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Hochschild cohomology of Sullivan algebras and mapping spaces

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Abstract. Let $f : X \to Y$ be a map between simply connected spaces having the homotopy of finite type CW-complexes, where $H^*(Y, \mathbb{Q})$ is finite dimensional and $\phi : (\wedge V, d) \to (B, d)$ a Sullivan model of f. We consider (B, d) as a module over $\wedge V$ via the mapping ϕ. Let $\mathrm{map}(X, Y; f)$ denote the component of f in the space of mappings from X to Y. In this paper we show that there is a canonical injection $\pi_*(\Omega\, \mathrm{map}(X, Y; f))\otimes\mathbb{Q} \to HH^*(\wedge V; B)$.

Keywords: Hochschild cohomology; Mapping space; L_∞ algebra

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1. Introduction

We work in the rational homotopy setting for which the standard reference is [6]. In this section we fix notation and recall a few facts on the Hochschild cohomology of an algebra. All vector spaces and algebras are taken over a field $\mathbb{k}$ of characteristic 0.

Definition 1. A lower graded vector space V is a direct sum of vector spaces, that is, $V = \oplus_i V_i$, where $i \in \mathbb{Z}$. We say that element $a \in V_i$ is homogeneous of degree i and we write $|a| = i$ and $V = V_\bullet$ is lower or homologically graded. If $V = \oplus_{i\geq 0} V_i$, then V is said to be non negatively graded. In the same way $V^\bullet = \oplus_i V^i$ is called cohomologically

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graded. We use the standard convention $V^i := V_{-i}$. Hence if $V = \oplus_{i\geq 0} V^i$, the dual space of V is denoted $V^{\#} = \prod_i \mathrm{Hom}(V^i, \mathbb{k}) = \prod_i \mathrm{Hom}(V_{-i}, \mathbb{k})$ has a lower non negative grading.

Definition 2. A morphism of graded vector spaces $f : V \to W$ of degree r, is a family of linear maps $f_n : V_n \to W_{n+r}$.

Let (M, d) be a differential (A, d)-bimodule. The Hochschild cohomology of A with coefficients in M is defined as $\mathrm{Ext}_{A^e}(A, M)$ where A is an $A^e = A \otimes A^{op}$-module under the action $(a_1 \otimes a_2)a = (-1)^{|a|\,|a_2|} a_1 a a_2$, where $a, a_1, a_2 \in A$.

Let $(P, d_P) \to (A, d)$ be a semifree resolution of A as an A^e-module [5], and (M, d_M) an A^e-differential module. Then $HH^*(A; M) := \mathrm{Ext}_{A^e}(A, M)$ is the homology of the complex $(\mathrm{Hom}_{A^e}(P, M), D)$, where the differential is defined by

$$(Df)(x) = d_M f(x) - (-1)^{|f|} f(d_P x). \tag{1}$$

In the sequel we work in the category of commutative differential graded algebras (cdga's for short). This implies that left (or right) modules have a natural bimodule structure. Let $f : A \to B$ be a morphism of cdga's. Then B is considered as an A-module by the action induced by f.

Our aim is to study the structure of $HH^*(A; B)$. Let $(\wedge V, d)$ be a Sullivan algebra, and $m : (\wedge V \otimes \wedge V, d' = d \otimes 1 + 1 \otimes d) \to (\wedge V, d)$ the multiplication. Then there is a quasi isomorphism

$$(\wedge V \otimes \wedge V \otimes \wedge \bar{V}, D) \to (\wedge V, d)$$

making the following diagram commutative.

$$\begin{array}{ccc} (\wedge V \otimes \wedge V, d') & \xrightarrow{\ m\ } & (\wedge V, d) \\ \big\downarrow \iota & \overset{p}{\underset{\simeq}{\nearrow}} & \\ (\wedge V \otimes \wedge V \otimes \wedge \bar{V}, D). & & \end{array}$$

Moreover $\bar{V}^n = V^{n+1}$ and the differential D is defined by

$$D(\bar{v}) = v \otimes 1 - 1 \otimes v + \alpha,\ \alpha \in \wedge V \otimes \wedge V \otimes \wedge^+ \bar{V},$$

and ι is the canonical inclusion [6, §15]. The quasi isomorphism

$$(\wedge V \otimes \wedge V \wedge \otimes \bar{V}, D) \xrightarrow{p} (\wedge V, d)$$

is a semifree resolution of $(\wedge V, d)$ as a $\wedge V \otimes \wedge V$-module [5,10]. Therefore, for any $\wedge V$-module M, $HH^*(\wedge V; M)$ is the homology of the complex

$$(\mathrm{Hom}_{\wedge V \otimes \wedge V}(\wedge V \otimes \wedge V \otimes \wedge \bar{V}, M), D),$$

where the differential is defined by (1).

We consider the cdga $(\wedge V \otimes \wedge \bar{V}, \tilde{D})$ where $Dv = dv$, $\tilde{D}(\bar{v}) = -S(dv)$ and S is the unique derivation on $\wedge V \otimes \wedge \bar{V}$ defined by $Sv = \bar{v}$ and $S\bar{v} = 0$. It is obtained as a push out in the diagram below.

$$\begin{array}{ccc} (\wedge V \otimes \wedge V, d') & \xrightarrow{\ \iota\ } & (\wedge V \otimes \wedge V \otimes \wedge \bar{V}, D) \\ \big\downarrow m & & \big\downarrow m' \\ (\wedge V, d) & \longrightarrow & (\wedge V \otimes \wedge \bar{V}, \tilde{D}). \end{array}$$

Moreover, the composition with m' yields an isomorphism of complexes

$$\mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge \bar{V}, M) \xrightarrow{\cong} \mathrm{Hom}_{\wedge V \otimes \wedge V}(\wedge V \otimes \wedge V \otimes \wedge \bar{V}, M).$$

As $\tilde{D}(\wedge V \otimes \wedge^n \bar{V}) \subset \wedge V \otimes \wedge^n \bar{V}$, hence each $(\mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge^n \bar{V}, M), \tilde{D})$ is a sub cochain complex [8]. This gives a Hodge type decomposition of the Hochschild cohomology

$$HH^*(\wedge V; M) = \oplus_{n \geq 0} HH^*_{(n)}(\wedge V; M)$$

for any $\wedge V$-differential module (M, d) [11,7].

Let $f : X \to Y$ be a map between simply connected spaces having the homotopy of finite type CW-complexes and assume that $H^*(Y, \mathbb{Q})$ is finite dimensional. Let $\phi : (\wedge V, d) \to (B, d)$ be a cdga model of f. We consider (B, d) as a module over $\wedge V$ via the mapping ϕ. Denote by $\mathrm{map}(X, Y; f)$ the component of f in the space of mappings from X to Y. In this paper we show the following result.

Theorem 3. *There is a canonical injection*

$$\pi_*(\Omega\, \mathrm{map}(X, Y; f)) \otimes \Bbbk \to HH^*(\wedge V; B).$$

Moreover $\pi_*(\mathrm{map}(X, Y; f)) \otimes \Bbbk \cong HH^*_{(1)}(\wedge V; B)$.

The result is a generalization of the inclusion $\pi_*(\Omega\, \mathrm{map}(X, X; 1_X)) \otimes \Bbbk \to HH^*(\wedge V; \wedge V)$. See [7, Theorem 2] and [9, Theorem 1.1].

2. $\mathbf{L_\infty}$-MODELS OF MAPPING SPACES

The notion of L_∞ algebra was introduced by Lada [14] and L_∞ models of mapping spaces were used by Félix et al. in [3,4]. We remind here their definition.

Definition 4. A permutation $\sigma \in S_k$ is called an $(i, k-i)$ shuffle if $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(k)$ where $i = 1, \ldots, n$. For graded objects $x_1, \ldots, x_k$, the Koszul sign $\epsilon(\sigma)$ is determined by

$$x_1 \wedge \cdots \wedge x_k = \epsilon(\sigma) x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(k)}.$$

It depends not only of the permutation σ but also on degrees of $x_1, \ldots, x_k$.

Definition 5. An L_∞-algebra or a strongly homotopy Lie algebra is a graded vector space $L = \oplus_i L_i$ with maps $\ell_k := [, \ldots,] : L^{\otimes k} \to L$ of degree $k-2$ such that

(1) ℓ_k is graded skew symmetric, that is, for a k-permutation σ

$$\ell_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = \mathrm{sgn}(\sigma)\epsilon(\sigma)\ell_k(x_1, \ldots, x_k),$$

where $\mathrm{sgn}(\sigma)$ is the sign of σ,

(2) There are generalized Jacobi identities

$$\sum_{i+j=k+1} \sum_{\sigma} \epsilon(\sigma)(-1)^{i(k-i)} \ell_j(\ell_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(k)}) = 0,$$

where the second summation extends to all $(i, k-i)$ shuffles of the symmetric group S_k.

In particular if $\ell_k = 0$ for $k \geq 3$, one recovers the notion of differential graded Lie algebra (L, d) where $[x, y] := \ell_2(x, y)$ and $dx = \ell_1(x)$.

There is a 1-1 correspondence between L_∞ structures on L and codifferentials $d_n : \wedge^m(sL) \to \wedge^{m-n+1}(sL)$ of degree -1 on the coalgebra $\wedge sL$, such that $d^2 = 0$, where $d = d_1 + d_2 + \cdots + d_n + \ldots$ [14].

Definition 6 (*[12]*)**.** Let (A, μ) be a commutative algebra and $D : A \to A$ an operator. Define multi-brackets on A as follows.

$F_D^1(a) = Da$
$F_D^n(a_1, \ldots, a_n) = \mu((D \otimes 1)(a_1 \otimes 1 - 1 \otimes a_1) \ldots (a_n \otimes 1 - 1 \otimes a_n))$.

Then D is called an operator of order n if $F_D^{n+1} = 0$.

There is a generalization of multi-brackets to non commutative algebras that is due to Akman [1].

Definition 7. A Gerstenhaber algebra is a graded commutative algebra $A = \oplus_i A_i$ together with a bracket

$$A_i \otimes A_j \to A_{i+j+1}, \quad a \otimes b \mapsto \{a, b\},$$

such that sL is a graded Lie algebra and the bracket acts like a derivation of algebras. That is, for $a, b, c \in A$,

(1) $\{a, b\} = -(-1)^{(|a|+1)(|b|+1)}\{b, a\}$,
(2) $\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|+1)(|b|+1)}\{b, \{a, c\}\}$,
(3) $\{a, bc\} = \{a, b\}c + (-1)^{|b|(|a|+1)}b\{a, c\}$.

Definition 8. A Batalin–Vilkovisky algebra (BV-algebra for short) is a graded commutative algebra A, together with an operator $\Delta : A_i \to A_{i+1}$ of order 2 and of square 0.

Any BV-algebra (A, Δ) is a Gerstenhaber algebra with the bracket defined by

$$\{a, b\} = (-1)^{|a|}(\Delta(ab) - \Delta(a)b - (-1)^{|a|}a\Delta(b)).$$

Definition 9 (*[13,2]*)**.** A commutative BV_∞-algebra is a graded commutative algebra $A = \oplus_{i \in \mathbb{Z}} A_i$ together with an operator $D = \sum_{i \geq 1} D_i$ such that $D^2 = 0$ and each D_n is an operator of order n and of degree $2n - 3$.

From the relation $D^2 = 0$, one gets $D_1^2 = 0$, hence D_1 is a differential on the algebra A. Moreover $D_1 D_2 + D_2 D_1 = 0$, therefore D_2 induces an action on the homology $H_*(A, D_1)$ which induces a BV-algebra structure [13]. If $D_i = 0$ for all $i \geq 3$, then $(A, D_1 + D_2)$ is called a differential BV-algebra.

Definition 10. Let $\phi : (A, d) \to (B, d)$ be a morphism of cdga's. A ϕ-derivation of degree k is a linear mapping $\theta : A^n \to B^{n-k}$ such that $\theta(ab) = \theta(a)\phi(b) + (-1)^{k|a|}\phi(a)\theta(b)$. We denote by $\mathrm{Der}_n(A, B; \phi)$ the vector space of ϕ-derivations of degree n and by $\mathrm{Der}(A, B; \phi) = \oplus_n \mathrm{Der}_n(A, B; \phi)$ the $\mathbb{Z}$-graded vector space of all ϕ-derivations. The differential on $\mathrm{Der}(A, B; \phi)$ is defined by $\delta\theta = d\theta - (-1)^k \theta d$.

If $A = B$ and $\phi = 1_A$, then we get the Lie algebra of derivations $\operatorname{Der} A$, where the Lie bracket is the commutator bracket. If V is finite, then $\operatorname{Der}(\wedge V) \cong \wedge V \otimes V^{\#}$. We have the following result for ϕ-derivations.

Proposition 11. *Let $\phi : (\wedge V, d) \to (B, d)$ be a surjective morphism between cdga's where V is finite dimensional and $I = \operatorname{Ker} \phi$. Then* $\operatorname{Der}(\wedge V, B; \phi) \cong \wedge V / I \otimes V^{\#}$.

Proof. Let $\{v_1, \dots, v_k\}$ be a basis of V. In $\operatorname{Der}(\wedge V, B; \phi)$, we denote by $(v_i, 1)$ the ϕ-derivation θ_i such that $\theta_i(v_i) = \delta_{ij}$. We observe that $v_i^{\#}$ corresponds to the derivation $\theta_i = (v_i, 1)$. Let θ be a ϕ-derivation. Then $\theta(v_i) = b_i$, where $b_i \in B$. As ϕ is surjective, there exist $a_i \in \wedge V$ such that $\phi(a_i) = b_i$. Hence $\theta = \sum_i a_i \theta_i = a_i v_i^{\#}$. By the first isomorphism theorem $\operatorname{Der}(\wedge V, B; \phi) \cong \wedge V / I \otimes V^{\#}$. □

Define $\widetilde{\operatorname{Der}}(A, B; \phi)$ as follows.

$$\widetilde{\operatorname{Der}}(A, B; \phi)_i = \begin{cases} \operatorname{Der}(A, B; \phi)_i, & i > 1, \\ \{\theta \in \operatorname{Der}_1(A, B; \phi) : \delta\theta = 0\}, & i = 1. \end{cases}$$

Let $A = \wedge V$ and $\theta_1, \dots, \theta_k \in \widetilde{\operatorname{Der}}(\wedge V, B; \phi)$ be ϕ-derivations of respective degrees $n_1, \dots, n_k$, define

$$[\theta_1, \dots, \theta_k](v) = (-1)^{\eta(k)} \sum \sum_{i_1, \dots, i_k} \epsilon \phi(v_1 \dots \hat{v}_{i_1} \dots \hat{v}_{i_k} \dots v_m) \theta_1(v_{i_1}) \dots \theta_k(v_{i_k}),$$

where $dv = \sum v_1 \dots v_m$, $\eta(j) = n_1 + \cdots + n_k - 1$, and ϵ is the corresponding Koszul sign of the permutation

$$(v_1, \dots, v_m) \to (v_1, \dots, \hat{v}_{i_1}, \dots, \hat{v}_{i_k}, \dots, v_m, v_{i_1}, \dots, v_{i_k}).$$

We note that $[\theta_1, \dots, \theta_k]$ is of degree $n_1 + \cdots + n_k - 1$. Now define linear maps ℓ_k of degree $k - 2$ on $s^{-1}\widetilde{\operatorname{Der}}(\wedge V, B; \phi)$ by

$$\ell_1(s^{-1}\theta) = -s^{-1}\delta\theta, \quad \ell_k(s^{-1}\theta_1, \dots, s^{-1}\theta_k) = (-1)^{\epsilon_k} s^{-1}[\theta_1, \dots, \theta_k],$$

where $\epsilon_k = \frac{k(k-1)}{2} + \sum_{i=1}^{k-1} (k - i)|\theta_i|$ [4].

Proposition 12 (*Lemma 3.3,[4]*). *If $\phi : \wedge V \to B$ is a Sullivan model of a mapping $f : X \to Y$ between simply connected spaces and V is finite dimensional, then $(s^{-1}\widetilde{\operatorname{Der}}(\wedge V, B; \phi), \ell_k)$ is an L_∞ model of* $\operatorname{map}(X, Y; f)$.

Theorem 13. *Let $(\wedge V, d) \to (B, d)$ be a cdga model of map $f : X \to Y$ between 1-connected spaces of finite type where Y is finite dimensional.*

(1) *Then there is a natural isomorphism*

$$\Gamma : \pi_*(\Omega \operatorname{map}(X, Y; f)) \otimes \mathbb{Q} \to HH^*_{(1)}(\wedge V; B),$$

(2) *Moreover the following diagram commutes:*

$$\begin{array}{ccc} \pi_*(\operatorname{aut}_1 Y) \otimes \mathbb{Q} & \longrightarrow & \pi_*(\operatorname{map}(X, Y; f)) \otimes \mathbb{Q} \\ \downarrow & & \downarrow \\ HH^*(\wedge V; \wedge V) & \longrightarrow & HH^*(\wedge V; B). \end{array}$$

Proof of the theorem. Before we prove the theorem, we need a generalization of derivations.

Definition 14. Let A be a commutative cochain algebra and M a differential A-module (considered here as an A-bimodule). A derivation θ from A to M of degree k is a linear map $\theta : A^* \to M^{*-k}$ such that $\theta(ab) = \theta(a)b + (-1)^{k|a|}a\theta(b)$.

It is easily seen that if $\theta : A \to M$ is derivation and $f : M \to N$ a morphism of A-bimodules, then the composition $f \circ \theta : A \to N$ is a derivation.

Let $(\wedge V, d)$ be a Sullivan model of a simply connected space. Define $\bar{V} = sV$, that is, $\bar{V}^n = V^{n+1}$. A Sullivan model of the loop space map(S^1, X) is given by $(\wedge(V \oplus \bar{V}), \tilde{D})$, the cdga defined in Section 1. For recall, $\tilde{D}v = dv$, $\tilde{D}\bar{v} = -S(dv)$ where S is the unique derivation defined by $Sv = \bar{v}$ and $S\bar{v} = 0$ [6].

Consider the linear map $S : (\wedge V, d) \to (\wedge V \otimes \bar{V}, D)$ defined $Sv = \bar{v}$ and extended S as a derivation in the sense of Definition 14. As $S(dv) = -D(Sv)$, then $Sd + DS = 0$, then S is a morphism of differential modules of upper degree -1.

We define a map

$$\Phi : \mathrm{Hom}_{\wedge V}(\wedge V \otimes \bar{V}, B) \to \mathrm{Der}(\wedge V, B; \phi)$$

such that $\Phi(f)$ is the following composition mapping

$$\wedge V \xrightarrow{S} \wedge V \otimes \bar{V} \xrightarrow{f} B,$$

that is, $\Phi(f)(v) = f(\bar{v})$.

Lemma 15. *The map Φ commutes with differentials.*

Proof. Let $f \in \mathrm{Hom}_{\wedge V}(\wedge V \otimes \bar{V}, \wedge V)$.

$$\begin{aligned}(Df)(\bar{v}) &= d(f(\bar{v})) - (-1)^{|f|} f(D(\bar{v})) \\ &= d(f(\bar{v})) + (-1)^{|f|}(f(sdv)),\end{aligned}$$

hence $(\Phi(Df))(v) = d(f(\bar{v})) + (-1)^{|f|}(f(sdv))$.

On the other hand

$$\begin{aligned}(D\Phi(f))(v) &= d(\Phi(f)(v)) - (-1)^{|\Phi(f)|}\Phi(f)(dv) \\ &= d(f(sv)) + (-1)^{|f|} f(sdv).\end{aligned}$$

Hence Φ is a morphism of chain complexes. □

Moreover, there are isomorphisms of vector spaces $\mathrm{Hom}_{\wedge V}(\wedge V \otimes \bar{V}, B) \cong \mathrm{Hom}(\bar{V}, B) \cong \mathrm{Der}(\wedge V, B)$. Hence Φ is bijective. Therefore

$$H_*(s^{-1}\,\mathrm{Der}(\wedge V, B)) \cong HH^*_{(1)}(\wedge V, B) \rightarrowtail HH^*(\wedge V, B).$$

Remark 16. It was shown that if L is an L_∞-algebra, then $\wedge s^{-1}L$ is a BV_∞ algebra [2]. It would be interesting to find a link between the BV_∞-algebra $\wedge s^{-1}L$ and $HH^*(\wedge V; B)$.

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