NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 48 (2018), 1-10

ON CERTAIN SERIES OF HECKE-TYPE

Heng Huat Chan and Zhi-Guo Liu

(Received September 19, 2017)

Abstract. Around 2004, J. Lovejoy [**8**] proved three Hecke-type series identities using Bailey pairs. In this article, we prove Lovejoy's identities using transformation formulas for q-series discovered by Z.G. Liu in 2013. Some new Hecke-type series are also derived. Our approach also allows us to derive some new Hecke-type identities.

1. Introduction

Let $n \in \mathbf{Z}^+$, $a, q \in \mathbf{C}$ with $|q| < 1$. Let

$$(a;q)_0 = 1,\, (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$$

and

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n = \prod_{k=0}^{\infty}(1-aq^k).$$

Let $a_1, a_2, \dots, a_m$ be complex numbers and define

$$(a_1, a_2, \dots, a_m; q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n,$$

where n is a non-negative integer or ∞. The basic hypergeometric series ${}_r\phi_s$ is defined by

$${}_r\phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,z\right) = \sum_{n=0}^{\infty}\frac{(a_1,\dots,a_r;q)_n}{(q,b_1,\dots,b_s;q)_n}\left((-1)^n q^{n(n-1)/2}\right)^{1+s-r} z^n.$$

In this article, we say that a series is a Hecke-type series if it is of the form

$$\sum_{n=1}^{\infty}\sum_{j=-n+1}^{n} C_{n,j}(q)q^{an^2-bj^2},$$

where $a, b \in \mathbf{Q} - \{0\}$ and $C_{n,j}(q)$ are polynomials in q.

2010 *Mathematics Subject Classification* 05A30, 33D15, 11E25.
Key words and phrases: q-series, Hecke–type series, Rogers' summation formula.
Liu was supported by the National Science Foundation of China (Grant No. 11571114) and Science and Technology Commission of Shanghai Municipality (Grant No. 13dz2260400).

In [**8**], J. Lovejoy used Bailey pairs to prove three identities involving certain Hecke-type series. They are

$$\sum_{n=1}^{\infty}\sum_{j=-n+1}^{n}(-1)^{n+j+1}q^{2n^2-j^2}=\sum_{n=1}^{\infty}\frac{(q;q)_{n-1}}{(-q;q)_n}(-1)^nq^{n(n+1)/2}, \tag{1.1}$$

$$\sum_{n=1}^{\infty}\sum_{j=-n+1}^{n}(-1)^{n+j+1}q^{3n^2-2j^2}=\sum_{n=1}^{\infty}\frac{(q;q)_{2n-1}}{(-q^2;q^2)_n}q^n, \tag{1.2}$$

and

$$\sum_{n=1}^{\infty}\sum_{j=-n+1}^{n}(-1)^{n+j+1}q^{3n^2-j^2}=\sum_{n=1}^{\infty}\frac{(q^2;q^2)_{n-1}}{(-q;q)_{2n}}(-1)^nq^{n(n+1)}. \tag{1.3}$$

In this article, we give new proofs of (1.1), (1.2) and (1.3) using the following identities of Z.G. Liu.

Theorem 1.1. [**6**, Theorem 1.12][1] *For* $\max\{|uab/q|, |qv/c|, |ua|, |ub|\} < 1$, *we have*

$$\frac{(uq, uab/q;q)_\infty}{(ua, ub;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}q/a, & q/b, & v\\ & c, & d\end{matrix};q;\frac{uab}{q}\right) \tag{1.4}$$
$$=\sum_{n=0}^{\infty}\frac{(1-uq^{2n})(u,q/a,q/b,qu/c;q)_n}{(1-u)(q,ua,ub,c;q)_n}(uabc)^nq^{n^2-2n}$$
$$\times\,{}_3\phi_2\left(\begin{matrix}q^{-n}, & uq^n, & d/v\\ & d, & qu/c\end{matrix};q,\frac{qv}{c}\right).$$

Theorem 1.2. [**6**, p. 2089] *For* $\max\{|uab/q|, |ua|, |ub|\} < 1$, *we have*

$$\frac{(uq, uab/q;q)_\infty}{(ua, ub;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}q/a, & q/b, & v\\ & c, & d\end{matrix};q;\frac{uab}{q}\right) \tag{1.5}$$
$$=\sum_{n=0}^{\infty}\frac{(1-uq^{2n})(u,q/a,q/b;q)_n}{(1-u)(q,ua,ub;q)_n}(-uab)^nq^{n^2-2n}$$
$$\times\,{}_3\phi_2\left(\begin{matrix}q^{-n}, & uq^n, & v\\ & c, & d\end{matrix};q,q\right).$$

Our method not only provides a systematic approach to proving (1.1), (1.2) and (1.3) but also leads to their analogues, namely,

$$\sum_{n=1}^{\infty}\sum_{j=-n+1}^{n}(1-q^n)^2(-1)^{n+j+1}q^{2n^2-n-j^2}=\sum_{n=1}^{\infty}\frac{(q;q)_n}{(-q;q)_n}(-1)^nq^{n(n-1)/2}, \tag{1.6}$$

$$\sum_{n=1}^{\infty}\sum_{j=-n+1}^{n}(-1)^{j+1}q^{2n^2-j^2}=\sum_{n=1}^{\infty}\frac{(q;q)_{2n-1}}{(-q;q)_{2n}}q^n, \tag{1.7}$$

and

$$\sum_{n=1}^{\infty}\sum_{j=-n+1}^{n}(1-q^{2n})^2(-1)^{n+j+1}q^{3n^2-2n-j^2}=\sum_{n=1}^{\infty}\frac{(q^2;q^2)_n}{(-q;q)_{2n}}(-1)^nq^{n^2-n}. \tag{1.8}$$

[1] There is a misprint in [**6**] and the power q^{n^2-n} should be replaced by q^{n^2-2n}.

In order to prove identities such as (1.1) and (1.3), we first need to establish some identities which are consequences of Theorem 1.1 and Theorem 1.2. These are given as follows.

Theorem 1.3. *If* $\max\{|ab/q|, |a|, |b|\} < 1$, *then*

$$\frac{(q, ab/q; q)_\infty}{(a, b; q)_\infty} \sum_{n=1}^{\infty} \frac{(q/a, q/b; q)_n}{(-q; q)_n} \left(\frac{ab}{q}\right)^n \tag{1.9}$$
$$= -\sum_{n=1}^{\infty} (1-q^n) \frac{(q/a, q/b; q)_n}{(a, b; q)_n} (ab)^n q^{3n(n+1)/2} \sum_{j=-n+1}^{n+1} (-1)^j q^{-j^2}.$$

Theorem 1.4. *For* $\max\{|ab/q^2|, |a|, |b|\} < 1$, *we have*

$$\frac{(q^2, ab/q^2; q^2)_\infty}{(a, b; q^2)_\infty} \sum_{n=1}^{\infty} \frac{(q^2/a, q^2/b; q^2)_n}{(-q; q)_{2n}} \left(\frac{ab}{q^2}\right)^n \tag{1.10}$$
$$= -\sum_{n=1}^{\infty} (1-q^{2n}) \frac{(q^2/a, q^2/b; q^2)_n}{(a, b; q^2)_n} (ab)^n q^{2n^2-3n} \sum_{j=-n+1}^{n} (-1)^j q^{-j^2}.$$

Theorem 1.3 will be established using Theorem 1.1 in Section 2. This is followed by proofs of (1.1), (1.2) and (1.6).

Theorem 1.4 will be derived using Theorem 1.2 in Section 3 and will be used to establish (1.3), (1.7) and (1.8).

We end this introduction by providing our motivation behind this work. In a recent work of L.Q. Wang and A.J. Yee [**9**], several new Hecke-Rogers type identities were discovered. All except one of the identities in [**9**] were proved using Theorem 1.1. The remaining identity was established using (1.1), an identity established by Lovejoy using Bailey pairs. Our intention to give a new proof of (1.1) is to provide readers of [**9**] a complete understanding of their work with only the knowledge of Theorem 1.1.

2. Proofs of Theorem 1.3, (1.1), (1.2) and (1.6)

We begin this section by proving Theorem 1.3 with the aid of Theorem 1.1.

Proof of (1.9). First, set $d = 0, v = q$ and $c = -q$ in (1.4) to conclude that

$$\frac{(uq, uab/q; q)_\infty}{(ua, ub; q)_\infty} \sum_{n=0}^{\infty} \frac{(q/a, q/b; q)_n}{(-q; q)_n} \left(\frac{uab}{q}\right)^n \tag{2.1}$$
$$= \sum_{n=0}^{\infty} \frac{(1-uq^{2n})(u, q/a, q/b, -u; q)_n}{(1-u)(q, ua, ub, -q; q)_n} (-uab)^n q^{n^2-n} {}_2\phi_1 \left(\begin{matrix} q^{-n} & uq^n \\ & -u \end{matrix}; q, -q \right).$$

We next rewrite (2.1) as

$$\frac{(uq, uab/q; q)_\infty}{(ua, ub; q)_\infty} + \frac{(uq, uab/q; q)_\infty}{(ua, ub; q)_\infty} \sum_{n=1}^{\infty} \frac{(q/a, q/b; q)_n}{(-q; q)_n} \left(\frac{uab}{q}\right)^n \tag{2.2}$$
$$= 1 + \sum_{n=1}^{\infty} \frac{(1-uq^{2n})(q/a, q/b, u, -u; q)_n}{(1-u)(q, ua, ub, -q; q)_n} (-uab)^n q^{n^2-n} {}_2\phi_1 \left(\begin{matrix} q^{-n} & uq^n \\ & -u \end{matrix}; q, -q \right).$$

Letting $u \to 1$ in (2.2), we deduce that

$$\frac{(q, ab/q; q)_\infty}{(a, b; q)_\infty} + \frac{(q, ab/q; q)_\infty}{(a, b; q)_\infty} \sum_{n=1}^{\infty} \frac{(q/a, q/b; q)_n}{(-q; q)_n} \left(\frac{ab}{q}\right)^n \tag{2.3}$$
$$= 1 + 2\sum_{n=1}^{\infty} \frac{(q/a, q/b; q)_n}{(a, b; q)_n} (-ab)^n q^{n^2-n} {}_2\phi_1 \left(\begin{matrix} q^{-n} & q^n \\ & -1 \end{matrix}; q, -q\right).$$

Next, observe that [**3**, (2.5)]

$$2\, {}_2\phi_1 \left(\begin{matrix} q^{-n} & q^n \\ & -1 \end{matrix}; q, -q\right) = 2q^{-n(n-1)/2} + (-1)^{n-1}(1-q^n)q^{n(n-1)/2} \sum_{j=-n+1}^{n-1} (-1)^j q^{-j^2} \tag{2.4}$$
$$= (1+q^n)q^{-n(n+1)/2} + (-1)^{n-1}(1-q^n)q^{n(n-1)/2} \sum_{j=-n+1}^{n} (-1)^j q^{-j^2}.$$

Substituting (2.4) to the right hand side of (2.3), we deduce that

$$\frac{(q, ab/q; q)_\infty}{(a, b; q)_\infty} + \frac{(q, ab/q; q)_\infty}{(a, b; q)_\infty} \sum_{n=1}^{\infty} \frac{(q/a, q/b; q)_n}{(-q; q)_n} \left(\frac{ab}{q}\right)^n \tag{2.5}$$
$$= 1 + \sum_{n=1}^{\infty} \frac{(q/a, q/b; q)_n}{(a, b; q)_n} \left(-\frac{ab}{q}\right)^n q^{n(n-1)/2}(1+q^n)$$
$$- \sum_{n=1}^{\infty} (1-q^n) \frac{(q/a, q/b; q)_n}{(a, b; q)_n} (ab)^n q^{3n(n+1)/2} \sum_{j=-n+1}^{n} (-1)^j q^{-j^2}.$$

Using Rogers' ${}_6\phi_5$ summation formula [**4**, (2.7.1)]

$$\sum_{n=0}^{\infty} \frac{(1-uq^{2n})(u; q)_n (q/a, q/b, q/c; q)_n}{(1-u)(ua, ub, uc, q; q)_n} \left(\frac{uabc}{q^2}\right)^n \tag{2.6}$$
$$= \frac{(uq, uab/q, ubc/q, uac/q; q)_\infty}{(ua, ub, uc, uabc/q^2; q)_\infty}$$

with $u \to 1$ and $c = 0$, we find that the first two terms of the right hand side of (2.5) can be written as

$$1 + \sum_{n=1}^{\infty} \frac{(q/a, q/b; q)_n}{(a, b; q)_n} \left(-\frac{ab}{q}\right)^n q^{n(n-1)/2}(1+q^n) = \frac{(q, ab/q; q)_\infty}{(a, b; q)_\infty}. \tag{2.7}$$

Identity (2.7) shows that the first two terms of the right hand side of (2.5) cancel with the first term of its left hand side. This completes the proof of Theorem 1.9. □

Next, we use Theorem 1.3 to derive (1.1), (1.2) and (1.6).

Proof of (1.1)**.** We divide both sides of (1.9) by $1 - q/a$ and then let $a \to q$ to deduce that

$$\sum_{n=1}^{\infty} \frac{(q/b; q)_n (q; q)_{n-1}}{(-q; q)_n} b^n = -\sum_{n=1}^{\infty} \frac{(q/b; q)_n}{(b; q)_n} b^n q^{(3n^2-n)/2} \sum_{j=-n+1}^{n} (-1)^j q^{-j^2}. \tag{2.8}$$

Letting $b \to 0$ in (2.8) and simplifying, we complete the proof of (1.1). □

Proof of (1.2). Identity (1.2) is established by replacing q by q^2 and setting $b = q$ in (2.8). □

Remark 2.1. Identities (1.9) and (2.8) can be viewed as a two-variable extension and a one-variable extension of (1.1) and (1.2), respectively.

Proof of (1.6). Differentiating (2.8) with respect to b, we deduce that

$$\sum_{n=1}^{\infty} \frac{(q/b;q)_n (q;q)_{n-1}}{(-q;q)_n} b^n \left(\frac{1}{b-q} + \frac{1}{b-q^2} + \cdots + \frac{1}{b-q^n} \right) \tag{2.9}$$
$$= -\sum_{n=1}^{\infty} \sum_{j=-n+1}^{n} \frac{(q/b;q)_n}{(b;q)_n} b^n q^{(3n^2-n)/2-j^2}$$
$$\times \left(\frac{1}{b-q} + \frac{1}{b-q^2} + \cdots + \frac{1}{b-q^n} + \frac{1}{1-b} + \frac{q}{1-bq} + \cdots + \frac{q^{n-1}}{1-bq^{n-1}} \right).$$

Letting $b = 0$ in (2.9) and using the identities

$$\frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^n} = \frac{(1-q^n)q^{-n}}{1-q},$$

and

$$-\frac{1}{q} - \frac{1}{q^2} - \cdots \frac{1}{q^n} + 1 + q + \cdots + q^{n-1} = -\frac{(1-q^n)^2 q^{-n}}{1-q},$$

we complete the proof of (1.6). □

3. Proofs of Theorem 1.4, (1.3), (1.7) and (1.8)

We begin this section by proving Theorem 1.4 with Theorem 1.2 and Rogers' ${}_6\phi_5$ summation.

Proof of Theorem 1.4. In (1.10), we replace q by q^2, followed by the substitutions $v = q^2, c = -q$ and $d = -q^2$ to deduce that

$$\frac{(uq^2, uab/q^2;q^2)_\infty}{(ua,ub;q^2)_\infty} + \frac{(uq^2, uab/q^2;q^2)_\infty}{(ua,ub;q^2)_\infty} \sum_{n=1}^{\infty} \frac{(q^2/a, q^2/b;q^2)_n}{(-q;q)_{2n}} \left(\frac{uab}{q^2}\right)^n = \tag{3.1}$$
$$1 + \sum_{n=1}^{\infty} \frac{(1-uq^{4n})(u, q^2/a, q^2/b;q^2)_n}{(1-u)(q^2, ua, ub;q^2)_n} (-uab)^n q^{n^2-3n} {}_3\phi_2 \left(\begin{matrix} q^{-2n}, & uq^{2n}, & q^2 \\ & -q, & -q^2 \end{matrix} ; q^2, q^2 \right).$$

Now, letting $u \to 1$ in (3.1), we deduce that

$$\frac{(q^2, ab/q^2;q^2)_\infty}{(a,b;q^2)_\infty} + \frac{(q^2, ab/q^2;q^2)_\infty}{(a,b;q^2)_\infty} \sum_{n=1}^{\infty} \frac{(q^2/a, q^2/b;q^2)_n}{(-q;q)_{2n}} \left(\frac{ab}{q^2}\right)^n = \tag{3.2}$$
$$1 + \sum_{n=1}^{\infty} (1+q^{2n}) \frac{(q^2/a, q^2/b;q^2)_n}{(a,b;q^2)_n} (-ab)^n q^{n^2-3n} {}_3\phi_2 \left(\begin{matrix} q^{-2n}, & q^{2n}, & q^2 \\ & -q, & -q^2 \end{matrix} ; q^2, q^2 \right).$$

Combining (3.2) with the identity [**3**, (6.6)]

$$(1+q^{2n})\,{}_3\phi_2\left(\begin{matrix}q^{-2n}, & q^{2n}, & q^2\\ & -q, & -q^2\end{matrix};q^2,q^2\right)$$
$$=(1+q^{2n})+(-1)^{n-1}(1-q^{2n})q^{n^2}\sum_{j=-n+1}^{n}(-1)^j q^{-j^2},$$

we deduce that

$$\frac{(q^2,ab/q^2;q^2)_\infty}{(a,b;q^2)_\infty}+\frac{(q^2,ab/q^2;q^2)_\infty}{(a,b;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(q^2/a,q^2/b;q^2)_n}{(-q;q)_{2n}}\left(\frac{ab}{q^2}\right)^n \tag{3.3}$$
$$=1+\sum_{n=1}^{\infty}(1+q^{2n})\frac{(q^2/a,q^2/b;q^2)_n}{(a,b;q^2)_n}(-ab)^n q^{n^2-3n}$$
$$-\sum_{n=1}^{\infty}(1-q^{2n})\frac{(q^2/a,q^2/b;q^2)_n}{(a,b;q^2)_n}(ab)^n q^{2n^2-3n}\sum_{j=-n+1}^{n}(-1)^j q^{-j^2}.$$

Using the Rogers summation formula (2.6) with $u=1$, $c=0$ and q replaced by q^2, we find that the first term on the left hand side of (3.3) is equal to the sum of the first two terms on the right hand side of (3.3) and the proof of (1.10) is complete. □

We now prove (1.3), (1.7) and (1.8) by using (1.10).

Proof of (1.3)**.** Divide both sides of (1.10) by $1-q^2/a$, followed by letting $a\to q^2$ to deduce that

$$\sum_{n=1}^{\infty}\frac{(q^2;q^2)_{n-1}(q^2/b;q^2)_n}{(-q;q)_{2n}}b^n \tag{3.4}$$
$$=-\sum_{n=1}^{\infty}(1-q^{2n})\frac{(q^2;q^2)_{n-1}(q^2/b;q^2)_n}{(b,q^2;q^2)_n}b^n q^{2n^2-n}\sum_{j=-n+1}^{n}(-1)^j q^{-j^2}.$$

Substituting $b=0$ in (3.4), we conclude the proof of (1.3). □

Proof of (1.7)**.** Let $b=q$ in (3.4) and (1.7) follows immediately. □

Remark 3.1. Identities (1.10) and (3.4) can be viewed as a two-variable and a one-variable extension of both (1.3) and (1.7), respectively.

Proof of (1.8)**.** Differentiating (3.4) with respect to b, we deduce that

$$\sum_{n=1}^{\infty}\frac{(q^2/b;q^2)_n(q^2;q^2)_{n-1}}{(-q;q)_{2n}}b^n\left(\frac{1}{b-q^2}+\frac{1}{b-q^4}+\cdots+\frac{1}{b-q^{2n}}\right) \tag{3.5}$$
$$=-\sum_{n=1}^{\infty}\sum_{j=-n+1}^{n}\frac{(q^2/b;q^2)_n}{(b;q^2)_n}b^n q^{2n^2-n-j^2}$$
$$\times\left(\frac{1}{b-q^2}+\frac{1}{b-q^4}+\cdots+\frac{1}{b-q^{2n}}+\frac{1}{1-b}+\frac{q^2}{1-bq^2}+\cdots+\frac{q^{2n-2}}{1-bq^{2n-2}}\right).$$

Letting $b = 0$ in (3.5) and using the identities

$$\frac{1}{q^2} + \frac{1}{q^4} + \cdots + \frac{1}{q^{2n}} = \frac{(1-q^{2n})q^{-2n}}{1-q^2},$$

and

$$-\frac{1}{q^2} - \frac{1}{q^4} - \cdots \frac{1}{q^{2n}} + 1 + q^2 + \cdots + q^{2n-2} = -\frac{(1-q^{2n})^2 q^{-2n}}{(1-q^2)},$$

we complete the proof of (1.8). □

4. Conclusion

In the previous sections, we established identities such as (1.1) using Liu's identities (1.4), (1.5) and Rogers' formula (2.6). The approach presented can further be modified to produce other identities associated with Hecke-type series. We end this article with proofs of some of these identities, some of which are new. For simplicity, let

$$S_n(q) = \sum_{j=-n}^{n} (-1)^j q^{-j^2}.$$

Theorem 4.1. *For* $\max\{|a|, |b|, |ab/q|\} < 1$, *we have*

$$\begin{aligned} &\frac{(q, ab/q; q)_\infty}{(a, b; q)_\infty} \sum_{n=0}^{\infty} \frac{(q/a, q/b; q)_n (ab/q)^n}{(-q; q)_n} \qquad (4.1)\\ &= \sum_{n=0}^{\infty} \frac{(q/a, q/b; q)_n (ab)^n q^{(3n^2-n)/2}}{(a, b; q)_n} \left\{1 - \frac{(a-q^{n+1})(b-q^{n+1})q^{2n}}{(1-aq^n)(1-bq^n)}\right\} S_n(q). \end{aligned}$$

Theorem 4.1 was first proved by Liu [**7**, Proposition 6.14]. We now give another proof of Theorem 4.1 using Theorem 1.1.

Proof of Theorem 4.1. Setting $d = 0$ and $v = q$ in (1.4), we immediately find that

$$\begin{aligned} &\frac{(qu, abu/q; q)_\infty}{(au, bu; q)_\infty} \sum_{n=0}^{\infty} \frac{(q/a, q/b; q)_n (uab/q)^n}{(c; q)_n} \qquad (4.2)\\ &= \sum_{n=0}^{\infty} \frac{(1-uq^{2n})(u, q/a, q/b, qu/c; q)_n}{(1-u)(q, ua, ub, c; q)_n} (uabc)^n q^{n^2-2n} \\ &\quad \times {}_2\phi_1\left(\begin{matrix} q^{-n}, & uq^n \\ & qu/c \end{matrix}; q, q^2/c\right). \end{aligned}$$

Letting $c = -q$ and $u \to 1$ in (4.2), we conclude that

$$\begin{aligned} &\frac{(q, ab/q; q)_\infty}{(a, b; q)_\infty} \sum_{n=0}^{\infty} \frac{(q/a, q/b; q)_n (uab/q)^n}{(c; q)_n} \qquad (4.3)\\ &= 1 + 2\sum_{n=1}^{\infty} \frac{(q/a, q/b; q)_n}{(a, b; q)_n} (-ab)^n q^{n^2-n} \\ &\quad \times {}_2\phi_1\left(\begin{matrix} q^{-n}, & q^n \\ & -1 \end{matrix}; q, -q\right). \end{aligned}$$

To complete the proof of Theorem 4.1, we recall Andrews' identity [**3**, (2.5)] which states that

$$2\ {}_2\phi_1\left(\begin{matrix} q^{-n}, & q^n \\ & -1 \end{matrix}; q, -q\right) = (-1)^n q^{n(n-1)/2}\left(q^n S_n(q) - S_{n-1}(q)\right). \tag{4.4}$$

Substituting (4.4) into (4.3) and simplifying, we complete the proof of (4.1). □

Setting $a = b = 0$ in (4.1), we obtain the following identity due to Andrews [**1**, (6.1)], [**5**, (8.16)]:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n} = \frac{1}{(q;q)_\infty}(-1)^j(1-q^{4n+2})q^{n(5n+1)/2-j^2}.$$

Multiplying both sides of (4.1) by $1-a$ and then letting $a \to 1, b \to 0$, we derive the identity

$$\begin{aligned}&\sum_{n=0}^{\infty}(-1)^n \frac{(q;q)_n}{(-q;q)_n} q^{n(n-1)/2} \\ &= \sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^{n+j}(1-q^n+q^{3n+1}-q^{4n+2})q^{2n^2-j^2},\end{aligned}$$

which is Proposition 6.16 of [**7**]. Using Theorems 1.1, we can also prove the following.

Theorem 4.2. *For* $\max\{|qa|, |qb|, |ab|\} < 1$, *we have*

$$\begin{aligned}&\frac{(q, ab : q)_\infty}{(qa, qb; q)_\infty}\left\{1 + 2\sum_{n=1}^{\infty}\frac{(q/a, q/b; q)_n (ab)^n}{(q;q)_n(1+q^n)}\right\} \\ &= \sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^j(1-q^{2n+1})\frac{(q/a, q/b; q)_n}{(qa, qb; q)_n}(ab)^n q^{n(n-1)/2+j^2}.\end{aligned} \tag{4.5}$$

Proof. Setting $c = -q, d = 0, u = q$ and $v = -1$ in Theorems 1.1, we find that

$$\begin{aligned}&\frac{(q, ab; q)_\infty}{(qa, qb; q)_\infty}\left\{1 + 2\sum_{n=1}^{\infty}\frac{(q/a, q/b; q)_n (ab)^n}{(q;q)_n(1+q^n)}\right\} \\ &= \sum_{n=0}^{\infty}\frac{(1-q^{2n+1})(q/a, q/b; q)_n}{(qa, qb; q)_n}(-ab)^n q^{n^2} \\ &\quad \times {}_2\phi_1\left(\begin{matrix} q^{-n}, & q^{n+1} \\ & -q \end{matrix}; q, 1\right).\end{aligned} \tag{4.6}$$

Letting $b = d = 0, c = -q$ in [**6**, (2.1)] and then letting $\alpha \to 1$, we find that

$${}_2\phi_1\left(\begin{matrix} q^{-n}, & q^{n+1} \\ & -q \end{matrix}; q, 1\right) = (-1)^n q^{-n(n+1)/2}\sum_{j=-n}^{n}(-1)^j q^{j^2}. \tag{4.7}$$

(The factor $(-1)^n$ is missing in Propositions 2.4, 2.5, 2.6 and Theorem 4.10 of [**6**].)

Substituting (4.7) into the right hand side of (4.6), we complete the proof of Theorem 4.2. □

Setting $a = b = 0$ in (4.5), we immediately deduce that

$$1 + 2\sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1+q^n)(q;q)_n} = (q;q)_\infty^{-1} \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^j q^{j^2+(3n^2+n)/2}. \tag{4.8}$$

Letting $b = 0$, followed by $a \to 1$ in (4.5), we find that

$$1 + 2\sum_{n=1}^{\infty} (-1)^n \frac{q^{n(n+1)/2}}{1+q^n} = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^{n+j}(1-q^{2n+1})q^{n^2+j^2}. \tag{4.9}$$

It appears that (4.8) and (4.9) are new.

There are very few identities involving Hecke-type series given in (4.9). A similar identity

$$\sum_{n=1}^{\infty} \frac{q^n(q;q^2)_n}{(-q;q^2)_n(1+q^{2n})} = \sum_{n=1}^{\infty} \sum_{j=-n+1}^{n} (-1)^j q^{n^2+j^2}$$

was recently discovered and proved by Wang and Yee [**9**, Proof of Theorem 1.1].

5. Acknowledgments

We are extremely grateful to the referee for his/her helpful comments and suggestions. Part of this work was written when the first author was a visitor at Max Planck Institute for Mathematics in Bonn. He would like to thank Pieter Moree for his invitation and warm hospitality.

References

[1] G.E. Andrews, *The fifth and seventh order mock theta functions*, Trans. Am. Math. Soc., **293** (1986) 113–134.

[2] G.E. Andrews, F.J. Dyson and D. Hickerson, *Partitions and indefinite quadratic forms*, Invent. Math., **91** (1988) 391–407.

[3] G.E. Andrews, *q-Orthogonal polynomials, Rogers–Ramanujan identities and mock theta functions*, Proc. Steklov Inst. Math., **276** (2012) 21–32.

[4] G. Gasper and M. Rahman, *Basic Hypergeometric Series, 2nd edn.*, Cambridge Univ. Press, Cambridge, 2004.

[5] Z.-G, Liu, *An expansion formula for q-series and applications*, Ramanujan J., **6** (2002) 429–447.

[6] Z.-G. Liu, *On the q-derivative and q-series expansions*, Int. J. Number Theory, **9** (2013) 2069–2089.

[7] Z.-G. Liu, *A q-series expansion formula and the Askey-Wilson polynomials*, Ramanujan J., **30** (2013) 193–210.

[8] J. Lovejoy, *Overpartitions and real quadratic fields*, J .Number Theory, **106** (2004) 178–186.

[9] L.Q. Wang and A.J. Yee, *Some Hecke-Rogers type identities*, preprint.

Heng Huat Chan
Main address:
Department of Mathematics,
National University of Singapore,
Block S17,
10 Lower Kent Ridge Road,
Singapore 119076, Singapore

Second address:
University of Vienna,
Faculty of Mathematics,
Oskar-Morgenstern-Platz 1,
1090 Wien,
Austria

Third address:
Max Planck Institute for Mathematics,
Vivatsgasse 7,
53111 Bonn,
Germany

matchh@nus.edu.sg

Zhi-Guo Liu
School of Mathematical Sciences,
East China Normal University,
500 Dongchuan Road,
Shanghai 200241, P. R. China
zgliu@math.ecnu.edu.cn