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# Rigidity and reconstruction for graphs

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**Abstract.** The edge reconstruction conjecture of Harary (1964) states that a finite graph $G$ can be reconstructed up to isomorphism from the multiset of its edge-deleted subgraphs $G - e$ (with $e$ running over the edges of $G$). We put this conjecture in the framework of measure-theoretic rigidity, revealing the importance of the lengths of labeled closed walks for the problem.

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## 1. Introduction

Can a network be reconstructed from subnetworks? W. T. Tutte called such reconstruction problems in graphs the mathematical equivalent of "an archeologist trying to assemble broken fragments of pottery to find the shape and pattern of an ancient vase" [17, p. 106]. Nowadays, we might add the problem of phylogenetic network reconstruction [18].

To give a mathematically more precise description, let $G = (V, E)$ denote a graph with vertex set $V$ and edge set $E$. The *edge deck* $\mathcal{D}^e(G)$ of $G$ is the multi-set of isomorphism classes of all edge-deleted subgraphs $G - e$ (for $e \in E$) of $G$. Harary [9] conjectured in 1964 that graphs on at least four edges are edge-reconstructible, i.e., determined up to isomorphism by their edge deck. This is the *edge reconstruction conjecture* (ERC), the analogue for edges of the famous vertex reconstruction conjecture (VRC) of Kelly and Ulam that every graph on at least three vertices is determined by its vertex deck (compare [2]). Despite interesting partial results—most prominently, Vladimír Müller [14] has shown that the ERC holds for graphs in which $|E| \geq |V| \log_2 |V|$—the conjecture still appears to be wide open.

Our modest contribution in this paper is to put the conjecture in the framework of fractal geometry. More precisely, following Coornaert [3], we will use measure-theoretic rigidity to describe a graph uniquely up to isomorphism by a class of measures on the boundary of a regular tree. Let $b$ denote the first Betti number of a connected graph $G$, i.e., $b = |E| - |V| + 1$ is the minimal number of edges that need to be removed from $G$ so the remaining graph is a tree. Assume $b \geq 2$, and consider the action of the fundamental group of $G$ (isomorphic to a free group $\mathbb{F}_b$ of rank $b$) on the boundary $\partial\mathcal{T}$ (i.e., the space of ends) of the universal covering tree $\mathcal{T}$ of $G$. Rigidity then says that the graph is uniquely determined by this space *with* a Patterson–Sullivan measure, up to absolute continuity. We make this precise and reformulate it slightly (in terms of push forwards of the measure to the boundary of the Cayley graph of $\mathbb{F}_b$) in Theorem 2.8 below. We then describe the measure precisely in terms of spectral invariants related to the edge adjacency operator (Theorem 3.4). In this way, we arrive at our two main new results. First of all, combining the two previous theorems (that are essentially rephrasings of known results) with our own results in [6], we reformulate the ERC as "growth rate" property of lengths of corresponding loops in the two graphs (Theorem 4.2). In a subsequent Theorem 5.4 we relate the ERC to the reconstruction of the lengths of overlaps of certain basic walks in the graph, this time without reference to measure theory.

**Remark.** A side effect of relating the ERC to measures is that it might open the road to formulate analogues of the reconstruction question in other parts of geometry where measure theoretic rigidity is available. We leave this as a future research problem, but we would like to mention two contexts where at least a rigidity theorem is known.

(1) Predating the rigidity theorem for graphs is the following version for compact Riemann surfaces $X$ and $Y$ of genus $g \geq 2$: they are isomorphic if and only if there exists an absolutely continuous self-homeomorphism of the boundary $S^1 = \partial\Delta$ of the Poincaré disk $\Delta$ (their universal covering) that is equivariant with respect to the action of their respective fundamental groups (cf. e.g. [12]).

(2) One may also extend rigidity from graphs to curves over non-archimedean fields, but the boundary homomorphisms need to respect nonlinear relations between harmonic measures [5].

## 2. Dynamics on the boundary of the universal covering tree

In this section, we formulate a precise rigidity theorem for graphs, in terms of measures induced by Patterson–Sullivan measure on a "universal" topological dynamical system ("universal" in that it only depends on the first Betti number of the graph).

**Notation 2.1.** Let $G$ denote a connected graph with vertex set $V$ and edge set $E$. Assume that the first Betti number $b$ satisfies $b > 1$ and that $G$ does not have ends, i.e., vertices of degree 1. Let $\mathcal{T}$ denote the universal covering tree of $G$, so that $G$ is the quotient of $\mathcal{T}$ by its fundamental group $\Gamma \cong \mathbb{F}_b$, a free group of rank $b$. By assumption, $\mathcal{T}$ has no end-vertices and is locally compact. Let $\partial\mathcal{T}$ denote the (topological) space of ends of $\mathcal{T}$, on which $\Gamma$ acts. The space $\partial\mathcal{T}$ consists of equivalence classes of half lines in $\mathcal{T}$, where two half lines are equivalent if they differ in only finitely many edges. Fixing a base point $x_0 \in \mathcal{T}$, the space $\partial\mathcal{T}$ consists of all half lines $p\colon \mathbf{N} \to V(\mathcal{T})$ with $p(0) = x_0$, and for all $n \geq 1$, $p(n) \neq p(n-1)$ and $p(n-1) \neq p(n+1)$. The Borel sigma-algebra of $\partial\mathcal{T}$ is spanned by a basis of clopen sets, the *cylinder sets* $\mathrm{Cyl}_{x_0}(f)$, where $f$ runs through the edges $f \in E(\mathcal{T})$. Here, a cylinder set $\mathrm{Cyl}_{x_0}(f)$ consists of classes of half-lines that originate from $x_0$ and pass through $f$.

**Remark 2.2.** The above description of $\partial\mathcal{T}$ is that of the so-called *visual boundary* of $\mathcal{T}$. It is also possible to define $\partial\mathcal{T}$ as the *hyperbolic boundary* $\partial\mathcal{T} = \overline{\mathcal{T}} - \mathcal{T}$, where $\overline{\mathcal{T}}$ is a metric completion of $\mathcal{T}$ in a suitable hyperbolic metric on $\mathcal{T}$. The two notions coincide in our case, so we will use them interchangeably; cf. [4, Chapter 2].

**Notation 2.3.** Let $\mathrm{Cay}(\mathbb{F}_b)$ denote the Cayley graph of $\mathbb{F}_b$, for any chosen symmetrization of a set of generators $g_1, \dots, g_b$. This is a $2b$-regular tree. Let $C := \partial\mathrm{Cay}(\mathbb{F}_b)$ denote its boundary. The Borel sigma-algebra of $C = \partial\mathrm{Cay}(\mathbb{F}_b)$ is spanned by the cylinder sets $\mathrm{Cyl}(g) := \mathrm{Cyl}_1(g)$ for $g \in \mathbb{F}_b - \{1\}$, given as the set of limits of reduced words that begin with $g$.

The next lemma describes the "universality" of the action of $\mathbb{F}_b$ on $C$.

**Lemma 2.4.** *The tree $\mathcal{T}$ is quasi-isometric to* $\mathrm{Cay}(\mathbb{F}_b)$*, and there is a topological conjugacy $(\Phi_G, \alpha_G)\colon (\partial\mathcal{T}, \Gamma) \to (C, \mathbb{F}_b)$ of dynamical systems, i.e.,*

(1) $\Phi_G\colon \partial\mathcal{T} \to C$ *is a homeomorphism*;

(2) $\alpha_G\colon \Gamma \to \mathbb{F}_b$ *is a group isomorphism*;

(3) *The equivariance $\Phi_G(\gamma x) = \alpha_G(\gamma)\Phi_G(x)$ holds for all $x \in \partial\mathcal{T}$ and $\gamma \in \Gamma$.*

*Proof* (see, e.g., [4, Chapter 4, Theorem 4.1]). Choose a base point $x_0 \in \mathcal{T}$ and an isomorphism $\alpha: \mathbb{F}_b \to \Gamma$. Now

$$\varphi: \mathrm{Cay}(\mathbb{F}_b) \longrightarrow \mathcal{T}, \quad g \longmapsto \alpha(g)(x_0)$$

is a quasi-isometry and extends to a boundary homeomorphism

$$\Phi: C \longrightarrow \partial\mathcal{T}, \quad \lim g \longmapsto \lim \alpha(g)(x_0)$$

(loc. cit., 2.2). The resulting limit map is obviously equivariant with respect to the group isomorphism $\alpha$, since for $h \in \mathbb{F}_b$ and $\lim g \in C$, we have

$$\begin{aligned}\Phi(h \lim g) &= \Phi(\lim hg) \\ &= \lim \alpha(h)\alpha(g)(x) \\ &= \alpha(h) \lim \alpha(g)(x_0) \\ &= \alpha(h)\Phi(\lim g).\end{aligned}$$

Hence we can set $\alpha_G = \alpha^{-1}$ and $\Phi_G = \Phi^{-1}$. □

**Definition 2.5.** Let $d(\cdot,\cdot)$ denote the distance between the vertices of $\mathcal{T}$ (i.e., such that $d(v,w) = 1$ if $v$ and $w$ are adjacent). For $\xi \in \partial\mathcal{T}$ and $x, y \in \mathcal{T}$ the *Busemann function* is defined by

$$B_\xi(x,y) = \lim_{\substack{z \in \mathcal{T} \\ z \to \xi}} (d(x,z) - d(y,z)).$$

A family of positive finite Borel measures $\{\mu_x\}_{x \in \mathcal{T}}$ on $\partial\mathcal{T}$ is called $\Gamma$-*conformal of dimension* $\delta$ if it satisfies the following properties:

(1) the family $(\mu_x)_x$ is $\Gamma$-equivariant, i.e., $\mu_{\gamma x} = (\gamma^{-1})_* \mu_x$, for all $x \in \mathcal{T}, \gamma \in \Gamma$;

(2) for all $x, y \in \mathcal{T}$ the Radon–Nikodym derivative of $\mu_x$ with respect to $\mu_y$ exists and equals $d\mu_x/d\mu_y(\xi) = e^{-\delta B_\xi(x,y)}$.

Observe that if $\xi \in \mathrm{Cyl}_x(f)$, then

$$B_\xi(x, o(f)) = d(x,\xi) - d(o(f),\xi) = d(x, o(f)) \tag{1}$$

is constant in $\xi$. We consider such measures only up to scaling by a global constant.

**Remark 2.6.** Families of $\Gamma$-conformal measures *exist*: let $\mu_x = \mu_{G,x}$ denote the family of Patterson–Sullivan measures for the action of $\Gamma$ on $\mathcal{T}$, based at some point $x \in \mathcal{T}$, defined as the weak limit of measures

$$\mu_x = \lim_{s \to \log(\lambda)} \frac{\sum_{\gamma \in \Gamma} e^{-sd(x,\gamma x)} \delta_{\gamma x}}{\sum_{\gamma \in \Gamma} e^{-sd(x,\gamma x)}},$$

for suitable (unique) $\lambda$ (actually, $\lambda = \lambda_{\mathrm{PF}}$ is the Perron–Frobenius eigenvalue of the edge-adjacency operator $T$ defined in the next section), and $\delta_x$ is the Dirac delta measure at $x$; the dimension of the measures $\mu_x$ is $\log \lambda$.

**Notation 2.7.** Recall that two measures $\mu_1$ and $\mu_2$ on a space $Y$ are called *mutually absolutely continuous* if $\mu_1(A) = 0 \iff \mu_2(A) = 0$ for all measurable $A \subseteq Y$. A measurable map $\varphi\colon (X, \mu_X) \to (Y, \mu_Y)$ between measure spaces is absolutely continuous if $\Phi_* \mu_X$ and $\mu_Y$ are mutually absolutely continuous on $Y$. Here, the *push-forward measure* $\varphi_* \mu_X$ is defined by $\varphi_* \mu_X(A) := \mu_X(\varphi^{-1}(A))$ for measurable $A \subseteq Y$.

We have the following measure-theoretic rigidity theorem.

**Theorem 2.8.** *Let $G$ denote a graph of minimal degree $\geq 3$ and first Betti number $b > 1$, with universal covering tree $\mathcal{T}$ and fundamental group $\Gamma$, and let $\mu$ denote a $\Gamma$-conformal measure on $\mathcal{T}$. Let $(\mathcal{T}', \Gamma', \mu')$ denote the same data associated to another graph $G'$ of minimal degree $\geq 3$ with the same first Betti number $b > 1$. Then $G$ and $G'$ are isomorphic if and only if the push-forward measures $\Phi_{G*}\mu$ and $\Phi_{G'*}\mu'$ on $C$ have the same dimension (in the sense of Definition 2.5) and are mutually absolutely continuous.*

*Proof.* If $G$ and $G'$ are isomorphic, there is nothing to prove. For the converse direction, from [5, Theorem 2.7] we recall measure-theoretic rigidity for graphs (this is a slight variation on the case of graphs with the *same* covering trees, proven by Coornaert in [3]): the graphs $G$ (corresponding to $(\partial\mathcal{T}, \Gamma, \mu)$) and $G'$ (corresponding to $(\partial\mathcal{T}', \Gamma', \mu')$) are isomorphic if and only if there exists

(1) a group isomorphism $\alpha\colon \Gamma \to \Gamma'$ and

(2) a homeomorphism $\varphi\colon \partial\mathcal{T} \to \partial\mathcal{T}'$

such that

(a) $\varphi$ is $\alpha$-equivariant, i.e., we have $\varphi(\gamma x) = \alpha(\gamma)\varphi(x)$, for all $x \in \partial\mathcal{T}, \gamma \in \Gamma$;

(b) the measures $\mu$ and $\mu'$ have the same dimension;

(c) $\varphi$ is absolutely continuous with respect to $\mu$ and $\mu'$.

The theorem is a reformulation of this result by passing to the fixed space $(C, \mathbb{F}_b)$. Set $\varphi := \Phi_{G'}^{-1} \circ \Phi_G$ and $\alpha := \alpha_{G'}^{-1} \circ \alpha_G$. If $\Phi_{G*}\mu$ and $\Phi_{G'*}\mu'$ are absolutely continuous and have the same dimension, then $(\varphi, \alpha)$ satisfy the five listed conditions (1)-(2) and (a)-(c), so $G$ and $G'$ are isomorphic. $\square$

**Remark 2.9.** In [7], rigidity for graphs is formulated in terms of a quantum statistical mechanical system on the boundary operator algebra for the action of $\mathbb{F}_b$ on its Cayley graph.

## 3. The measure and the $T$-operator

In this section, we show that the push-forward of the Patterson–Sullivan measure can be described in terms of the Perron–Frobenius eigenspace of a certain "edge adjacency operator".

**Definition 3.1.** If $e = \{v_1, v_2\} \in E$, we denote by $\vec{e} = (v_1, v_2)$ the edge $e$ with a chosen orientation, and by $\bar{e} = (v_2, v_1)$ the same edge with the inverse orientation to that of $\vec{e}$. Let $o(\vec{e}) = v_1$ denote the origin of $\vec{e}$ and $t(\vec{e}) = v_2$ its end point. The *edge adjacency matrix* $T = T_G$ (compare [16]) is defined as follows. Let $\mathbf{E}$ denote the set of oriented edges of $G$ for any possible choice of orientation, so $|\mathbf{E}| = 2|E|$. Then $T$ is defined to be the $2|E| \times 2|E|$ matrix, in which the rows and columns are indexed by $\mathbf{E}$, and

$$T_{\vec{e}_1,\vec{e}_2} = \begin{cases} 1 & \text{if } t(\vec{e}_1) = o(\vec{e}_2) \text{ but } \vec{e}_2 \neq \bar{e}_1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $b \geq 2$, $T$ is an irreducible non-negative matrix, hence, by Perron–Frobenius theory, it has a unique Perron–Frobenius eigenvalue $\lambda_{\mathrm{PF}}$ that is real of maximal modulus, and a corresponding Perron–Frobenius eigenvector $\mathbf{p}$, unique up to scaling, with all entries non-zero.

**Definition 3.2.** Fix a base point $x_0 \in \mathcal{T}$ and let $v_0$ denote the corresponding vertex in the graph $G$. Choose a spanning tree $\mathcal{B}$ for $G$, and let $\{e_1, \dots, e_b\}$ denote the set of edges outside $\mathcal{B}$; choose an orientation on $e_i$. Let $\{\gamma_1, \dots, \gamma_b\}$ denote a set of generators for the fundamental group of $G$, seen as closed walks based at $v_0$ through an isomorphism $\Gamma \to \pi_1(G, v_0)$, such that $\gamma_i$ is a closed walk that passes through $\mathcal{B}$ and $\gamma_i$, but not through $\gamma_j$ for $j \neq i$. Choose an isomorphism $\alpha \colon \mathbb{F}_b \to \Gamma$ and set $g_i := \alpha^{-1}(\gamma_i)$ as generators for $\mathbb{F}_b$. For $\gamma \in \Gamma$, let $\ell(\gamma)$ denote the *length* of the closed walk $\gamma$. If $g \in \mathbb{F}_b$, we define the *final edge* $\tau(g) \in \mathbf{E}$ of $g$ as follows:

$$\tau(g) = \begin{cases} \vec{e}_i & \text{if } g_i \text{ is the final letter of } g, \\ \bar{e}_i & \text{if } g_i^{-1} \text{ is the final letter of } g, \end{cases}$$

where $g$ is written as a reduced word in the alphabet $\{g_i\}$. We also define the *lifted final edge* $\hat{\tau}(g) \in \mathcal{T}$ of $g$ to be the last occurrence of a lift of $\tau(g)$ from $G$ to $\mathcal{T}$ on the directed path from $x_0$ to $\alpha(g)(x_0)$ in $\mathcal{T}$.

**Lemma 3.3.** *For any $g \in \mathbb{F}_b - \{1\}$, we have $\Phi_G^{-1}(\mathrm{Cyl}(g)) = \mathrm{Cyl}_{x_0}(\hat{\tau}(g))$.*

*Proof.* Let $g \in \mathbb{F}_b - \{1\}$ and let $g_i$ denote the final letter of $g$. Since $\mathrm{Cyl}(g)$ consists of half infinite words starting $gg_j \dots$ with $g_j \neq g_i^{-1}$, under $\Phi_G^{-1}$, this is mapped to half infinite sequences of edges in $\mathcal{T}$, in which there can some backtracking (if there is cancellation of edges between $g_i$ and $g_j$), but never beyond $\hat{\tau}(g)$. Since $\partial\mathcal{T}$ is the universal cover of $G$, backtracking occurs precisely up to $\hat{\tau}(g)$, and $\Phi_G^{-1}(\mathrm{Cyl}(g))$ is the cylinder set of $\hat{\tau}(g)$. $\square$

We now give an intrinsic formula for the push-forward of Patterson–Sullivan measure to the boundary $C$ of the Cayley graph, purely in terms of data related to the original graph:

**Theorem 3.4.** *The push-forward measure $\mu = \Phi_{G*}\mu_{x_0}$ on $C$ is characterised (up to scaling by a global constant) by*

$$\mu(\mathrm{Cyl}(g)) = \lambda_{\mathrm{PF}}^{-\ell(\alpha(g)) + d_{G-\tau(g)}(v_0, o(\tau(g)))} \cdot \mathbf{p}_{\tau(g)}, \tag{2}$$

*where $\lambda_{\mathrm{PF}}$ is the Perron–Frobenius eigenvalue, and $\mathbf{p}$ a Perron–Frobenius eigenvector for $T$ (unique up to scaling).*

*Proof.* We argue as in [10, 3.13 & 4.2 & 4.3]. The conformal dimension of the Patterson–Sullivan measures is $\log \lambda_{\mathrm{PF}}$. Suppose that $\vec{e}$ runs through $\mathbf{E}$, and $\vec{e}'$ runs through a set of lifts of $\vec{e}$ to $\mathcal{T}$, where $x_e = o(\vec{e}')$ is the origin of the lift $\vec{e}'$. Define a vector $w \in \mathbf{R}^{2|E|}$ by $w_{\vec{e}} := \mu_{x_e}(\mathrm{Cyl}_{x_e}(\vec{e}'))$. Then $w$ satisfies the equation $Tw = \lambda_{\mathrm{PF}} \cdot w$, by the conformality of the measure and using (1) to move from adjacent edges back to the original edge. Since $w$ is non-negative, it is unique (up to global scaling; by Perron–Frobenius theory), so, up to scaling, equal to $\mathbf{p}$. By the previous lemma, we have $\mu(\mathrm{Cyl}(g)) = \mu_{x_0}(\mathrm{Cyl}_{x_0}(\hat{\tau}(g))$. The conformality property (2) from Definition 2.5 implies that

$$\mu_{x_0}(\mathrm{Cyl}_{x_0}(\hat{\tau}(g)) = \lambda_{\mathrm{PF}}^{-d(x_0, o(\hat{\tau}(g)))} \cdot \mu_{o(\hat{\tau}(g))}(\mathrm{Cyl}(\hat{\tau}(g))).$$

Now we have just seen that $\mu_{o(\hat{\tau}(g))}(\mathrm{Cyl}(\hat{\tau}(g))) = \mathbf{p}_{\tau(g)}$, and again, conformality and (1) imply that

$$d(x_0, o(\hat{\tau}(g))) = \ell(\alpha(g)) - d_{G-\tau(g)}(v_0, o(\tau(g))). \qquad \square$$

## 4. Measure-theoretic rigidity and the ERC

In this section, we explore the relation between measures on $C$, lengths of loops, and the ERC.

**Notation 4.1.** The *average degree* of a graph $G = (V, E)$ is

$$\bar{d} := \sum_{v \in V} \frac{\deg(v)}{|V|} = 2|E|/|V|.$$

Recall that the result of Müller [14] on ERC is valid for graphs with $\bar{d} \geq 2 \log_2 |V|$. The next theorem gives a conditional result with a constant lower bound on the average degree.

**Theorem 4.2.** *Suppose $G$ and $G'$ are graphs with average degree $\bar{d} > 4$, no degree 1 vertices, and with identical edge decks $\mathcal{D}^e(G) = \mathcal{D}^e(G')$. Then $G$ and $G'$ are isomorphic if and only if there is a* ***length-preserving*** *group isomorphism*

$$\beta \colon \pi_1(G) \to \pi_1(G')$$

(*for chosen base points*); *i.e., such that $\ell(\gamma) = \ell(\beta(\gamma))$ for all $\gamma \in \pi_1(G)$.*

*Proof.* One direction is clear, so let us assume such a length-preserving map $\beta \colon \pi_1(G) \to \pi_1(G')$ exists. The assumed equality of edge decks allows us to identify some invariants of the graphs.

- Since the edge-decks are equal, $G$ and $G'$ have the *same first Betti number* $b = |E| - |V| + 1$ (alternatively, $b = \operatorname{rk} \pi_1^{\text{ab}}(G) = \operatorname{rk} \pi_1^{\text{ab}}(G')$). Furthermore, since $|E| > 2|V|$, we have $b > |V| + 1 \geq 2$.
- The graphs $G$ and $G'$ have the *same Perron–Frobenius eigenvalue* $\lambda_{\text{PF}}$ ([6, Theorem 1] assuming $\bar{d} > 4$). Furthermore, $\lambda_{\text{PF}} > 1$. Indeed, under the given conditions (no degree one vertices and $\bar{d} > 4$), the matrix of $T$ is irreducible positive [6, Notation 7], so the Perron–Frobenius theorem (e.g., [13, p. 673]) applies, and since $T$ has 1 as eigenvalue with multiplicity $b > 1$ [6, Proposition 18], $\lambda_{\text{PF}}$, which is the simple and maximal real eigenvalue of $T$, satisfies $\lambda_{\text{PF}} > 1$.

Let us now give a description of sets of measure zero. Since the measure $\mu$ on $C$ corresponding to $G$ is a Borel measure on a metric space, it is (outer) regular. Hence $\mu(A) = 0$ is equivalent to the existence of a countable family of open sets $\mathcal{U}_i$ with $A \subseteq \mathcal{U}_i$ and $\mu(\mathcal{U}_i) \to 0$. Since an open set is a union of basic open sets, we can write $\mathcal{U}_i$ as a union of cylinder sets, and because cylinder sets are either contained in each other or disjoint, we can write $\mathcal{U}_i = \bigsqcup_j \mathrm{Cyl}(w_{ij})$ for some sequence of words $\mathbf{w} := (w_{ij})$ in chosen symmetrized generators $g_i \in \mathbb{F}_b$; hence $\mu(\mathcal{U}_i) = \sum_j \mu(\mathrm{Cyl}(w_{ij}))$. Conversely, every such construction based on a sequence of words $\mathbf{w}$ produces a set of measure zero. We now express the measure of the cylinder set using Formula (2). Recall that all entries of the Perron–Frobenius eigenvector $\mathbf{p}$ are strictly positive [13, p. 673]. Notice that the second term in the exponent of $\lambda_{\mathrm{PF}}$ of Formula (2) is bounded by the diameter of the graph. This means that $\mu(\mathcal{U}_i) \to 0$ is equivalent to the condition $\mathbf{C}_\alpha(\mathbf{w})$ on $\mathbf{w}$ given by

$$\mathbf{C}_\alpha(\mathbf{w})\colon \sum_j \lambda_{\mathrm{PF}}^{-\ell(\alpha(w_{ij}))} \longrightarrow 0 \quad \text{as } i \to +\infty, \tag{3}$$

where, as before, we have chosen a base point $v_0$ and isomorphism

$$\alpha\colon \mathbb{F}_b \to \pi_1(G, v_0).$$

Now let $\mu'$ denote the corresponding measure on $G'$. Set

$$\alpha' = \beta \circ \alpha\colon \mathbb{F}_b \longrightarrow \pi_1(G').$$

To apply Theorem 2.8, notice that the dimensions of $\mu$ and $\mu'$ are the same (equal to $\log \lambda_{\mathrm{PF}}$). Hence the only condition needed to guarantee that the graphs are isomorphic is that $\mu(A) = 0 \iff \mu'(A) = 0$ for all measurable $A$ in $C$. By the above discussion, this is equivalent to $\mathbf{C}_\alpha(\mathbf{w}) \iff \mathbf{C}_{\alpha'}(\mathbf{w})$ for all $\mathbf{w} = (w_{ij})$ with $w_{ij} \in \mathbb{F}_b$. But since $\beta$ is length-preserving, we have $\ell(\alpha(w_{ij})) = \ell(\beta \circ \alpha(w_{ij})) = \ell(\alpha'(w_{ij}))$ and the result follows. □

**Remark 4.3.** The length of a loop equals the distance in the universal covering tree $\mathcal{T}$ between a lift of the chosen base point and the end point of a lift of the loop. Since, by the Švarc–Milnor's Lemma [4, Chapter 4, Proposition 4.4] all $\mathcal{T}$ are quasi-isometric for fixed $b$ (to $\mathrm{Cay}(\mathbb{F}_b)$), we deduce that there exist non-zero constants $\kappa_1, \kappa_2$ such that $\kappa_1^{-1}\ell(\gamma) - \kappa_2 \le \ell(\beta(\gamma)) \le \kappa_1 \ell(\gamma) + \kappa_2$ for all $\gamma \in \pi_1(G)$. In particular, for fixed $b$, a sequence $\ell(\gamma_i)$ is divergent if and only if $\ell(\beta(\gamma_i))$ is so.

**Remark 4.4.** Only knowledge of $b$ and $\lambda_{\mathrm{PF}}$ is not enough to reconstruct a graph; the smallest non-isomorphic connected graphs (in terms of number of vertices) with minimal degree 2 and the same Ihara zeta function (hence, the same $\lambda_{\mathrm{PF}}$ and $b$) have 8 vertices and 14 edges (and $b = 7$) and are displayed on p. 569 of [15]. We checked that these two graphs have totally disjoint edge decks.

## 5. Reconstruction of closed walks lengths and the ERC

We give a direct and elementary proof of the result alluded to in the previous section, that knowing the "structure of lengths on the space of closed walks" determines the graph uniquely, formulated precisely in Theorem 5.4 below in terms of overlap lengths of loops in special spanning trees of the graph.

Let $G$ denote a connected graph with minimal degree $\delta \geq 2$, without cut vertices (i.e., vertices whose removal results in a disconnected graph) in which every element of the edge deck is connected and has at most one vertex of degree $\delta - 1$. Fix an element $H = G - e$ of the edge deck of $G$ that contains at least one vertex $\alpha$ of degree $\delta - 1$. The missing edge connects $\alpha$ to another (to-be-reconstructed) vertex $\omega \in H$.

**Remark 5.1.** The assumptions are not restrictive in view of the ERC. The edge deck determines the vertex deck (Greenwell [8], compare [2, 6.13]), so if VRC is known for some graph, then ERC also holds for such graphs. Since VRC (hence ERC) is known for disconnected graphs (Kelly [11], compare [2, 4.6]), we can assume that $G$ is connected. Also, since VRC (hence ERC) is known for graphs with a cut vertex without pendant vertices (Bondy, [1]) and we assume that $G$ has no pendant vertices (since $\delta \geq 2$), we can assume that all cards in the edge deck are connected: if such a card is disconnected, any of the end points of the missing edge would be a cut vertex of $G$. Hence any card $H = G - e$ has first Betti number $b_1(H) = b - 1$. Finally, if $H$ contains another vertex of degree $\delta - 1$, then this vertex is $\omega$, and the problem is solved. Hence we can assume that all vertices except $\alpha$ have degree at least $\delta$ in $H$.

Orient the missing edge $\vec{e}$ such that it has origin $o(\vec{e}) = \alpha$. Now let $\gamma_0$ denote an embedded closed walk from $\alpha$ to $\alpha$ through $\vec{e}$ of minimal length $\ell_0$, that passes through $e$ exactly once. Such a closed walk exists: since $H$ is connected, there exists a shortest path $P$ in $H$ from $\omega$ to $\alpha$, which we can close by adding the edge $e$.

**Lemma 5.2.** *The length $\ell_0$ is edge-reconstructible.*

*Proof.* Indeed, $\ell_0 = \min\{r \colon S_r(G) - S_r(G - e) > 0\}$, where $S_r(G)$ is the number of subgraphs of a graph $G$ isomorphic to the cycle graph $C_r$.

Now $\ell_0 < |E|$. Indeed, the length of the path $P$ is at most $|E(H)|$. In case $P$ has length $|E(H)|$, then $H$ is itself a path, and $G$ is a cycle graph. But a cycle graph with at least four edges has two adjacent vertices of minimal degree, which we assume is not the case. Now for $r < |E|$, $S_r(G)$ is edge-reconstructible by Kelly's Lemma [2, 6.6]. □

In the minimal case where $\delta = 2$, i.e., if $\deg_H \alpha = 1$, we make some replacements: we denote by $H$ the graph $H - \alpha$; we denote by $\alpha$ the vertex of the new $H$ that corresponds to the (unique) vertex adjacent to the original $\alpha$ in the original $H$, and we replace $\ell_0$ by $\ell_0 - 1$. After these replacements, we can assume that $H$ had minimal degree $\geq 2$.

We now prove a technical lemma about the existence of certain spanning trees, that will afterwards allow us to construct specific generators of the fundamental group of $H$.

**Lemma 5.3.** *If a vertex $v \in V$ is not equal or adjacent to $\alpha$ in $H$ and has degree $\deg_H v \geq 3$, then there exist oriented edges $\vec{e}_1$ and $\vec{e}_v$ in $H$, such that $t(\vec{e}_1) = \alpha$, $t(\vec{e}_v) = v$, and such that there exists a spanning tree $\mathbb{T}$ for $H$ with $e_1 \notin \mathbb{T}$ and $e_v \notin \mathbb{T}$. If $\tilde{v}$ is a vertex adjacent to $v$, we can furthermore guarantee that $o(\vec{e}_v) \neq \tilde{v}$.*

*If $v \in V$ is not equal but adjacent to $\alpha$ and $\deg_H \alpha \geq 3$, then there exist oriented edges $\vec{e}_1$ and $\vec{e}_v$ in $H$, such that $t(\vec{e}_1) = \alpha$, $t(\vec{e}_v) = v$, $o(\vec{e}_v) \neq \alpha$, and such that there exists a spanning tree $\mathbb{T}$ for $H$ with $e_1 \notin \mathbb{T}$ and $e_v \notin \mathbb{T}$.*

*Proof.* It suffices to prove that there are edges $e_1$ incident to $\alpha$, $e_v$ incident to $v$, $e_1 \neq e_v$ such that $H - e_1 - e_v$ is connected, since then we can set $\mathbb{T}$ to be the spanning tree of $H - e_1 - e_v$. Note that the number of connected components of $H - v$ is at most 2; the missing edge $e$ connects at most two components and the graph $G$ has no cut-vertex by assumption. Since $\deg v \geq 3$ there is at least one connected component $C$ of $H - v$ such that there are two edges connecting $v$ and $C$.

First, assume that $\alpha$ is not adjacent to $v$. If $\alpha \notin C$, but in the other component $C'$, then let $e_v$ be one of the edges connecting $v$ and $C$, and let $e_1$ be any edge incident to $\alpha$ such that $H - e_1$ is connected. Note that this is possible since if there is an edge $\tilde{e}$ incident to $\alpha$ contained in $C'$ such that $C' - \tilde{e}$ is not connected, $\alpha$ would have been a cut-vertex in $G$. If $\alpha \in C$ and $C - \alpha$ is connected, then let $e_v$ be one of the edges connecting $v$ and $C$ and set $e_1$ to be any edge incident with $\alpha$. If $C - \alpha$ is not connected, then any of the connected components of $C - \alpha$ is connected to $v$, since otherwise $\alpha$ would have been a cut-vertex. Pick $e_v$ and $e_1$ such that they connect $v$ and $\alpha$ with different connected components of $C - \alpha$ respectively. It is clear that in both cases the choice can be made such that $o(\vec{e}_v) \neq \tilde{v}$ for any vertex $\tilde{v} \neq \alpha$ adjacent to $v$.

Next, assume that $\alpha$ is adjacent to $v$ and that $\deg_H \alpha \geq 3$. If $\alpha \notin C$, then again pick one of edges connecting $v$ and $C$ as $e_v$ and an edge incident to $\alpha$ but not to $v$ as $e_1$. It is clear that the choice can be made such that $o(\vec{e}_v) \neq \tilde{v}$ for any vertex $\tilde{v} \neq \alpha$ adjacent to $v$. If $\alpha \in C$, then pick for $e_v$ an edge connecting $v$ with $C$ but not incident with $\alpha$ in $H$. Since $\deg_H \alpha \geq 3$ there is an edge in incident with $\alpha$ but not with $o(\vec{e}_v)$ and not with $v$. Pick this edge as $e_1$. □

Fix an arbitrary vertex $v \in V - \{\alpha\}$ and assume that either $v$ is not adjacent to $\alpha$ and $\deg_H v \geq 3$, or $\deg_H \alpha \geq 3$ and $v$ is adjacent to $\alpha$. Let $\mathbb{T}$ denote a corresponding tree constructed in the previous lemma (with respect to a choice of auxiliary vertex $\tilde{v}$ as indicated). Now $EH - E\mathbb{T}$ consists of $b-1$ edges $e_1, \ldots, e_{b-1}$, one of which is $e_v$. Create a basis for the fundamental group of $H$ based at $\alpha$ consisting of closed walks $\gamma_i$ for $i = 1, \ldots, b-1$ that start in $\alpha$, reach $\vec{e}_i$ via $E\mathbb{T}$, go through $\vec{e}_i$ and then back to $\alpha$ through $E\mathbb{T}$. Note that the length of these closed walks can be read off from $H$. Let $\gamma_v$ be the closed walk corresponding to $e_v$.

If $\gamma$ and $\gamma'$ are two closed walks based at $\alpha$, we denote by $s_v(\gamma, \gamma') = s(\gamma, \gamma')$ the length of the path that two closed walks have in common at the start of the walk.

**Theorem 5.4.** *The reconstruction conjecture holds for a graph $G$ with minimal degree $\geq 2$ if we can reconstruct the values $s_v(\gamma_0, \gamma_i)$ for $i = 1, \ldots, b-1$ and for each vertex $v$ of degree $\geq 3$.*

*Proof.* We start with two lemmas.

**Lemma 5.5.** *Assume we know the lengths $s_v(\gamma_0, \gamma_i)$ as in Theorem 5.4. If $\gamma$, expressed as a reduced word in the generators $\{\gamma_i\}_{i=0}^{b-1}$, is an arbitrary closed walk in $G$, based at $\alpha$, then the length of $\gamma$ is edge-reconstructible.*

*Proof.* We know the lengths of all the generating closed walks $\gamma_i$; for $i > 0$, this length can be read off from $H$, and for $i = 0$, we know this length by construction. We also know the lengths of the overlaps $s(\gamma_i^{\pm 1}, \gamma_j^{\pm 1})$; for $i > 0$ and $j > 0$, $s(\gamma_i^{\pm 1}, \gamma_j^{\pm 1})$ can be read off from $H$, and for $i = 0$ or $j = 0$, we know this length by assumption. In a reduced word in $\gamma_i$ and their inverses, no entire closed walk can cancel, since each closed walk contains an edge ($e_i$ for $i > 1$ or $e$ for $\gamma_0$) that does not occur in any other generator. Thus, knowing the overlap between generators and their inverses, we know the length of any closed walk given as a word in the generators and their inverses. Concretely:

$$\ell(\gamma_{i_1}^{\pm 1} \ldots \gamma_{i_n}^{\pm 1}) = \sum_{j=1}^{n} \ell(\gamma_{i_j}^{\pm 1}) - 2 \sum_{j=1}^{n-1} s(\gamma_{i_j}^{\mp 1}, \gamma_{i_{j+1}}^{\pm 1}),$$

where the $\gamma_{i_j}^{\pm}$ is the inverse of $\gamma_{i_j}^{\mp}$. △

**Lemma 5.6.** *Assume we know the lengths $s_v(\gamma_0, \gamma_i)$ as in Theorem 5.4. Let $L[\vec{e}_v]_r$ denote the number of closed walks of length $r$ starting at $\alpha$ that first pass through $\vec{e}_v$ exactly once, and end in $\overleftarrow{e}$, without passing through $e$ before that. If* $\deg v \geq 3$, *then the numbers $L[\vec{e}_v]_r$ are edge-reconstructible.*

*Proof.* The number $L[\vec{e}_v]_r$ is the number of words in $\gamma_i^{\pm 1}$ of total length $r$ in which $\gamma_v$ (the generator through $e_v$) occurs once, and ends (in order of composition) with $\gamma_0^{-1}$, which also occurs exactly once. △

**Remark 5.7.** In Figure 1 we depict a walk as in the lemma. To reconstruct $L[\vec{e}_v]_r$, one needs to check only the (finitely many) words of word length $\leq r$ in the given generators.

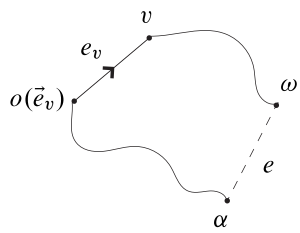


Figure 1. A closed walk starting at $\alpha$, going through $e_v$ and ending at $\overleftarrow{e}$

We proceed with the proof of Theorem 5.4. Now let $v$ run through the vertices of $G$ not equal to $\alpha$. Observe that if $\omega$ is adjacent to $\alpha$ in $H$ then $\deg_H \alpha \geq 3$. Define $D[\vec{e}_v] := \min\{r \colon L[\vec{e}_v]_r \neq 0\}$. There are four cases.

(1) If $\deg v \geq 3$, then $\omega = v$ exactly if the minimal $r$ for which $L[\vec{e}_v]_r \neq 0$ equal two more then the distance between $o(\vec{e}_v)$ and $\alpha$ in $H - e_v$:

$$D[\vec{e}_v] = 2 + d_{H-e_v}(o(\vec{e}_v), \alpha).$$

Since we have reconstructed $L[\vec{e}_v]_r$, we have found $\omega$.

(2) If $\deg v = 2$ and both its neighbouring vertices have degree 2 as well, then $v = \omega$, since we can assume that $G$ has no two adjacent vertices of degree 2.

(3) If $\deg v = 2$ and $v$ has two neighbouring vertices $v_1$ and $v_2$ of degree $\geq 3$, choose $\vec{e}_{v_1}$ and $\vec{e}_{v_2}$ such that $o(\vec{e}_{v_1}) \notin \{v, \alpha\}$ and $o(\vec{e}_{v_2}) \notin \{v, \alpha\}$. Then $\omega = v$ exactly if

$$D[\vec{e}_{v_i}] = 3 + d_{H-e_{v_i}}(o(\vec{e}_{v_i}), \alpha) \quad \text{for } i = 1 \text{ and } i = 2.$$

(4) If $\deg v = 2$, and $v$ has exactly one neighbouring vertex $v_1$ of degree two, then $v_1$ has a neighbouring vertex $v_2$ of degree $\geq 3$ and $v$ has a neighbouring vertex $v_3$ of degree $\geq 3$. Again, choose $\vec{e}_{v_2}$ and $\vec{e}_{v_3}$ such that $o(\vec{e}_{v_2}) \notin \{v, \alpha\}$, and $o(\vec{e}_{v_3}) \notin \{v, \alpha\}$. Now $\omega = v$ exactly if

$$D[\vec{e}_{v_i}] = \begin{cases} 4 + d_{H-e_{v_2}}(o(\vec{e}_{v_2}), \alpha) & \text{if } i = 2, \\ 3 + d_{H-e_{v_3}}(o(\vec{e}_{v_3}), \alpha) & \text{if } i = 3. \end{cases}$$

Hence in all cases, we have reconstructed the missing vertex $\omega$. Recall that if $\delta = 2$, we had changed the meaning of $H, \alpha$ and $\ell_0$, but after having followed the above procedure to reconstruct $\omega$, in this case, the graph $G$ is found by adding a once-subdivided edge between $\alpha$ and $\omega$ in $H$. □

**Remark 5.8.** Tom Kempton suggested to encode the "length overlaps" in a $b \times b$ matrix $O_G$ defined by $O_G(i,j) := \ell(\gamma_i) - s_{v_0}(\gamma_i, \gamma_j)$ (for a chosen base point $b_0$, and for a metric graph $G$ in which edges of the graph can be assigned real positive lengths). Typical questions to ask are (a) what happens to the matrix when the base point is changed? (b) which real positive matrices are the $O$-matrix of a genuine metric graph? (c) when do two matrices correspond to the same graph? (d) Is there interesting dynamics on the "moduli space" of matrices genuinely corresponding to graphs (up to equivalence given by graph isomorphism)?

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## References

[1] J. A. Bondy, On Ulam's conjecture for separable graphs. *Pacific J. Math.* **31** (1969), no. 2, 281–288. MR 0262098 Zbl 0187.45603

[2] J. A. Bondy, A graph reconstructor's manual. In A. D. Keedwell (ed.), *Surveys in combinatorics, 1991.* Papers from the Thirteenth British Combinatorial Conference held at the University of Surrey, Guildford, July 1991. London Mathematical Society Lecture Note Series, 166. Cambridge University Press, Cambridge, 1991, 221–252. MR 1161466 Zbl 0741.05052

[3] M. Coornaert, Rigidité ergodique de groupes d'isométries d'arbres. *C. R. Acad. Sci. Paris Sér.* I Math. **315** (1992), no. 3, 301–304. MR 1179724 Zbl 0759.28012

[4] M. Coornaert, T. Delzant, and A. Papadopoulos, *Géométrie et théorie des groupes*. Lecture Notes in Mathematics, 1441. Springer-Verlag, Berlin, 1990. MR 1075994 Zbl 0727.20018

[5] G. Cornelissen and J. Kool, Measure-theoretic rigidity for Mumford curves. *Ergodic Theory Dynam. Systems* **33** (2013), no. 3, 851–869. MR 3062904 Zbl 1307.14029

[6] G. Cornelissen and J. Kool, Edge reconstruction of the Ihara zeta function. *Electron. J. Combin.* **25** (2018), no. 2, Paper #P2.26, 22 pp. MR 3814260 Zbl 06873185

[7] G. Cornelissen and M. Marcolli, Graph reconstruction and quantum statistical mechanics. *J. Geom. Phys.* **72** (2013), 110–117. MR 3073902 Zbl 1278.05154

[8] D. L. Greenwell, Reconstructing graphs. *Proc. Amer. Math. Soc.* **30** (1971), 431–433. MR 0286699 Zbl 0221.05076

[9] F. Harary, On the reconstruction of a graph from a collection of subgraphs. In M. Fiedler (ed.), *Theory of graphs and its applications.* Proceedings of the Symposium held in Smolenice in June 1963. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1964 47–52. MR 0175111 Zbl 0161.43404

[10] I. Kapovich and T. Nagnibeda, The Patterson–Sullivan embedding and minimal volume entropy for outer space. *Geom. Funct. Anal.* **17** (2007), no. 4, 1201–1236. MR 2373015 Zbl 1135.20031

[11] P. J. Kelly, A congruence theorem for trees. *Pacific J. Math.* **7** (1957), 961–968. MR 0087949 Zbl 0078.37103

[12] T. Kuusalo, Boundary mappings of geometric isomorphisms of Fuchsian groups. *Ann. Acad. Sci. Fenn. Ser.* A I (1973), no. 545, 7 pp. MR 0342692 Zbl 0272.30023

[13] C. Meyer, *Matrix analysis and applied linear algebra*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000. MR 1777382 Zbl 0962.15001

[14] V. Müller, The edge reconstruction hypothesis is true for graphs with more than $n \cdot \log_2 n$ edges. *J. Combinatorial Theory Ser.* B **22** (1977), no. 3, 281–283. MR 0485584 Zbl 0319.05127

[15] A. Setyadi and C. K. Storm, Enumeration of graphs with the same Ihara zeta function. *Linear Algebra Appl.* **438** (2013), no. 1, 564–572. MR 2993401 Zbl 1257.05064

[16] A. Terras, *Zeta functions of graphs*. Cambridge Studies in Advanced Mathematics, 128. Cambridge University Press, Cambridge, 2011. MR 2768284 Zbl 1206.05003

[17] W. T. Tutte, *Graph theory as I have known it*. Oxford Lecture Series in Mathematics and its Applications, 11. The Clarendon Press, Oxford University Press, New York, 1998. MR 1635397 Zbl 0915.05041

[18] L. van Iersel and V. Moulton, Leaf-reconstructibility of phylogenetic networks. *SIAM J. Discrete Math.* **32** (2018), no. 3, 2047–2066. MR 3840884 Zbl 1392.05104

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