dots, z_n).$$

This is not a MLE, since the observations are generated from the inhomogeneous model, whereas q_b^ν is the likelihood function of the homogeneous model. Roughly, we assume the following (for a precise definition see Section 3.1):

- 1.) The transition matrix of the hidden finite state space Markov chain is irreducible and satisfies a continuity condition w.r.t. the parameters.
- 2.) The observable and non-observable random variables $(Z_n)_{n \in \mathbb{N}}$ and $(Y_n)_{n \in \mathbb{N}}$ are “close” to each other in a suitable sense.
- 3.) The homogeneous HMM is well behaving, such that observations of $(Y_n)_{n \in \mathbb{N}}$ would lead to a consistent MLE.

We show that if the Z_n approximate the Y_n reasonably well (see the condition (C1) in Section 3.1) the estimator $\theta_{\nu,n}^{\text{QML}}$ provides also a reasonable way for approximating “the true” parameter θ^* . If the model satisfies all conditions, see Section 3.1, then Theorem 1, states that

$$\theta_{\nu,n}^{\text{QML}} \rightarrow \theta^* \quad \text{a.s., as } n \rightarrow \infty.$$

Hence the QMLE is consistent. As a consequence we obtain under an additional assumption that also the MLE is consistent, $\theta_{\nu,n}^{\text{ML}} \rightarrow \theta^*$ almost surely, as $n \rightarrow \infty$. For a Poisson model and linear Gaussian model we specify Theorem 1, see Section 4. In the DHMM described in (2) and (3) we obtain consistency of the QMLE whenever $\beta_n = \mathcal{O}(n^{-q})$ for some $q > 0$. In Section 5 we reconsider the approximating condition 2.), precisely stated in Section 3.1, provide an outlook to possible extensions and discuss asymptotic normality of the estimators.

Literature review and connection to our work. The study of maximum likelihood estimation in homogeneous hidden Markov models has a long history and was initiated by Baum and Petrie, see [9, 10], who proved strong consistency of the MLE for finite state spaces S and G . Leroux extends this result to general observation spaces in [12]. These consistency results rely on ergodic theory for stationary processes which is not applicable in our setting since the process we observe is not stationary. More precisely, it was shown that the relative entropy rate converges for any parameter θ in the parameter space Θ using an ergodic theorem for sub-additive processes. There are further extensions also to Markov chains on general state spaces, but under stronger assumptions, see [13, 14, 15, 16, 17]. A breakthrough has been achieved by Douc et al. [11] who used the concept of exponential separability. This strategy allows one to bound the relative entropy rate directly.

Although the state space of the Markov chain is more general than in our setting, we cannot apply the results of [11] due to the inhomogeneity of the observation, but we use the same approach to show our consistency statements.

The investigation of strong consistency of maximum likelihood estimation in inhomogeneous HMMS is less developed. In [18] and [19] the MLE in inhomogeneous Markov switching models is studied. There, the transition probabilities are also influenced by the observations, but the inhomogeneity there is different from the time-dependent inhomogeneity considered in our work, since the conditional law is not changing over time.

Related to strong consistency, as considered here, is the investigation of asymptotic normality (as it provides weak consistency). For homogeneous HMMS asymptotic normality has been shown for example in [14, 20]. In [19], also, asymptotic normality for the MLE in Markov switching models is studied whereas in [21] asymptotic normality of M -estimators in more general inhomogeneous situations is

considered. However, the QMLE we suggest and analyze does not satisfy the assumptions imposed there. In Section 5.4 and in Appendix B we provide and discuss necessary conditions to achieve asymptotic normality for the QMLE by adapting the approach of [21].

To ease readability Section 6 is devoted to the proofs of our main results. In particular, we draw the connection between asymptotic mean stationary processes and inhomogeneous hidden Markov models.

2. SETUP AND NOTATION

We denote the finite state space of $(X_n)_{n \in \mathbb{N}}$ by $S = \{1, \dots, K\}$ and \mathcal{S} denotes the power set of S . Furthermore, let (G, m) be a Polish space with metric m and corresponding Borel σ -field $\mathcal{B}(G)$. The measurable space $(G, \mathcal{B}(G))$ is equipped with a σ -finite reference measure λ . Throughout the whole work we consider parametrized families of DHMMs (see Definition 1) with compact parameter space $\Theta \subset \mathbb{R}^d$ for some $d \in \mathbb{N}$. For this let $(\mathbb{P}_\theta)_{\theta \in \Theta}$ be a sequence of probability measures on a measurable space (Ω, \mathcal{F}) such that for each parameter θ the distribution of $(X_n, Y_n, Z_n): (\Omega, \mathcal{F}, \mathbb{P}_\theta) \rightarrow S \times G \times G$ is specified by

- an initial distribution ν on S and a $K \times K$ transition matrix $P_\theta = (P_\theta(s, t))_{s, t \in S}$ of the Markov chain $(X_n)_{n \in \mathbb{N}}$, such that

$$\mathbb{P}_\theta(X_n = s) = \nu P_\theta^{n-1}(s), \quad s \in S,$$

where $\nu P_\theta^0 = \nu$ and for $n > 1$,

$$\nu P_\theta^{n-1}(s) = \sum_{s_1, \dots, s_{n-1} \in S} P_\theta(s_{n-1}, s) \prod_{i=1}^{n-2} P_\theta(s_i, s_{i+1}) \nu(s_1), \quad s \in S;$$

(Here and elsewhere we use the convention that $\prod_{i=1}^0 a_i = 1$ for any sequence $(a_i)_{i \in \mathbb{N}} \subset \mathbb{R}$.)

- and by the conditional distribution $Q_{\theta, n}$ of (Y_n, Z_n) given $X_n = s$, that is,

$$\mathbb{P}_\theta((Y_n, Z_n) \in C \mid X_n = s) = Q_{\theta, n}(s, C), \quad C \in \mathcal{B}(G^2)$$

which satisfies that there are conditional density functions $f_\theta, f_{\theta, n}: S \times G \rightarrow [0, \infty)$ w.r.t. λ , such that

$$\mathbb{P}_\theta(Y_n \in A \mid X_n = s) = Q_{\theta, n}(s, A \times G) = \int_A f_\theta(s, y) \lambda(dy), \quad A \in \mathcal{B}(G),$$

$$\mathbb{P}_\theta(Z_n \in B \mid X_n = s) = Q_{\theta, n}(s, G \times B) = \int_B f_{\theta, n}(s, z) \lambda(dz), \quad B \in \mathcal{B}(G).$$

Here the distribution of Y_n given $X_n = s$ is independent of n , whereas the distribution of Z_n given $X_n = s$ depends through $f_{\theta, n}$ also explicitly on n .

By $\mathcal{P}(S)$ we denote the set of probability measures on S . To indicate the dependence on the initial distribution, say $\nu \in \mathcal{P}(S)$, we write \mathbb{P}_θ^ν instead of just \mathbb{P}_θ . To shorten the notation, let $X = (X_n)_{n \in \mathbb{N}}$, $Y = (Y_n)_{n \in \mathbb{N}}$ and $Z = (Z_n)_{n \in \mathbb{N}}$. Further, let $\mathbb{P}_\theta^{\nu, Y}$ and $\mathbb{P}_\theta^{\nu, Z}$ be the distributions of Y and Z on $(G^\mathbb{N}, \mathcal{B}(G^\mathbb{N}))$, respectively.

The ‘‘true’’ underlying model parameter will be denoted as $\theta^* \in \Theta$ and we assume that the transition matrix P_{θ^*} possesses a unique invariant distribution $\pi \in \mathcal{P}(S)$. We have access to a finite length observation of Z . Then, the problem is to find a consistent estimate of θ^* on the basis of the observations without observing $(X_n, Y_n)_{n \in \mathbb{N}}$. Consistency of the estimator of θ^* is limited up to equivalence classes

in the following sense. Two parameters $\theta_1, \theta_2 \in \Theta$ are equivalent, written as $\theta_1 \sim \theta_2$, iff there exist two stationary distributions $\mu_1, \mu_2 \in \mathcal{P}(S)$ for $P_{\theta_1}, P_{\theta_2}$, respectively, such that $\mathbb{P}_{\theta_1}^{\mu_1, Y} = \mathbb{P}_{\theta_2}^{\mu_2, Y}$. For the rest of the work assume that each $\theta \in \Theta$ represents its equivalence class.

For an arbitrary finite measure ν on (S, \mathcal{S}) , $t \in \mathbb{N}$, $x_{t+1} \in S$ and $z_1, \dots, z_t \in G$ define

$$p_{\theta}^{\nu}(x_{t+1}; z_1, \dots, z_t) := \sum_{x_1, \dots, x_t \in S} \nu(x_1) \prod_{i=1}^t f_{\theta, i}(x_i, z_i) P_{\theta}(x_i, x_{i+1}),$$

$$p_{\theta}^{\nu}(z_1, \dots, z_t) := \sum_{x_{t+1} \in S} p_{\theta}^{\nu}(x_{t+1}; z_1, \dots, z_t).$$

If ν is a probability measure on (S, \mathcal{S}) , then $p_{\theta}^{\nu}(z_1, \dots, z_n)$ is the likelihood of the observations $(Z_1, \dots, Z_n) = (z_1, \dots, z_n) \in G^n$ for the inhomogeneous HMM $(X_n, Z_n)_{n \in \mathbb{N}}$ with parameter $\theta \in \Theta$ and $X_1 \sim \nu$. Although there are no observations of Y available, we define similar quantities for $(Y_1, \dots, Y_n) = (y_1, \dots, y_n) \in G^n$ by

$$q_{\theta}^{\nu}(x_{t+1}, y_1, \dots, y_t) := \sum_{x_1, \dots, x_t \in S} \nu(x_1) \prod_{i=1}^t f_{\theta}(x_i, y_i) P_{\theta}(x_i, x_{i+1}),$$

$$q_{\theta}^{\nu}(y_1, \dots, y_t) := \sum_{x_{t+1} \in S} q_{\theta}^{\nu}(x_{t+1}, y_1, \dots, y_t).$$

3. ASSUMPTIONS AND MAIN RESULT

Assume for a moment that observations y_1, \dots, y_n of Y_1, \dots, Y_n are available. Then the log-likelihood function of q_{θ}^{ν} , with initial distribution $\nu \in \mathcal{P}(S)$, is given by

$$\log q_{\theta}^{\nu}(y_1, \dots, y_n).$$

In our setting we do not have access to observations of Y , but have access to “contaminated” observations z_1, \dots, z_n of Z_1, \dots, Z_n . Based on these observations define a quasi-log-likelihood function

$$\ell_{\nu, n}^{\mathcal{Q}}(\theta) := \log q_{\theta}^{\nu}(z_1, \dots, z_n),$$

i.e., we plug the contaminated observations into the likelihood of Y_1, \dots, Y_n . Now we approximate θ^* by $\theta_{\nu, n}^{\text{QML}}$ which is the QMLE, that is,

$$(4) \quad \theta_{\nu, n}^{\text{QML}} \in \operatorname{argmax}_{\theta \in \Theta} \ell_{\nu, n}^{\mathcal{Q}}(\theta).$$

In addition, we are interested in the “true” MLE of a realization z_1, \dots, z_n of Z_1, \dots, Z_n . For this define the log-likelihood function

$$\ell_{\nu, n}(\theta) := \log p_{\theta}^{\nu}(z_1, \dots, z_n),$$

which leads to the MLE $\theta_{\nu, n}^{\text{ML}}$ given by

$$(5) \quad \theta_{\nu, n}^{\text{ML}} \in \operatorname{argmax}_{\theta \in \Theta} \ell_{\nu, n}(\theta).$$

Under certain structural assumptions we prove that the QMLE from (4) is consistent. By adding one more condition this result can be used to verify that the MLE from (5) is also consistent.

3.1. Structural conditions. We prove consistency of the QMLE $\theta_{\nu, n}^{\text{QML}}$ and the MLE $\theta_{\nu, n}^{\text{ML}}$ under the following structural assumptions:

Irreducibility and continuity of X .

- (P1) The transition matrix P_{θ^*} is irreducible.
- (P2) The parametrization $\theta \mapsto P_\theta$ is continuous.

Proximity of Y and Z .

- (C1) There exists $p > 1$ such that for any $s \in S$ and $\varepsilon > 0$ we have

$$\mathbb{P}_{\theta^*}(m(Z_n, Y_n) \geq \varepsilon \mid X_n = s) = \mathcal{O}(n^{-p}).$$

(Recall that m is the metric on G .)

- (C2) There exists an integer $k \in \mathbb{N}$ such that

$$(6) \quad \mathbb{P}_{\theta^*}^{\pi} \left(\prod_{i=1}^{k-1} \max_{s \in S} \frac{f_{\theta^*,i}(s, Z_i)}{f_{\theta^*}(s, Z_i)} < \infty \right) = 1,$$

$$\mathbb{E}_{\theta^*}^{\pi} \left[\max_{s' \in S} \frac{f_{\theta^*,n}(s', Z_n)}{f_{\theta^*}(s', Z_n)} \mid X_n = s \right] < \infty, \quad \forall s \in S, n \geq k,$$

and

$$(7) \quad \limsup_{n \rightarrow \infty} \mathbb{E}_{\theta^*}^{\pi} \left[\max_{s' \in S} \frac{f_{\theta^*,n}(s', Z_n)}{f_{\theta^*}(s', Z_n)} \mid X_n = s \right] \leq 1, \quad \forall s \in S.$$

- (C3) For every $\theta \in \Theta$ with $\theta \not\sim \theta^*$, there exists a neighborhood \mathcal{E}_θ of θ such that there exists an integer $k \in \mathbb{N}$ with

$$(8) \quad \mathbb{P}_{\theta^*}^{\pi} \left(\prod_{i=1}^{k-1} \sup_{\theta' \in \mathcal{E}_\theta} \max_{s \in S} \frac{f_{\theta',i}(s, Z_i)}{f_{\theta'}(s, Z_i)} < \infty \right) = 1,$$

$$\mathbb{E}_{\theta^*}^{\pi} \left[\sup_{\theta' \in \mathcal{E}_\theta} \max_{s' \in S} \frac{f_{\theta',n}(s', Z_n)}{f_{\theta'}(s', Z_n)} \mid X_n = s \right] < \infty, \quad \forall s \in S, n \geq k,$$

and

$$(9) \quad \lim_{n \rightarrow \infty} \left(\mathbb{E}_{\theta^*}^{\pi} \left[\sup_{\theta' \in \mathcal{E}_\theta} \max_{s' \in S} \frac{f_{\theta',n}(s', Z_n)}{f_{\theta'}(s', Z_n)} \mid X_n = s \right] \right) = 1, \quad \forall s \in S.$$

Remark 1. (C1) guarantees in particular that $m(Z_n, Y_n)$ converges \mathbb{P}_{θ^*} -a.s. to zero whereas (C2) ensures that the ratio of $p_{\theta^*}^{\nu}(z_1, \dots, z_n)$ and $q_{\theta^*}^{\nu}(z_1, \dots, z_n)$ does not diverge exponentially or faster. Assumption (C3) is needed to carry over the consistency of the QMLE to the MLE. In particular it implies that for all $\theta \not\sim \theta^*$ the ratio of $p_{\theta}^{\nu}(z_1, \dots, z_n)$ and $q_{\theta}^{\nu}(z_1, \dots, z_n)$ does not diverge exponentially or faster uniformly in \mathcal{E}_θ .

Well behaving HMM. It is plausible that we are only able to prove consistency in the case where the unobservable sequence Y would lead to a consistent estimator of θ^* , itself. To guarantee that this is indeed the case we assume:

- (H1) For all $s \in S$ let $\mathbb{E}_{\theta^*}^{\pi} [|\log f_{\theta^*}(s, Y_1)|] < \infty$.
- (H2) For every $\theta \in \Theta$ with $\theta \not\sim \theta^*$, there exists a neighborhood \mathcal{U}_θ of θ such that

$$\mathbb{E}_{\theta^*}^{\pi} \left[\sup_{\theta' \in \mathcal{U}_\theta} (\log f_{\theta'}(s, Y_1))^+ \right] < \infty \quad \text{for all } s \in S.$$

- (H3) The mappings $\theta \mapsto f_\theta(s, y)$ and $\theta \mapsto f_{\theta,n}(s, y)$ are continuous for any $s \in S$, $n \in \mathbb{N}$ and $y \in G$.
- (H4) For all $s \in S$ and $n \in \mathbb{N}$ let $\mathbb{E}_{\theta^*}^{\pi} [|\log f_{\theta^*,n}(s, Z_n)|] < \infty$.

Remark 2. *The conditions (H1)–(H3) coincide with the assumptions in [11, Sect. 3.2.] for finite state models and guarantee that the MLE for θ^* based on observations of Y is consistent. The condition (H4) is an additional regularity assumption required for the inhomogeneous setting.*

3.2. Consistency theorem. Now we formulate our main results about the consistency of the QMLE and the MLE.

Theorem 1. *Assume that the irreducibility and continuity conditions (P1), (P2), the proximity conditions (C1), (C2) and the well behaving HMM conditions (H1)–(H4) are satisfied. Further, let the initial distribution $\nu \in \mathcal{P}(S)$ be strictly positive if and only if π is strictly positive. Then*

$$\theta_{\nu,n}^{\text{QML}} \rightarrow \theta^*, \quad \mathbb{P}_{\theta^*}^{\pi}\text{-a.s.}$$

as $n \rightarrow \infty$.

Note that condition (C3) is not required in the previous statement. We only need it to prove the consistency of the MLE $\theta_{\nu,n}^{\text{ML}}$.

Corollary 1. *Assume that the setting and conditions of Theorem 1 and (C3) are satisfied. Then*

$$\theta_{\nu,n}^{\text{ML}} \rightarrow \theta^*, \quad \mathbb{P}_{\theta^*}^{\pi}\text{-a.s.}$$

as $n \rightarrow \infty$.

4. APPLICATION

We consider two models where we explore the structural assumptions from Section 3.1 explicitly. The Poisson model, see Section 4.1, illustrates a simple example with countable observation space. The linear Gaussian model is an extension of the model introduced in (1) and (2) to multivariate and possibly correlated observations.

4.1. Poisson DHMM. For $i = 1, \dots, K$ let $\lambda_{\theta^*}^{(i)} > 0$ and define the vector $\lambda_{\theta^*} = (\lambda_{\theta^*}^{(1)}, \dots, \lambda_{\theta^*}^{(K)})$. Conditioned on X the non-observed homogeneous sequence $Y = (Y_n)_{n \in \mathbb{N}}$ is an independent sequence of Poisson-distributed random variables with parameter $\lambda_{\theta^*}^{(X_n)}$. In other words, given $X_n = x_n$ we have $Y_n \sim \text{Poi}(\lambda_{\theta^*}^{(x_n)})$. Here $\text{Poi}(\alpha)$ denotes the Poisson distribution with expectation $\alpha > 0$. The observed sequence $Z = (Z_n)_{n \in \mathbb{N}}$ is determined by

$$Z_n = Y_n + \varepsilon_n,$$

where $(\varepsilon_n)_{n \in \mathbb{N}}$ is an independent sequence of random variables with $\varepsilon_n \sim \text{Poi}(\beta_n)$. Here $(\beta_n)_{n \in \mathbb{N}}$ is a sequence of positive real numbers satisfying for some $p > 1$ that

$$(10) \quad \beta_n = \mathcal{O}(n^{-p}).$$

We also assume that $(\varepsilon_n)_{n \in \mathbb{N}}$ is independent of Y and that the parameter θ determines the transition matrix P_{θ} and the intensity λ_{θ} continuously. Note that the observation space is given by $G = \mathbb{N} \cup \{0\}$ equipped with the counting measure λ . Figure 3 illustrates the empirical mean square error of approximations of the MLEs.

To obtain the desired consistency of the two estimators we need to check the conditions (P1), (P2), (C1)–(C3) and (H1)–(H4):

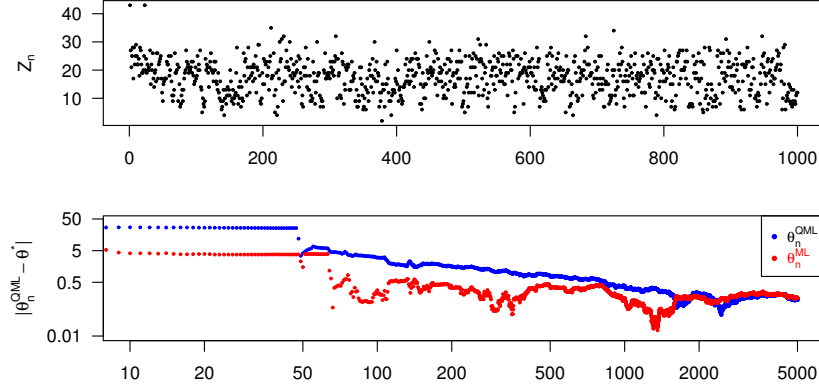


FIGURE 3. Exemplary trajectory of observations of the Poisson model from Section 4.1 (above) and Euclidean norm of the difference of θ^* and the estimators based on a single trajectory of observations (below). Here $K = 2$, $n = 5 \cdot 10^3$ and $\theta^* = (10, 20, 0.8, 0.1)$. The parameter θ^* determines $\lambda_{\theta^*} = (10, 20)$ and the “true” transition matrix by $P_{\theta^*}(1, 1) = 0.8$, $P_{\theta^*}(2, 1) = 0.1$. The inhomogeneous noise is driven by an intensity $\beta_n = 40 n^{-1.01}$.

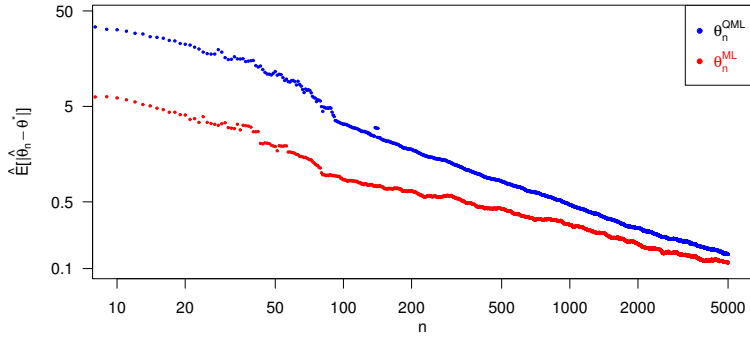


FIGURE 4. Empirical mean of the Euclidean norm of the difference of θ^* and the estimators based on 100 i.i.d. replications of the DHMM. Here $K = 2$, $n = 5 \cdot 10^3$ and $\theta^* = (10, 20, 0.8, 0.1)$. The parameter θ^* determines $\lambda_{\theta^*} = (10, 20)$ and the “true” transition matrix by $P_{\theta^*}(1, 1) = 0.8$, $P_{\theta^*}(2, 1) = 0.1$. The inhomogeneous noise is driven by an intensity $\beta_n = 40 n^{-1.01}$.

To (P1) and (P2): By the assumptions in this scenario those conditions are satisfied.

To (H1)–(H4): For $\theta \in \Theta$, $s \in S$ and $y \in G$ we have

$$\begin{aligned} |\log f_\theta(s, y)| &= -\log \left(\frac{(\lambda_\theta^{(s)})^y}{y!} \exp(-\lambda_\theta^{(s)}) \right) = -y \log(\lambda_\theta^{(s)}) + \log(y!) + \lambda_\theta^{(s)} \\ &\leq -y \log(\lambda_\theta^{(s)}) + y^2 + \lambda_\theta^{(s)}. \end{aligned}$$

Hence

$$\begin{aligned} &\mathbb{E}_{\theta^*}^\pi [|\log f_{\theta^*}(s, Y_1)|] \\ &\leq -\log(\lambda_{\theta^*}^{(s)}) \sum_{s=1}^K \pi(s) \lambda_{\theta^*}^{(s)} + \sum_{s=1}^K \pi(s) \left((\lambda_{\theta^*}^{(s)})^2 + \lambda_{\theta^*}^{(s)} \right) + \lambda_{\theta^*}^{(s)} < \infty \end{aligned}$$

and (H1) is verified. A similar calculation gives (H4). Condition (H2) follows simply by $(\log f_\theta(s, y))^+ = 0$. Condition (H3) follows by the continuity in the parameter of the probability function of the Poisson distribution and the continuity of the mapping $\theta \mapsto (P_\theta, \lambda_\theta)$.

To (C1) – (C3): For any $\delta > 0$ and any $s \in S$ we have

$$\mathbb{P}_\theta^\pi (|Z_n - Y_n| \geq \delta \mid X_n = s) = \mathbb{P}_\theta^\pi (|\varepsilon_n| \geq \delta) \leq 1 - \mathbb{P}_\theta^\pi (\varepsilon_n = 0) = 1 - \exp(-\beta_n).$$

From (10) it follows that

$$1 - \exp(-\beta_n) = \mathcal{O}(n^{-p}),$$

which proves (C1). Observe that for any $s \in S, z \in G$ we have

$$\max_{s \in S} \frac{f_{\theta^*, n}(s, z)}{f_{\theta^*}(s, z)} = \max_{s \in S} \frac{(\beta_n + \lambda_{\theta^*}^{(s)})^z}{(\lambda_{\theta^*}^{(s)})^z} \exp(-\beta_n) = (a_n)^z \exp(-\beta_n),$$

with $a_n = \max_{s \in S} \frac{\beta_n + \lambda_{\theta^*}^{(s)}}{\lambda_{\theta^*}^{(s)}}$. Now we verify (C2) with $k = 1$. For all $n \in \mathbb{N}$ and $s \in S$ we have

$$\begin{aligned} \mathbb{E}_{\theta^*}^\pi \left[\max_{s' \in S} \frac{f_{\theta^*, n}(s', Z_n)}{f_{\theta^*}(s', Z_n)} \mid X_n = s \right] &= \mathbb{E}_{\theta^*}^\pi [a_n^{Z_n} \exp(-\beta_n) \mid X_n = s] \\ &= \exp \left((\lambda_{\theta^*}^{(s)} + \beta_n)(a_n - 1) - \beta_n \right) < \infty. \end{aligned}$$

Fix $s \in S$, and note that

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \mathbb{E}_{\theta^*}^\pi \left[\max_{s' \in S} \frac{f_{\theta^*, n}(s', Z_n)}{f_{\theta^*}(s', Z_n)} \mid X_n = s \right] \\ &= \limsup_{n \rightarrow \infty} \exp \left((\lambda_{\theta^*}^{(s)} + \beta_n)(a_n - 1) - \beta_n \right) = 1. \end{aligned}$$

The last equality follows by the fact that $\lim_{n \rightarrow \infty} a_n = 1$ and $\lim_{n \rightarrow \infty} \beta_n = 0$. Condition (C3) follows by similar arguments.

The application of Theorem 1 and Corollary 1 leads to the following result.

Corollary 2. *For any initial distribution $\nu \in \mathcal{P}(S)$ which is strictly positive if and only if π is strictly positive, we have for the Poisson DHMM if (10) holds for some $p > 1$ that*

$$\theta_{\nu, n}^{\text{QML}} \rightarrow \theta^*, \quad \mathbb{P}_{\theta^*}^\pi \text{-a.s.}$$

and

$$\theta_{\nu,n}^{\text{ML}} \rightarrow \theta^*, \quad \mathbb{P}_{\theta^*}^{\pi} \text{-a.s.}$$

as $n \rightarrow \infty$.

4.2. Multivariate linear Gaussian DHMM. For $i = 1, \dots, K$ let $\mu_{\theta^*}^{(i)} \in \mathbb{R}^M$, $\Sigma_{\theta^*}^{(i)} \in \mathbb{R}^{M \times M}$ with full rank, where $M \in \mathbb{N}$. Define $\mu_{\theta^*} = (\mu_{\theta^*}^{(1)}, \dots, \mu_{\theta^*}^{(K)})$ as well as $\Sigma_{\theta^*} = (\Sigma_{\theta^*}^{(1)}, \dots, \Sigma_{\theta^*}^{(K)})$. The sequences $Y = (Y_n)_{n \in \mathbb{N}}$ and $Z = (Z_n)_{n \in \mathbb{N}}$ are defined by

$$\begin{aligned} Y_n &= \mu_{\theta^*}^{(X_n)} + \Sigma_{\theta^*}^{(X_n)} V_n \\ Z_n &= Y_n + \varepsilon_n. \end{aligned}$$

Here $(V_n)_{n \in \mathbb{N}}$ is an i.i.d. sequence of random vectors with $V_n \sim \mathcal{N}(0, I)$, where $I \in \mathbb{R}^{M \times M}$ denotes the identity matrix, and $(\varepsilon_n)_{n \in \mathbb{N}}$ is a sequence of independent random vectors with $\varepsilon_n \sim \mathcal{N}(0, \beta_n^2 I)$, where $(\beta_n)_{n \in \mathbb{N}}$ is a positive real-valued sequence satisfying for some $q > 0$ that

$$(11) \quad \beta_n = \mathcal{O}(n^{-q}).$$

Here we also assume that the mapping $\theta \mapsto (P_\theta, \mu_\theta, \Sigma_\theta)$ is continuous. Furthermore, note that $G = \mathbb{R}^M$ and λ is the M -dimensional Lebesgue measure. Figures 5 illustrates the empirical mean square error of approximations of the MLEs.

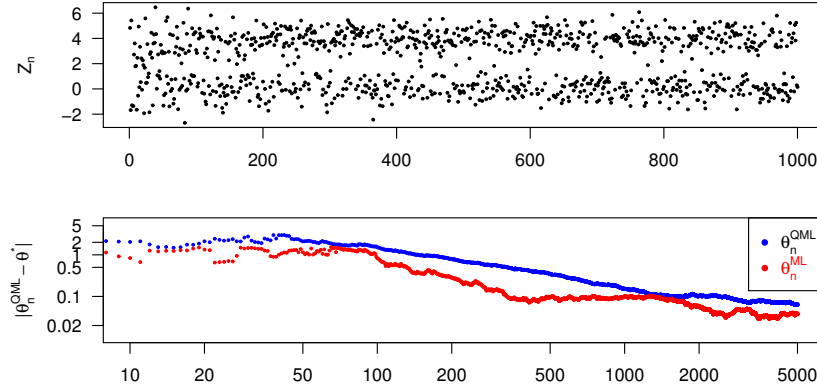


FIGURE 5. Exemplary trajectory of observations of the linear Gaussian model from Section 4.2 (above) and Euclidean norm of the difference of θ^* and the estimators based on a single trajectory of observations (below). Here $M = 1$, $K = 2$, $n = 5 \cdot 10^3$ and $\theta^* = (0, 4, 0.5, 0.5, 0.4, 0.5)$. The parameter θ^* determines $\mu_{\theta^*} = (0, 4)$, $\Sigma_{\theta^*} = (0.5, 0.5)$ and the “true” transition matrix by $P_{\theta^*}(1, 1) = 0.4$, $P_{\theta^*}(2, 1) = 0.5$. The inhomogeneous noise is driven by an intensity $\beta_n = 10 n^{-0.75}$.

To obtain consistency of the two estimators we need to check the conditions (P1), (P2), (C1)–(C3) and (H1)–(H4):

To (P1) and (P2): By definition of the model this conditions are satisfied.

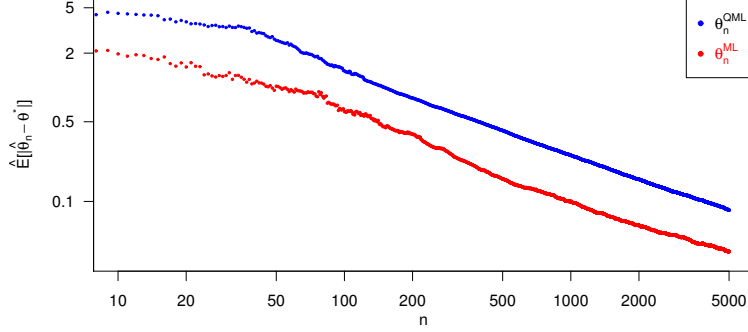


FIGURE 6. Empirical mean of the Euclidean norm of the difference of θ^* and the estimators norm based on 100 i.i.d. replications of the DHMM. Here $M = 1$, $K = 2$, $n = 5 \cdot 10^3$ and $\theta^* = (0, 4, 0.5, 0.5, 0.4, 0.5)$. The parameter θ^* determines $\mu_{\theta^*} = (0, 4)$, $\Sigma_{\theta^*} = (0.5, 0.5)$ and the “true” transition matrix by $P_{\theta^*}(1, 1) = 0.4$, $P_{\theta^*}(2, 1) = 0.5$. The inhomogeneous noise is driven by an intensity $\beta_n = 10 n^{-0.75}$.

To (H1)–(H4): For a matrix $A \in \mathbb{R}^{M \times M}$ denote $A^2 = AA^T$ and $A^{-2} = (A^2)^{-1}$. Note that for $s \in S$, $\theta \in \Theta$ and $y, z \in G$ we have by

$$f_{\theta}(s, y) = \frac{(2\pi)^{-M/2}}{\det\left(\left(\Sigma_{\theta}^{(s)}\right)^2\right)^{1/2}} \exp\left(-\frac{1}{2}(y - \mu_{\theta}^{(s)})^T \left(\Sigma_{\theta}^{(s)}\right)^{-2} (y - \mu_{\theta}^{(s)})\right),$$

$$f_{\theta, n}(s, z) = \frac{(2\pi)^{-M/2}}{\det\left(\left(\Sigma_{\theta}^{(s)}\right)^2 + \beta_n^2 I\right)^{1/2}} \times \exp\left(-\frac{1}{2}(z - \mu_{\theta}^{(s)})^T \left(\left(\Sigma_{\theta}^{(s)}\right)^2 + \beta_n^2 I\right)^{-1} (z - \mu_{\theta}^{(s)})\right).$$

Further, observe that $\det\left(\left(\Sigma_{\theta}^{(s)}\right)^2\right) > 0$ for all $s \in S$. For some constant $C_1 > 0$ we have

$$\mathbb{E}_{\theta^*}^{\pi} [|\log f_{\theta}(s, Y_1)|] \leq C_1 + \mathbb{E}_{\theta^*}^{\pi} \left[\frac{1}{2} (Y_1 - \mu_{\theta}^{(s)})^T \left(\Sigma_{\theta}^{(s)}\right)^{-2} (Y_1 - \mu_{\theta}^{(s)}) \right] < \infty,$$

since for each $i, j \in \{1, \dots, M\}$ we have $\mathbb{E}_{\theta^*}^{\pi} [Y_1^{(i)} Y_1^{(j)}] < \infty$ with the notation $Y_1 = (Y_1^{(1)}, \dots, Y_1^{(M)})$. By this estimate (H1) and (H2) follows easily. Condition (H4) follows by similar arguments. More detailed, we have that β_n^2 is finite and converges to zero as well, as that there exists a constant $C_2 > 0$ such that

$$\mathbb{E}_{\theta^*}^{\pi} [|\log f_{\theta^*, n}(s, Z_n)|] \leq C_2 + \mathbb{E}_{\theta^*}^{\pi} \left[\frac{1}{2} (Z_n - \mu_s)^T \left(\left(\Sigma_{\theta}^{(s)}\right)^2 + \beta_n^2 I\right)^{-1} (Z_n - \mu_s) \right].$$

For all $n \in \mathbb{N}$ the right-hand side of the previous inequality is finite, since for each $i, j \in \{1, \dots, M\}$ we have $\mathbb{E}_{\theta^*}^\pi \left[Z_n^{(i)} Z_n^{(j)} \right] < \infty$, with $Z_n = (Z_n^{(1)}, \dots, Z_n^{(M)})$. Finally condition (H3) is satisfied by the continuity of the conditional density and the continuity of the mapping $\theta \mapsto (P_\theta, \mu_\theta, \Sigma_\theta)$.

To (C1) – (C3): Here m is the Euclidean metric in \mathbb{R}^M such that $|\varepsilon_n| = m(Y_n, Z_n)$. Fix some $r > 0$ with $r/q > 1$ and observe that for any $\delta > 0$ and $s \in S$ we have

$$\mathbb{P}_{\theta^*}^\pi (m(Y_n, Z_n) > \delta \mid X_n = s) = \mathbb{P}_{\theta^*}^\pi (|\varepsilon_n| > \delta) = \mathbb{P}_{\theta^*}^\pi (\beta_n^r |N|^r > \delta^r) \leq \frac{\mathbb{E}_{\theta^*}^\pi [|N|^r] \beta_n^r}{\delta^r},$$

where $N \sim \mathcal{N}(0, I)$. By the fact that $\mathbb{E}_{\theta^*}^\pi [|N|^r] < \infty$ and (11) we obtain that condition (C1) is satisfied with $p = r/q > 1$.

The requirement of (6) of (C2) holds for any $k \in \mathbb{N}$, since the density of normally distributed random vectors is strictly positive and finite. Observe that

$$\max_{s \in S} \frac{f_{\theta, n}(s, Z_n)}{f_\theta(s, Z_n)} \leq C_n \max_{s \in S} \exp \left(-\frac{1}{2} (Z_n - \mu_\theta^{(s)})^T \left((\Sigma_\theta^{(s)})^2 + \beta_n^2 I \right)^{-1} - (\Sigma_\theta^{(s)})^{-2} \right) (Z_n - \mu_\theta^{(s)}),$$

with

$$C_n := \max_{s \in S} \frac{\left(\det \left((\Sigma_\theta^{(s)})^2 \right) \right)^{1/2}}{\left(\det \left((\Sigma_\theta^{(s)})^2 + \beta_n^2 I \right) \right)^{1/2}}.$$

Note that $\lim_{n \rightarrow \infty} C_n = 1$. Since for an invertible matrix $A \in \mathbb{R}^{M \times M}$, $A \mapsto A^{-1}$ is continuous and $\Sigma_{\theta^*}^s$ has full rank, it follows that

$$\lim_{n \rightarrow \infty} \left((\Sigma_\theta^{(s)})^2 + \beta_n^2 I \right)^{-1} = (\Sigma_\theta^{(s)})^{-2}.$$

f Set $(\Sigma_\theta^{(s)})_n^2 := (\Sigma_\theta^{(s)})^2 + \beta_n^2 I$ and define $B_n = B_{n, s} := (\Sigma_\theta^{(s)})^{-2} - (\Sigma_\theta^{(s)})_n^{-2}$. Note that the entries of B_n converge to zero when n goes to infinity.

Further, by the fact that $(B_n)_{n \in \mathbb{N}}$ is a sequence of symmetric, positive definite matrices there exist sequences of orthogonal matrices $(U_n)_{n \in \mathbb{N}} \subset \mathbb{R}^{M \times M}$ and diagonal matrices $(D_n)_{n \in \mathbb{N}} \subset \mathbb{R}^{M \times M}$ such that

$$B_n = U_n^T D_n^{1/2} D_n^{1/2} U_n.$$

Of course, U_n and D_n depend on s . We define a sequence of random vectors $(W_{n, s})_{n \in \mathbb{N}}$ by setting $W_{n, s} := U_n D_n^{1/2} (Z_n - \mu_\theta^{(s)})$, such that

$$\begin{aligned} & (Z_n - \mu_\theta^{(s)})^T \left((\Sigma_\theta^{(s)})^{-2} - ((\Sigma_\theta^{(s)})^2 + \beta_n^2 I)^{-1} \right) (Z_n - \mu_\theta^{(s)}) \\ &= (Z_n - \mu_\theta^{(s)})^T B_n (Z_n - \mu_\theta^{(s)}) = W_{n, s}^T W_{n, s}. \end{aligned}$$

The random variable Z_i conditioned on $X_i = x$ is normally distributed with mean $\mu_\theta^{(x)}$ and covariance matrix $(\Sigma_\theta^{(x)})_n^2$. Hence $W_{i, s}$, conditioned on $X_i = x$, satisfies

$$W_{i, s} \sim \mathcal{N}(\tilde{\mu}_i, A_i),$$

with

$$\tilde{\mu}_i = U_i^T D_i^{1/2} (\mu_\theta^{(x)} - \mu_\theta^{(s)})$$

and

$$A_i = U_i^T D_i^{1/2} (\Sigma_\theta^{(x)})_i^2 (U_i^T D_i^{1/2})^T.$$

Since A_i is symmetric and positive definite, we find sequences of orthogonal matrices $(U'_n)_{n \in \mathbb{N}}$ and diagonal matrices $(D'_n)_{n \in \mathbb{N}}$ depending on x and s such that

$$A_i = U'_i D_i'^{1/2} D_i'^{1/2} U_i'^T.$$

Let $(N_i)_{i \in \mathbb{N}}$ be an i.i.d. sequence of random vectors with $N_i \sim \mathcal{N}(0, I)$ and denote $N_i = (N_i^{(1)}, \dots, N_i^{(M)})$. Then

$$\begin{aligned} W_{i,s}^T W_{i,s} &= |W_{i,s}|^2 \stackrel{\mathcal{D}}{=} \left| U'_i D_i'^{1/2} (N_i + D_i'^{-1/2} U_i'^T \tilde{\mu}_i) \right|^2 \\ &= \left| D_i'^{1/2} (N_i + D_i'^{-1/2} U_i'^T \tilde{\mu}_i) \right|^2 = \sum_{j=1}^M D_i'(j, j) \left(N_i^{(j)} + (D_i'^{-1/2} U_i'^T \tilde{\mu}_i)^{(j)} \right)^2. \end{aligned}$$

Recall that for a chi-squared distribution with one degree of freedom and non-centrality parameter $\gamma > 0$ the moment generating function in t , with $t < 1/2$, is given by $\frac{\exp(\gamma t / (1-2t))}{(1-2t)^{1/2}}$. Hence, for any $t < \min_{j=1, \dots, M} D_i'(j, j)^{-1}$ with non-centrality parameter $(D_i'^{-1/2} U_i'^T \tilde{\mu}_i)^{(j)}$ at $\frac{t}{2} D_i'(j, j)$ it is well-defined and we obtain

$$\begin{aligned} &\mathbb{E}_{\theta^*}^\pi \left[\exp \left(\frac{t}{2} W_{i,s'}^T W_{i,s'} \right) \mid X_i = s \right] \\ &= \prod_{j=1}^M (1 - 2(\frac{t}{2}) D_i'(j, j))^{-1/2} \exp \left(\frac{(D_i'^{-1/2} U_i'^T \tilde{\mu}_i)^{(j)} (\frac{t}{2}) D_i'(j, j)}{1 - 2(\frac{t}{2}) D_i'(j, j)} \right) \\ &= \prod_{j=1}^M (1 - t D_i'(j, j))^{-1/2} \exp \left(\frac{(D_i'^{-1/2} U_i'^T \tilde{\mu}_i)^{(j)} (\frac{t}{2}) D_i'(j, j)}{1 - t D_i'(j, j)} \right) \rightarrow 1 \end{aligned}$$

as $i \rightarrow \infty$, since $\lim_{i \rightarrow \infty} D_i'(j, j) = 0$ for all $j = 1, \dots, M$. We can choose k sufficiently large, such that $K < \min_{j=1, \dots, M} D_i'(j, j)^{-1}$ for all $i \geq k$. We find that

$$\begin{aligned} &\mathbb{E}_{\theta^*}^\pi \left[\max_{s' \in S} \exp \left(\frac{1}{2} W_{k,s'}^T W_{k,s'} \right) \mid X_k = s \right] \leq \mathbb{E}_{\theta^*}^\pi \left[\sum_{s' \in S} \exp \left(\frac{1}{2} W_{k,s'}^T W_{k,s'} \right) \mid X_k = s \right] \\ &\leq \prod_{s' \in S} \left(\mathbb{E}_{\theta^*}^\pi \left[\exp \left(\frac{K}{2} W_{k,s'}^T W_{k,s'} \right) \mid X_k = s \right] \right)^{1/K}, \end{aligned}$$

where we used the generalized Hölder inequality in the last estimate. Then, by taking the limit superior we obtain that the right-hand side of the previous inequality goes to one for $k \rightarrow \infty$ such that (C2) holds. Condition (C3) can be verified similarly.

The application of Theorem 1 and Corollary 1 leads to the following result.

Corollary 3. *For any initial distribution $\nu \in \mathcal{P}(S)$ which is strictly positive if and only if π is strictly positive, we have for the multivariate Gaussian DHMM satisfying (11) for some $q > 0$ that*

$$\theta_{\nu, n}^{\text{QML}} \rightarrow \theta^*, \quad \mathbb{P}_{\theta^*}^\pi \text{-a.s.}$$

and

$$\theta_{\nu, n}^{\text{ML}} \rightarrow \theta^*, \quad \mathbb{P}_{\theta^*}^\pi \text{-a.s.}$$

as $n \rightarrow \infty$.

Remark 3. For $K = 2$ and $M = 1$ we have the model of the conductance level of ion channel data with varying voltage provided in the introduction, see Figure 1 and (2) and (3). The previous corollary states the desired consistency of the considered MLEs in that setting. A data analysis of the ion channel recordings of the underlying DHMM will be done in a separate paper.

5. DISCUSSION AND LIMITATIONS

In this section we discuss four aspects. First, having the models from Section 4 in mind, one might consider a hybrid case, that is, e.g. if the non-observed sequence Y is Poisson distributed and the inhomogeneous noise is normally distributed. We discuss where our approach fails here and provide a strategy how to resolve this issue. Second, one might ask whether the proximity assumptions formulated in Section 3.1 can be relaxed. We provide a simple example where (C1) is not satisfied and $\theta_{\nu,n}^{\text{QML}}$ is not consistent anymore. Third, we discuss the restriction of considering only hidden Markov chains on finite state spaces. Finally, we comment and discuss conditions which lead to asymptotic normality of the QMLE.

5.1. Hybrid model.

The hidden sequences X and Y of the DHMM are defined as in Section 4.1. The observed sequence $Z = (Z_n)_{n \in \mathbb{N}}$ is given by

$$Z_n = Y_n + \varepsilon_n,$$

where $(\varepsilon_n)_{n \in \mathbb{N}}$ is an independent sequence of random variables with $\varepsilon_n \sim \mathcal{N}(0, \beta_n^2)$ and a $(\beta_n)_{n \in \mathbb{N}} \subset (0, \infty)$ satisfies $\lim_{n \rightarrow \infty} \beta_n^2 = 0$. In other words, on the Poisson random variable Y_n we add Gaussian time-dependent noise.

The main issue is that the observed sequence Z takes values in \mathbb{R} whereas Y takes values in $\mathbb{N} \cup \{0\}$. Consider $G = \mathbb{R}$ equipped with the reference measure $\lambda(\cdot) = \mathcal{L}(\cdot) + \sum_{i=0}^{\infty} \delta_i(\cdot)$. Here $\mathcal{L}(\cdot)$ denotes the Lebesgue measure and $\delta_i(\cdot)$ the Dirac-measure at point $i \in \mathbb{N}$. The conditional density $f_{\theta,n}$ w.r.t. λ is given by

$$f_{\theta,n}(s, z) = \begin{cases} \sum_{j=0}^{\infty} \frac{\lambda_{\theta}^{(s)}}{j!} \exp(-\lambda_{\theta}^{(s)}) \frac{1}{(2\pi\beta_n^2)^{1/2}} \exp\left(-\frac{(z-j)^2}{2\beta_n^2}\right) & z \in \mathbb{R} \setminus \mathbb{N} \\ 0 & z \in \mathbb{N}. \end{cases}$$

One can verify that (C2) is not satisfied in this scenario. In general, assumption (C2) is difficult to handle, whenever the support of f_{θ} is strictly “smaller” than the support of $f_{\theta,n}$. We mention a possible strategy to resolve this problem:

- (1) Transform the observed sequence to a sequence $\tilde{Z} = (\tilde{Z}_n)_{n \in \mathbb{N}}$, such that the support of the corresponding conditional density coincides with the support of f_{θ} . For example, this might be done by rounding to the nearest natural number, that is, $\tilde{Z}_n = \lfloor Z_n + 0.5 \rfloor$.
- (2) Prove that the QMLE $\tilde{\theta}_{\nu,n}^{\text{QML}}$, based on \tilde{Z} , is consistent. (For example, by applying Theorem 1.)
- (3) Prove that $\theta_{\nu,n}^{\text{QML}} - \tilde{\theta}_{\nu,n}^{\text{QML}} \rightarrow 0$, $\mathbb{P}_{\theta^*}^{\pi}$ a.s. as $n \rightarrow \infty$.

A similar strategy might be used to obtain consistency for the MLE.

5.2. Proximity assumption. We show that in general one cannot weaken the proximity assumption from Section 3.1.

We provide an example, which does not satisfy (C1) and show that $\theta_{\nu,n}^{\text{QML}}$ is not strongly consistent for the approximation of θ^* .

Example 1. Consider the linear Gaussian model of Section 4.2 in the case $m = 1$ and $K = 1$ with $\theta^* = (0, 1)$. The parameter θ^* determines the mean $\mu_{\theta^*} = 0$ and the variance $\sigma_{\theta^*}^2 = 1$. Let $\beta = \lim_{n \rightarrow \infty} \beta_n > 0$ and $\theta_0 = (0, 1 + \beta^2)$. Note that

$$\varepsilon_n \xrightarrow{\mathcal{D}} N,$$

as $n \rightarrow \infty$, where $N \sim \mathcal{N}(0, \beta^2)$. This contradicts the conclusion of Lemma 1 below and therefore assumption (C1) is not satisfied. Further we have

$$\lim_{n \rightarrow \infty} \mathbb{E}_{\theta^*}^{\pi} [\log f_{\theta}(1, Z_n)] = -\frac{1}{2} \log(2\pi\sigma^2) - \frac{1 + \mu^2 + \beta^2}{2\sigma^2},$$

which implies that

$$\lim_{n \rightarrow \infty} \mathbb{E}_{\theta^*}^{\pi} [\log f_{\theta_0}(1, Z_n)] > \lim_{n \rightarrow \infty} \mathbb{E}_{\theta^*}^{\pi} [\log f_{\theta^*}(1, Z_n)].$$

For any $\theta \in \Theta$ we have that

$$n^{-1} \log q_{\theta}(Z_1, \dots, Z_n) = n^{-1} \sum_{i=1}^n \log f_{\theta}(1, Z_i) \rightarrow \lim_{n \rightarrow \infty} \mathbb{E}_{\theta^*}^{\pi} [\log f_{\theta}(1, Z_n)].$$

In fact, for any closed set $C \subset \Theta$ with $\theta_0 \notin C$ we have that

$$\lim_{n \rightarrow \infty} n^{-1} \log q'_{\theta_0}(Z_1, \dots, Z_n) > \limsup_{\theta \in C} \log q'_{\theta}(Z_1, \dots, Z_n)$$

and therefore $\theta_n^{\text{QML}} \rightarrow \theta_0$ a.s., see Lemma 7 and Theorem 6.

5.3. Finite state space of the hidden Markov chain. A generalization of the consistency results of maximum likelihood estimation to scenarios with general state space of the hidden Markov chain might be of interest. There are mainly two reasons why we assume that \mathcal{S} is finite:

- (1) Our main motivation comes from the DHMM which models the conductance levels of ion channel data with finite \mathcal{S} .
- (2) The requirements one needs to impose get more technical. In particular, our conditions on the “irreducibility and continuity of X ” as well as the “well behaving HMM” from Section 3.1 become more difficult on general state spaces. It seems that the assumptions (A1)-(A6) of [11] are sufficient, but then in the proof of Theorem 1 we cannot argue with Lemma 6 anymore. This lemma can also be adapted to the more general scenario as in [11, Lemma 13], but then involves an additional term.

5.4. Asymptotic normality of the QMLE. Under additional conditions one can obtain asymptotic normality of the MLE by applying the work of [21]. The requirements to obtain this result for the QMLE are similar. Namely, let $\theta_{\nu, n}^{\text{QML}}$ be strongly consistent, which is guaranteed under the assumptions of Theorem 1, and assume that

- the mixing condition (M),
- the CLT guaranteeing condition (CLT),
- as well as the uniform convergence condition (UC),

formulated in Appendix B, do hold. Then, one can prove

$$\sqrt{n} G_n^{-1/2} F_n(\theta_{\nu, n}^{\text{QML}} - \theta^*) \xrightarrow{\mathcal{D}} N,$$

where $N \sim \mathcal{N}(0, I)$, with the identity matrix $I \in \mathbb{R}^{d \times d}$,

$$\begin{aligned} G_n &:= \frac{1}{n} \text{Cov}_{\theta^*}^{\pi} (S_n(\theta^*)), \\ F_n &:= -\frac{1}{n} \mathbb{E}_{\theta^*}^{\pi} \left[\left(\frac{\partial}{\partial \theta'} S_n(\theta') \Big|_{\theta'=\theta^*} \right)^T \right], \\ S_n(\theta) &:= \frac{\partial}{\partial \theta'} \log q_{\theta'}^{\nu} (Z_1, \dots, Z_n) \Big|_{\theta'=\theta}, \end{aligned}$$

and the covariance matrix of $S_n(\theta^*)$ denoted by $\text{Cov}_{\theta^*}^{\pi} (S_n(\theta^*))$. The proof of this fact is technical and follows the approach of [21] by applying additional non-trivial arguments. The main issue of the result is the condition

$$(12) \quad \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} |\mathbb{E}_{\theta^*}^{\pi} (S_n(\theta^*))|_1 = 0,$$

formulated in (CLT) in Appendix B with $|\cdot|_1$ being the ℓ_1 -norm. It guarantees that the limiting distribution of S_n has mean zero, which is automatically satisfied for the corresponding quantity of the MLE. Hence, the assumptions of [21] for asymptotic normality of the MLE simplify to (M), (CLT), (UC) of Appendix B, where q_{θ}^{ν} and Z_1, \dots, Z_n has to be replaced by p_{θ}^{ν} and Y_1, \dots, Y_n , respectively, without the requirement to check (12). However, for the QMLE the crucial problem is that we are unfortunately not able to verify (12) in the applications presented above.

6. PROOFS AND AUXILIARY RESULTS

We prove some results that specify the proximity of Y and Z .

Lemma 1. *Under the assumption formulated in (C1), we have*

$$(13) \quad \mathbb{P}_{\theta}^{\nu} \left(\lim_{n \rightarrow \infty} m(Z_n, Y_n) = 0 \right) = 1.$$

for any $\theta \in \Theta$ and $\nu \in \mathcal{P}(S)$.

Proof. By (C1) we obtain for any $\varepsilon > 0$ that

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{P}_{\theta}^{\nu} (m(Z_n, Y_n) \geq \varepsilon) &= \sum_{n=1}^{\infty} \sum_{k=1}^K \mathbb{P}_{\theta}^{\nu} (m(Z_n, Y_n) \geq \varepsilon, X_n = k) \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^K \mathbb{P}_{\theta}^{\nu} (X_n = k) \mathbb{P}_{\theta} (m(Z_n, Y_n) \geq \varepsilon \mid X_n = k) \\ &\leq \sum_{n=1}^{\infty} \max_{k \in S} \mathbb{P}_{\theta} (m(Z_n, Y_n) \geq \varepsilon \mid X_n = k) < \infty. \end{aligned}$$

By the Borel-Cantelli lemma we obtain the desired almost sure convergence of $m(Z_n, Y_n)$ to zero. \square

In [11] the consistency of the maximum likelihood estimation for homogeneous HMMS under weak conditions is verified. We use the following result of them, which verifies that the relative entropy rate exists.

Theorem 2 ([11, Theorem 9]). *Assume that the conditions (P1) and (H1) are satisfied. Then, there exists an $\ell(\theta^*) \in \mathbb{R}$, such that*

$$(14) \quad \ell(\theta^*) = \lim_{n \rightarrow \infty} \mathbb{E}_{\theta^*}^{\pi} [n^{-1} \log q_{\theta^*}^{\pi} (Y_1, \dots, Y_n)]$$

and

$$(15) \quad \ell(\theta^*) = \lim_{n \rightarrow \infty} n^{-1} \log q_{\theta^*}^{\nu}(Y_1, \dots, Y_n), \quad \mathbb{P}_{\theta^*}^{\pi}\text{-a.s.}$$

for any probability measure $\nu \in \mathcal{P}(S)$ which is strictly positive if and only if π is strictly positive.

In the proof of the previous result one essentially uses the generalized Shannon-McMillan-Breiman theorem for stationary processes proven by Barron et. al in [22]. Additionally, we also use a version of the generalized Shannon-McMillan-Breiman theorem for asymptotic mean stationary processes, also proven in [22]. In the following we provide basic definitions to apply this result, for a detailed survey let us refer to [23].

Definition 2. Let (Ω, \mathcal{F}) be a measurable space equipped with a probability measure \mathbb{Q} and let $T: \Omega \rightarrow \Omega$ be a measurable mapping. Then

- \mathbb{Q} is *ergodic*, if for every $A \in \mathcal{I}$ either $\mathbb{Q}(A) = 0$ or $\mathbb{Q}(A) = 1$. Here \mathcal{I} denotes the σ -algebra of the invariant sets, that are, the sets $A \in \mathcal{F}$ satisfying $T^{-1}(A) = A$.
- \mathbb{Q} is called *asymptotically mean stationary* (a.m.s.) if there is a probability measure $\bar{\mathbb{Q}}$ on (Ω, \mathcal{F}) , such that for all $A \in \mathcal{F}$ we have

$$\frac{1}{n} \sum_{j=1}^n \mathbb{Q}(T^{-j}A) \xrightarrow{n \rightarrow \infty} \bar{\mathbb{Q}}(A).$$

We call $\bar{\mathbb{Q}}$ *stationary mean* of \mathbb{Q} .

- a probability measure $\hat{\mathbb{Q}}$ on (Ω, \mathcal{F}) *asymptotically dominates* \mathbb{Q} if for all $A \in \mathcal{F}$ with $\hat{\mathbb{Q}}(A) = 0$ holds

$$\lim_{n \rightarrow \infty} \mathbb{Q}(T^{-n}A) = 0.$$

We need the following equivalence from [24]. The result follows also by virtue of [25, Theorem 2, Theorem 3 and the remark after the proof of Theorem 3].

Lemma 2. Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space and $T: \Omega \rightarrow \Omega$ be a measurable mapping. Then, the following statements are equivalent:

- The probability measure \mathbb{Q} is a.m.s. with stationary mean $\bar{\mathbb{Q}}$.
- There is a stationary probability measure $\hat{\mathbb{Q}}$, which asymptotically dominates \mathbb{Q} .

In our inhomogeneous HMM situation (Ω, \mathcal{F}) is the space $G^{\mathbb{N}}$ equipped with the product σ -field $\mathcal{B} = \otimes_{i \in \mathbb{N}} \mathcal{B}(G)$. The transformation $T: G^{\mathbb{N}} \rightarrow G^{\mathbb{N}}$ is the left time shift, that is, for $A \in \mathcal{B}$ and $i \in \mathbb{N}$ we have

$$(16) \quad T^{-i}(A) = \{(z_1, z_2, \dots) \in G^{\mathbb{N}} : (z_{1+i}, z_{2+i}, \dots) \in A\}.$$

Finally $\mathbb{Q} = \mathbb{P}_{\theta^*}^{\pi, Z}$. In this setting we have the following result:

Theorem 3. Let us assume that condition (C1) is satisfied. Then $\mathbb{P}_{\theta^*}^{\pi, Z}$ is a.m.s. with stationary mean $\mathbb{P}_{\theta^*}^{\pi, Y}$.

Proof. An intersection-stable generating system of the σ -algebra \mathcal{B} is the union over any finite index set $J \subset \mathbb{N}$ of cylindrical set systems

$$\mathcal{Z}_J := \{\rho_J^{-1}(A_1 \times \dots \times A_{|J|}) \mid A_j \in \mathcal{B}(G) \text{ open}\},$$

where $\rho_J: G^{\mathbb{N}} \rightarrow G^{|J|}$ is the canonical projection to J , that is, $\rho_J((a_i)_{i \in \mathbb{N}}) = (a_j)_{j \in J}$. By the uniqueness theorem of finite measures it is sufficient to prove for an arbitrary finite index set $J \subset \mathbb{N}$ that for any $B \in \mathcal{Z}_J$ we have

$$(17) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \mathbb{P}_{\theta^*}^{\pi, Z}(T^{-i}(B)) = \mathbb{P}_{\theta^*}^{\pi, Y}(B).$$

Fix a finite index set $J = \{j_1, \dots, j_k\} \subset \mathbb{N}$ and note that $(G^{|J|}, m_J)$, with the metric

$$m_J(a, b) = \sum_{j=1}^{|J|} m(a_j, b_j), \quad a = (a_1, \dots, a_{|J|}), \quad b = (b_1, \dots, b_{|J|}) \in G^{|J|},$$

is a metric space. Here it is worth to mention that the σ -algebra $\bigotimes_{j \in J} \mathcal{B}(G)$ coincides with the σ -algebra generated by the open sets w.r.t. m_J . By Lemma 1 we obtain

$$(18) \quad \mathbb{P}_{\theta^*}^{\pi} \left(\lim_{i \rightarrow \infty} m_J((Y_{i+j_1}, \dots, Y_{i+j_k}), (Z_{i+j_1}, \dots, Z_{i+j_k})) = 0 \right) = 1.$$

Let $h: G^{|J|} \rightarrow \mathbb{R}$ be a bounded, uniformly continuous function, i.e., for any $\varepsilon > 0$ there is $\delta > 0$ such that for all $a, b \in G^{|J|}$ with $m_J(a, b) < \delta$ we have $|h(a) - h(b)| < \varepsilon$. Then, by the stationarity of Y , the boundedness of h and Fatou's lemma, we have

$$(19) \quad \begin{aligned} 0 &\leq \liminf_{i \rightarrow \infty} \mathbb{E}_{\theta^*}^{\pi} [|h(Z_{i+j_1}, \dots, Z_{i+j_k}) - h(Y_{j_1}, \dots, Y_{j_k})|] \\ &\leq \limsup_{i \rightarrow \infty} \mathbb{E}_{\theta^*}^{\pi} [|h(Z_{i+j_1}, \dots, Z_{i+j_k}) - h(Y_{i+j_1}, \dots, Y_{i+j_k})|] \\ &\leq \mathbb{E}_{\theta^*}^{\pi} \left[\limsup_{i \rightarrow \infty} |h(Z_{i+j_1}, \dots, Z_{i+j_k}) - h(Y_{i+j_1}, \dots, Y_{i+j_k})| \right]. \end{aligned}$$

By the uniform continuity of h we obtain

$$\lim_{i \rightarrow \infty} |h(z_{i+j_1}, \dots, z_{i+j_k}) - h(y_{i+j_1}, \dots, y_{i+j_k})| = 0$$

for all sequences $((z_{i+j_1}, \dots, z_{i+j_k})_{i \in \mathbb{N}}, ((y_{i+j_1}, \dots, y_{i+j_k})_{i \in \mathbb{N}}) \subset G^{|J|}$ which satisfy

$$\lim_{i \rightarrow \infty} m_J((z_{i+j_1}, \dots, z_{i+j_k}), (y_{i+j_1}, \dots, y_{i+j_k})) = 0.$$

Then, by using (18) we obtain

$$\mathbb{E}_{\theta^*}^{\pi} \left[\limsup_{i \rightarrow \infty} |h(Z_{i+j_1}, \dots, Z_{i+j_k}) - h(Y_{i+j_1}, \dots, Y_{i+j_k})| \right] \leq 0,$$

such that (by (19)) we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{\theta^*}^{\pi} [h(Z_{i+j_1}, \dots, Z_{i+j_k})] = \mathbb{E}_{\theta^*}^{\pi} [h(Y_{j_1}, \dots, Y_{j_k})].$$

Finally, by [26, Theorem 1.2] we have for any $A \in \bigotimes_{j \in J} \mathcal{B}(G)$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \mathbb{P}_{\theta^*}^{\pi} ((Z_{i+j_1}, \dots, Z_{i+j_k}) \in A) = \mathbb{P}_{\theta^*}^{\pi} ((Y_{j_1}, \dots, Y_{j_k}) \in A),$$

which implies (17) for any $B \in \mathcal{Z}_J$. \square

Apart of the fact that we need the previous result to apply [22, Theorem 3] it has also the following two useful consequences.

Corollary 4. *Assume that condition (C1) is satisfied. Then $\mathbb{P}_{\theta^*}^{\pi, Z}$ is ergodic.*

Proof. From [12, Lemma 1] it follows that $\mathbb{P}_{\theta^*}^{\pi, Y}$ is ergodic. Then, the assertion is implied by Theorem 3 and [23, Lemma 7.13], which essentially states that $\mathbb{P}_{\theta^*}^{\pi, Y}$ is ergodic if and only if $\mathbb{P}_{\theta^*}^{\pi, Z}$ is ergodic. \square

Corollary 5. *Assume that condition (C1) is satisfied and let $k \in \mathbb{N}$. Then, for any $g: G^k \rightarrow \mathbb{R}$ with $\mathbb{E}_{\theta^*}^{\pi} [|g(Y_1, \dots, Y_k)|] < \infty$ we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n g(Z_{j+1}, \dots, Z_{j+k}) = \mathbb{E}_{\theta^*}^{\pi} [g(Y_1, \dots, Y_k)], \quad \mathbb{P}_{\theta^*}^{\pi} \text{-a.s.}$$

Proof. By the a.m.s. property and the ergodicity of $\mathbb{P}_{\theta^*}^{\pi, Z}$ the assertion is implied by [23, Theorem 8.1.]. \square

For $z = (z_i)_{i \in \mathbb{N}} \in G^{\mathbb{N}}$ and $k, m \in \mathbb{N}$ with $k < m$ we use $z_{k:m}$ to denote a segment of z . Specifically let $z_{k:m} = (z_k, \dots, z_m)$. Let $\lambda_k = \bigotimes_{i=1}^k \lambda$ be the product measure of λ with itself, i.e., the measurable space $(G^k, \bigotimes_{i=1}^k \mathcal{B}(G))$ is equipped with reference measure λ_k . Now define

$$p_{\theta^*}^{\pi}(z_{1:k} \mid z_{k+1:m}) := \frac{p_{\theta^*}^{\pi}(z_{1:m})}{\int_{G^k} p_{\theta^*}^{\pi}(z_{1:m}) \lambda_k(dz_{1:k})}.$$

We aim to apply [22, Theorem 3]. For this we need the concept of conditional mutual information.

Definition 3. For $k, m, n \in \mathbb{N}$ define the (k, m, n) -conditional mutual information of Z by

$$I_{k,m}^Z(n) := \mathbb{E}_{\theta^*}^{\pi} \left[\log \left(\frac{p_{\theta^*}^{\pi}(Z_{1:k} \mid Z_{k+1:k+m+n})}{p_{\theta^*}^{\pi}(Z_{1:k} \mid Z_{k+1:k+m})} \right) \right].$$

Remark 4. *Observe that the (k, m, n) -conditional mutual information of Z coincides with the definition of the conditional mutual information of $Z_{k+m+1:k+m+n}$ and $Z_{1:k}$ given $Z_{k+1:k+m}$ in [22, p. 1296]. Note that by [22, Lemma 3] it is known that $I_{k,m}^Z := \lim_{n \rightarrow \infty} I_{k,m}^Z(n)$ exists.*

Lemma 3. *Assume that condition (H4) is satisfied. Then, for every $k, m \in \mathbb{N}$ we have $I_{k,m}^Z := \lim_{n \rightarrow \infty} I_{k,m}^Z(n) < \infty$.*

Proof. For $n \in \mathbb{N}$ we obtain

$$I_{k,m}^Z(n) \leq \mathbb{E}_{\theta^*}^{\pi} [|\log p_{\theta^*}^{\pi}(Z_{1:k} \mid Z_{k+1:k+m})|] + \mathbb{E}_{\theta^*}^{\pi} [|\log p_{\theta^*}^{\pi}(Z_{1:k} \mid Z_{k+1:k+m+n})|].$$

For $1 \leq k < j$ we have by using $\int_{G^k} \prod_{i=1}^k f_{\theta^*,i}(s_i, z_i) \lambda_k(dz_{1:k}) = 1$ that

$$\begin{aligned} p_{\theta^*}^{\pi}(Z_{1:j}) &= \sum_{s_1, \dots, s_k \in S} \pi(s_1) \prod_{i=1}^k f_{\theta^*,i}(s_i, Z_i) \prod_{i=1}^{k-1} P_{\theta^*}(s_i, s_{i+1}) \\ &\times \sum_{s_{k+1}, \dots, s_{j+1} \in S} P_{\theta^*}(s_k, s_{k+1}) \prod_{\ell=k+1}^j f_{\theta^*,\ell}(s_{\ell}, Z_{\ell}) P_{\theta^*}(s_{\ell}, s_{\ell+1}) \end{aligned}$$

$$\leq \max_{x_1, \dots, x_k \in S} \prod_{i=1}^k f_{\theta^*, i}(x_i, Z_i) \int_{G^k} p_{\theta^*}^{\pi}(z_{1:k}, Z_{k+1:j}) \lambda_k(dz_{1:k}).$$

By (H4) this leads to

$$\mathbb{E}_{\theta^*}^{\pi} \left[\left| \log p_{\theta^*, k|j}^{\pi}(Z_{1:k} | Z_{k+1:j}) \right| \right] \leq \max_{x_1, \dots, x_k \in S} \sum_{i=1}^k \mathbb{E}_{\theta^*}^{\pi} [|\log(f_{\theta^*, i}(x_i, Z_i))|] < \infty,$$

which gives $I_{k,m}^Z(n) < \infty$ for any $n \in \mathbb{N}$ and implies the assertion. \square

The next result is a consequence of the generalized Shannon-McMillan-Breiman theorem for asymptotic mean stationary processes, see [22, Theorem 3].

Theorem 4. *Assume that the conditions (P1), (C1), (H1) and (H4) are satisfied. Then*

$$\lim_{n \rightarrow \infty} n^{-1} \log p_{\theta^*}^{\pi}(Z_1, \dots, Z_n) = \ell(\theta^*) \quad \mathbb{P}_{\theta^*}^{\pi}\text{-a.s.}$$

(Recall that $\ell(\theta^*)$ is given by (14).)

Proof. Theorem 3 shows that $\mathbb{P}_{\theta^*}^{\pi, Z}$ is a.m.s. with stationary mean $\mathbb{P}_{\theta^*}^{\pi, Y}$. Theorem 2 yields

$$\lim_{n \rightarrow \infty} n^{-1} \log q_{\theta^*}^{\pi}(Y_1, \dots, Y_n) = \ell(\theta^*) \quad \mathbb{P}_{\theta^*}^{\pi}\text{-a.s.}$$

Lemma 3 guarantees that $I_{k,m}^Z < \infty$ for all $k, m \in \mathbb{N}$. Then, the statement follows by [22, Theorem 3]. \square

We need some auxiliary lemmas that ensure that the ratio of $p_{\theta^*}^{\nu}(z_1, \dots, z_n)$ and $q_{\theta^*}^{\nu}(z_1, \dots, z_n)$ does not diverge exponentially or faster.

Lemma 4. *Assume that condition (C2) is satisfied. Then, with $k \in \mathbb{N}$ from (C2), we have*

$$\limsup_{n \rightarrow \infty} n^{-1} \log \left(\mathbb{E}_{\theta^*}^{\pi} \left[\prod_{i=k}^n \max_{s \in S} \frac{f_{\theta^*, i}(s, Z_i)}{f_{\theta^*}(s, Z_i)} \right] \right) \leq 0.$$

Proof. The assertion follows from

$$\begin{aligned} & \limsup_{n \rightarrow \infty} n^{-1} \log \left(\mathbb{E}_{\theta^*}^{\pi} \left[\prod_{i=k}^n \max_{s \in S} \frac{f_{\theta^*, i}(s, Z_i)}{f_{\theta^*}(s, Z_i)} \right] \right) \\ &= \limsup_{n \rightarrow \infty} n^{-1} \log \left(\mathbb{E}_{\theta^*}^{\pi} \left[\mathbb{E}_{\theta^*}^{\pi} \left[\prod_{i=k}^n \max_{s \in S} \frac{f_{\theta^*, i}(s, Z_i)}{f_{\theta^*}(s, Z_i)} \mid X_k, \dots, X_n \right] \right] \right) \\ &= \limsup_{n \rightarrow \infty} n^{-1} \log \left(\mathbb{E}_{\theta^*}^{\pi} \left[\prod_{i=k}^n \mathbb{E}_{\theta^*}^{\pi} \left[\max_{s \in S} \frac{f_{\theta^*, i}(s, Z_i)}{f_{\theta^*}(s, Z_i)} \mid X_i \right] \right] \right) \\ &\leq \limsup_{n \rightarrow \infty} n^{-1} \log \left(\mathbb{E}_{\theta^*}^{\pi} \left[\prod_{i=k}^n \max_{s' \in S} \mathbb{E}_{\theta^*}^{\pi} \left[\max_{s \in S} \frac{f_{\theta^*, i}(s, Z_i)}{f_{\theta^*}(s, Z_i)} \mid X_i = s' \right] \right] \right) \\ &= \limsup_{n \rightarrow \infty} n^{-1} \sum_{i=k}^n \max_{s' \in S} \log \left(\mathbb{E}_{\theta^*}^{\pi} \left[\max_{s \in S} \frac{f_{\theta^*, i}(s, Z_i)}{f_{\theta^*}(s, Z_i)} \mid X_i = s' \right] \right) \leq 0, \end{aligned}$$

where the last line follows from assumption (C2), especially (7). \square

By the same arguments as in the previous lemma we obtain the following result.

Lemma 5. *Assume that condition (C3) is satisfied. Then, for $k \in \mathbb{N}$ and \mathcal{E}_θ from (C3), we have*

$$\lim_{n \rightarrow \infty} n^{-1} \log \left(\mathbb{E}_{\theta^*}^\pi \left[\prod_{i=k}^n \sup_{\theta' \in \mathcal{E}_\theta} \max_{s \in S} \frac{f_{\theta', i}(s, Z_i)}{f_{\theta'}(s, Z_i)} \right] \right) \leq 0.$$

The next result allows us to carry the limit from Theorem 4 over, to the case where we keep the finite trajectory of Z , but consider $q_{\theta^*}^\nu$ instead of $p_{\theta^*}^\pi$ for suitable $\nu \in \mathcal{P}(S)$.

Theorem 5. *Assume that the conditions (P1), (H1), (H4), (C1) and (C2) are satisfied. Then*

$$\lim_{n \rightarrow \infty} n^{-1} \log q_{\theta^*}^\nu(Z_1, \dots, Z_n) = \ell(\theta^*) \quad \mathbb{P}_{\theta^*}^\pi\text{-a.s.}$$

for any probability measure $\nu \in \mathcal{P}(S)$ which is strictly positive if and only if π is strictly positive.

Proof. From Theorem 4 it follows that

$$(20) \quad \lim_{n \rightarrow \infty} n^{-1} \log p_{\theta^*}^\pi(Z_1, \dots, Z_n) = \ell(\theta^*) \quad \mathbb{P}_{\theta^*}^\pi\text{-a.s.}$$

and by using (C2) we first show

$$(21) \quad \lim_{n \rightarrow \infty} n^{-1} \log q_{\theta^*}^\pi(Z_1, \dots, Z_n) = \ell(\theta^*) \quad \mathbb{P}_{\theta^*}^\pi\text{-a.s.}$$

For any $\varepsilon > 0$ we obtain by Markov's inequality that

$$\begin{aligned} \mathbb{P}_{\theta^*}^\pi \left(n^{-1} \log \left(\frac{q_{\theta^*}^\pi(Z_1, \dots, Z_n)}{p_{\theta^*}^\pi(Z_1, \dots, Z_n)} \right) \geq \varepsilon \right) &= \mathbb{P}_{\theta^*}^\pi \left(\frac{q_{\theta^*}^\pi(Z_1, \dots, Z_n)}{p_{\theta^*}^\pi(Z_1, \dots, Z_n)} \geq \exp(n\varepsilon) \right) \\ &\leq \exp(-n\varepsilon) \cdot \mathbb{E}_{\theta^*}^\pi \left[\frac{q_{\theta^*}^\pi(Z_1, \dots, Z_n)}{p_{\theta^*}^\pi(Z_1, \dots, Z_n)} \right]. \end{aligned}$$

By the fact that $\mathbb{E}_{\theta^*}^\pi \left[\frac{q_{\theta^*}^\pi(Z_1, \dots, Z_n)}{p_{\theta^*}^\pi(Z_1, \dots, Z_n)} \right] = 1$, the Borel-Cantelli Lemma implies

$$\limsup_{n \rightarrow \infty} n^{-1} \log \left(\frac{q_{\theta^*}^\pi(Z_1, \dots, Z_n)}{p_{\theta^*}^\pi(Z_1, \dots, Z_n)} \right) \leq 0 \quad \mathbb{P}_{\theta^*}^\pi\text{-a.s.}$$

This leads by (20) to

$$(22) \quad \limsup_{n \rightarrow \infty} n^{-1} \log q_{\theta^*}^\pi(Z_1, \dots, Z_n) \leq \ell(\theta^*) \quad \mathbb{P}_{\theta^*}^\pi\text{-a.s.}$$

Observe that

$$\frac{p_{\theta^*}^\pi(Z_1, \dots, Z_n)}{q_{\theta^*}^\pi(Z_1, \dots, Z_n)} \leq \prod_{i=1}^n \max_{s \in S} \frac{f_{\theta^*, i}(s, Z_i)}{f_{\theta^*}(s, Z_i)}.$$

Then, with the $k \in \mathbb{N}$ from (C2), in particular (6), it follows that

$$\begin{aligned} \limsup_{n \rightarrow \infty} n^{-1} \log \left(\frac{p_{\theta^*}^\pi(Z_1, \dots, Z_n)}{q_{\theta^*}^\pi(Z_1, \dots, Z_n)} \right) &\leq \limsup_{n \rightarrow \infty} n^{-1} \log \left(\prod_{i=1}^n \max_{s \in S} \frac{f_{\theta^*, i}(s, Z_i)}{f_{\theta^*}(s, Z_i)} \right) \\ &= \limsup_{n \rightarrow \infty} n^{-1} \left(\log \left(\prod_{i=1}^{k-1} \max_{s \in S} \frac{f_{\theta^*, i}(s, Z_i)}{f_{\theta^*}(s, Z_i)} \right) + \log \left(\prod_{i=k}^n \max_{s \in S} \frac{f_{\theta^*, i}(s, Z_i)}{f_{\theta^*}(s, Z_i)} \right) \right) \\ &= \limsup_{n \rightarrow \infty} n^{-1} \log \left(\prod_{i=k}^n \max_{s \in S} \frac{f_{\theta^*, i}(s, Z_i)}{f_{\theta^*}(s, Z_i)} \right) \quad \mathbb{P}_{\theta^*}^\pi\text{-a.s.} \end{aligned}$$

Again, for any $\varepsilon > 0$ we obtain by Markov's inequality that

$$\begin{aligned} & \mathbb{P}_{\theta^*}^{\pi} \left(n^{-1} \log \left(\prod_{i=k}^n \max_{s \in S} \frac{f_{\theta^*,i}(s, Z_i)}{f_{\theta^*}(s, Z_i)} \right) \geq \varepsilon \right) \\ &= \mathbb{P}_{\theta^*}^{\pi} \left(\prod_{i=k}^n \max_{s \in S} \frac{f_{\theta^*,i}(s, Z_i)}{f_{\theta^*}(s, Z_i)} \geq \exp(n\varepsilon) \right) \leq \frac{\mathbb{E}_{\theta^*}^{\pi} \left[\prod_{i=k}^n \max_{s \in S} \frac{f_{\theta^*,i}(s, Z_i)}{f_{\theta^*}(s, Z_i)} \right]}{\exp(n\varepsilon)} \\ &= \exp \left(n \left(n^{-1} \log \left(\mathbb{E}_{\theta^*}^{\pi} \left[\prod_{i=k}^n \max_{s \in S} \frac{f_{\theta^*,i}(s, Z_i)}{f_{\theta^*}(s, Z_i)} \right] \right) - \varepsilon \right) \right). \end{aligned}$$

By Lemma 4, the Borel-Cantelli Lemma yields

$$\limsup_{n \rightarrow \infty} n^{-1} \log \left(\prod_{i=k}^n \max_{s \in S} \frac{f_{\theta^*,i}(s, Z_i)}{f_{\theta^*}(s, Z_i)} \right) \leq 0 \quad \mathbb{P}_{\theta^*}^{\pi}\text{-a.s.}$$

which leads to

$$\limsup_{n \rightarrow \infty} n^{-1} \log \left(\frac{p_{\theta^*}^{\pi}(Z_1, \dots, Z_n)}{q_{\theta^*}^{\pi}(Z_1, \dots, Z_n)} \right) \leq 0 \quad \mathbb{P}_{\theta^*}^{\pi}\text{-a.s.}$$

This implies

$$(23) \quad \liminf_{n \rightarrow \infty} n^{-1} \log \left(\frac{q_{\theta^*}^{\pi}(Z_1, \dots, Z_n)}{p_{\theta^*}^{\pi}(Z_1, \dots, Z_n)} \right) \geq 0 \quad \mathbb{P}_{\theta^*}^{\pi}\text{-a.s.}$$

By (22) and (23) we obtain (21).

Next we prove the statement of the theorem using (21). For any $n \in \mathbb{N}$ observe that

$$(24) \quad \frac{q_{\theta^*}^{\pi}(Z_1, \dots, Z_n)}{q_{\theta^*}^{\nu}(Z_1, \dots, Z_n)} = \frac{\sum_{s_1, \dots, s_{n+1} \in S} \nu(s_1) \frac{\pi(s_1)}{\nu(s_1)} \prod_{i=1}^n f_{\theta^*}(s_i, Z_i) P_{\theta^*}(s_i, s_{i+1})}{\sum_{s_1, \dots, s_{n+1} \in S} \nu(s_1) \prod_{i=1}^n f_{\theta^*}(s_i, Z_i) P_{\theta^*}(s_i, s_{i+1})} \leq \max_{s \in S} \frac{\pi(s)}{\nu(s)} < \infty,$$

where the finiteness follows by the fact that ν is strictly positive if and only if π is strictly positive. By using (24) we also obtain

$$(25) \quad \frac{q_{\theta^*}^{\pi}(Z_1, \dots, Z_n)}{q_{\theta^*}^{\nu}(Z_1, \dots, Z_n)} \geq \min_{s \in S} \frac{\pi(s)}{\nu(s)} > 0.$$

Then

$$\begin{aligned} & \limsup_{n \rightarrow \infty} n^{-1} \log q_{\theta^*}^{\nu}(Z_1, \dots, Z_n) \\ &= \limsup_{n \rightarrow \infty} n^{-1} \left(\log \left(\frac{q_{\theta^*}^{\nu}(Z_1, \dots, Z_n)}{q_{\theta^*}^{\pi}(Z_1, \dots, Z_n)} \right) + \log q_{\theta^*}^{\pi}(Z_1, \dots, Z_n) \right) \\ &\leq \limsup_{n \rightarrow \infty} n^{-1} \left(\max_{s \in S} \frac{\pi(s)}{\nu(s)} + \log q_{\theta^*}^{\pi}(Z_1, \dots, Z_n) \right) = \ell(\theta^*) \end{aligned}$$

and by (25) we similarly have

$$\liminf_{n \rightarrow \infty} n^{-1} \log q_{\theta^*}^{\nu}(Z_1, \dots, Z_n) \geq \ell(\theta^*).$$

By the previous two inequalities the assertion follows. \square

Before we come to the proof of our main result, Theorem 1, we provide a lemma which is essentially used and proven in [11]. In our setting the formulation and the statement slightly simplifies compared to [11, Lemma 13], since we only consider finite state spaces S .

Lemma 6. *Let δ be the counting measure on S . Assume that the conditions (P1), (P2) and (H1) - (H3) are satisfied. Then, for any $\theta \in \Theta$ with $\theta \not\sim \theta^*$, there exists a natural number n_θ and a real number $\eta_\theta > 0$ such that $B(\theta, \eta_\theta) \subseteq \mathcal{U}_\theta$ and*

$$(26) \quad \frac{1}{n_\theta} \mathbb{E}_{\theta^*}^\pi \left[\sup_{\theta' \in B(\theta, \eta_\theta)} \log q_{\theta'}^\delta(Y_1, \dots, Y_{n_\theta}) \right] < \ell(\theta^*).$$

Here $B(\theta, \eta) \subseteq \Theta$ is the Euclidean ball of radius $\eta > 0$ centered at $\theta \in \Theta$.

Proof. The result follows straightforward from [11, Theorem 12] and the arguments in the proof of [11, Lemma 13]. \square

Systematically, the proof of Theorem 1 follows the same line of arguments as the proof of [11, Theorem 1]. However, let us point out that the scenario is very different:

- We consider the QMLE $\theta_{\nu, n}^{\text{QML}}$ instead of the MLE.
- The arguments we use heavily rely on the a.m.s. property of $\mathbb{P}_{\theta^*}^Z$.

Proof of Theorem 1. By the standard approach to prove consistency, see Lemma 7 and Theorem 6, Theorem 5 and the fact that

$$q_{\theta_n}^\nu(Z_1, \dots, Z_n) \geq q_{\theta^*}^\nu(Z_1, \dots, Z_n) \quad \forall n \in \mathbb{N}$$

it is sufficient to prove for any closed set $C \subseteq \Theta$ with $\theta^* \notin C$ that

$$\limsup_{n \rightarrow \infty} \sup_{\theta' \in C} n^{-1} \log q_{\theta'}^\nu(Z_1, \dots, Z_n) < \ell(\theta^*) \quad \mathbb{P}_{\theta^*}^\pi\text{-a.s.}$$

Note that, with η_θ defined in Lemma 6, the set $\{B(\theta, \eta_\theta), \theta \in C\}$ is a cover of C . As Θ is compact, C is also compact and thus admits a finite subcover $\{B(\theta_i, \eta_{\theta_i}), \theta_i \in C, i = 1, \dots, N\}$. Hence it is enough to verify

$$(27) \quad \limsup_{n \rightarrow \infty} \sup_{\theta' \in B(\theta, \eta_\theta) \cap C} n^{-1} \log q_{\theta'}^\nu(Z_1, \dots, Z_n) < \ell(\theta^*) \quad \mathbb{P}_{\theta^*}^\pi\text{-a.s.}$$

for any $\theta \not\sim \theta^*$.

Let us fix $\theta \not\sim \theta^*$ and let η_θ as well as n_θ as in Lemma 6. Observe that for any $\theta' \in \Theta$ and any $1 \leq m \leq n$ we have

$$(28) \quad q_{\theta'}^\nu(z_1, \dots, z_n) \leq q_{\theta'}^\nu(z_1, \dots, z_{m-1}) q_{\theta'}^\delta(z_m, \dots, z_n),$$

$$(29) \quad q_{\theta'}^\delta(z_1, \dots, z_n) \leq q_{\theta'}^\delta(z_1, \dots, z_{m-1}) q_{\theta'}^\delta(z_m, \dots, z_n),$$

and define $g_{\theta', m, n}^*(z_m, \dots, z_n) := \prod_{i=m}^n \max_{s \in S} f_{\theta'}(s, z_i)$ as well as $i(n) := \lfloor n/n_\theta \rfloor$.

By using those definitions, and by (28) and (29) we obtain for sufficiently large $n \in \mathbb{N}$ that

$$\begin{aligned} \ell_{\nu, n}^{\text{Q}}(\theta') &\leq \frac{1}{n_\theta} \sum_{r=1}^{n_\theta} \ell_{\nu, r}^{\text{Q}}(\theta') + \log q_{\theta'}^\delta(Z_{r+1}, \dots, Z_n) \\ &\leq \frac{1}{n_\theta} \sum_{r=1}^{n_\theta} \log g_{\theta', 1, r}^*(Z_1, \dots, Z_r) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{n_\theta} \sum_{r=1}^{n_\theta} \sum_{k=1}^{i(n)-1} \log q_{\theta'}^\delta(Z_{n_\theta(k-1)+r+1}, \dots, Z_{n_\theta k+r}) \\
& + \frac{1}{n_\theta} \sum_{r=1}^{n_\theta} \log g_{\theta', n_\theta(i(n)-1)+r+1, n}^*(Z_{n_\theta(i(n)-1)+r+1}, \dots, Z_n) \\
& = \frac{1}{n_\theta} \sum_{r=1}^{n_\theta} \log g_{\theta', 1, r}^*(Z_1, \dots, Z_r) \\
& + \frac{1}{n_\theta} \sum_{r=1}^{n_\theta(i(n)-1)} \log q_{\theta'}^\delta(Z_{r+1}, \dots, Z_{n_\theta+r}) \\
& + \frac{1}{n_\theta} \sum_{r=1}^{n_\theta} \sum_{k=n_\theta(i(n)-1)+r+1}^n \max_{s \in S} \log f_{\theta'}(s, Z_k).
\end{aligned}$$

Observe that for $1 \leq r \leq n_\theta$ holds $n_\theta(i(n)-1)+r \geq n-2n_\theta$. Hence we can further estimate the last average and obtain

$$\begin{aligned}
\sup_{\theta' \in B(\theta, \eta_\theta) \cap C} \ell_{\nu, n}^Q(\theta') & \leq \frac{1}{n_\theta} \sum_{r=1}^{n_\theta} \sup_{\theta' \in B(\theta, \eta_\theta) \cap C} \log g_{\theta', 1, r}^*(Z_1, \dots, Z_r) \\
& + \frac{1}{n_\theta} \sum_{r=1}^{n_\theta(i(n)-1)} \sup_{\theta' \in B(\theta, \eta_\theta) \cap C} \log q_{\theta'}^\delta(Z_{r+1}, \dots, Z_{n_\theta+r}) \\
& + \sum_{k=n-2n_\theta+1}^n \sup_{\theta' \in B(\theta, \eta_\theta) \cap C} \max_{s \in S} \log (f_{\theta'}(s, Z_k))^+.
\end{aligned}$$

We multiply both sides of the previous inequality by n^{-1} and consider the limit $n \rightarrow \infty$ of each sum on the right-hand side. In particular we show that the right-hand side is smaller than $\ell(\theta^*)$ which verifies (27).

To the first sum: By the fact that $\int_G f_{\theta'}(s, z) \lambda(dz) = 1$, for any $s \in S$ we conclude $\lambda(\{z \in G : f_{\theta'}(s, z) = \infty\}) = 0$. Hence

$$\mathbb{P}_{\theta^*}^\pi (f_{\theta'}(s, Z_i) = \infty) = 0,$$

and (H3) implies

$$\mathbb{P}_{\theta^*}^\pi \left(\sup_{\theta' \in B(\theta, \eta_\theta) \cap C} \log g_{\theta', 1, r}^*(Z_1, \dots, Z_r) = \infty \right) = 0 \quad \forall r \in \mathbb{N}.$$

This leads to

$$\lim_{n \rightarrow \infty} \frac{1}{n} \frac{1}{n_\theta} \sum_{r=1}^{n_\theta} \sup_{\theta' \in B(\theta, \eta_\theta) \cap C} \log g_{\theta', 1, r}^*(Z_1, \dots, Z_r) = 0 \quad \mathbb{P}_{\theta^*}^\pi\text{-a.s.}$$

To the second sum: By the fact that $i(n)/n \rightarrow n_\theta^{-1}$ as $n \rightarrow \infty$, Lemma 6 and Corollary 4 we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{n} \frac{1}{n_\theta} \sum_{r=1}^{n_\theta(i(n)-1)} \sup_{\theta' \in B(\theta, \eta_\theta) \cap C} \log q_{\theta'}^\delta(Z_{r+1}, \dots, Z_{n_\theta+r})$$

$$= \frac{1}{n_\theta} \mathbb{E}_{\theta^*}^\pi \left[\sup_{\theta' \in B(\theta, \eta_\theta) \cap C} \log p_{\theta'}^\delta(Y_1, \dots, Y_{n_\theta}) \right] < \ell(\theta^*).$$

To the third sum: By assumption (H2) it follows that

$$\mathbb{E}_{\theta^*}^\pi \left[\sup_{\theta' \in \mathcal{U}_\theta} \max_{s \in S} (\log f_\theta(s, Y_1))^+ \right] \leq \sum_{s \in S} \mathbb{E}_{\theta^*}^\pi \left[\sup_{\theta' \in \mathcal{U}_\theta} (\log f_\theta(s, Y_1))^+ \right] < \infty$$

and by Corollary 4 we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sup_{\theta' \in B(\theta, \eta_\theta)} \max_{s \in S} \log(f_{\theta'}(s, Z_k))^+ = \mathbb{E}_{\theta^*}^\pi \left[\sup_{\theta' \in B(\theta, \eta_\theta)} \max_{s \in S} (\log f_\theta(s, Y_1))^+ \right].$$

Hence

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=n-2n_\theta+1}^n \sup_{\theta' \in B(\theta, \eta_\theta)} \max_{s \in S} \log(f_{\theta'}(s, Z_k))^+ = 0 \quad \mathbb{P}_{\theta^*}^\pi\text{-a.s.}$$

and the proof is complete \square

As a consequence of the proof of Theorem 1 we are able to prove consistency for the MLE under condition (C3).

Proof of Corollary 1. We use the same strategy as in the proof of Theorem 1. By Theorem 4 it follows that

$$\lim_{n \rightarrow \infty} n^{-1} \log p_{\theta^*}^\pi(Z_1, \dots, Z_n) = \ell(\theta^*) \quad \mathbb{P}_{\theta^*}^\pi\text{-a.s.}$$

For $\theta \not\sim \theta^*$, we chose $\kappa_\theta \leq \eta_\theta$, where η_θ is defined in Lemma 6, such that $B(\theta, \kappa_\theta) \subset \mathcal{E}_\theta$. As explained in the proof of Theorem 1, it is sufficient to verify for any closed set $C \subseteq \Theta$ with $\theta^* \notin C$ that

$$(30) \quad \limsup_{n \rightarrow \infty} \sup_{\theta' \in B(\theta, \kappa_\theta) \cap C} n^{-1} \log p_{\theta'}^\nu(Z_1, \dots, Z_n) < \ell(\theta^*) \quad \mathbb{P}_{\theta^*}^\pi\text{-a.s.}$$

With $k \in \mathbb{N}$ from condition (C3) we obtain by using (8) that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \sup_{\theta' \in B(\theta, \kappa_\theta) \cap C} n^{-1} \log \left(\frac{p_{\theta'}^\nu(Z_1, \dots, Z_n)}{q_{\theta'}^\nu(Z_1, \dots, Z_n)} \right) \\ & \leq \limsup_{n \rightarrow \infty} \sup_{\theta' \in B(\theta, \kappa_\theta) \cap C} n^{-1} \log \left(\prod_{i=1}^n \max_{s \in S} \frac{f_{\theta', i}(s, Z_i)}{f_{\theta'}(s, Z_i)} \right) \\ & = \limsup_{n \rightarrow \infty} \sup_{\theta' \in B(\theta, \kappa_\theta) \cap C} n^{-1} \log \left(\prod_{i=k}^n \max_{s \in S} \frac{f_{\theta', i}(s, Z_i)}{f_{\theta'}(s, Z_i)} \right) \\ & \leq \limsup_{n \rightarrow \infty} n^{-1} \log \left(\prod_{i=k}^n \sup_{\theta' \in B(\theta, \kappa_\theta) \cap C} \max_{s \in S} \frac{f_{\theta', i}(s, Z_i)}{f_{\theta'}(s, Z_i)} \right). \end{aligned}$$

By the same arguments as for proving (23) in the proof of Theorem 5 we get that

$$\mathbb{P}_{\theta^*}^\pi \left(n^{-1} \log \left(\prod_{i=k}^n \sup_{\theta' \in B(\theta, \kappa_\theta) \cap C} \max_{s \in S} \frac{f_{\theta', i}(s, Z_i)}{f_{\theta'}(s, Z_i)} \right) \geq \varepsilon \right) \leq \exp(n(c_n - \varepsilon)),$$

with

$$c_n := \limsup_{n \rightarrow \infty} n^{-1} \log \left(\mathbb{E}_{\theta^*}^\pi \left[\prod_{i=k}^n \sup_{\theta' \in B(\theta, \kappa_\theta) \cap C} \max_{s \in S} \frac{f_{\theta', i}(s, Z_i)}{f_{\theta'}(s, Z_i)} \right] \right).$$

Assumption (C3), in particular Lemma 5, and the Borel Cantelli lemma implies that

$$\mathbb{P}_{\theta^*}^{\pi} \left(\limsup_{n \rightarrow \infty} \sup_{\theta' \in B(\theta, \kappa_{\theta}) \cap C} n^{-1} \log \left(\frac{p_{\theta'}^{\nu}(Z_1, \dots, Z_n)}{q_{\theta'}^{\nu}(Z_1, \dots, Z_n)} \right) \leq 0 \right) = 1.$$

Similarly, it follows that

$$\mathbb{P}_{\theta^*}^{\pi} \left(\limsup_{n \rightarrow \infty} \sup_{\theta' \in B(\theta, \kappa_{\theta}) \cap C} n^{-1} \log \left(\frac{q_{\theta'}^{\nu}(Z_1, \dots, Z_n)}{p_{\theta'}^{\nu}(Z_1, \dots, Z_n)} \right) \leq 0 \right) = 1,$$

which implies

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \sup_{\theta' \in B(\theta, \kappa_{\theta}) \cap C} n^{-1} \log p_{\theta'}^{\nu}(Z_1, \dots, Z_n) \\ &= \limsup_{n \rightarrow \infty} \sup_{\theta' \in B(\theta, \kappa_{\theta}) \cap C} n^{-1} \log q_{\theta'}^{\nu}(Z_1, \dots, Z_n). \end{aligned}$$

Finally the assertion follows from (27). \square

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APPENDIX A. STRONG CONSISTENCY

We follow the classical the approach of Wald, see [27], adopted to quasi likelihood estimation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and (G, \mathcal{G}) be a measurable space. Assume that $\Theta \subseteq \mathbb{R}^d$ and let $|\cdot|$ be the d -dimensional Euclidean norm.

Theorem 6 (Strong consistency). *Let $(W_n)_{n \in \mathbb{N}}$ be a sequence of random variables mapping from $(\Omega, \mathcal{F}, \mathbb{P})$ to (G, \mathcal{G}) . For any $n \in \mathbb{N}$ let $h_n : \Theta \times G^n \rightarrow [0, \infty)$ be a measurable function. Assume that there exists an element $\theta^* \in \Theta$ such that for any closed $C \subset \Theta$ with $\theta^* \notin C$ and all $n \in \mathbb{N}$, we have*

$$(31) \quad \limsup_{n \rightarrow \infty} \sup_{\theta \in C} \frac{h_n(\theta, W_1, \dots, W_n)}{h_n(\theta^*, W_1, \dots, W_n)} = 0 \quad \mathbb{P}\text{-a.s.}$$

Let $(\hat{\theta}_n)_{n \in \mathbb{N}}$ be a sequence of random variables mapping from $(\Omega, \mathcal{F}, \mathbb{P})$ to Θ such that

$$(32) \quad \exists c > 0 \ \& \ n_0 \in \mathbb{N} \quad \forall n \geq n_0 : \frac{h_n(\hat{\theta}_n, W_1, \dots, W_n)}{h_n(\theta^*, W_1, \dots, W_n)} \geq c, \quad \mathbb{P}\text{-a.s.}$$

Then

$$\lim_{n \rightarrow \infty} |\hat{\theta}_n - \theta^*| = 0 \quad \mathbb{P}\text{-a.s.}$$

Proof. For arbitrary $\varepsilon > 0$ define

$$\begin{aligned} A_{\varepsilon}^{(1)} &:= \left\{ \omega \in \Omega : \limsup_{n \rightarrow \infty} |\hat{\theta}_n(\omega) - \theta^*| > \varepsilon \right\}, \\ A_{\varepsilon}^{(2)} &:= \left\{ \omega \in \Omega : \limsup_{n \rightarrow \infty} \sup_{\theta : |\theta - \theta^*| \geq \varepsilon} \frac{h_n(\theta, W_1(\omega), \dots, W_n(\omega))}{h_n(\hat{\theta}_n(\omega), W_1(\omega), \dots, W_n(\omega))} \geq 1 \right\}, \end{aligned}$$

$$A_\varepsilon^{(3)} := \left\{ \omega \in \Omega : \limsup_{n \rightarrow \infty} \sup_{\theta: |\theta - \theta^*| \geq \varepsilon} \frac{h_n(\theta, W_1(\omega), \dots, W_n(\omega))}{h_n(\theta^*, W_1(\omega), \dots, W_n(\omega))} \geq c \right\}.$$

Note that $A_\varepsilon^{(1)} \subseteq A_\varepsilon^{(2)} \subseteq A_\varepsilon^{(3)}$, where the last inclusion follows by (32). Hence, by (31) we have $\mathbb{P}(A_\varepsilon^{(3)}) = 0$ so that

$$\mathbb{P}(A_\varepsilon^{(1)}) = \mathbb{P} \left(\limsup_{n \rightarrow \infty} |\hat{\theta}_n - \theta^*| > \varepsilon \right) = 0,$$

which implies the assertion. \square

The following lemma is useful to verify condition (31).

Lemma 7. *Let $(W_n)_{n \in \mathbb{N}}$ be a sequence of random variables mapping from $(\Omega, \mathcal{F}, \mathbb{P})$ to (G, \mathcal{G}) and, as in Theorem 6, for any $n \in \mathbb{N}$ let $h_n : \Theta \times G^n \rightarrow [0, \infty)$ be a measurable function. Assume that there is an element $\theta^* \in \Theta$ such that for any closed $C \subset \Theta$ with $\theta^* \notin C$ we have*

(33)

$$\limsup_{n \rightarrow \infty} \sup_{\theta \in C} \frac{1}{n} \log h_n(\theta, W_1, \dots, W_n) < \lim_{n \rightarrow \infty} \frac{1}{n} \log h_n(\theta^*, W_1, \dots, W_n) \quad \mathbb{P}\text{-a.s.}$$

provided that the limit on the right hand-side exists. Then condition (31) is satisfied.

Proof. Obviously (33) implies

$$\log \left(\limsup_{n \rightarrow \infty} \sup_{\theta \in C} \left[\frac{h_n(\theta, W_1, \dots, W_n)}{h_n(\theta^*, W_1, \dots, W_n)} \right]^{1/n} \right) < 0.$$

This leads to

$$\limsup_{n \rightarrow \infty} \sup_{\theta \in C} \left[\frac{h_n(\theta, W_1, \dots, W_n)}{h_n(\theta^*, W_1, \dots, W_n)} \right]^{1/n} < 1$$

from which (31) follows. \square

APPENDIX B. ASSUMPTIONS FOR ASYMPTOTIC NORMALITY

For the MLE to achieve a statement about asymptotic normality one can apply the theory for M -estimators developed by Jensen in [21]. Before we are able to formulate assumptions which lead to asymptotic normality of $\theta_{\nu, n}^{\text{QML}}$ we need some further notations. Recall that $B(\theta^*, \delta)$ is the Euclidean ball of radius $\delta > 0$ centered at $\theta^* \in \Theta$. For $v \in \mathbb{R}^d$ let $|v|_1$ be the ℓ_1 -norm. Consider a sequence of functions $(a_i)_{i \in \mathbb{N}}$ with $a_i : \Theta \times S \times S \times G \rightarrow \mathbb{R}$. We say that $(a_i)_{i \in \mathbb{N}}$ belongs to the class C_k if there exist a sequence of functions $(a_i^0)_{i \in \mathbb{N}}$, with $a_i^0 : G \rightarrow [0, \infty)$, a constant $\delta_0 > 0$ and a constant $K < \infty$ such that for all $i \in \mathbb{N}$,

$$\sup_{s_1, s_2 \in S, \theta \in B(\theta^*, \delta_0)} |a_i(\theta, s_1, s_2, z)| \leq a_i^0(z) \quad \forall z \in G \quad \text{and} \quad \mathbb{E}_{\theta^*}^\pi [a_i^0(Z_i)^k] \leq K.$$

Furthermore, $(a_i)_{i \in \mathbb{N}}$ belongs to the class $C_{k,m}$ if $(a_i)_{i \in \mathbb{N}} \in C_k$, there exist a sequence of functions $(\bar{a}_i)_{i \in \mathbb{N}}$, with $\bar{a}_i : G \rightarrow [0, \infty)$, and $\delta_0 > 0$ such that for all $\theta \in B(\theta^*, \delta_0)$, for all $s_1, s_2 \in S$ and for all $i \in \mathbb{N}$,

$$|a_i(\theta, s_1, s_2, z) - a_i(\theta^*, s_1, s_2, z)| \leq |\theta - \theta^*| \bar{a}_i(z) \quad \forall z \in G \quad \text{and} \quad \mathbb{E}_{\theta^*}^\pi [\bar{a}_i(Z_i)^m] \leq K.$$

For a positive semi-definite, symmetric matrix $A \in \mathbb{R}^{d \times d}$ let $\lambda_{\min}(A)$ to be the smallest eigenvalue of A . For any $\theta \in \Theta$ define the gradient

$$S_n(\theta) := \frac{\partial}{\partial \theta'} \log q_{\theta'}^\nu(Z_1, \dots, Z_n) \Big|_{\theta'=\theta}$$

and note that with random vectors

$$(34) \quad \psi_i(\theta) := \begin{cases} \frac{\partial}{\partial \theta'} \log(P_{\theta'}(X_{i-1}, X_i) f_{\theta'}(X_i, Z_i)) \Big|_{\theta'=\theta}, & i \geq 2, \\ \frac{\partial}{\partial \theta'} \log(\nu(X_1) f_{\theta'}(X_1, Z_1)) \Big|_{\theta'=\theta}, & i = 1, \end{cases}$$

a simple calculation reveals $S_n(\theta) = \sum_{i=1}^n \mathbb{E}_\theta^\nu(\psi_i(\theta) \mid Z_1, \dots, Z_n)$. The following three conditions are needed to adapt the proof of the asymptotic normality for the MLE of [21] to the QMLE:

Mixing.

(M) There is a constant $c_0 > 0$ such that

$$c_0 \leq P_{\theta^*}(s_1, s_2) \quad \forall s_1, s_2 \in S.$$

Central Limit Theorem.

(CLT) Assume that

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} |\mathbb{E}_{\theta^*}^\pi(S_n(\theta^*))|_1 = 0,$$

and that $(\psi_i)_{i \in \mathbb{N}} \in C_3$. Furthermore, there exist constants $c_1 > 0$ and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ holds

$$\lambda_{\min} \left(\frac{1}{n} \text{Cov}_{\theta^*}^\pi(S_n(\theta^*)) \right) \geq c_1,$$

where $\text{Cov}_{\theta^*}^\pi(S_n(\theta^*))$ denotes the covariance matrix of $S_n(\theta^*)$.

Uniform Convergence.

(UC) Let $F_n \in \mathbb{R}^{d \times d}$ be defined by

$$F_n := -\frac{1}{n} \mathbb{E}_{\theta^*}^\pi \left[\left(\frac{\partial}{\partial \theta'} S_n(\theta') \Big|_{\theta'=\theta^*} \right)^T \right].$$

Assume that there exist constants $c_2 > 0$ and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ holds $\lambda_{\min}(F_n) \geq c_2$. Furthermore, assume that $(\psi_i)_{i \in \mathbb{N}}$ is of class C_4 and for any $r = 1, \dots, d$ we have that $(\partial \psi_i / \partial \theta_r)_{i \in \mathbb{N}}$ is of class $C_{3,1}$.

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