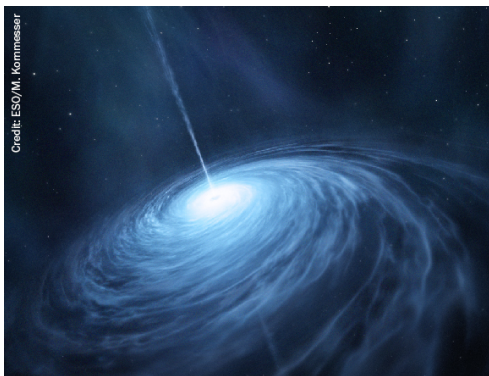**PAPER • OPEN ACCESS**

Teukolsky formalism for nonlinear Kerr perturbations

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Teukolsky formalism for nonlinear Kerr perturbations

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Abstract

We develop a formalism to treat higher order (nonlinear) metric perturbations
of the Kerr spacetime in a Teukolsky framework. We first show that solutions
to the linearized Einstein equation with nonvanishing stress tensor can be
decomposed into a pure gauge part plus a zero mode (infinitesimal perturbation
of the mass and spin) plus a perturbation arising from a certain scalar ('Debye–
Hertz') potential, plus a so-called 'corrector tensor'. The scalar potential is a
solution to the spin -2 Teukolsky equation with a source. This source, as
well as the tetrad components of the corrector tensor, are obtained by solving
certain decoupled ordinary differential equations involving the stress tensor.
As we show, solving these ordinary differential equations reduces simply to
integrations in the coordinate r in outgoing Kerr–Newman coordinates, so
in this sense, the problem is reduced to the Teukolsky equation with source,
which can be treated by a separation of variables ansatz. Since higher order
perturbations are subject to a linearized Einstein equation with a stress
tensor obtained from the lower order perturbations, our method also applies
iteratively to the higher order metric perturbations, and could thus be used
to analyze the nonlinear coupling of perturbations in the near-extremal Kerr
spacetime, where weakly turbulent behavior has been conjectured to occur.
Our method could also be applied to the study of perturbations generated by
a pointlike body traveling on a timelike geodesic in Kerr, which is relevant to
the extreme mass ratio inspiral problem.

(Some figures may appear in colour only in the online journal)

Contents

1. Introduction

Metrics describing a perturbed Kerr black hole play an important role in gravitational physics, for instance in order to model the ringdown phase after a black hole merger, or an extreme mass ratio inspiral. Their analysis also presents nontrivial mathematical challenges: one may wish to prove the expected decay properties of the perturbations (see e.g. [1–3] and references for recent progress directly relevant to this work), or to understand the complicated but highly special geometrical structure of the perturbation equations on the Kerr background (see e.g. the monograph [4]).

In the case of linear perturbations (i.e. solutions to the linearized Einstein equation around Kerr), one has the Teukolsky formalism [5, 6], which in effect reduces the problem to the study of a linear scalar wave equation. Furthermore, this equation can be solved by a separation of variables ansatz. Combined, these two features dramatically simplify the analysis of linear perturbations. As originally derived, Teukolsky's equation applies to the extreme components ψ_0 or ψ_4 of the linearized Weyl tensor, see section 2. However, it was discovered soon afterwards [7, 8] that the equation also applies directly to the metric perturbation itself in a sense. In fact, if one makes an ansatz of the form $h_{ab} = \text{Re}\,\mathcal{S}^\dagger_{ab}\Phi$, where $\mathcal{S}^\dagger_{ab}$ is a certain second order differential operator (see (25)) constructed using geometrical objects of the background Kerr geometry, then if the complex 'Hertz potential' Φ is a solution to the source-free (adjoint) Teukolsky equation, h_{ab} is a solution to the source-free linearized Einstein equation. Furthermore, it is generally accepted—and argued more carefully in section 3—that this ansatz in a sense covers all solutions with the exception only of pure gauge ones and 'zero modes', by which one means linear perturbations towards a Kerr black hole with different mass and/or spin.

While the Weyl tensor components are sufficient to extract the information about gravitational radiation at null infinity, it is also highly desirable to know directly the metric perturbation itself. For instance, in the self-force approach [9, 10] to extreme mass ratio inspiral, the

first step is to determine the linearized metric perturbation resulting from a stress tensor of a point particle on a geodesic in the Kerr background. If one is interested in higher order perturbations, one has to solve, for the nth order perturbed metric, the linearized Einstein equation with a source determined by the metric perturbations up to order $(n-1)$. Both of these problems involve solving the linearized Einstein equation with a nonvanishing source. But in the presence of a source, the Hertz-potential ansatz $h_{ab} = \text{Re}\, \mathcal{S}^{\dagger}_{ab}\Phi$ is generically no longer consistent! Furthermore, while the metric perturbation can be extracted from the perturbed Weyl scalars ψ_0 or ψ_4 for perturbations satisfying the source free linearized Einstein equation by means of the Teukolsky–Starobinsky relations, this is no longer true for the Einstein equation with source. Thus, it would seem that the Teukolsky formalism is not useful to study higher order perturbations of Kerr (for example), and the mileage gained from the highly special properties of the background seems lost again.

In this paper, we propose a way around this problem, which takes advantage of the very special geometric properties of Kerr (and in fact, algebraically special solutions). Our starting point is the linearized Einstein equation with source[3],

$$(\mathcal{E}h)_{ab} = T_{ab}, \tag{1}$$

where $\mathcal{E}$ is the linearized Einstein operator on the Kerr background (17). In applications, T_{ab} would be e.g. the stress tensor of a point source on a geodesic in Kerr, or the nonlinear terms in the equation for the nth order gravitational perturbation. Instead of the Hertz-potential ansatz, which is generically inconsistent with a nontrivial source T_{ab}, we will argue that the metric perturbation can be decomposed in the following manner[4] (see section 2 for the precise definition of the operators $\mathcal{S}, \mathcal{O}, \mathcal{T}$)

$$h_{ab} = (\mathcal{L}_{\xi}g)_{ab} + \dot{g}_{ab} + x_{ab} + \text{Re}\,(\mathcal{S}^{\dagger}\Phi)_{ab}, \tag{2}$$

where:

- $(\mathcal{L}_{\xi}g)_{ab} = \nabla_a\xi_b + \nabla_b\xi_a$ is a pure gauge perturbation (irrelevant).
- $\dot{g}_{ab} = \frac{\mathrm{d}}{\mathrm{d}s}g^{M(s),a(s)}_{ab}|_{s=0}$ is a zero mode, i.e. perturbation to another metric in the Kerr family $g^{M,a}_{ab}$, where $(M(0), a(0)) = (M, a)$ are the mass/spin parameter of the given Kerr background $(\mathcal{M}, g_{ab})$.
- x_{ab} is called a *corrector* field. As we will show, its nontrivial tetrad components (see (55)) are obtained from T_{ab} by solving three decoupled *ordinary* differential equations (see (107)) along a congruence of outgoing null geodesics aligned with a principal null direction.
- Φ is a complex potential which is a solution to the adjoint Teukolsky equation with a source,

$$\mathcal{O}^{\dagger}\Phi = \eta. \tag{3}$$

The source η is determined in terms of T_{ab} by an equation of the form $(\mathcal{T}^{\dagger}\eta)_{ab} = T_{ab} - (\mathcal{E}x)_{ab}$. As we will show, this equation can be solved by integrating one *ordinary* differential equation along a congruence of outgoing null geodesics aligned with a principal null direction (see (108) and (91)).

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[3] To lighten the notation, we absorb the conventional prefactor of $8\pi G$ into T_{ab}.

[4] To be precise, this decomposition holds under certain technical assumptions detailed in the main text. The essentially only unproven hypothesis is the absence of 'purely outgoing algebraically special perturbations' with real frequency. This restriction, however, plays no role for the main application we have in mind which is a retarded solution generated by a compact source.

The difficult step in our algorithm is thus to solve the adjoint Teukolsky equation with source $\mathcal{O}^\dagger \Phi = \eta$, a scalar wave equation, which can be dealt with by a separation of variables ansatz [6]. By contrast, the determination of the source η and the tetrad components of the corrector field x_{ab} involves the comparatively easy task of solving scalar *ordinary* differential equations depending on T_{ab}. Thus, our method represents a drastic simplification compared to the nonseparated tensor equation (1), taking advantage to the maximum possible extent of the special geometric features of Kerr. Before presenting the details of our algorithm in the following sections, let us mention three potential applications of our method, which were the motivation of our analysis.

1.1. Point sources

Point sources for the linearized Einstein equation are described by a distributional stress tensor supported on a timelike geodesic γ,

$$T_{ab}(x) = m \int_{-\infty}^{\infty} \dot{\gamma}_a(t)\dot{\gamma}_b(t)\delta(x - \gamma(t)) \, \mathrm{d}t \, . \tag{4}$$

Since our corrector field x_{ab} is obtained by integrating linear ordinary differential equations sourced by appropriate components of T_{ab} along a congruence of outward geodesics from the past horizon to future null infinity, the corrector will only be nonzero for those geodesics intersecting γ. Furthermore, if we impose zero boundary conditions on x_{ab} on $\mathscr{H}^-$, it follows that x_{ab} is supported in each spatial slice along a 'string' attached to the point source going out to infinity. The locus of this string is where the null geodesics intersecting γ pierce the slice (see figure 1). Thus, our analysis shows that the perturbation h_{ab} is defined in terms of a Hertz potential (as well as possibly trivial zero mode and gauge perturbations), *plus* a corrector x_{ab} supported on a semi-infinite string emanating at the point particle going to infinity. If instead vanishing boundary conditions on x_{ab} are imposed at $\mathscr{I}^+$ rather than $\mathscr{H}^-$, then the string goes to the horizon rather than to infinity.

The occurrence of such 'Dirac-type strings' has been observed also by a number of previous authors [11] who have studied the equation (1) for point sources by a somewhat different approach. In their approach, the exterior of the black hole is divided into two domains $\mathscr{M}_+$ and $\mathscr{M}_-$ separated by an interface $\mathscr{S}$ containing the particle orbit. In each domain, an ansatz $h_{ab}^\pm = \mathrm{Re}\,(\mathcal{S}^\dagger \Phi^\pm)_{ab}$ is made for the metric, and then the existence of a distributional source is encoded into suitable junction conditions relating Φ^+ to Φ^- on the interface $\mathscr{S}$ and in particular at the worldline γ in the form of differentiability and continuity requirements. The perturbations obtained in this manner can be categorized [12] based on the form of the singularity into the 'half-string' classes and the 'full-string' class. A 'no-string' gauge may be formed by joining together the two regular sides of two opposite half-string perturbations along the interface $\mathscr{S}$ through the particle. Such a perturbation, introduced by Friedman and collaborators in [13, 14], avoids the stringlike singularities, but has instead a gauge discontinuity (and also delta-function distributions [12]) on the interface $\mathscr{S}$.

Our construction scheme by contrast avoids the consideration of an interface, but the metric perturbation involves the corrector piece x_{ab}, which has a delta-function-like distribution along a string emanating from the particle. Furthermore, our potential Φ will be a solution to the adjoint Teukolsky equation with a source (3) that will be supported along the string. Our construction thereby gives a rather transparent conceptual understanding for the occurrence of such Dirac-type strings that have been observed in other approaches. It is also conceivable
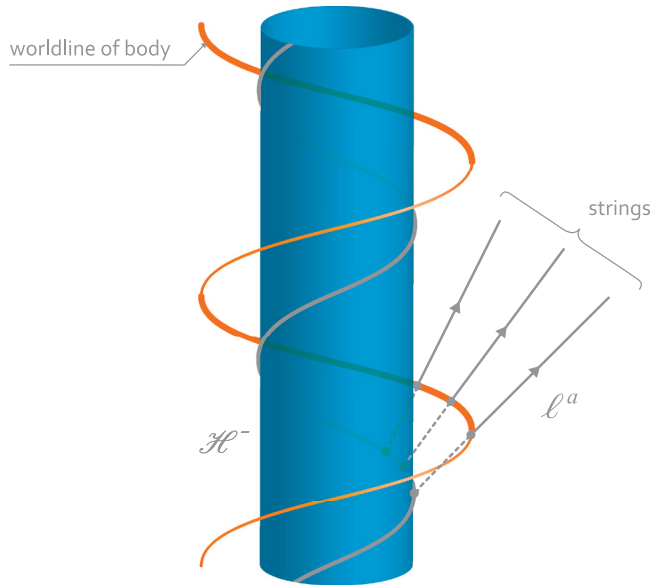
Figure 1. Worldline of a body orbiting a Kerr black hole. The corrector tensor x_{ab} is supported on the semi-infinite strings extending from the worldline to infinity.

that our construction, which avoids the consideration of an interface, might provide a viable alternative scheme for numerical integrations.

1.2. Nonlinear perturbations: decay and turbulent behavior

By iterating our prescription, we can recursively obtain the nth order gravitational perturbation up to gauge as $h_{ab}^{(n)} = g_{ab}^{(n)} + x_{ab}^{(n)} + \mathrm{Re}\,(\mathcal{S}^{\dagger}\Phi^{(n)})_{ab}$ in terms of an n-order potential subject to a sourced adjoint Teukolsky equation $\mathcal{O}^{\dagger}\Phi^{(n)} = \eta^{(n)}$, where $\eta^{(n)}$, $x_{ab}^{(n)}$ are determined by transport equations (ODEs) from the nonlinear terms $T_{ab}^{(n)}$ in the nth order Einstein tensor (involving $h_{ab}^{(1)}, \ldots, h_{ab}^{(n-1)}$), and where $g_{ab}^{(n)}$ is an nth order perturbation to another Kerr black hole. While a thorough discussion is outside the scope of the present paper, we here want to ask whether higher order perturbations could be physically relevant in astrophysical situations.

At the linear level, an important aspect of characterizing black hole perturbations is the determination of a quantized set of complex frequencies associated with certain mode solutions to Teukolsky's equation called quasinormal modes [15]. Quasinormal modes are the natural resonances of black hole spacetimes and carry information about the parameters of binary merger remnants. Rapidly spinning Kerr black holes exhibit a special family of weakly damped quasinormal frequencies, which in a sense 'corotate' with the horizon. These modes each have real part proportional to the horizon frequency and become arbitrarily long-lived in the extremal limit [16, 17]. At linear order, the long-lived modes all get coherently excited, and their collective behavior leads to interesting physical effects. Near the horizon, the superposition of modes results in a transient growth of tidal fields as measured by infalling observers. Away from the horizon, the mode sum yields a transient power-law tail, slowing the asymptotic decay [17, 18]. Furthermore, as they have commensurate real part of frequency, these modes can be *nonlinearly* in a state of resonance and can readily transfer energy amongst each other. As such, the nonlinear coupling between multiple modes is conjectured to lead

to turbulent cascades exhibiting a Kolmogorov-type scaling [19]. In this way, such nonlinear features could in principle lead to a unique signature in the ringdown signals of highly spinning black holes, as measured by future gravitational wave detectors.

Generically, however, an arbitrary perturbation to Kerr will extract angular momentum through the mechanism of superradiance, the wave analog of the Penrose process, see e.g. [20]. For nonlinear mode-coupling effects to become important, it is, therefore, necessary that they develop faster than (a) the time scale for linear decay of the quasinormal modes, and (b) the spin-down time scale of the black hole. We now give a heuristic argument, concluding that as long as the perturbation amplitude and surface gravity are appropriately chosen, both of these conditions can be satisfied.

We first estimate the spin-down time scale of the black hole[5]. Suppose we have a mode perturbation to Kerr of amplitude A, frequency ω, and azimuthal number m. The energy and angular momentum fluxes into the black hole scale as [21]

$$\dot{M} \sim M^2 \omega(\omega - m\Omega_H)A^2, \qquad \dot{J} \sim M^2 m(\omega - m\Omega_H)A^2, \tag{5}$$

where the overdot denotes differentiation with respect to ingoing Kerr time and Ω_H is the angular velocity of the outer horizon. For short,

$$\dot{M} \sim \omega_f \omega_n A^2, \qquad \dot{J} \sim M m \omega_n A^2, \tag{6}$$

with dimensionless frequencies

$$\omega_f = M\omega, \qquad \omega_n = M(\omega - m\Omega_H). \tag{7}$$

The fractional change in the extremality parameter (surface gravity) $\kappa = \sqrt{1 - J^2/M^4}$ due to absorption at the horizon is given by

$$
\begin{aligned}
M\frac{\dot{\kappa}}{\kappa} &= \frac{2J^2}{M^2\kappa^2}\left(\frac{\dot{M}}{M^2} - \frac{\dot{J}}{2MJ}\right) \\
&= \frac{2J^2}{M^2\kappa^2}(\omega - m\Omega_H)A^2\left(\omega - \frac{M}{2J}m\right) \\
&= \frac{2J^2}{M^2\kappa^2}(\omega - m\Omega_H)A^2\left(\omega - m\Omega_H + O(\kappa)\right).
\end{aligned}
\tag{8}
$$

In the last line, we have used that $\Omega_H = M/(2J) + O(\kappa)$. The long-lived 'near-horizon' modes of nearly extremal Kerr have $\omega - m\Omega_H = O(\kappa)$, so this term does not change the order of the final term. Therefore, for near-horizon modes, we have

$$M\frac{\dot{\kappa}}{\kappa} \sim \frac{A^2}{\kappa^2}\omega_n^2. \tag{9}$$

Thus $A\omega_n \ll \kappa$ suffices to make the fractional change in κ small over a typical 'fast' time scale M. For near-horizon modes, with $\omega_n \sim \kappa$, the spin-down time scale goes as $\tau_{\text{spin-down}} \sim M/A^2$.

The time scale for linear decay of a near-horizon mode, meanwhile, is $\tau_{\text{decay}} \sim M/\kappa$. Thus, for nonlinear mode-coupling to become important, the perturbation amplitude must be such that the mode-coupling time scale $\tau_{\text{mode-coupling}} \lesssim \tau_{\text{spin-down}}, \tau_{\text{decay}}$.

The mode-coupling time scale depends on whether the fundamental interaction involves a three- or four-mode coupling [22]. Three-mode interactions are stronger, but their presence depends on the dispersion relation of the modes and any associated selection rules, analysis of

---

[5] We thank Gralla for a discussion in which we made this estimate.

Table 1. Criteria for long-lived quasinormal mode coupling. Conditions are given for three and four mode interactions such that the mode coupling time scales are longer than the linear decay and spin down time scales.

	Three-mode $\tau \sim M/A$	four-mode $\tau \sim M/A^2$
Spin-down $\tau \sim M/A^2$	$A \ll 1$	—
Decay $\tau \sim 1/\kappa$	$\kappa \ll A$	$\kappa \ll A^2$

which we do not attempt here. Four-mode interactions are more generic. Three-mode interactions have an interaction time scale that goes as $1/A$, whereas four-mode interactions go as $1/A^2$. The overall strength of the interaction depends on the precise overlap integrals, which would have to be studied using the framework developed in [23]. For now, we assume this goes as $1/M$. By comparing these time scales to those for dissipation and spin-down, we obtain constraints on the size of the perturbation compared to the spin of the black hole in order for nonlinear mode-coupling effects to become important (see table 1).

To summarize this discussion, we conclude that if the leading interaction involves three modes, then provided $1 \gg A \gg \kappa$, nonlinear coupling could become important for near-extremal black holes. If the leading interaction involves four modes and $A^2 \gg \kappa$, then the interaction time scale is the same as the spin-down time scale, and such nonlinear effects are possible, although only marginally so. We will analyze this in detail in a future paper [24].

1.3. Perturbative quantum gravity

Our method should also bring substantial simplifications to perturbative quantum gravity on the Kerr spacetime. For linear perturbations, the corrector tensor x_{ab} is zero, and our results give a decomposition $h_{ab} = \dot{g}_{ab} + \text{Re}\,(\mathcal{S}^{\dagger}\Phi)_{ab}$ up to gauge. It turns out that, with respect to the canonical symplectic form W for general relativity (see (30)), the zero mode perturbations $\dot{g}_{ab}$ (finitely many degrees of freedom) are symplectically orthogonal to the Hertz-potential perturbations[6] $(\mathcal{S}^{\dagger}\Phi)_{ab}$, and thus can be 'quantized' independently. Thus, up to the zero mode perturbations (quantization of a finite-dimensional phase space), the quantization of the gravitational field reduces at linearized order to the quantization of the Hertz-potential field. For these, we have the naturally induced symplectic form $\tilde{W}[\Phi_1, \Phi_2] = W[\text{Re}\mathcal{S}^{\dagger}\Phi_1, \text{Re}\mathcal{S}^{\dagger}\Phi_2]$, which may be used as the basis for the quantization of the field Φ.

For the higher perturbative orders, we should be able to proceed in a recursive manner just as one does when defining the higher orders of the interacting Klein–Gordon quantum field in ϕ^4-theory on a curved spacetime, say, see [26, 27]. The only difference should be that the sources of the equations for the nth order perturbation are determined by a certain transport equation as described in section 5, and that in addition to the Hertz-potential quantum field Φ, we need to keep track of the corrector x_{ab} at each perturbation order, also as described in section 5. We leave the details to a future investigation.

This paper is organized as follows. In section 2 we review the essential parts of the GHP formalism for the convenience of the reader. In section 3 we study the decomposition (2) for solutions to the homogeneous linearized Einstein equations. In section 4, we generalize this to solutions to the linearized Einstein equations with source. In section 5 we point out the fairly obvious application to higher order (nonlinear) perturbations. Some formulae are relegated to the appendix.

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[6] This follows, e.g. from proposition 3.1 of [25] and the fact that $\mathcal{T}\dot{g} = 0$.

Conventions: We use the $(+ - --)$ signature convention for the metric and the conventions of [20] otherwise. We also make extensive use of the Geroch–Held–Penrose (GHP)-formalism [28] and the associated standard notation.

2. GHP formalism and Teukolsky equation

Our construction makes heavy use of the special geometric features of the background Kerr geometry. The key simplifying feature behind most arguments is in fact the algebraically special nature of the background geometry and, of course, the vacuum Einstein equation

$$R_{ab} = 0, \tag{10}$$

assumed throughout this article. We recall (see [20] for details) that the Weyl tensor C_{abcd} of a 4 dimensional Lorentz metric generically has four distinct 'principal null directions' at each point, where a principal null direction is a null vector k^a such that

$$0 = k_{[f} C_{a]bc[d} k_{e]} k^b k^c. \tag{11}$$

Definition 1. A 4D spacetime is called algebraically special (or Petrov type II) if two principal null directions coincide at every point, $k_1^a = k_2^a = l^a$, and in this case $C_{abc[d} l_{e]} l^b l^c = 0$. An algebraically special spacetime is called Petrov type D if there is another pair of coinciding principal null directions $k_3^a = k_4^a = n^a$.

In an algebraically special spacetime, it is true that l^a is tangent to a congruence of shear free null-geodesics by the Goldberg–Sachs theorem [29]. Examples of such spacetimes are the Kerr- and Robinson–Trautman families of metrics [30]. The former ones are actually of type D, while in the latter the twist of l^a vanishes. In order to take full advantage of the algebraically special property, it is useful to consider null tetrads $(l^a, n^a, m^a, \bar{m}^b)$ such that l^a is aligned with the principal null direction (and n^a with the other one in type D). Our conventions for the tetrad are: $l^a n_a = 1$, $m^a \bar{m}_a = -1$ and all other inner product vanish. From the null tetrad, we define as usual the Weyl components,

$$
\begin{aligned}
\Psi_0 &= -C_{abcd} \, l^a m^b l^c m^d, \\
\Psi_1 &= -C_{abcd} l^a n^b l^c m^d, \\
\Psi_2 &= -\tfrac{1}{2} C_{abcd} (l^a n^b l^c n^d + l^a n^b m^c \bar{m}^d), \\
\Psi_3 &= -C_{abcd} l^a n^b \bar{m}^c n^d, \\
\Psi_4 &= -C_{abcd} \, n^a \bar{m}^b n^c \bar{m}^d,
\end{aligned}
\tag{12}
$$

and the spin coefficients $\rho, \rho', \sigma, \sigma', \tau, \tau', \kappa, \kappa', \beta, \beta', \epsilon, \epsilon'$; see appendix B. The real part of ρ corresponds to the expansion of the null-geodesic congruence tangent to l^a, the imaginary part to the twist, and the real and imaginary parts of σ correspond to the shear of l^a. κ measures the failure of the congruence to be geodesic in general, so by the Goldberg–Sachs theorem [29] and its corollaries,

$$\kappa = \sigma = \Psi_0 = \Psi_1 = 0. \tag{13}$$

The corresponding primed quantities correspond to sending $l^a \to n^a$ and $m^a \to \bar{m}^a$. By choosing n^a appropriately, one can set further quantities to zero, but there are different options:

(n1) In any type II spacetime, we can perform a null rotation $l^a \to l^a, m^a \to m^a + Al^a, n^a \to A\bar{m}^a + \bar{A}m^a + A\bar{A}l^a$, under which τ transforms as $\tau \to \tau + A\rho$. Then, if $\rho \neq 0$, we can adjust A so that $\tau = 0$, and this choice fixes the null vector n^a up to a rescaling. Making further use of the Einstein field equation, one then sees that $\tau' = \sigma' = 0$ in addition to (13); see [31] for details.

(n2) In type D, we may alternatively choose n^a to be the second principal null direction, which results in the primed version of the Goldberg–Sachs theorem $\kappa' = \sigma' = \Psi_3 = \Psi_4 = 0$ in addition to (13). In Kerr, an example of such a tetrad is the Kinnersley frame; see appendix B.

Except for the case of Schwarzschild spacetime, one must in general choose between these two possibilities, and we shall make use of either one of them in the following depending on the purpose.

The GHP formalism [28] allows one to handle in a very efficient manner the equations obtained for the spin coefficients and Weyl components. Furthermore, that formulation has a geometrical basis which renders its equations automatically invariant under the remaining permissible rescalings of the tetrad $(l^a, n^a, m^a, \bar{m}^b)$. These rescalings consist of (a) a local boost, sending $l^a \to \lambda\bar{\lambda}l^a, n^a \to \lambda^{-1}\bar{\lambda}^{-1}n^a$, or (b) a local rotation $m^a \to \lambda\bar{\lambda}^{-1}m^a$, where λ is a nonzero complex number depending on the spacetime point. A GHP quantity η is said to have weights (p, q) if transforms as

$$\eta \to \lambda^p\bar{\lambda}^q\eta \quad \Longleftrightarrow: \quad \eta \overset{\circ}{=} (p, q), \tag{14}$$

under a combined local boost and rotation.

More mathematically speaking, the construction can be seen as follows. Given two null *directions*—rather than vectors—aligned with the given l^a, n^a at each point, one can define the bundle of *all* null frames $(l^a, n^a, m^a, \bar{m}^b)$ over $\mathscr{M}$ such that l^a, n^a point along the prescribed directions and such that $m^a, \bar{m}^a$ are positively oriented and span the complexified orthogonal complement of the plane determined by l^a, n^a. This set is seen to define a principal fibre bundle, $\mathscr{P}$, over $\mathscr{M}$ with structure (gauge) group the boosts and rotations, i.e. $G = \mathbb{R}_+ \times SO(2)$, isomorphic also to the multiplicative group $G = \mathbb{C}_\times$ of nonzero complex numbers; see [32] for the basic definitions related to the notion of a principal fibre bundle. The gauge group G acts precisely by local boosts and rotations on $\mathscr{P}$. $G = \mathbb{C}_\times$ also acts by multiplication with $\lambda^p\bar{\lambda}^q$ on the 1-dimensional vector space $\mathbb{C}$, and this gives a representation $\pi_{p,q}$. The GHP quantities $\eta\overset{\circ}{=}(p, q)$ are precisely the sections of the so-called associated complex line bundle $\mathscr{L}^{p,q} = \mathscr{P} \ltimes_{\pi_{p,q}} \mathbb{C}$. We may also consider 'mixed' tensorial/GHP objects of GHP weight (p, q) and tensorial rank (r, s), which are sections in the vector bundles $\mathscr{L}^{p,q} \otimes T^{r,s}\mathscr{M}$.

The key idea of the GHP formalism is to view the spin coefficients $\rho, \rho', \kappa, \kappa', \tau, \tau', \sigma, \sigma'$ as GHP quantities of the appropriate weight as specified in appendix B. Likewise, the tetrad components of an ordinary tensor are GHP quantities of an appropriate weight. For instance, if ξ_a is a covector field then $\xi_l = \xi_a l^a \overset{\circ}{=} (1, 1)$, or $\xi_n = \xi_a n^a \overset{\circ}{=} (-1, -1)$, or $\rho \overset{\circ}{=} (1, 1)$, etc. The remaining spin coefficients $\beta, \beta', \epsilon, \epsilon'$ by contrast are used to define a covariant derivative operator (i.e. a connection) called Θ_a of the vector bundle $\mathscr{L}^{p,q} \otimes T^{r,s}\mathscr{M}$, where $T^{r,s}\mathscr{M} = T\mathscr{M}^{\otimes r} \otimes T^*\mathscr{M}^{\otimes r}$. Its definition is

$$\Theta_a = \nabla_a - \frac{1}{2}(p + q)n^b\nabla_a l_b + \frac{1}{2}(p - q)\bar{m}^b\nabla_a m_b$$
$$= \nabla_a + l_a(p\epsilon' + q\bar{\epsilon}') + n_a(-p\epsilon - q\bar{\epsilon}) - m_a(p\beta' - q\bar{\beta}) - \bar{m}_a(-p\beta + q\bar{\beta}'). \tag{15}$$

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Ordinary tensor fields are identified with sections of $\mathscr{L}^{0,0} \otimes T^{r,s}\mathscr{M}$, i.e. mixed GHP-tensor fields of weight $(0,0)$. The GHP-covariant directional derivatives along the tetrad legs are denoted traditionally by

$$\text{þ} = \Theta_l, \quad \text{þ}' = \Theta_n, \quad \text{ð} = \Theta_m, \quad \text{ð}' = \Theta_{\bar{m}}. \tag{16}$$

These operators shift the GHP weights by the amounts $\text{þ} : (1,1), \text{þ}' : (-1,-1)$, $\text{ð} : (1,-1), \text{ð}' : (-1,1)$. Apart from mathematical elegance, this geometric viewpoint is useful because we may encounter null tetrads that behave in a singular way at certain points of $\mathscr{M}$, such as the horizon, infinity, the north pole of a sphere, etc. The components of a mixed GHP quantity in such a singular null frame will therefore also be singular, but such a singularity is obviously absent in the geometrical viewpoint where it is simply ascribed to a bad choice of gauge. By working throughout with invariantly-defined GHP quantities, the formalism therefore automatically takes care of such artificial singularities.

As noted by Wald [33], the essence of the Teukolsky formalism [5, 6] and its extension by [7, 8] may be succinctly encoded in an operator relation. Let $(\mathscr{M}, g_{ab})$ be a type II vacuum solution. To state the operator relation, we introduce the linearized Einstein operator,

$$(\mathcal{E}h)_{ab} = \frac{1}{2}\left[-\nabla^c\nabla_c h_{ab} - \nabla_a\nabla_b h^c{}_c + 2\nabla^c\nabla_{(a}h_{b)c} + g_{ab}\left(\nabla^c\nabla_c h^d{}_d - \nabla^c\nabla^d h_{cd}\right)\right], \tag{17}$$

as well as partial differential operators $\mathcal{S}, \mathcal{T}, \mathcal{O}$ defined as follows. $\mathcal{S}, \mathcal{T}$ act on symmetric rank-two covariant tensor fields and give a GHP scalar of weight $(4,0)$,

$$\begin{aligned}
\mathcal{S}T = (\text{ð} - \bar{\tau}' - 4\tau)\left[(\text{þ} - 2\bar{\rho})T_{lm} - (\text{ð} - \bar{\tau}')T_{ll}\right] \\
+ (\text{þ} - \bar{\rho} - 4\rho)\left[(\text{ð} - 2\bar{\tau}')T_{lm} - (\text{þ} - \bar{\rho})T_{mm}\right],
\end{aligned} \tag{18a}$$

$$\begin{aligned}
\mathcal{T}h = \frac{1}{2}(\text{ð} - \bar{\tau}')(\text{ð} - \bar{\tau}')h_{ll} + \frac{1}{2}(\text{þ} - \bar{\rho})(\text{þ} - \bar{\rho})h_{mm} \\
- \frac{1}{2}\left[(\text{þ} - \bar{\rho})(\text{ð} - 2\bar{\tau}') + (\text{ð} - \bar{\tau}')(\text{þ} - 2\bar{\rho})\right]h_{(lm)}.
\end{aligned} \tag{18b}$$

$\mathcal{O}$, the wave operator appearing in Teukolsky's master equation [5, 6], takes a $(4,0)$ GHP scalar to another $(4,0)$ GHP scalar and is defined by

$$\mathcal{O}\eta = 2\left[(\text{þ} - 4\rho - \bar{\rho})(\text{þ}' - \rho') - (\text{ð} - 4\tau - \bar{\tau}')(\text{ð}' - \tau') - 3\Psi_2\right]\eta. \tag{19}$$

The operator relation is

$$\mathcal{O}\mathcal{T} = \mathcal{S}\mathcal{E}, \tag{20}$$

and can be exploited as follows. The quantity $\mathcal{T}h = \psi_0$ is by construction equal to the linearization of the Weyl component Ψ_0. (We denote the perturbed values of the Weyl scalars Ψ_n given in (12) as ψ_n.) Then, if h_{ab} is a solution to the linearized field equation $\mathcal{E}h_{ab} = 0$, it follows that ψ_0 solves the homogeneous Teukolsky equation $\mathcal{O}\psi_0 = 0$. We can also take the formal adjoint of this operator relation and use that $\mathcal{E}^\dagger = \mathcal{E}$. (The formal adjoint[7] $\mathcal{P}^\dagger$ of a partial differential operator $\mathcal{P}$ taking sections in $\mathscr{L}^{p,q} \otimes T^{r,s}\mathscr{M}$ to sections in $\mathscr{L}^{p',q'} \otimes T^{r',s'}\mathscr{M}$ is a partial differential operator taking sections in $\mathscr{L}^{-p',-q'} \otimes T^{s',r'}\mathscr{M}$ to sections in $\mathscr{L}^{-p,-q} \otimes T^{s,r}\mathscr{M}$. It is defined uniquely by the condition that

$$(\mathcal{P}^\dagger\eta')^{a...b}{}_{c...d}\eta_{a...b}{}^{c...d} - \eta'^{a...b}{}_{c...d}(\mathcal{P}\eta)_{a...b}{}^{c...d} = \nabla_a w^a \tag{21}$$

---

[7] By contrast with the conventions in quantum mechanics, the operation † is linear, rather than anti-linear. This is because no sesqui-linear inner product is used in the definition.

for some vector field w^a constructed locally out of η, η' and their derivatives of GHP weight zero.) This results in the relation

$$\mathcal{T}^\dagger \mathcal{O}^\dagger = \mathcal{E} \mathcal{S}^\dagger. \tag{22}$$

As a consequence, suppose $\Phi \overset{\circ}{=} (-4, 0)$ satisfies $\mathcal{O}^\dagger \Phi = 0$. Then $h_{ab} = (\mathcal{S}^\dagger \Phi)_{ab}$ is a complex-valued solution to the linearized Einstein equation, $\mathcal{E} h_{ab} = 0$. By taking the real part and using that $\mathcal{E}$ is a real operator, we thereby get a real tensor field satisfying the linearized Einstein equation from the 'Hertz potential' Φ. The adjoint Teukolsky operator $\mathcal{O}^\dagger$ is explicitly

$$
\begin{aligned}
\mathcal{O}^\dagger \Phi &= 2 \left[(\text{\th}' - \rho')(\text{\th} + 3\rho) - (\text{\dh}' - \tau')(\text{\dh} + 3\tau) - 3\Psi_2 \right] \Phi \\
&= 2 \left[\left(\frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2 \theta \right) \frac{\partial^2 \Phi}{\partial t^2} + \frac{4Mar}{\Delta} \frac{\partial^2 \Phi}{\partial t \partial \phi} + 4 \left(\frac{M(r^2 - a^2)}{\Delta} - r - \mathrm{i} a \cos \theta \right) \frac{\partial \Phi}{\partial t} \right. \\
&\quad + \left(\frac{a^2}{\Delta} - \frac{1}{\sin^2 \theta} \right) \frac{\partial^2 \Phi}{\partial \phi^2} - \Delta^2 \frac{\partial}{\partial r} \left(\Delta^{-1} \frac{\partial \Phi}{\partial r} \right) - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Phi}{\partial \theta} \right) \\
&\quad \left. + 4 \left(\frac{a(r - M)}{\Delta} + \frac{\mathrm{i} \cos \theta}{\sin^2 \theta} \right) \frac{\partial \Phi}{\partial \phi} + \left(4 \cot^2 \theta + 2 \right) \Phi \right],
\end{aligned}
\tag{23}
$$

where the second line is the coordinate expression in the Kinnersley tetrad in the Kerr spacetime, see appendix B. The adjoint operators $\mathcal{T}^\dagger, \mathcal{S}^\dagger$ are explicitly

$$
\begin{aligned}
(\mathcal{T}^\dagger \eta)_{ab} &= \frac{1}{2} l_a l_b (\text{\dh} - \tau)(\text{\dh} - \tau)\eta + \frac{1}{2} m_a m_b (\text{\th} - \rho)(\text{\th} - \rho)\eta \\
&\quad - \frac{1}{2} l_{(a} m_{b)} \{ (\text{\dh} + \bar\tau' - \tau)(\text{\th} - \rho) + (\text{\th} - \rho + \bar\rho)(\text{\dh} - \tau) \}\eta
\end{aligned}
\tag{24}
$$

and

$$
\begin{aligned}
(\mathcal{S}^\dagger \Phi)_{ab} &= -l_a l_b (\text{\dh} - \tau)(\text{\dh} + 3\tau)\Phi - m_a m_b (\text{\th} - \rho)(\text{\th} + 3\rho)\Phi \\
&\quad + l_{(a} m_{b)} \{ (\text{\th} - \rho + \bar\rho)(\text{\dh} + 3\tau) + (\text{\dh} - \tau + \bar\tau')(\text{\th} + 3\rho) \}\Phi.
\end{aligned}
\tag{25}
$$

As one sees from this expression, the real perturbation $h_{ab} = \mathrm{Re}\,(\mathcal{S}^\dagger \Phi)_{ab}$ automatically is in a gauge where

$$h_{ab} l^b = 0 = g^{ab} h_{ab} \qquad (= h_{ab} m^a \bar{m}^b). \tag{26}$$

We follow the tradition in which this gauge is referred to as 'ingoing radiation gauge' (IRG), although in a general type II vacuum spacetime this terminology has little physical meaning.

3. Decomposition of h_{ab} for homogeneous equation

Although we are generally interested in the inhomogeneous linearized Einstein equation (1), we will in this section first inspect the homogeneous equation

$$(\mathcal{E} h)_{ab} = 0. \tag{27}$$

As we have recalled, on a type II vacuum background, if Φ is a (IRG) Hertz potential, i.e. a weight $(-4, 0)$ GHP scalar satisfying the (adjoint) homogeneous Teukolsky equation $\mathcal{O}^\dagger \Phi = 0$, then the metric perturbation $h_{ab} = \mathrm{Re}\,(\mathcal{S}^\dagger \Phi)_{ab}$ is a solution to the source-free linearized Einstein equation. If we add to such a solution a pure gauge perturbation $(\mathcal{L}_\xi g)_{ab} = \nabla_a \xi_b + \nabla_b \xi_a$ and a zero mode perturbation $\dot{g}_{ab} = \frac{\mathrm{d}}{\mathrm{d}s} g_{ab}^{M(s), a(s)}$, we obtain a new solution,

$$h_{ab} = \text{Re}\,(\mathcal{S}^\dagger \Phi)_{ab} + (\mathcal{L}_\xi g)_{ab} + \dot{g}_{ab}, \tag{28}$$

and we would now like to argue that, in a sense, this is the most general solution in the case of the Kerr background, or more precisely, when $(\mathscr{M}, g_{ab})$ is the exterior region of the black hole in the Kerr spacetime.

We shall give an argument for (28) following that given by Prabhu and Wald [25] in the case of Schwarzschild, with suitable modifications for Kerr. Their argument is of a conceptual nature as it is embedded in the phase-space (Hamiltonian) formulation of general relativity. In its basic form, it applies to the sub-extremal family of Kerr black holes $|a| < M$, although later we will need to impose, for technical reasons, the more stringent condition $|a| \ll M$, which should be unnecessary. One begins by defining the symplectic current for vacuum general relativity. Let h_{ab}, h'_{ab} be two symmetric tensor fields, not necessarily solutions to the linearized Einstein equations. By the definition of the adjoint, since $\mathcal{E}^\dagger = \mathcal{E}$, we have

$$(\mathcal{E}h')^{ab} h_{ab} - (\mathcal{E}h)^{ab} h'_{ab} = \nabla_a w^a, \tag{29}$$

where w^a is in the present context called the 'symplectic current'. Evidently, it is conserved if h_{ab}, h'_{ab} solve the linearized Einstein equation. The explicit form of w^a can be found e.g. in [34]. Following that reference, one defines the symplectic form of general relativity to be

$$W[h, h'] = \int_\Sigma \nu^a w_a(h, h'), \tag{30}$$

where Σ is a Cauchy surface with unit future pointing normal ν^a, stretching from the bifurcation surface $\mathscr{S} = \mathscr{H}^- \cap \mathscr{H}^+$ to spatial infinity. Since w^a is conserved, the definition of W is invariant under deformations of Σ leaving $\mathscr{S}$ and spatial infinity fixed. If ξ^a is a smooth vector field vanishing at $\mathscr{S}$ and approaching as $r \to \infty$ either a time translation, spatial translation, or rotation [in the asymptotically Cartesian coordinate system (t, x_1, x_2, x_3) determined by (t, r, θ, ϕ)], the symplectic inner products with the corresponding pure gauge perturbation $(\mathcal{L}_\xi g)_{ab}$,

$$W[h, \mathcal{L}_\xi g] = \begin{cases} \dot{M} & \text{if } \xi^a \to \partial_t \\ -\dot{P}_i & \text{if } \xi^a \to \partial_i \\ \dot{J}_i & \text{if } \xi^a \to \epsilon_{ijk} x_k \partial_j, \end{cases} \tag{31}$$

define the perturbed ADM mass, linear momentum, respectively angular momentum, see [34].

From a solution to the Einstein equation we may determine its initial data (q_{ab}, p^{ab}), where q_{ab} is a smooth symmetric tensor field on Σ representing the intrinsic metric, and where p^{ab} is a smooth symmetric tensor density on Σ representing the conjugate momentum (related to the perturbed extrinsic curvature), see [20]. The corresponding quantities for the linearized perturbation are denoted by $(\dot{q}_{ab}, \dot{p}^{ab})$, where here and in the following, an overdot refers to a first order perturbed quantity, and not a time derivative. In terms of these, the symplectic form reads

$$W[\dot{q}, \dot{p}; \dot{q}', \dot{p}'] = \int_\Sigma \dot{p}^{ab} \dot{q}'_{ab} - \dot{p}^{ab'} \dot{q}_{ab}. \tag{32}$$

The symplectic form is automatically finite on the space of square integrable pairs $(\dot{q}_{ab}, \dot{p}^{ab})$. We denote the corresponding L^2-type norm by $\| \cdot \|$, so

$$\|(\dot{p}, \dot{q})\|^2 = \int_\Sigma \dot{p}^{ab} \dot{p}_{ab} (-q)^{-1/2} + \dot{q}^{ab} \dot{q}_{ab} (-q)^{+1/2}. \tag{33}$$

If $\mathscr{X}$ is a linear subspace of $L^2 \times L^2$ of such pairs, we denote by $\overline{\mathscr{X}}$ its closure in the norm $\| \cdot \|$.

Following section 4.1 of [35] we consider several such linear subspaces in this $L^2 \times L^2$ space of (unconstrained) initial data. We let $\mathscr{W}$ be the space of all gauge perturbations $(\mathcal{L}_\xi g)_{ab}$ (identified with initial data) generated by smooth vector fields ξ^a that become an asymptotic translation or rotation at infinity and whose projection onto the bifurcation surface $\mathscr{S}$ vanishes. Further, we let $\mathscr{V} = (\mathscr{W})^\perp$, where $\perp$ denotes the symplectic orthogonal complement. Finally, we can introduce within $\mathscr{V}$ the subspace $\mathscr{V} \cap \mathscr{U}$, where $\mathscr{U} \subset C^\infty \times C^\infty$ is a certain intersection of weighted Sobolev spaces. It is shown in section 4.1 of [35] that: (i) Initial data from $\mathscr{V} \cap \mathscr{U}$ satisfy the linearized constraints, (ii) have $\dot{M} = \dot{P} = \dot{J} = 0$, (iii) $\mathscr{U} \cap \mathscr{V}$ is dense in the closed subspace $\mathscr{V}$, (iv) satisfy $\dot{q}_{ab}, \dot{p}^{ab} = o(r^{-3/2})$ in an asymptotically Cartesian coordinates system as $r \to \infty$, with derivatives falling off faster by corresponding powers of r^{-1} and (v)

$$\dot{\varepsilon} = \dot{\vartheta}_l = \dot{\vartheta}_n = 0, \quad \text{on } \mathscr{S} \tag{34}$$

where ε is the induced area element on the bifurcation surface $\mathscr{S}$ and $\dot{\vartheta}_{n,l}$ the perturbed expansions in the directions n^a, l^a respectively. The second two conditions physically mean that the perturbation is in a gauge in which the horizons $\mathscr{H}^\pm$ stay in a fixed position, whereas the first is just a gauge condition that can be imposed consistent with $\dot{M} = \dot{J} = 0$ in view of the first law of black hole mechanics, see [35] for further explanations.

Finally, let $\mathscr{Y}$ denote the space of perturbations $h_{ab} = \text{Re} \, (\mathcal{S}^\dagger \Phi)_{ab}$ generated by smooth GHP scalar fields Φ of type $(-4, 0)$ such that $\mathcal{O}^\dagger \Phi = 0$ and such that Φ has compact initial data on Σ. Prabhu and Wald [25] now argue as follows. Suppose h_{ab} is a smooth solution to the linearized Einstein equations with initial data $(\dot{q}_{ab}, \dot{p}^{ab}) \in \mathscr{V} \cap \mathscr{U}$, and suppose one could show that the relation

$$W[h, \text{Re} \, \mathcal{S}^\dagger \Phi] = 0 \qquad \forall \text{Re} \, (\mathcal{S}^\dagger \Phi)_{ab} \in \mathscr{Y} \tag{35}$$

implies that h_{ab} is pure gauge perturbation in $\mathscr{W}$. In formulas, this means

$$(\mathscr{Y})^\perp \cap (\mathscr{U} \cap \mathscr{V}) \subset \mathscr{W}. \tag{36}$$

Now take the symplectic complement $\perp$ of this relation. We can use $(\mathscr{U} \cap \mathscr{V})^\perp = \mathscr{V}^\perp$ in view of (iii), and further that $\mathscr{V}^\perp = \mathscr{W}^{\perp\perp} = \overline{\mathscr{W}}$, by definition. Then we obtain $\overline{(\mathscr{Y} + \mathscr{W})} \supset \mathscr{V}$. Thus, in view of (i)–(v), we would learn that:

Conclusion. [25] *Any smooth perturbation with* $\dot{M} = \dot{P} = \dot{J} = 0$*, with (34) and the falloff for* $r \to \infty$ *implied by the Sobolev norm of* $\mathscr{U}$ *(basically* $\dot{q}_{ab}, \dot{p}^{ab} = o(r^{-3/2})$*) can be approximated in the norm* L^2 *on initial data by the sum of a Hertz potential perturbation* $\text{Re} \, (\mathcal{S}^\dagger \Phi)_{ab}$ *and a pure gauge perturbation* $\mathcal{L}_\xi g_{ab}$*. Since perturbations with a given nonzero* $\dot{M}, \dot{P}, \dot{J}$ *can be obtained by adding a perturbation* $\dot{g}_{ab}$ *towards another, perhaps boosted/rotated, Kerr black hole this gives an argument—and precise sense—in which (28) holds true.*

Furthermore, [25] have given arguments that (36) must in fact hold in the Schwarzschild background. Thus, in the Schwarzschild spacetime, this completes the argument. We now give a corresponding argument for the Kerr metrics with sufficiently small $|a| \ll M$. First, it has been demonstrated in [35] by gluing methods that $\mathscr{V} \cap \mathscr{U}$ has the same closure as those constrained initial data in $\mathscr{V}$ that are smooth and vanish outside some value $r > r_0$ on Σ. We may therefore restrict to such solutions, and call the corresponding space of initial data $\mathscr{V}_0$. Secondly, [25] have shown that in a general type II background, the relation (35) implies

$\mathcal{T}h = 0$, so $\psi_0 = 0$. The argument is then completed by the following theorem, which shows that h_{ab} must be pure gauge.

Theorem 1. *Let h_{ab} be a solution to the linearized Einstein equation on a Kerr background with $|a| \ll M$, with initial data $(\dot{q}_{ab}, \dot{p}^{ab}) \in \mathcal{V}_0$, having $\psi_0 = 0$. Then also $\psi_4 = 0$ and $h_{ab} = \mathcal{L}_\xi g_{ab}$ is a pure gauge perturbation.*

Remarks.

(1) The proof does not show explicitly that the gauge vector field must be in $\mathcal{W}$, i.e. it need not have vanishing projection onto the bifurcation surface. This means that the space for which the approximation claimed in the Conclusion could be slightly smaller than $\mathcal{V}$, but it will always contain e.g. perturbations with compact support on the initial data surface Σ.

(2) Before we give the proof of this theorem, we point out that Wald [36] has given an argument for smooth mode solutions. Our proof thus generalizes that of Wald in the sense that our solutions need not be mode solutions.

(3) The restriction to $|a| \ll M$ comes about because we are using results on the large time decay of solutions to the Teukolsky equation from [2, 37], which hold under this condition. Concretely, we use that integrals of the type $\int_{\{r_1 < r < r_2, u_1 < u < u_2\}} |\Psi_2^{-4/3}\psi_4|^2 \sin\theta \mathrm{d}u \mathrm{d}r \mathrm{d}\varphi_* \mathrm{d}\theta$ are uniformly bounded in u_2 for fixed u_1 and $r_+ < r_1 < r_2 < \infty$. Our proof would generalize to all $|a| < M$ if such a result could be generalized to this range[8].

Proof. By the Teukolsky–Starobinsky identities which hold in a general type D background (see e.g. [38, 39]),

$$\text{þ}^4 \Psi_2^{-4/3}\psi_4 = \eth'^4 \Psi_2^{-4/3}\psi_0 - 3 \text{Ł}_\xi \bar{\psi}_0, \tag{37a}$$

$$\text{þ}'^4 \Psi_2^{-4/3}\psi_0 = \eth^4 \Psi_2^{-4/3}\psi_4 + 3 \text{Ł}_\xi \bar{\psi}_4, \tag{37b}$$

where $\xi^a = \Psi_2^{-1/3}(\tau' m^a - \tau \bar{m}^a - \rho' l^a + \rho n^a)$ (which is $M^{-1/3}\partial_t$ in Kerr), and where Ł is the GHP-covariant Lie derivative [40]. Because $\psi_0 = 0$, these relations evidently give strong constraints on ψ_4. In the remainder of the proof we will argue that, in fact, we learn $\psi_4 = 0$.

From (37a), we first conclude in view of $\text{þ}\rho = \rho^2$ that

$$\Psi_2^{-4/3}\psi_4 = \alpha_0^\circ + \alpha_1^\circ \frac{1}{\rho} + \alpha_2^\circ \frac{1}{\rho^2} + \alpha_3^\circ \frac{1}{\rho^3}. \tag{38}$$

Here and in the following, a degree mark $^\circ$ on a quantity indicates that this quantity is annihilated by þ, so x° means that $\text{þ}x^\circ = 0$. For the solution $\Psi_2^{-4/3}\psi_4$ to the adjoint Teukolsky equation, we then use the following consequence of the Teukolsky–Starobinsky identities, see [38],

$$\text{þ}'^4 \bar{\Psi}_2^{-4/3} \text{þ}^4 \Psi_2^{-4/3}\psi_4 = \eth'^4 \bar{\Psi}_2^{-4/3} \eth^4 \Psi_2^{-4/3}\psi_4 - 9 \bar{\text{Ł}}_\xi \text{Ł}_\xi \psi_4. \tag{39}$$

---

[8] To be precise, this is not a GHP-invariant statement. We here mean the outgoing Kerr–Newman coordinates and e.g. the Kinnersley frame. An invariant statement would be $\int_{\{r_1 < r < r_2, u_1 < u < u_2\}} \rho^8 |\Psi_2^{-4/3}\psi_4|^2 \, \mathrm{d}vol_g < \infty$.

Using (38), the left side vanishes. Since $Ł_\xi \Psi_2 = 0$ in Kerr, this gives

$$0 = \{\Psi_2^{-4/3}\eth'^4 \bar\Psi_2^{-4/3}\eth^4 - 9Ł_\xi Ł_\xi\}\left(\alpha_0^\circ + \alpha_1^\circ \frac{1}{\rho} + \alpha_2^\circ \frac{1}{\rho^2} + \alpha_3^\circ \frac{1}{\rho^3}\right). \qquad (40)$$

To take full advantage of $\flat\alpha_i^\circ = 0$ in this equation and in several similar instances in the rest of the paper, we employ a formalism invented by Held [31, 41]. His formalism has three main ingredients. The first is a set of new operators, denoted by $\tilde\eth, \tilde\eth'$ and $\tilde\flat'$ which replace $\eth, \eth'$ and $\flat'$ and contrary to the latter have the property that $\tilde\eth x^\circ, \tilde\eth' x^\circ$ and $\tilde\flat' x^\circ$ are quantities annihilated by $\flat$. The second is a list of identities, generated from the Einstein equations and Bianchi identities, expressing any background quantity in terms of ρ and quantities annihilated by $\flat$, together with a table of how the new operators act on these. These two features enable one to write any expression appearing in (40) as a Laurent polynomial in ρ with coefficients that are annihilated by $\flat$. The third ingredient is a simple lemma showing that if GHP quantities a_i° satisfy

$$a_0^\circ + a_1^\circ \rho + \cdots + a_n^\circ \rho^n = 0 \quad \Longrightarrow \quad a_i^\circ = 0. \qquad (41)$$

(Proof: Divide by ρ^n, then act with $\flat^n$ and use repeatedly $\flat\rho = \rho^2$ to obtain $a_0^\circ = 0$. Substitute this relation and iterate the process.) We begin by recalling the precise definition of the new operators [41] on weight (p, q) GHP quantities,

$$\tilde\flat' = \flat' - \bar\tau\tilde\eth - \tau\tilde\eth' + \tau\bar\tau\left(\frac{q}{\bar\rho} + \frac{p}{\rho}\right) + \frac{1}{2}\left(\frac{q\bar\Psi_2}{\bar\rho} + \frac{p\Psi_2}{\rho}\right), \qquad (42a)$$

$$\tilde\eth = \frac{1}{\bar\rho}\eth + \frac{q\tau}{\rho}, \qquad (42b)$$

$$\tilde\eth' = \frac{1}{\rho}\eth' + \frac{p\bar\tau}{\bar\rho}. \qquad (42c)$$

Using the vacuum Einstein equations in GHP form, [41] next shows that on a type D background of the Case II in Kinnersley's classification [42] (which includes Kerr) with a null tetrad satisfying (n2),

$$\rho' = \rho'^\circ\bar\rho - \frac{1}{2}\Psi_2^\circ\rho^2 - (\tilde\eth\bar\tau^\circ + \frac{1}{2}\Psi_2^\circ)\rho\bar\rho - \tau^\circ\bar\tau^\circ\rho^2\bar\rho, \qquad (43a)$$

$$\tau' = -\bar\tau^\circ\rho^2, \qquad (43b)$$

$$\tau = \tau^\circ\rho\bar\rho, \qquad (43c)$$

$$\Psi_2 = \Psi_2^\circ\rho^3, \qquad (43d)$$

with $\tau^\circ \overset{\circ}{=}(-1, -3)$, $\Psi_2^\circ \overset{\circ}{=}(-3, -3)$, $\rho'^\circ \overset{\circ}{=}(-2, -2)$, together with

$$\frac{1}{\bar\rho} = \Omega^\circ + \frac{1}{\rho}, \qquad (44)$$

Table 2. Action of Held's operators on GHP background quantities in type D spacetimes of Kinnersley's Case II, in a tetrad satisfying (n2).

	$\tilde{\text{þ}}'$	$\tilde{\eth}$	$\tilde{\eth}'$
ρ	$\rho^2\rho'^\circ - \frac{1}{2}\rho(\rho\Psi_2^\circ + \bar{\rho}\bar{\Psi}_2^\circ) - \rho^3\bar{\rho}\tau^\circ\bar{\tau}^\circ$	$\tau^\circ\rho^2$	$-\bar{\tau}^\circ\rho^2$
Ω°	0	$2\tau^\circ$	$-2\bar{\tau}^\circ$
ρ'°	0	0	0
τ°	0	0	$\frac{1}{2}(\rho'^\circ + \bar{\rho}'^\circ)\Omega^\circ + \frac{1}{2}(\Psi_2^\circ - \bar{\Psi}_2^\circ)$
Ψ_2°	0	0	0

with $\Omega^\circ \hat{=} (-1, -1)$. These identities are complemented by identities for the action of the new operators on ρ and on $\Omega^\circ, \rho'^\circ, \tau^\circ, \Psi_2^\circ$ given in the table 2 extracted from [41].

Now we consider (40), replacing the GHP operators in terms of Held's operators (42), and expanding all quantities using (43). Each of the resulting terms is given by a product of integer powers of ρ and $\bar{\rho}$ times a quantity annihilated by þ. We then divide by $\bar{\rho}^2$ in order to get rid of any positive power of $\bar{\rho}$, and for the negative powers of $\bar{\rho}$ we substitute (44). Multiplying by a suitable power ρ, the resulting equation is then of the general form (41), with coefficients a_i° built from finite sums and products of $\Omega^\circ, \rho'^\circ, \tau^\circ, \Psi_2^\circ, \alpha_i^\circ$, and their Held-derivatives $\tilde{\eth}, \tilde{\eth}'$, and $\tilde{\text{þ}}'$. By (41), we get $a_i^\circ = 0$, and this gives a set of conditions to be satisfied by the 4 coefficient (distributions) $\alpha_i^\circ, i = 0, 1, 2, 3$.

If we now consider the leading identity $a_0^\circ = 0$, or equivalently the limit of equation (40) at $\mathscr{I}^+$, then the results of table 2 (and $\rho = -(r - ia\cos\theta)^{-1} \to 0$ towards $\mathscr{I}^+$, with $r \to \infty$ at fixed outgoing Kerr–Newman coordinate u, see (B.4a)) lead to the equation

$$0 = \{\tilde{\eth}'^4\tilde{\eth}^4 - 9(\Psi_2^\circ)^2\tilde{\text{þ}}'^2\}\alpha_3^\circ. \tag{45}$$

Since $\text{þ} = l^a\nabla_a - p\epsilon - q\bar{\epsilon}$ and $\epsilon = 0$ in Kerr in the Kinnersley frame, and since l^a is asymptotically tangent to a congruence of outgoing null geodesics, a GHP quantity annihilated by þ may be viewed geometrically as a function—or rather a section of the appropriate line bundle—on $\mathscr{I}^+$ (or $\mathscr{H}^-$); see figure 2. The outgoing Kerr–Newman coordinates (u, θ, φ_*) as in (B.4a) are good coordinates on $\mathscr{I}^+$ (or $\mathscr{H}^-$) away from the poles, so as already said, we may think of GHP quantities annihilated by þ in the Kinnersley gauge as functions of (u, θ, φ_*). In this frame, the Held operators $\tilde{\eth}, \tilde{\eth}'$ and $\tilde{\text{þ}}'$ are

$$\tilde{\eth} = -\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\theta} + i\csc\theta\frac{\partial}{\partial\varphi_*} + ia\sin\theta\frac{\partial}{\partial u} - \frac{1}{2}(p - q)\cot\theta\right),$$

$$\tilde{\eth}' = -\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\theta} - i\csc\theta\frac{\partial}{\partial\varphi_*} - ia\sin\theta\frac{\partial}{\partial u} + \frac{1}{2}(p - q)\cot\theta\right), \tag{46}$$

$$\tilde{\text{þ}}' = \frac{\partial}{\partial u},$$

and $\Psi_2^\circ = M$.

We now would like to take the Fourier transform of α_3° in u. In order to be able to do this, we must know the behavior of these functions for $|u| \to \infty$. At our disposal, we have our knowledge that the initial data of h_{ab} lie in the space $\mathscr{V}_0$ (having in particular their support on a Cauchy surface for $r < r_0$), as well as (38). Due to the essentially trivial dependence upon r of (38) which enters only in $\rho = -(r - ia\cos\theta)^{-1}$, we can conclude that $\alpha_3^\circ = 0$ when

$u < u(r_0)$, since $\text{\th}^3(\Psi_2^{-4/3}\psi_4) = 6\alpha_3^\circ$ vanishes for $r > r_0$ on Σ. When $u \to +\infty$, we cannot argue in this manner because $\Psi_2^{-4/3}\psi_4$ does not necessarily vanish near r_+. But we can for example use the decay results by [2] (theorem 10.1, equation (233)), see also [37] (theorem 2), showing that[9] integrals of the type $I_n(u_2) = \int_{\{r_1<r<r_2,u_1<u<u_2\}} r^n|\Psi_2^{-4/3}\psi_4|^2 \sin\theta\,du\,dr\,d\varphi_*\,d\theta$ are uniformly bounded in u_2 for fixed n,u_1 and some $r_+ < r_1, r_2 < \infty$. It then follows from the special structure of (38) (using that $\alpha_j^\circ = \alpha_j^\circ(u,\theta,\varphi_*)$) that if we take a suitable linear combination of finitely many I_n's picking out only the contribution $\int_{u_1<u<u_2} |\alpha_3^\circ|^2 \sin\theta\,du\,d\varphi_*\,d\theta$ after performing the r-integrals, that α_3° must be an L^2-function in u. The mentioned results by [2, 37] so far only cover the case that $|a| \ll M$, which is the origin of this restriction in our theorem.

Assuming that this is the case, we can do Fourier-transforms at our hearts content and write

$$\alpha_3^\circ(u,\theta,\varphi_*) = \sum_{l,m}\int_{-\infty}^{\infty} d\omega\; {}_{-2}\hat{\alpha}_{3,lm}^\circ(\omega)\, {}_{-2}S_{\omega lm}(\theta)e^{-i\omega u+im\varphi_*}, \tag{47}$$

where ${}_{-2}\hat{\alpha}_{3,lm}^\circ(\omega)$ is a distribution which is in L^2 with respect to ω, and where ${}_{-2}S_{\omega lm}(\theta)$ are the normalized eigenfunctions of the angular spin $s = -2$ Teukolsky master equation [5, 6]. Using either the explicit form (46) and that of the angular Teukolsky equation, or going back to the form (39) and using the identities for the eigenfunctions of the radial Teukolsky equation given e.g. in [4], one can see that (dropping the subscripts l, m, ω)

$$\{\tilde{\eth}'^4\tilde{\eth}^4 - 9(\Psi_2^\circ)^2\tilde{\text{\th}}'^2\}[S(\theta)e^{-i\omega u+im\varphi_*}] = C^2 S(\theta)e^{-i\omega u+im\varphi_*}, \tag{48}$$

where the Starobinsky constant $C \equiv {}_{-2}C_{lm}(\omega)$ is given by

$$C^2 = \frac{1}{16}\{\lambda^2(\lambda+2)^2 - 8\omega^2\lambda[A^2(5\lambda+6) - 12a^2] + 144\omega^2(M^2 + \omega^2A^4)\}, \tag{49}$$

where λ is the eigenvalue of the spheroidal harmonic ${}_{-2}S_{\omega lm}(\theta)$ and where $A = a^2 - am/\omega$. Putting together (45) and (48), we therefore learn that

$$_{-2}C_{lm}(\omega)^2\,\hat{\alpha}_3^\circ(\omega) = 0. \tag{50}$$

If[10] $\omega \mapsto {}_{-2}C_{lm}(\omega)$ does not vanish anywhere on the real axis for any choice of l, m, then this equation implies $\alpha_3^\circ(\omega) = 0$ and then obviously $\alpha_3^\circ = 0$ in view of (47). We can then substitute this result into (40), and repeat the same argument, leading now to $\alpha_2^\circ = 0$, and subsequently to $\alpha_i^\circ = 0$ for all i. So, $\psi_4 = 0$ in view of (38), in addition to $\psi_0 = 0$. Thus, h_{ab} is a perturbation towards another type D metric. These have been analyzed by [36] who has shown that h_{ab} is, modulo gauge, either a perturbation towards another Kerr metric, or a perturbation towards a rotating C-metric or a NUT metric [44] (in total 4 real linear independent perturbations). However, we are assuming that h_{ab} has vanishing initial data for $r > r_0$. Thus, such a

---

[9] Their basic variable is related to ours by $\alpha^{[-2]} = \Psi_2^{-4/3}\psi_4$. When considering their energy norms such as $\bar{\mathbb{E}}_{\tilde{\Sigma}_\tau}(\alpha^{[-2]})$ appearing in their theorem 10.1, one must take into account that $\Delta^{-2}\alpha^{[-2]}$ is smooth at $\mathscr{H}^+$ in the case considered here, and the relations $l^a = (r^2+a^2)L^a/\Delta$, $n^a = (r^2+a^2)\underline{L}^a/\Sigma$ between the Kinnersley tetrad and the vector fields $L^a, \underline{L}^a$ employed by these authors. Furthermore, one should note that on $\mathscr{H}^+$, $\Delta^{-1}\underline{L}^a$ and L^a have finite limits.

[10] In Schwarzschild, the separation constant $\lambda = (l-1)(l+2)$ is independent of ω and ${}_{-2}C_{lm}(\omega) = 0$ manifestly has no solutions on the real ω-axis by inspection. The complex solutions correspond to the algebraically special perturbations of Schwarzschild found previously by Couch and Newman [43].

solution would have vanishing perturbed curvature invariants (e.g. ψ_2) outside the domain of dependence of the set where the initial data are not zero, which is not the case for the perturbations towards a C- or NUT metric. So the proof would be complete provided we knew that $_{-2}C_{lm}(\omega)$ does not vanish anywhere on the real ω-axis for any choice of l, m.

In Kerr, there can be solutions to $_{-2}C_{lm}(\omega) = 0$ in the complex ω-plane[11] [4], but it is not known to us whether there could be any solutions on the real axis. Whichever way, one can show that any solutions to $_{-2}C_{lm}(\omega) = 0$ on the real axis, or in fact complex plane, must be isolated if they do exist. This follows because the separation constant λ, being obtained from a Sturm–Liouville problem for the angular operator appearing in the separated Teukolsky equation, with coefficients that are polynomial in ω, must depend analytically on ω. In other words, $\hat{\alpha}_3^\circ(\omega)$ is a sum of delta-functions and their derivatives peaked at the the countable set $\{\omega_j\} \subset \mathbb{R}$ where $_{-2}C_{lm}(\omega_j) = 0$. But this is impossible for an L^2 function and we conclude again that $\alpha_3^\circ = 0$. Then we proceed in the same manner to successively show that all $\alpha_i^\circ = 0$. The proof is complete. $\square$

4. Decomposition of h_{ab} for inhomogeneous equation

Now we would like to study solutions h_{ab} to the sourced linearized Einstein equation

$$(\mathcal{E}h)_{ab} = T_{ab}, \tag{51}$$

where $\mathcal{E}$ is the linearized Einstein operator (17). The first part of our analysis will be entirely local and works in any type II spacetime and with any, possibly distributional, T_{ab}. The final part of the analysis involves global features of Kerr, and to avoid difficult issues of analytical nature distracting from the main points, we will assume that the the source T_{ab} is smooth and of compact support. Of course, for there to be any solutions h_{ab}, we must have $\nabla^a T_{ab} = 0$.

Ideally, we would like to be able to write the perturbation up to gauge as $h_{ab} = \text{Re}\,(\mathcal{S}^\dagger \Phi)_{ab}$ in terms of a Hertz potential in order to take advantage of the special features of type II backgrounds. However, it is easy to see that this will not be possible for a generic T_{ab}. Indeed, such a h_{ab} is automatically in the ingoing radiation gauge $h_{nl} = h_{ll} = h_{ml} = h_{m\bar{m}} = 0$, and the ll component of (51) together with (A.1) for $(\mathcal{E}h)_{ll}$ then gives $T_{ll} = 0$. Thus, if $T_{ll} \neq 0$ somewhere—as happens, e.g. for the stress tensor of a point particle (4)—then the perturbation can definitely not be written as $h_{ab} = \text{Re}\,(\mathcal{S}^\dagger \Phi)_{ab}$ in terms of a Hertz potential (up to gauge).

The main idea of this paper is that we may nevertheless write $h_{ab} = \text{Re}\,(\mathcal{S}^\dagger \Phi)_{ab} + x_{ab}$ up to gauge for a relatively simple 'corrector tensor' x_{ab} determined from T_{ab}. Furthermore, the Hertz potential Φ will be a solution to the adjoint Teukolsky equation with a certain source also determined from T_{ab}. To begin, it is easy to see that without the use of any Einstein equation, a metric perturbation h_{ab} may be put in a gauge where $h_{ab}l^b = 0$. Indeed, if h_{ab} is not in this gauge to begin with, we can solve for a gauge vector field ξ^a satisfying

$$(h_{ab} - \mathcal{L}_\xi g_{ab})l^a = 0. \tag{52}$$

This equation is equivalent to the GHP equations,

$$
\begin{aligned}
2\text{þ}\xi_l &= h_{ll}, \\
(\text{þ} + \bar{\rho})\xi_m + (\text{ð} + \bar{\tau}')\xi_l &= h_{lm}, \\
\text{þ}\xi_n + \text{þ}'\xi_l + (\tau + \bar{\tau}')\xi_{\bar{m}} + (\bar{\tau} + \tau')\xi_m &= h_{ln}.
\end{aligned}
\tag{53}
$$

---

[11] As shown by Chandrasekhar [4], such frequencies correspond to algebraically special perturbations. The point is that these cannot be in L^2.

Since $\text{þ} = l^a \nabla_a - p\epsilon - q\bar{\epsilon}$, the first equation is an ordinary differential equation (ODE) for ξ_l along the orbits of l^a. Substituting a solution ξ_l into the second equation, this similarly gives an ODE for ξ_m along the orbits of l^a. Finally substituting both solutions ξ_l, ξ_m into the third equation gives an ODE for ξ_n along the orbits of l^a. Having determined the null-tetrad components $(\xi_l, \xi_n, \xi_m, \xi_{\bar{m}})$ of ξ_a we can redefine $h_{ab} \to h_{ab} - \mathcal{L}_\xi g_{ab}$ in order to obtain a perturbation such that $h_{ab}l^b = 0$.

The corrector field x_{ab} is now determined in such a way that

$$(T_{ab} - \mathcal{E}x_{ab})l^b = 0, \tag{54}$$

with the idea to eliminate any l component from T_{ab}. Our ansatz for x_{ab} is

$$x_{ab} = 2m_{(a}\bar{m}_{b)}x_{m\bar{m}} - 2l_{(a}\bar{m}_{b)}x_{nm} - 2l_{(a}m_{b)}x_{n\bar{m}} + l_al_bx_{nn}, \tag{55}$$

where $x_{n\bar{m}} = \bar{x}_{nm}$, so there are 4 real independent components encoded in $x_{m\bar{m}}, x_{mn}, x_{nn}$ by which we attempt to satisfy the 4 real independent equation (54). We now first transvect (54) with l^a and use the ll component of the Einstein operator $\mathcal{E}$, (A.1), to obtain

$$\{\text{þ}(\text{þ} - \rho - \bar{\rho}) + 2\rho\bar{\rho}\}x_{m\bar{m}} = T_{ll}. \tag{56}$$

Next, we transvect (54) with m^a and use the ml component of the Einstein operator $\mathcal{E}$, (A.4), to obtain:

$$\frac{1}{2}\{\text{þ}(\text{þ} - 2\rho) + 2\bar{\rho}(\rho - \bar{\rho})\}x_{nm}$$
$$= T_{lm} - \frac{1}{2}\{(\text{þ} + \rho - \bar{\rho})(\eth + \bar{\tau}' - \tau) + 2\bar{\tau}'(\text{þ} - 2\rho) - (\eth - \tau - \bar{\tau}')\bar{\rho} + 2\rho\tau\}x_{m\bar{m}}. \tag{57}$$

Finally, we transvect (54) with n^a and use the nl component of the Einstein operator $\mathcal{E}$, (A.3), to obtain:

$$\frac{1}{2}\{\rho(\text{þ} - \rho) + \bar{\rho}(\text{þ} - \bar{\rho})\}x_{nn}$$
$$= T_{ln} - \frac{1}{2}\{(\eth' + \tau' - \bar{\tau})(\eth - \tau + \bar{\tau}') + (\eth'\eth - \tau\tau' - \bar{\tau}\bar{\tau}' + \tau\bar{\tau}) - (\Psi_2 + \bar{\Psi}_2)$$
$$+ (\text{Þ}' - 2\rho')\bar{\rho} + (\text{þ} - 2\bar{\rho})\rho' + \rho(3\text{Þ}' - 2\bar{\rho}') + \bar{\rho}'(3\text{þ} - 2\rho)$$
$$- 2\text{Þ}'\text{þ} + 2\rho\bar{\rho}' + 2\eth'(\tau) - \tau\bar{\tau}\}x_{m\bar{m}}$$
$$- \frac{1}{2}\{(\text{þ} - 2\rho)(\eth' - \bar{\tau}) + (\tau' + \bar{\tau})(\text{þ} + \bar{\rho}) - 2(\eth' - \tau')\rho - 2\bar{\tau}\text{þ}\}x_{nm}$$
$$- \frac{1}{2}\{(\text{þ} - 2\bar{\rho})(\eth - \tau) + (\bar{\tau}' + \tau)(\text{þ} + \rho) - 2(\eth - \bar{\tau}')\bar{\rho} - 2\tau\text{þ}\}x_{n\bar{m}}. \tag{58}$$

To simplify, we could set $\tau = \tau' = \sigma' = 0$ by an appropriate choice of null tetrad; see (n1). The system of equations (56)–(58) is now solved as follows. First, since $\text{þ} = l^a \nabla_a - p\epsilon - q\bar{\epsilon}$, the first equation (56) is an ODE for $x_{m\bar{m}}$ along the orbits of l^a, see figure 2 for the exterior region of Kerr. We take a solution $x_{m\bar{m}}$ and substitute it into the second equation (57), which thereby becomes an ODE for x_{nm} along the orbits of l^a. Finally, we take the solutions $x_{m\bar{m}}, x_{nm}$ and substitute them into the third equation (58), which thereby becomes is an ODE for x_{nn} along the orbits of l^a. We can thereby obtain x_{ab} by solving three scalar ODEs, a comparatively manageable task in practice.
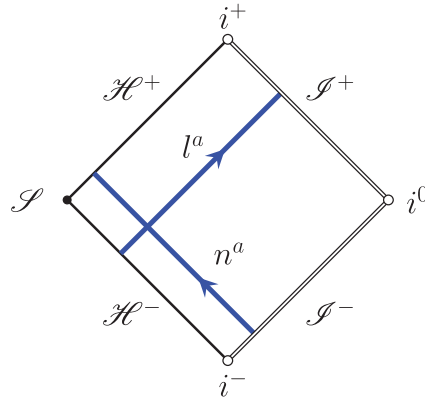
Figure 2. Orbits of the principal null vector field l^a and of n^a in the exterior region of Kerr.

Having determined x_{ab}, we now adjust $h_{ab} \to h_{ab} - x_{ab}$. The adjusted h_{ab} still has $h_{ab}l^b = 0$ and additionally satisfies

$$(\mathcal{E}h)_{ab} = S_{ab} \equiv T_{ab} - \mathcal{E}x_{ab} . \tag{59}$$

What x_{ab} has thereby achieved is to remove the l components from the new source S_{ab}, i.e. $S_{ab}l^b = 0$. At this stage, we can apply a result by [45] showing that there exists a gauge vector field η^a such that after a gauge transformation $h_{ab} \to h_{ab} - \mathcal{L}_\eta g_{ab}$, the perturbation h_{ab} satisfying (59) is in ingoing radiation gauge, $h_{nl} = h_{ll} = h_{ml} = h_{m\bar{m}} = 0$. To summarize, we have obtained from the original perturbation satisfying (51) by a shift $h_{ab} \to h_{ab} - \mathcal{L}_\xi g_{ab} - \mathcal{L}_\eta g_{ab} - x_{ab}$ a new perturbation in ingoing radiation gauge satisfying (59). Furthermore, the new source S_{ab} has no l components. Since we have described a constructive procedure to obtain x_{ab}, S_{ab} from the original source T_{ab}, we can from now work with the shifted h_{ab} satisfying (59), remembering to add x_{ab} back at the end.

At this stage, we would like to show that the shifted h_{ab} satisfying (59) is of the form $h_{ab} = \text{Re}\,(\mathcal{S}^\dagger \Phi)_{ab}$, for some potential $\Phi \stackrel{\circ}{=} (-4, 0)$ satisfying the $\mathcal{O}^\dagger$ Teukolsky equation with some source to be determined. From the definition of $\mathcal{S}^\dagger$ (25), this means that Φ should simultaneously satisfy the three equations,

$$h_{\bar{m}\bar{m}} = -\frac{1}{2}(\mathrm{þ} - \rho)(\mathrm{þ} + 3\rho)\Phi, \tag{60}$$

$$h_{\bar{m}n} = -\frac{1}{4}\{(\mathrm{þ} - \rho + \bar{\rho})(\eth + 3\tau) + (\eth - \tau + \bar{\tau}')(\mathrm{þ} + 3\rho)\}\Phi, \tag{61}$$

$$h_{nn} = -\frac{1}{2}(\eth - \tau)(\eth + 3\tau)\Phi + \text{c.c.}. \tag{62}$$

It is natural to use the first equation (60) in order to define Φ. Since (60) can be viewed as a second order ODE along the geodesic tangent to l^a, integration leaves us with two constants per geodesic. A change of those integration constants entails the change

Table 3. Action of Held's operators on GHP background quantities in type II spacetimes in a tetrad satisfying (n1), taken from [31].

	$\tilde{\mathrm{p}}'$	$\tilde{\eth}$	$\tilde{\eth}'$
ρ	$\rho^2\rho'^\circ - \frac{1}{2}\rho(\rho\Psi_2^\circ + \bar{\rho}\bar{\Psi}_2^\circ)$	0	$\rho^2\tilde{\eth}'\Omega^\circ$
Ω°	$\bar{\rho}'^\circ - \rho'^\circ$	$\tilde{\eth}\,\Omega^\circ$	$\tilde{\eth}'\Omega^\circ$
ρ'°	$\tilde{\eth}\,\kappa'^\circ$	$\tilde{\eth}\,\rho'^\circ$	$-\Psi_3^\circ - \Omega^\circ\kappa'^\circ$
κ'°	$\tilde{\mathrm{p}}'\kappa'^\circ$	$\tilde{\eth}\,\kappa'^\circ$	$-\Psi_4^\circ$
Ψ_4°	$\tilde{\mathrm{p}}'\Psi_4^\circ$	$\tilde{\mathrm{p}}'\Psi_3^\circ$	$\tilde{\eth}'\Psi_4^\circ$
Ψ_3°	$\tilde{\eth}\Psi_4^\circ$	$\tilde{\mathrm{p}}'\Psi_2^\circ$	$\tilde{\eth}'\Psi_3^\circ$
Ψ_2°	$\tilde{\eth}\Psi_3^\circ$	0	$\tilde{\eth}'\Psi_2^\circ$

$$\Phi \to \Phi + A^\circ + \frac{B^\circ}{\rho^3}, \tag{63}$$

where $A^\circ \hat{=} (-4,0)$, $B^\circ \hat{=} (-1,3)$, and, as throughout this article, a degree $^\circ$ marks a quantity annihilated by p. We shall have to use this freedom momentarily when trying to satisfy the second equation (61). In order to understand this relation, we consider the $l\bar{m}$ component of the Einstein equation (59), $(\mathcal{E}h)_{l\bar{m}} = 0$; see (A.4). Into this equation, we substitute (60), and we use the following GHP operator identity:

$$\frac{1}{2}\{\mathrm{p}(\mathrm{p} - 2\bar{\rho}) + 2\rho(\bar{\rho} - \rho)\}\{(\mathrm{p} - \rho + \bar{\rho})(\eth + 3\tau) + (\eth - \tau + \bar{\tau}')(\mathrm{p} + 3\rho)\}$$
$$= \{(\mathrm{p} - \rho)(\eth - \bar{\tau}' + \tau) - 2\tau\rho\}(\mathrm{p} - \rho)(\mathrm{p} + 3\rho), \tag{64}$$

which follows from the GHP commutators, Bianchi identity, and background Einstein equation in any type II spacetime[12]. Then, if y is defined as the left hand side of (61) minus the right side, we get

$$\{\mathrm{p}(\mathrm{p} - 2\bar{\rho}) + 2\rho(\bar{\rho} - \rho)\}y = 0. \tag{65}$$

This is again a second order ODE along the geodesics tangent to l^a, which integrates using $\mathrm{p}\rho = \rho^2$ to

$$y = \bar{\rho}(\bar{\rho} + \rho)a^\circ + \bar{\rho}(\bar{\rho} - 2\rho)\frac{b^\circ}{\rho^3}, \tag{66}$$

where $a^\circ \hat{=} (-3,-1)$, $b^\circ \hat{=} (0,2)$ are undetermined GHP scalars annihilated by p. In order to compensate these, we now use (63), under which y changes as

$$y \to y + \frac{1}{4}\{(\mathrm{p} - \rho + \bar{\rho})(\eth + 3\tau) + (\eth - \tau + \bar{\tau}')(\mathrm{p} + 3\rho)\}\left\{A^\circ + \frac{B^\circ}{\rho^3}\right\}. \tag{67}$$

On the right side, we now substitute for the GHP operators, $\eth, \eth'$ and p Held's operators $\tilde{\eth}, \tilde{\eth}'$, and $\tilde{\mathrm{p}}'$ (42), and we use the table 3 of operations in type II spacetimes and the following definitions/results (68) taken from [31], taking advantage also of the special tetrad choice described in (n1):

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[12] It is the $l\bar{m}$ component of (22).

$$\rho' = \bar{\rho}\rho'^{\circ} - \frac{1}{2}\rho(\rho + \bar{\rho})\Psi_2^{\circ},$$

$$\kappa' = \kappa'^{\circ} - \rho\Psi_3^{\circ} - \frac{1}{2}\rho^2\tilde{\eth}'\Psi_2^{\circ} - \frac{1}{2}\rho^3\Psi_2^{\circ}\tilde{\eth}'\Omega^{\circ},$$

$$\Psi_3 = \rho^2\Psi_3^{\circ} + \rho^3\tilde{\eth}'\Psi_2^{\circ} + \frac{1}{2}3\rho^4\Psi_2^{\circ}\tilde{\eth}'\Omega^{\circ} \tag{68}$$

$$\Psi_4 = \rho\Psi_4^{\circ} + \rho^2\tilde{\eth}'\Psi_3^{\circ} + \rho^3(\Psi_3^{\circ}\tilde{\eth}'\Omega^{\circ} + \frac{1}{2}\tilde{\eth}'\tilde{\eth}'\Psi_2^{\circ}),$$

$$+ \rho^4(\frac{1}{2}3\tilde{\eth}'\Psi_2^{\circ}\tilde{\eth}'\Omega^{\circ} + \frac{1}{2}\Psi_2\tilde{\eth}\tilde{\eth}\Omega^{\circ}) + \frac{1}{2}3\rho^5\Psi_2^{\circ}(\tilde{\eth}'\Omega^{\circ})^2.$$

Then it is found after a calculation that the change (67) will set y as in (66) to zero, provided A°, B° can be chosen so that

$$\tilde{\eth}A^{\circ} = -2a^{\circ}, \quad \tilde{\eth}B^{\circ} = -2b^{\circ}. \tag{69}$$

We shall postpone the discussion of these equations until after theorem 2 and assume that they can be satisfied. Then both (61) and (60) hold, and we can turn our attention to the final equation to be satisfied by Φ, (62). We consider the ln component of the Einstein equation (59), $(\mathcal{E}h)_{ln} = 0$; see (A.3). Into this equation, we substitute (60), (61), and we use the following GHP operator identity:

$$\{\rho(\flat - \rho) + \bar{\rho}(\flat - \bar{\rho})\}(\eth - \tau)(\eth + 3\tau)$$
$$= -\{(\flat - 2\bar{\rho})(\eth - \tau) + (\bar{\tau}' + \tau)(\flat + \rho) - 2(\eth - \bar{\tau}')\bar{\rho} - 2\tau\flat\} \tag{70}$$
$$\times \{(\flat - \rho + \bar{\rho})(\eth + 3\tau) + (\eth - \tau + \bar{\tau}')(\flat + 3\rho)\} +$$
$$\{(\eth - \tau)(\eth - \bar{\tau}') - \bar{\tau}(\tau - \bar{\tau}')\}(\flat - \rho)(\flat + 3\rho),$$

which follows from the GHP commutators, Bianchi identity, and background Einstein equation in any type II spacetime[13]. Then, if z is defined as the left hand side of (62) minus the right side, we get:

$$\{\rho(\flat - \rho) + \bar{\rho}(\flat - \bar{\rho})\}z = 0. \tag{71}$$

This is a first order ODE along the geodesics tangent to l^a, which integrates using $\flat\rho = \rho^2$ to:

$$z = (\bar{\rho} + \rho)c^{\circ}, \tag{72}$$

where $c^{\circ} \stackrel{\circ}{=} (1, 1)$ is a real, undetermined GHP scalar annihilated by $\flat$. So the remaining task is to eliminate this c° by whatever freedom we have left. As noted by [45] there exist gauge vector fields ζ^a such that after a gauge transformation $h_{ab} \rightarrow h_{ab} - \mathcal{L}_\zeta g_{ab}$, the ingoing radiation gauge perturbation h_{ab} satisfying (59) is still in ingoing radiation gauge, $h_{nl} = h_{ll} = h_{ml} = h_{m\bar{m}} = 0$. These so called 'residual gauge vector fields' can be characterized as follows [45]. Denoting by $\zeta_l, \zeta_n, \zeta_m, \zeta_{\bar{m}}$ the covariant tetrad components of ζ^a in a tetrad satisfying (n1), we have

$$\zeta_l = \zeta_l^{\circ},$$

$$\zeta_n = \zeta_n^{\circ} + \frac{1}{2}(\Psi_2^{\circ}\rho + \bar{\Psi}_2\bar{\rho})\zeta_l^{\circ} + \frac{1}{2}(\frac{1}{\rho} + \frac{1}{\bar{\rho}})\tilde{\flat}'\zeta_l^{\circ}, \tag{73}$$

$$\zeta_{\bar{m}} = \zeta_{\bar{m}}^{\circ}\frac{1}{\rho} - \tilde{\eth}'\zeta_l^{\circ},$$

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[13] It is the ln component of (22).

where $\zeta_l^\circ, \zeta_n^\circ, \zeta_m^\circ$ are GHP quantities annihilated by þ which must satisfy

$$\tilde{\text{þ}}' \zeta_l^\circ = -\frac{1}{2}(\tilde{\eth}' \zeta_m^\circ + \tilde{\eth}\, \zeta_{\bar{m}}^\circ),$$

$$\zeta_n^\circ = \frac{1}{2}(\tilde{\eth}'\tilde{\eth} + \tilde{\eth}\tilde{\eth}' - \rho'^\circ - \bar{\rho}'^\circ)\zeta_l^\circ - \frac{1}{2}(\tilde{\eth}' \zeta_m^\circ - \tilde{\eth}\, \zeta_{\bar{m}}^\circ). \tag{74}$$

These conditions leave ζ_m° undetermined, and we shall now use this freedom to remove c°. First, under a residual gauge transformation $h_{ab} \to h_{ab} - \mathcal{L}_\zeta g_{ab}$, the Hertz potential Φ as defined above changes to

$$\Phi \to \Phi - (\tilde{\eth}'\tilde{\eth}' \zeta_l^\circ + \tilde{\eth}' \Omega^\circ \zeta_{\bar{m}}^\circ)\frac{1}{\rho} + \tilde{\eth}' \zeta_{\bar{m}}^\circ \frac{1}{\rho^2}, \tag{75}$$

whereas h_{nn} changes to

$$h_{nn} \to h_{nn} - 2\tilde{\text{þ}}' \zeta_n^\circ - \tilde{\text{þ}}' \{(\Psi_2^\circ \rho + \bar{\Psi}_2^\circ \bar{\rho})\zeta_l^\circ + (\frac{1}{\rho} + \frac{1}{\bar{\rho}})\tilde{\text{þ}}' \zeta_l^\circ\}$$

$$+ (\Psi_2^\circ \rho^2 + \bar{\Psi}_2^\circ \bar{\rho}^2)\{\zeta_n^\circ + \frac{1}{2}(\Psi_2^\circ \rho + \bar{\Psi}_2^\circ \bar{\rho})\zeta_l^\circ + \frac{1}{2}(\frac{1}{\rho} + \frac{1}{\bar{\rho}})\tilde{\text{þ}}' \zeta_l^\circ\}. \tag{76}$$

Now, z is defined as the left hand side of (62) minus the right side, and z is related to c° by (72). From this, it follows after a lengthy calculation using formulas from table 3 that, provided that we can chose $\zeta_l^\circ, \zeta_n^\circ, \zeta_m^\circ$ so that

$$c^\circ = -\frac{1}{4}(\tilde{\eth}\,\tilde{\eth}\,\tilde{\eth}'\tilde{\eth}' + \tilde{\eth}'\tilde{\eth}'\tilde{\eth}\,\tilde{\eth} - 2\tilde{\eth}\,\Psi_3^\circ - 2\tilde{\eth}'\bar{\Psi}_3^\circ)\zeta_l^\circ + (\Omega^\circ \rho'^\circ - \Psi_2^\circ)\tilde{\eth}' \zeta_m^\circ - (\Omega^\circ \bar{\rho}'^\circ + \bar{\Psi}_2^\circ)\tilde{\eth}\, \zeta_{\bar{m}}^\circ$$

$$-\frac{1}{2}\{\tilde{\eth}\Omega^\circ \bar{\rho}'^\circ - \Omega^\circ(\bar{\Psi}_3^\circ - \Omega^\circ \bar{\kappa}'^\circ) - \frac{1}{2}\tilde{\eth}\bar{\Psi}_2^\circ\}\zeta_{\bar{m}}^\circ + \frac{1}{2}\{\tilde{\eth}'\Omega^\circ \rho'^\circ - \Omega^\circ(\Psi_3^\circ + \Omega^\circ \kappa'^\circ) + \frac{1}{2}\tilde{\eth}'\Psi_2^\circ\}\zeta_m^\circ$$

$$-\frac{1}{4}\tilde{\eth}'\Omega^\circ \tilde{\eth}\, \zeta_{\bar{m}}^\circ + \frac{1}{4}\tilde{\eth}\Omega^\circ \tilde{\eth}'\tilde{\eth}' \zeta_m^\circ, \tag{77}$$

then performing the gauge transformation $h_{ab} \to h_{ab} - \mathcal{L}_\zeta g_{ab}$ with $\zeta_l^\circ, \zeta_n^\circ, \zeta_m^\circ$ subject to (74), will in effect cancel c°. Therefore, (62) will be satisfied. If we combine the gauge vector fields defined so far into $X^a = \eta^a + \xi^a + \zeta^a$, then we can summarize our findings to this point as follows.

Theorem 2. *Suppose h_{ab} satisfies the inhomogeneous linearized Einstein equation $(\mathcal{E}h)_{ab} = T_{ab}$ on a Petrov type II background, and suppose the symmetric tensor field x_{ab} is defined as (55), with tetrad components obtained by successively solving the three ODEs (56)–(58). Then there exists a gauge vector field X^a and a GHP scalar $\Phi \doteq (-4, 0)$ such that*

$$h_{ab} = x_{ab} + (\mathcal{L}_X g)_{ab} + \mathrm{Re}\,(\mathcal{S}^\dagger \Phi)_{ab}, \tag{78}$$

provided that we can solve for given $a^\circ, b^\circ, c^\circ$ the equations (69), (77) subject to (74).

Remark. We note that the transport operators appearing on the left side of equations (56)–(58) may be rewritten in any type II background as

$$\frac{1}{2}\{\rho(\text{þ} - \rho) + \bar{\rho}(\text{þ} - \bar{\rho})\} = \frac{1}{2}(\rho + \bar{\rho})^2 \text{þ} \frac{1}{\rho + \bar{\rho}}, \tag{79a}$$

$$\text{þ}(\text{þ} - \rho - \bar{\rho}) + 2\rho\bar{\rho} = \rho^2 \text{þ} \frac{\bar{\rho}}{\rho^3} \text{þ} \frac{\rho}{\bar{\rho}}, \tag{79b}$$
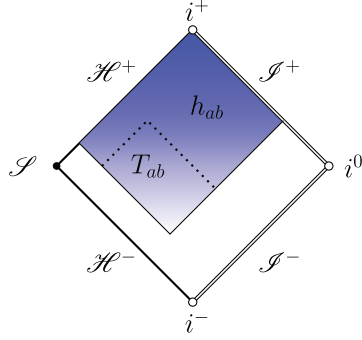
Figure 3. Supports of T_{ab} and h_{ab} in the exterior region of Kerr.

$$\frac{1}{2}\{\text{þ}(\text{þ} - 2\bar{\rho}) + 2\rho(\bar{\rho} - \rho)\} = \frac{\bar{\rho}}{2(\rho + \bar{\rho})}\text{þ}(\rho + \bar{\rho})^2\text{þ}\frac{1}{\bar{\rho}(\rho + \bar{\rho})}. \qquad (79c)$$

These identities follow from $\text{þ}\rho = \rho^2, \text{þ}\bar{\rho} = \bar{\rho}^2$ and facilitate the integration of the transport equations in adapted coordinates, see (107).

Now we investigate whether we can indeed solve for given $a^\circ, b^\circ, c^\circ$ the equations (69), (77) subject to (74). This will in general depend on the chosen type II background, and on the nature of the source T_{ab}. From now on, we assume that $(\mathcal{M}, g_{ab})$ is the exterior region of the Kerr black hole with $0 \leqslant |a| \leqslant M$. We also assume that T_{ab} has compact support and (for simplicity) that it is smooth. Furthermore, we assume that h_{ab} is the retarded solution. Then the support of h_{ab} satisfies $\text{supp}(h_{ab}) \subset J^+(\text{supp}(T_{ab}))$ in some gauge (e.g. the Geroch–Xanthopoulos gauge [46]), where J^+ is the causal future of a set, see figure 3. Furthermore, the linearized ADM/Bondi-mass and angular momentum vanish, $\dot{M} = \dot{J} = 0$.

In Kerr we have the relations $\tilde{\text{þ}}'\rho'^\circ = \tilde{\text{þ}}'\Omega^\circ = \kappa'^\circ = \tilde{\text{þ}}'\Psi_2^\circ = \tilde{\text{ð}}\Psi_2^\circ = \Psi_3^\circ = 0$ and both $\rho'^\circ, \Psi_2^\circ$ are real while Ω° is by definition imaginary. Furthermore, for any GHP-scalar x° annihilated by þ, we have $[\tilde{\text{þ}}', \tilde{\text{ð}}]x^\circ = 0 = [\tilde{\text{þ}}', \tilde{\text{ð}}']x^\circ$. This leads to significant simplifications in (77). Also, for any algebraically special spacetime where $\Omega^\circ = 0$ such as the Robinson–Trautman class or Schwarzschild, (77) simplifies to

$$c^\circ = -\frac{1}{4}(\tilde{\text{ð}}\tilde{\text{ð}}\tilde{\text{ð}}'\tilde{\text{ð}}' + \tilde{\text{ð}}'\tilde{\text{ð}}'\tilde{\text{ð}}\tilde{\text{ð}} - 2\tilde{\text{þ}}'\Psi_2^\circ - 2\tilde{\text{þ}}'\bar{\Psi}_2^\circ)\zeta_l^\circ - (\Psi_2^\circ + \bar{\Psi}_2^\circ)\tilde{\text{þ}}'\zeta_l^\circ. \qquad (80)$$

Furthermore, in Schwarzschild $M > 0$, $\tilde{\text{þ}}'\Psi_2^\circ = 0, \Psi_2^\circ = M, \tilde{\text{þ}}' = \partial/\partial u$ in outgoing Kerr–Newman coordinates (B.4a) $(u, r, \theta, \varphi_*)$ and c° can be viewed as a function on $\mathscr{I}^+$, i.e. of (u, θ, φ_*), vanishing for $u < u_0$ by construction. This equation can be solved by noting that it is like a (forward) fourth order heat equation with a source on $\mathbb{R} \times S^2$, with, say, trivial, initial data starting before the source. The operator replacing the Laplacian in the ordinary heat equation is $\frac{1}{4}(\tilde{\text{ð}}\tilde{\text{ð}}\tilde{\text{ð}}'\tilde{\text{ð}}' + \tilde{\text{ð}}'\tilde{\text{ð}}'\tilde{\text{ð}}\tilde{\text{ð}})$, where in Schwarzschild, $\tilde{\text{ð}}, \tilde{\text{ð}}'$ are given by (46) with $a = 0$.

These operators are equal to the 'spin-weighted' invariant operators on S^2 described from a geometric point of view in detail in [47]. More precisely, a cross section S^2 of $\mathscr{I}^+$ gives rise to bundles $\mathscr{E}^s \to S^2$, where $\mathscr{E}^s$ is the line bundle isomorphic to an appropriate restriction of $\mathscr{L}^{p,q}$ to this cross section with 'spin' $s = \frac{1}{2}(p - q)$, and the operators are invariantly defined elliptic operators $\tilde{\text{ð}}, \tilde{\text{ð}}'$ mapping sections of $\mathscr{E}^s$ to sections of $\mathscr{E}^{s+1}, \mathscr{E}^{s-1}$, respectively. From the relation between $\tilde{\text{ð}}, \tilde{\text{ð}}'$ and the spin-weighted spherical harmonics $_s Y_{\ell,\mu}$ [48], the eigenvalues of

this operator are explicitly found to be $\frac{1}{2}L(L+1)$ where $L = \ell(\ell+1), \ell = 0, 1, 2, \ldots$, and the eigenfunctions are the $_0Y_{\ell,\mu}$-harmonics. Thus, the 'heat equation' can be solved by decomposing into such harmonics. Once we have ζ_l°, the remaining components $\zeta_m^\circ, \zeta_n^\circ$ are obtainable from (74).

Furthermore, the first equation (69) can be solved for A° if and only if $a^\circ \overset{\circ}{=} (-3, -1)$ is orthogonal to the $_sY_{\ell,\mu}$ spin-weighted spherical harmonics with $s = -1, l = 1$. Now, the components of the perturbed Bondi angular momentum $\dot{J}$ of h_{ab} is obtained by taking r times the integral of $_{-1}Y_{1,\mu}h_{mn}$ over a sphere S^2 at constant u and letting $r \to \infty$, see e.g. [49], and we know that $\dot{J} = 0$. From this, it can be seen going through the definitions that a° is orthogonal to the $_{-1}Y_{1,\mu}, \mu = -1, 0, 1$ spin-weighted spherical harmonics. On the other hand, one can see that if we choose Φ as a solution to (60) with zero initial conditions on $\mathscr{H}^-$, say, then b° must be zero, because such a term in y as in (66) is inconsistent with the behavior [46] of h_{ab} at $\mathscr{I}^+$ in the Geroch–Xanthopoulos gauge. Thus, it is consistent to take $B^\circ = 0$.

In Minkowski, $M = 0$ (77) simplifies to:

$$c^\circ = -\frac{1}{4}(\tilde{\eth}\,\tilde{\eth}\,\tilde{\eth}'\tilde{\eth}' + \tilde{\eth}'\tilde{\eth}'\tilde{\eth}\,\tilde{\eth})\zeta_l^\circ. \tag{81}$$

The operator on the right side is invertible on the subspace orthogonal to the zero mode $\ell = 0$. The perturbed Bondi-mass of h_{ab} is obtained by taking r times the integral of h_{nn} over a sphere S^2 at constant u and letting $r \to \infty$, see e.g. [49]. Going through the definitions, it follows that if c° is not orthogonal to the constant function, then the perturbation h_{ab} would have a nonzero perturbed ADM (and Bondi-) mass $\dot{M}$, which cannot be. The equation (69) is solved as in Schwarzschild.

In Kerr, one can also solve the equations, but the argument is more involved. Thus, we shall now present an alternative argument which in effect bypasses equations (69) and (77). In the argument leading to the preceeding theorem, we go back to the definitions of the gauge vector field ξ^a and the tensor x_{ab}. We integrate the ODEs required to solve for these quantities, (53) and (56)–(58) outwards from $\mathscr{H}^-$ to $\mathscr{I}^+$ along the geodesics tangent to l^a; see figure 2. Then, due to the support properties of h_{ab}, T_{ab} (see figure 3) after passing from $h_{ab} \to h_{ab} - (\mathcal{L}_\xi g)_{ab} - x_{ab}$, we obtain a new h_{ab} that vanishes in an open neighborhood of $\mathscr{H}^-$. Now, we look at the ll component of the Einstein equation $(\mathcal{E}h)_{ll} = 0$ for this new metric. Equation (A.1) gives

$$0 = \{\text{\th}(\text{\th} - \rho - \bar{\rho}) + 2\rho\bar{\rho}\}h_{m\bar{m}} \equiv (\text{\th} - 2\rho)(\text{\th} + \rho - \bar{\rho})h_{m\bar{m}}, \tag{82}$$

which integrates to [45]

$$h_{m\bar{m}} = \bar{h}^\circ\frac{\rho}{\bar{\rho}} + h^\circ\frac{\bar{\rho}}{\rho} + (k^\circ + \bar{k}^\circ)(\rho + \bar{\rho}), \tag{83}$$

for undetermined GHP scalars h°, k° annihilated by þ. However, $h_{m\bar{m}} = 0$ near $\mathscr{H}^-$. By Held's technique we can write (83) as a polynomial equation in ρ as in (41) with coefficients a_i° annihilated by þ. Then these coefficients vanish near $\mathscr{H}^-$, and therefore all coefficients $a_i^\circ = 0$ globally since these quantities are annihilated by þ, which is a directional derivative along the orbits of l^a going from $\mathscr{H}^-$ to $\mathscr{I}^+$; see figure 2. One thereby easily finds $h^\circ = k^\circ = 0$, i.e. $h_{m\bar{m}} = 0$ globally and the metric is automatically in ingoing radiation gauge. The next step in the argument was to integrate the ODE (60) for Φ along the geodesics tangent to l^a going from $\mathscr{H}^-$ to $\mathscr{I}^+$. Again, we do this by choosing vanishing initial conditions for Φ. Then y as in (66) vanishes in a neighborhood of $\mathscr{H}^-$. By (44), we can express y as $\bar{\rho}^2\rho^{-3}$ times the polynomial $\Omega^\circ a^\circ \rho^4 + 2a^\circ \rho^3 - \Omega^\circ b^\circ \rho - b^\circ$ in ρ with coefficients annihilated by þ. Again, by Held's

basic lemma (41) a°, b° vanish globally, so $y = 0$ globally. Similarly, we show by the same type of argument that the GHP quantity c° appearing (72) in fact vanishes automatically, too.

We have shown:

Theorem 3. *Suppose h_{ab} is a retarded solution to the inhomogeneous linearized Einstein equation $(\mathcal{E}h)_{ab} = T_{ab}$ with smooth compactly supported T_{ab} on a Kerr background having $|a| \leqslant M$, and suppose the symmetric tensor field x_{ab} is defined as (55), with tetrad components obtained by successively solving the three ODEs (56)–(58). Then there exists a gauge vector field X^a and a GHP scalar $\Phi \overset{\circ}{=} (-4, 0)$ such that*

$$h_{ab} = x_{ab} + (\mathcal{L}_X g)_{ab} + \mathrm{Re}\,(\mathcal{S}^\dagger \Phi)_{ab}. \tag{84}$$

Remarks.

(1) As stated, our result is not symmetric under time-reflection because we have been working with a tetrad satisfying (n1) where l^a is a principal null direction while n^a is not (except in Schwarzschild). However, one can derive the same result also for a tetrad satisfying (n2) in Kerr, where l^a and n^a are both principal null directions (e.g. the Kinnersley tetrad, see appendix B). The derivation proceeds along the same lines, but some formulas, especially those related to the residual gauge vector field ζ^a, given in [45], are more complicated in this case.

(2) Solutions to the inhomogeneous linearized Einstein equation $(\mathcal{E}h)_{ab} = T_{ab}$ with other initial conditions differ from the retarded solution by a solution to the homogeneous equation. We have demonstrated in the previous section that a corresponding decomposition holds to arbitrary precision in a suitable sense, where there can now be an additional piece $\dot{g}_{ab}$ representing an infinitesimal perturbation towards another Kerr solution.

(3) The derivation of our result goes through also for distributional T_{ab} of compact support. In that case, X^a, Φ are distributional as well.

We now turn to the equation satisfied by Φ. To this end, we act with the linearized Einstein operator $\mathcal{E}$ (17) on (84), and we use the operator identity (22). This results in

$$\mathrm{Re}\,(\mathcal{T}^\dagger \mathcal{O}^\dagger \Phi)_{ab} = T_{ab} - (\mathcal{E}x)_{ab} \equiv S_{ab}. \tag{85}$$

Thus, if Φ solves the sourced adjoint Teukolsky equation

$$\mathcal{O}^\dagger \Phi = \eta \tag{86}$$

with η such that

$$\mathrm{Re}\,(\mathcal{T}^\dagger \eta)_{ab} = S_{ab}, \tag{87}$$

then h_{ab} as in (84) is a solution to the sourced linearized Einstein equation $(\mathcal{E}h)_{ab} = T_{ab}$, and as discussed, every solution arises in this way. We view (87) as the defining equation for η and we know that this equation must have solutions. In components, (87) is by (24)

$$S_{\bar{m}\bar{m}} = \frac{1}{4}(\text{\th} - \rho)(\text{\th} - \rho)\eta \tag{88}$$

$$S_{nn} = \frac{1}{2}(\eth - \tau)(\eth - \tau)\eta + \text{c.c.} \tag{89}$$

$$S_{\bar{m}n} = \frac{1}{4}\{(\eth + \bar{\tau}' - \tau)(\text{\th} - \rho) + (\text{\th} - \rho + \bar{\rho})(\eth - \tau)\}\eta. \tag{90}$$

The left side is in GHP form,

$$
\begin{aligned}
S_{mm} = {} & T_{mm} - \{(\text{\th} - \bar\rho)(\eth - \tau) - (\eth - \tau - \bar\tau')\bar\rho - \tau(\text{\th} + \rho) + \bar\tau'(\text{\th} - \rho + \bar\rho)\}x_{nm} \\
& - \{(\text{\th} - 2\bar\rho)\bar\sigma' + (\tau + \bar\tau')\eth + (\tau - \bar\tau')^2\}x_{m\bar m},
\end{aligned}
\tag{91}
$$

$$
\begin{aligned}
S_{nn} = {} & T_{nn} - \{(\eth' - \bar\tau)(\eth - \tau) + \bar\rho'(\text{\th} - \rho + \bar\rho) - (\text{\TH}' - \bar\rho')\bar\rho + \bar\Psi_2\}x_{nn} \\
& - \{-(\text{\TH}' - 3\rho')(\eth' + \tau' - \bar\tau) + \tau'\text{\TH}' - \rho'\eth' - \kappa'\text{\th} \\
& \quad + (\text{\th} - 2\rho + \bar\rho)\kappa' + (\eth - 3\tau + \bar\tau')\sigma' + \eth(\sigma') - \Psi_3\}x_{nm} \\
& - \{-(\text{\TH}' - 3\bar\rho')(\eth + \bar\tau' - \tau) + \bar\tau'\text{\TH}' - \bar\rho'\eth - \bar\kappa'\text{\th} \\
& \quad + (\text{\th} - 2\bar\rho + \rho)\bar\kappa' + (\eth' - 3\bar\tau + \tau')\bar\sigma' + \eth'(\bar\sigma') - \bar\Psi_3\}x_{n\bar m},
\end{aligned}
\tag{92}
$$

$$
\begin{aligned}
S_{n\bar m} = {} & T_{n\bar m} - \frac{1}{2}\{(\text{\th} - \rho + \bar\rho)(\eth' - \bar\tau) - (\eth' - 2\tau' + \bar\tau)\rho + \tau'(\text{\th} - \bar\rho)\}x_{nn} \\
& - \frac{1}{2}\{-\eth'(\eth' - 2\tau') + \sigma'(\text{\th} - 2\rho + 2\bar\rho) - 2\bar\tau(\tau' - \bar\tau)\}x_{nm} \\
& - \frac{1}{2}\{-(\text{\th} + \bar\rho)(\text{\TH}' - 2\bar\rho') + \rho'(\text{\th} + 2\rho - 2\bar\rho) - 4\rho\bar\rho' + 2\Psi_2 \\
& \quad + (\eth + \bar\tau')(\eth' - 2\bar\tau) - \tau'(\eth + \tau - 2\bar\tau') - \tau(\tau' - 4\bar\tau)\}x_{n\bar m} \\
& - \frac{1}{2}\{(\text{\TH}' + \rho' - \bar\rho')(\eth' - \tau' + \bar\tau) + 2\bar\tau(\text{\TH}' - 2\rho') - (\eth' - \tau' - \bar\tau)\bar\rho' + 2\rho'\tau' \\
& \quad + (\eth - \tau - \bar\tau')\sigma' + \sigma'\eth - \kappa'\text{\th} - \Psi_3\}x_{m\bar m},
\end{aligned}
\tag{93}
$$

using the linearized Einstein operator in GHP form given in appendix A.

$x_{nn}, x_{mn}, x_{m\bar m}$, which are obtained by successively solving in this order the three ODEs (56)–(58), can be chosen to vanish near $\mathscr{H}^-$ if we integrate those equations along the orbits of l^a outwards from $\mathscr{H}^-$ towards $\mathscr{I}^+$, with vanishing initial conditions, since T_{ab} has compact support. Then also $S_{nn}, S_{mn}, S_{m\bar m}$ vanish near $\mathscr{H}^-$. (88) is an ODE along the geodesics tangent to l^a, and we can pick a solution η vanishing near $\mathscr{H}^-$ if we integrate (88) outwards from $\mathscr{H}^-$ towards $\mathscr{I}^+$ with vanishing initial conditions. Any other solution is of the form $\eta + C^\circ + D^\circ\rho$. We substitute this into the remaining two equations (89), (90), and we apply Held's technique to write the resulting expressions involving C°, D° as a polynomial in ρ as in (41) with coefficients a_i° annihilated by $\text{\th}$. Then these polynomials vanish near $\mathscr{H}^-$, and therefore all coefficients $a_i^\circ = 0$ globally since these quantities are annihilated by $\text{\th}$, which is a directional derivative along the orbits of l^a going from $\mathscr{H}^-$ to $\mathscr{I}^+$. As a consequence, we find writing out the explicit expressions for a_i° in terms of $C^\circ, D^\circ, \Omega^\circ$ and their derivatives by $\tilde\eth, \tilde\eth'$ that (89), (90) are satisfied if and only if $\tilde\eth C^\circ = \tilde\eth D^\circ = 0$, so we may pick $C^\circ = D^\circ = 0$. Therefore, the source η is given by integrating (88) outwards from $\mathscr{H}^-$ to $\mathscr{I}^+$ with vanishing initial conditions.

We arrive at the main result of this paper:

Theorem 4. *Suppose h_{ab} is a retarded solution to the inhomogeneous linearized Einstein equation $(\mathcal{E}h)_{ab} = T_{ab}$ with smooth compactly supported T_{ab} on a Kerr background ($|a| \leqslant M$). Then $h_{ab} = x_{ab} + \mathrm{Re}\,(\mathcal{S}^\dagger\Phi)_{ab}$ up to gauge, where x_{ab} is defined as (55), with tetrad components obtained by successively solving the three ODEs (56)–(58) with vanishing initial conditions at $\mathscr{H}^-$. $\Phi \overset{\circ}{=} (-4, 0)$ is the retarded solution to the adjoint Teukolsky equation $\mathcal{O}^\dagger\Phi = \eta$. The source $\eta \overset{\circ}{=} (-4, 0)$ is obtained by solving the ODE (88) with vanishing initial conditions at $\mathscr{H}^-$, where S_{mm} is given by (91).*

Remarks.

(1) As is well known, the adjoint Teukolsky equation can formally be solved by a separation of variables ansatz [5, 6], i.e. $\Phi \hat{=} (-4, 0)$ may be represented in the Kinnersley frame (see appendix B) by

$$\Phi(t, r, \theta, \varphi) = \int_{-\infty}^{\infty} d\omega \sum_{\ell, m} {}_{-2}A_{lm}(\omega) {}_{-2}R_{\omega \ell m}(r) {}_{-2}S_{\omega \ell m}(\theta) e^{-i\omega t + im\varphi}, \qquad (94)$$

with radial function ${}_{-2}R_{\omega \ell m}(r)$ determined by the radial spin -2 Teukolsky equation [5, 6] with a source ${}_{-2}\eta_{\omega \ell m}(r)$ obtained from η by a similar decomposition, and with ${}_{-2}S_{\omega \ell m}(\theta)$ the spin-weighted spheroidal harmonics. This decomposition can presumably be established rigorously for a retarded solution using recent results by [50], who have established that the mode solutions are linearly independent on the real axis. For retarded propagation, as in the statement of the theorem, the radial function satisfies the conditions

$$\begin{aligned}
{}_{-2}R_{\omega \ell m} &\sim r^3 \, e^{i\omega r_*}, & r_* &\to +\infty, \\
{}_{-2}R_{\omega \ell m} &\sim \Delta(r)^2 \, e^{-ikr_*}, & r_* &\to -\infty,
\end{aligned} \qquad (95)$$

where Δ has the conventional meaning for the Kerr metric (see appendix B) and

$$k \equiv \omega - m\Omega_H. \qquad (96)$$

Here Ω_H is the angular frequency of the outer horizon $\Omega_H = a/(2Mr_+)$. Based on the result by [50], one should be able to prove that standard Laplace-transform expressions based on modes of the retarded propagator for $\mathcal{O}^\dagger$ (as given in, e.g. [15]) hold rigorously, thus giving the retarded solution in mode form.

(2) The four ODEs (56)–(58) and (88) defining x_{ab}, η involve the stress tensor components $T_{ll}, T_{nl}, T_{ml}, T_{mm}$ (in total six real components). The remaining four real components $T_{nn}, T_{mn}, T_{\bar{m}m}$ enter implicitly through the conservation law $\nabla^a T_{ab} = 0$ (four real equations).

(3) Different initial conditions for h_{ab} are encoded in different initial conditions for Φ plus a perturbation $\dot{g}_{ab}$ towards another metric in the Kerr family.

(4) The derivation of our result goes through also for distributional T_{ab} of compact support inside the chosen exterior region $\mathcal{M}$ of the Kerr spacetime. In that case, X^a, Φ, η are distributional as well. The compact support assumption about T_{ab} can to some extent be avoided. In fact, our arguments still go through if sufficiently many transverse derivatives of the components $T_{ll}, T_{nl}, T_{ml}, T_{mm}$ on $\mathcal{H}^-$ vanish and if T_{ab} has sufficiently fast decay towards past infinity for a retarded solution h_{ab} to exist.

(5) By construction, the perturbation $h_{ab} = x_{ab} + \mathrm{Re}\,(\mathcal{S}^\dagger \Phi)_{ab}$ is in a gauge where it vanishes in an open neighborhood of $\mathcal{H}^-$. It follows trivially that the perturbed expansion $\dot{\vartheta}_n$ (i.e. along n^a) vanishes on $\mathcal{H}^-$ and the perturbed expansion $\dot{\vartheta}_l$ vanishes on $\mathcal{H}^+$ in an open neighborhood of the bifurcation surface, $\mathcal{S}$ (for $|a| < M$). Then, by the perturbed Raychaudhuri equation on $\mathcal{H}^+$,

$$l^a \nabla_a \dot{\vartheta}_l = -\vartheta_l \dot{\vartheta}_l - 2\sigma_l^{ab} \dot{\sigma}_{l\,ab} - T_{ll}, \qquad (97)$$

since $T_{ll} = 0$ on $\mathcal{H}^+$ and $\vartheta_l = \sigma_l^{ab} = 0$ on $\mathcal{H}^+$ in the background, $\dot{\vartheta}_l$ vanishes on the entire future horizon $\mathcal{H}^+$. As explained in detail in [35], it follows that the perturbation is automatically in a gauge in which the position of $\mathcal{H}^\pm$ remain in their fixed (background) coordinate location to first order.

(6) All our equations and constructions remain valid if we work with n^a, the second repeated principal null direction, instead of l^a. The integrations necessary to solve the analogs of the ODEs (56)–(58) and (88) are now along n^a rather than l^a, going from $\mathscr{I}^-$ to $\mathscr{H}^+$, rather than from $\mathscr{H}^-$ to $\mathscr{I}^+$ (see figure 2). Thus, we should prescribe trivial initial conditions at $\mathscr{I}^-$ for these ODEs rather than at $\mathscr{H}^-$. The metric obtained in this way will be in the so-called 'outgoing radiation gauge'.

Let us finally clarify the relationship between the equation $\mathcal{O}^\dagger \Phi = \eta$ in theorem 4 and the inhomogeneous Teukolsky equation for the perturbed Weyl scalar ψ_0 already given in Teukolsky's original papers [5, 6]. First, we recall how this is derived: applying the operator relation (20) to a symmetric tensor h_{ab} such that $\mathcal{E}h_{ab} = T_{ab}$ gives the relation $\mathcal{O}\psi_0 = T_0$, where $\psi_0 = \mathcal{T}(h)$ is the perturbed 0-Weyl-scalar (12) and where $T_0 = \mathcal{S}(T)$ [see (18a)] is the source in Teukolsky's equation for ψ_0 [5, 6].

Now, we substitute for h_{ab} the decomposition $h_{ab} = x_{ab} + \mathrm{Re}\,(\mathcal{S}^\dagger \Phi)_{ab} + \mathcal{L}_X g_{ab} + \dot{g}_{ab}$ of theorem 4[14]. Since the perturbed Weyl scalar ψ_0 is gauge invariant on any background in which the corresponding background scalar $\Psi_0 = 0$, it follows that $\mathcal{T}(\mathcal{L}_X g) = 0$. Furthermore, $\mathcal{T}(x) = 0$, since the corrector field x_{ab} has vanishing lm, ll, mm components (see (18b) and (55)). Finally, $\mathcal{T}(\dot{g}) = 0$, since Ψ_0 remains zero for perturbations to other Kerr black holes. Therefore, $\psi_0 = \mathcal{T}\mathrm{Re}\,\mathcal{S}^\dagger \Phi$. On the other hand, in the proof of theorem 4, we have seen that $\mathrm{Re}\,(\mathcal{T}^\dagger \eta)_{ab} = S_{ab} = T_{ab} - (\mathcal{E}x)_{ab}$. Applying $\mathcal{S}$, as given in (18a), to this equation and using $\mathcal{S}\mathcal{E}x = \mathcal{O}\mathcal{T}x = 0$ in view of (20), we find $T_0 = \mathcal{S}\mathrm{Re}\,\mathcal{T}^\dagger \eta$. The expressions can be further simplified using (18a), (18b), (24), (25) and $\text{þ}\rho = \rho^2$, giving

$$\mathcal{S}\mathrm{Re}\,\mathcal{T}^\dagger \eta = -\tfrac{1}{4}\left(\text{þ}^2 - 4(\rho + \bar{\rho})\text{þ} + 12\rho\bar{\rho}\right)\text{þ}^2\bar{\eta} \text{ and}$$

$$\mathcal{T}\mathrm{Re}\,\mathcal{S}^\dagger \Phi = -\tfrac{1}{4}\text{þ}^4\bar{\Phi}$$

Thus, we arrive at the following theorem.

Theorem 5. *Let ψ_0 be the perturbed Weyl scalar associated with a metric perturbation $h_{ab} = x_{ab} + \mathrm{Re}\,(\mathcal{S}^\dagger \Phi)_{ab} + \mathcal{L}_X g_{ab}$ as in theorem 4, with Φ satisfying $\mathcal{O}^\dagger \Phi = \eta$ with source η as described in theorem 4. Then*

$$\mathcal{O}\psi_0 = T_0, \tag{98}$$

where

$$T_0 = -\frac{1}{4}\left(\text{þ}^2 - 4(\rho + \bar{\rho})\text{þ} + 12\rho\bar{\rho}\right)\text{þ}^2\bar{\eta}, \tag{99a}$$

$$\psi_0 = -\frac{1}{4}\text{þ}^4\bar{\Phi}. \tag{99b}$$

Remark. According to this theorem, alternatively, having obtained ψ_0 by solving the Teukolsky equation (98) in terms of $T_0 = \mathcal{S}(T)$, one may obtain Φ by integrating (99) along the orbits of l^a. For retarded initial conditions as in theorem 4, we should give zero initial conditions for this fourth order ODE at $\mathscr{H}^-$, see figure 2. Consideration of the Teukolsky equation $\mathcal{O}\psi_0 = T_0$ alone is nevertheless not sufficient to obtain the metric perturbation h_{ab}: it remains necessary to compute the corrector tensor x_{ab}.

---

[14] Note that in theorem 4, we are assuming retarded initial conditions. For other initial conditions, there would be another piece $\dot{g}_{ab}$ representing a perturbation towards another Kerr black hole, see remark (2) following theorem 3, which we can include here without problems.

5. Nonlinear perturbations of Kerr

So far we have limited the discussion to linearized perturbations. However, even in the absence of matter sources, the linearized metric perturbation acts as source for the second order perturbation and so on. Merely to count orders—but not implying any specific kind of convergence of the series—let us write

$$g_{ab} = g_{ab}^{(0)} + h_{ab}^{(1)} + h_{ab}^{(2)} + \cdots . \tag{100}$$

$g_{ab}^{(0)} = g_{ab}^{M,a}$ is the background Kerr metric. The equation for $h_{ab}^{(1)}$ is of course just the source free linearized Einstein equation in the absence of matter sources, whereas the higher order corrections for $n > 1$ satisfy an equation of the form

$$(\mathcal{E}h^{(n)})_{ab} = T_{ab}^{(n)}, \tag{101}$$

where $\mathcal{E}$ is the linearized Einstein operator in the Kerr-background $g_{ab}^{M,a}$ (17), and where the effective stress tensor $T_{ab}^{(n)}$ is built out of $h_{ab}^{(1)}, \ldots, h_{ab}^{(n-1)}$. For example, the source of the equation for $h_{ab}^{(2)}$ is given by minus the nonlinear (quadratic) terms in the second-order Einstein tensor

$$
\begin{aligned}
-T_{cd}^{(2)} = &- \frac{1}{2}(\nabla_b h^{(1)ab} - \frac{1}{2}g^{ab}\nabla_b h^{(1)})(2\nabla_{(d}h_{c)a}^{(1)} - \nabla_a h_{cd}^{(1)}) + \frac{1}{4}\nabla_c h^{(1)ab}\nabla_d h_{ab}^{(1)} \\
&+ \frac{1}{2}\nabla^b h^{(1)a}{}_c(\nabla_b h_{ad}^{(1)} - \nabla_a h_{bd}^{(1)}) \\
&+ \frac{1}{2}h^{(1)ab}(\nabla_c\nabla_d h_{ab}^{(1)} + \nabla_a\nabla_b h_{cd}^{(1)} - 2\nabla_{(d}\nabla_{|b|}h_{c)a}^{(1)}).
\end{aligned} \tag{102}
$$

In the perturbative setting, gauge transformations are formal diffeomorphisms $f = \text{Exp}(\xi^a)$ generated by a formal vector field

$$\xi^a = \xi^{(1)a} + \xi^{(2)a} + \xi^{(3)a} + \cdots \tag{103}$$

and correspond to $g_{ab} \to f^* g_{ab} = \exp(\mathcal{L}_\xi)g_{ab}$, giving, for instance

$$
\begin{aligned}
h_{ab}^{(1)} &\to h_{ab}^{(1)} + \mathcal{L}_{\xi^{(1)}} g_{ab}^{(0)} \\
h_{ab}^{(2)} &\to h_{ab}^{(2)} + \mathcal{L}_{\xi^{(2)}} g_{ab}^{(0)} + \mathcal{L}_{\xi^{(1)}}\mathcal{L}_{\xi^{(1)}} g_{ab}^{(0)} + \mathcal{L}_{\xi^{(1)}} h_{ab}^{(1)},
\end{aligned} \tag{104}
$$

etc. As described in detail in section 3, at zeroth order, we can apply the decomposition $h_{ab}^{(1)} = g_{ab}^{(1)} + (\mathcal{L}_{\xi^{(1)}} g^{(0)})_{ab} + \text{Re}\,(\mathcal{S}^\dagger\Phi^{(1)})_{ab}$, where $\Phi^{(1)}$ is a Hertz potential, i.e. a $(-4,0)$ solution to $\mathcal{O}^\dagger\Phi^{(1)} = 0$, and where in this section,

$$g_{ab}^{(n)} = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}s^n}g_{ab}^{M(s),a(s)} \tag{105}$$

is an nth order perturbation to another Kerr metric.

Now, if we could inductively apply at each order $n > 1$ the decomposition for $h_{ab}^{(n)}$ described in theorem 4 and the following remarks (see section 4), with an nth order gauge vector field $\xi^{(n)a}$ and $x_{ab}^{(n)}, \eta^{(n)}$ determined from $T_{ab}^{(n)}$, then we could write the metric g_{ab} as the formal series

$$g_{ab} = g_{ab}^{(0)} + \sum_{n=1}^{\infty}\left\{g_{ab}^{(n)} + \text{Re}\,(\mathcal{S}^\dagger\Phi^{(n)})_{ab}\right\} + \sum_{n=2}^{\infty}x_{ab}^{(n)} \tag{106}$$

with $\mathcal{O}^\dagger \Phi^{(n)} = \eta^{(n)}$, up to a pull back by the formal diffeomorphism generated by $\xi^a = \sum_{n=1}^{\infty} \xi^{(n)a}$. Since the correctors $x_{ab}^{(n)}$ and the GHP-sources $\eta^{(n)}$ are comparatively easy to obtain by solving ordinary transport equations along the orbits of l^a, the analysis of nonlinear perturbations has thereby effectively been reduced to solving a Teukolsky equation with source at each order. In view of the powerful methods/results already available in this setting—such as separation of variables method or the recent decay results by [1–3]—this is obviously a substantial simplification of matters.

Unfortunately, as stated, theorem 4 only holds for stress tensors of compact support, while the *n*th order stress tensors of the gravitational field will clearly not have this property. In order to make our arguments rigorous, we would have to make some approximation of $T_{ab}^{(n)}$ by stress tensors of compact support and then take a limit. In order to control the limit, we must understand the asymptotic behavior of the perturbations $h_{ab}^{(1)}, \ldots, h_{ab}^{(n-1)}$ for large r at finite u. These are determined by $\Phi^{(1)}, \ldots, \Phi^{(n-1)}$, which are solutions to the Teukolsky equation with a certain source inductively determined, and by $x_{ab}^{(2)}, \ldots, x_{ab}^{(n-1)}$, obtained by integrating certain transport equations as stated in theorem 4. The latter are easy to control once we can control in sufficient detail the asymptotic behavior of $\Phi^{(1)}, \ldots, \Phi^{(n-1)}$. For the Teukolsky equation without source, an essentially complete understanding of the asymptotic behavior (in terms of certain energy norms on the initial data) has recently been obtained by [1–3]. We expect that their results, suitably generalized to the Teukolsky equation with source, can be used to show rigorously the decomposition (106) to any arbitrary but finite order, thereby also establishing the asymptotic behavior of higher order gravitational perturbations on Kerr (without control over the convergence of the infinite sum (106)). However, such an analysis goes beyond the present work and is therefore postponed to another paper.

6. Summary and outlook

For the convenience of the reader, we summarize our integration scheme for the sourced linearized Einstein equation $(\mathcal{E}h)_{ab} = T_{ab}$, which consists of the following steps:

Step 1: For given T_{ab}, we integrate, in this order, the ODEs (56)–(58) to obtain $x_{m\bar{m}}, x_{nm}, x_{nn}$. In the frame (B.5) in outgoing Kerr–Newman coordinates (B.4) $(u, r, \theta, \varphi_*)$, where $\flat = \partial/\partial r$, they have the form (using equation (79)):

$$\rho^2 \frac{\partial}{\partial r} \left[\frac{\bar{\rho}}{\rho^3} \frac{\partial}{\partial r} \left(\frac{\rho}{\bar{\rho}} x_{m\bar{m}} \right) \right] = \text{r.h.s. of (56)}$$

$$\frac{\bar{\rho}}{2(\rho + \bar{\rho})} \frac{\partial}{\partial r} \left[(\rho + \bar{\rho})^2 \frac{\partial}{\partial r} \left(\frac{1}{\bar{\rho}(\rho + \bar{\rho})} x_{mn} \right) \right] = \text{r.h.s. of (57)} \qquad (107)$$

$$\frac{1}{2}(\rho + \bar{\rho})^2 \frac{\partial}{\partial r} \left(\frac{1}{\rho + \bar{\rho}} x_{nn} \right) = \text{r.h.s. of (58)}$$

with right hand sides involving T_{ll}, T_{ml}, T_{nl}, respectively and $\rho = -(r - \mathrm{i}a \cos\theta)^{-1}$. For $x_{m\bar{m}}, x_{nm}, x_{nn}$, we choose trivial initial conditions at the (past) event horizon $r = r_+$.

Step 2: We integrate the ODE (88) in order to obtain η. In the frame (B.5) in outgoing Kerr–Newman coordinates (B.4), this equation has the form:

$$\rho \frac{\partial^2}{\partial r^2} \left(\frac{1}{\rho} \eta \right) = \text{r.h.s. of (91)} \qquad (108)$$

with right hand side involving $T_{mm}, x_{m\bar{m}}, x_{nm}$, respectively. For η, we choose trivial initial conditions at $r = r_+$.

Step 3: Solve the adjoint Teukolsky equation $\mathcal{O}^\dagger \Phi = \eta$ with the desired initial conditions. One may attempt to this end a separation of variables ansatz

$$\Phi(u, r, \theta, \varphi_*) = \int_{-\infty}^{\infty} d\omega \sum_{\ell,m} {}_{-2}A_{lm}(\omega)\,{}_{-2}R_{\omega\ell m}(r)\,{}_{-2}S_{\omega\ell m}(\theta)e^{-i\omega u + im\varphi_*}, \quad (109)$$

with radial function ${}_{-2}R_{\omega\ell m}(r)$ determined by the radial spin -2 Teukolsky equation [5, 6], and with a source ${}_{-2}\eta_{\omega\ell m}(r)$ obtained from η by a similar decomposition. For instance, for a retarded perturbation h_{ab}, Φ should be a retarded solution to the adjoint Teukolsky equation, which is equivalent to the boundary conditions (95) on the modes. In such a case, it might be possible to establish the decomposition rigorously if one could justify rigorously the mode decomposition of the retarded propagator based on the standard Laplace-transform method e.g. using recent results by [50].

Step 4: The desired solution of the sourced linearized Einstein equation $(\mathcal{E}h)_{ab} = T_{ab}$ is obtained as $h_{ab} = x_{ab} + \text{Re}\,(\mathcal{S}^\dagger \Phi)_{ab}$, where x_{ab} is constructed from $x_{m\bar{m}}, x_{nm}, x_{nn}$ as in (55), and where $(\mathcal{S}^\dagger \Phi)_{ab}$ is as in (25).

By iterating this procedure, then provided theorem 4 can be extended to noncompact support sources, one can construct recursively the *n*th order gravitational perturbation $h_{ab}^{(n)}$ in terms of an *n*th order potential $\Phi^{(n)}$ solving a sourced Teukolsky equation and an *n*th order corrector tensor $x_{ab}^{(n)}$. These quantities are constructed from the nonlinear terms in the *n*th order Einstein tensor, which act as an effective stress tensor $T_{ab}^{(n)}$ in the *n*th iteration step.

A different path for constructing higher order (in fact second order) perturbations in a formalism based on the Teukolsky method was proposed some time ago by Campanelli and Lousto [51]. Their starting point is the wave equation for the Weyl tensor, which has the schematic form

$$\nabla^e \nabla_e C_{abcd} = \text{ terms quadratic in } C_{abcd} \quad (110)$$

in any Ricci flat ($R_{ab} = 0$) spacetime. By transvecting this tensor equation in all possible ways with the legs $(l^a, n^a, m^a, \bar{m}^a)$ of a Newman–Penrose tetrad, one obtains a system of equations involving the Weyl scalars (12) and the rotation coefficients (see appendix B). By developing the metric g_{ab} around Kerr, one obtains a Teukolsky type equation for the second order perturbed Weyl scalars sourced by terms involving the first order perturbed quantities. To get a closed system of equations, one must solve for the first order perturbed rotation coefficients, i.e. effectively the first order perturbed metric, in terms of the first order perturbed (extreme) Weyl scalars. This can be done, e.g. by well-known inversion formulas such as obtained by [52] at first order. However, the method breaks down as formulated at higher orders, since the inversion formulae no longer apply. It seems that our method, which works directly with the metric perturbation rather than Weyl scalars, is superior in this sense.

In the future, we would like to use the formalism developed in this paper in combination with the mode projection method outlined in [23] to study the interaction of quasinormal modes in the near horizon region of a near-extremal black hole [24]. It would also be an interesting project in our view to use our formalism in combination with results obtained by [1–3] in order to obtain decay properties of higher order gravitational perturbations in Kerr. What seems to be required here is primarily a detailed analysis of the decay properties for solutions to the Teukolsky equation with a source (step 3), since the transport equations (steps 1,2)

would then be fairly easy to treat. The method developed in this paper can also be applied to obtain the gravitational field of a point particle in the Kerr metric to linear order, and it would be interesting to compare this to the current alternative treatments of this problem. Finally, it might be possible to use our decompositions in order to obtain simplifications for perturbative quantum gravity off a Kerr background.

Acknowledgments

Appendix A. Linearized Einstein operator in GHP form

Here we quote from [45] the tetrad components of the linearized Einstein operator $(\mathcal{E}h)_{ab}$ (17) in GHP form[15]. It is assumed that the first null leg l^a of the null tetrad $(l^a, n^a, m^a, \bar{m}^a)$ is aligned with a principal null direction of a type II spacetime (so that $\kappa = \sigma = \Psi_0 = \Psi_1 = 0$ in view of the Goldberg–Sachs theorem). In a type D spacetime we have additionally the option (n2) described in section 2 to set further GHP scalars to zero, and in type II we can alternatively make the simplifications described in (n1) by an appropriate choice of n^a.

$$
\begin{aligned}
(\mathcal{E}h)_{ll} = {} & \{(\eth' - \tau')(\eth - \bar{\tau}') + \rho(\text{\th}' + \rho' - \bar{\rho}') - (\text{\th} - \rho)\rho' + \Psi_2\}h_{ll} \\
& + \{-(\rho + \bar{\rho})(\text{\th} + \rho + \bar{\rho}) + 4\rho\bar{\rho}\}h_{ln} \\
& + \{-(\text{\th} - 3\bar{\rho})(\eth' - \tau' + \bar{\tau}) + \bar{\tau}\text{\th} - \bar{\rho}\eth'\}h_{lm} \\
& + \{-(\text{\th} - 3\rho)(\eth + \tau - \bar{\tau}') + \tau\text{\th} - \rho\eth\}h_{l\bar{m}} \\
& + \{\text{\th}(\text{\th} - \rho - \bar{\rho}) + 2\rho\bar{\rho}\}h_{m\bar{m}},
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
(\mathcal{E}h)_{nn} = {} & \{2\kappa'\bar{\kappa}'\}h_{ll} \\
& + \{(\eth' - \bar{\tau})(\eth - \tau) + \bar{\rho}'(\text{\th} - \rho + \bar{\rho}) - (\text{\th}' - \bar{\rho}')\bar{\rho} + \bar{\Psi}_2\}h_{nn} \\
& + \{-(\rho' + \bar{\rho}')(\text{\th}' + \rho' + \bar{\rho}') + 4\rho'\bar{\rho}' - (\eth' - 2\bar{\tau})\bar{\kappa}' - (\eth - 2\tau)\kappa'\}h_{ln} \\
& + \{(\text{\th}' - \bar{\rho}')\kappa' + \kappa'(\text{\th}' - \rho' - \bar{\rho}') - \bar{\kappa}'\sigma'\}h_{lm} \\
& + \{(\text{\th}' - \rho')\bar{\kappa}' + \bar{\kappa}'(\text{\th}' - \bar{\rho}' - \rho') - \kappa'\bar{\sigma}'\}h_{l\bar{m}} \\
& + \{-(\text{\th}' - 3\rho')(\eth' + \tau' - \bar{\tau}) + \tau'\text{\th}' - \rho'\eth' - \kappa'\text{\th} \\
& \quad + (\text{\th} - 2\rho + \bar{\rho})\kappa' + (\eth - 3\tau + \bar{\tau}')\sigma' + \eth(\sigma') - \Psi_3\}h_{nm} \\
& + \{-(\text{\th}' - 3\bar{\rho}')(\eth + \bar{\tau}' - \tau) + \bar{\tau}'\text{\th}' - \bar{\rho}'\eth - \bar{\kappa}'\text{\th} \\
& \quad + (\text{\th} - 2\bar{\rho} + \rho)\bar{\kappa}' + (\eth' - 3\bar{\tau} + \tau')\bar{\sigma}' + \eth'(\bar{\sigma}') - \bar{\Psi}_3\}h_{n\bar{m}} \\
& + \{-(\eth' - 2\bar{\tau})\kappa' - \sigma'(\text{\th}' - \rho' + \bar{\rho}')\}h_{mm} \\
& + \{-(\eth - 2\tau)\bar{\kappa}' - \bar{\sigma}'(\text{\th}' - \bar{\rho}' + \rho')\}h_{\bar{m}\bar{m}} \\
& + \{\text{\th}'(\text{\th}' - \rho' - \bar{\rho}') + \kappa'(\tau - \bar{\tau}') + \bar{\kappa}'(\bar{\tau} - \tau') + 2\sigma'\bar{\sigma}' + 2\rho'\bar{\rho}'\}h_{m\bar{m}},
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
(\mathcal{E}h)_{ln} = {} & \frac{1}{2}\{\rho'(\text{Þ}' - \rho') + \bar{\rho}'(\text{Þ}' - \bar{\rho}') + (\eth - 2\bar{\tau}')\kappa' + (\eth' - 2\tau')\bar{\kappa}' + 2\sigma'\bar{\sigma}'\}h_{ll} \\
& + \frac{1}{2}\{\rho(\text{þ} - \rho) + \bar{\rho}(\text{þ} - \bar{\rho})\}h_{nn} \\
& + \frac{1}{2}\{-(\eth' + \tau' + \bar{\tau})(\eth - \tau - \bar{\tau}') - (\eth'\eth + 3\tau\tau' + 3\bar{\tau}\bar{\tau}') + 2(\bar{\tau} + \tau')\eth \\
& \quad + (\text{þ} - 2\bar{\rho})\rho' + (\text{Þ}' - 2\rho')\bar{\rho} - \bar{\rho}'(\text{þ} + \rho) - \rho(\text{Þ}' + \bar{\rho}') - \Psi_2 - \bar{\Psi}_2\}h_{ln} \\
& + \frac{1}{2}\{(\text{Þ}' - 2\bar{\rho}')(\eth' - \tau') + \bar{\tau}(\text{Þ}' + \rho' + \bar{\rho}') - \tau'(\text{Þ}' - \rho') \\
& \quad - (2\eth' - \bar{\tau})\bar{\rho}' - (\text{þ} - 2\bar{\rho})\kappa' + \sigma'(\tau - \bar{\tau}')\}h_{lm} \\
& + \frac{1}{2}\{(\text{Þ}' - 2\rho')(\eth - \bar{\tau}') + \tau(\text{Þ}' + \bar{\rho}' + \rho') - \bar{\tau}'(\text{Þ}' - \bar{\rho}') \\
& \quad - (2\eth - \tau)\rho' - (\text{þ} - 2\rho)\bar{\kappa}' + \bar{\sigma}'(\bar{\tau} - \tau')\}h_{l\bar{m}} \\
& + \frac{1}{2}\{(\text{þ} - 2\rho)(\eth' - \bar{\tau}) + (\tau' + \bar{\tau})(\text{þ} + \bar{\rho}) - 2(\eth' - \tau')\rho - 2\bar{\tau}\text{þ}\}h_{nm} \\
& + \frac{1}{2}\{(\text{þ} - 2\bar{\rho})(\eth - \tau) + (\bar{\tau}' + \tau)(\text{þ} + \rho) - 2(\eth - \bar{\tau}')\bar{\rho} - 2\tau\text{þ}\}h_{n\bar{m}} \\
& + \frac{1}{2}\{-(\eth' - \bar{\tau})(\eth' - \tau') + \bar{\tau}(\bar{\tau} - \tau') - \sigma'\rho\}h_{mm} \\
& + \frac{1}{2}\{-(\eth - \tau)(\eth - \bar{\tau}') + \tau(\tau - \bar{\tau}') - \bar{\sigma}'\bar{\rho}\}h_{\bar{m}\bar{m}} \\
& + \frac{1}{2}\{(\eth' + \tau' - \bar{\tau})(\eth - \tau + \bar{\tau}') + (\eth'\eth - \tau\tau' - \bar{\tau}\bar{\tau}' + \tau\bar{\tau}) - (\Psi_2 + \bar{\Psi}_2) \\
& \quad + (\text{Þ}' - 2\rho')\bar{\rho} + (\text{þ} - 2\bar{\rho})\rho' + \rho(3\text{Þ}' - 2\bar{\rho}') + \bar{\rho}'(3\text{þ} - 2\rho) \\
& \quad - 2\text{Þ}'\text{þ} + 2\rho\bar{\rho}' + 2\eth'(\tau) - \tau\bar{\tau}\}h_{m\bar{m}},
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
(\mathcal{E}h)_{lm} = {} & \frac{1}{2}\{(\text{Þ}' - \rho')(\eth - \bar{\tau}') + (\eth - \tau - 2\bar{\tau}')\bar{\rho}' - (\eth - \tau)\rho' + \tau(\text{Þ}' + \rho') \\
& \quad + \bar{\sigma}'(\eth' - \tau' + \bar{\tau}) + \bar{\Psi}_3 + \bar{\rho}\bar{\kappa}'\}h_{ll} \\
& + \frac{1}{2}\{-(\text{þ} - \rho + \bar{\rho})(\eth + \tau - \tau') - (\eth - 3\tau + \bar{\tau}')\bar{\rho} - 2\rho\bar{\tau}'\}h_{ln} \\
& + \frac{1}{2}\{-(\text{Þ}' + \bar{\rho}')(\text{þ} - 2\bar{\rho}) + \rho(\text{Þ}' + 2\rho' - 2\bar{\rho}') - 4\rho'\bar{\rho} + 2\Psi_2 \\
& \quad + (\eth' + \bar{\tau})(\eth - 2\bar{\tau}') - \tau(\eth' + \tau' - 2\bar{\tau}) - \tau'(\tau - 4\bar{\tau}')\}h_{lm} \\
& + \frac{1}{2}\{-\eth(\eth - 2\tau) - \bar{\sigma}'(\text{þ} + 2\bar{\rho} - 4\rho) - 2\bar{\tau}'(\tau - \tau')\}h_{l\bar{m}} \\
& + \frac{1}{2}\{\text{þ}(\text{þ} - 2\rho) + 2\bar{\rho}(\rho - \bar{\rho})\}h_{nm} \\
& + \frac{1}{2}\{-(\text{þ} - \bar{\rho})(\eth' - \tau' + \bar{\tau}) + 2\bar{\tau}\bar{\rho}\}h_{mm} \\
& + \frac{1}{2}\{(\text{þ} + \rho - \bar{\rho})(\eth + \bar{\tau}' - \tau) + 2\bar{\tau}'(\text{þ} - 2\rho) - (\eth - \tau - \bar{\tau}')\bar{\rho} + 2\rho\tau\}h_{m\bar{m}},
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
(\mathcal{E}h)_{n\bar{m}} = {} & \tfrac{1}{2}\{(\text{\th}' - \rho')\kappa' + \kappa'\text{\th}' + \bar{\kappa}'\sigma'\}h_{ll} \\
& + \tfrac{1}{2}\{(\text{\th} - \rho + \bar{\rho})(\eth' - \bar{\tau}) - (\eth' - 2\tau' + \bar{\tau})\rho + \tau'(\text{\th} - \bar{\rho})\}h_{nn} \\
& + \tfrac{1}{2}\{(-(\text{\th}' - \rho' + \bar{\rho}')(\eth' + \tau' - \bar{\tau}) - (\eth' - 3\tau' + \bar{\tau})\bar{\rho}' + (\eth - \tau + \bar{\tau}')\sigma' \\
& \quad - 2\sigma'\eth - \Psi_3 - 2\rho'\bar{\tau}\}h_{ln} \\
& + \{\sigma'(\rho' - 2\bar{\rho}') - \kappa'(\tau' - 2\bar{\tau}) + \tfrac{1}{2}\Psi_4\}h_{lm} \\
& + \tfrac{1}{2}\{(\text{\th}'(\text{\th}' - 2\rho') + -\kappa'(\eth - 2\tau + 2\bar{\tau}') + \bar{\kappa}'(\eth' - 4\tau' + 2\bar{\tau}) \\
& \quad + 2\bar{\rho}'(\rho' - \bar{\rho}') + 2\sigma'\bar{\sigma}'\}h_{l\bar{m}} \\
& + \tfrac{1}{2}\{-\eth'(\eth' - 2\tau') + \sigma'(\text{\th} - 2\rho + 2\bar{\rho}) - 2\bar{\tau}(\tau' - \bar{\tau})\}h_{nm} \\
& + \tfrac{1}{2}\{-(\text{\th} + \bar{\rho})(\text{\th}' - 2\bar{\rho}') + \rho'(\text{\th} + 2\rho - 2\bar{\rho}) - 4\rho\bar{\rho}' + 2\Psi_2 \\
& \quad + (\eth + \bar{\tau}')(\eth' - 2\bar{\tau}) - \tau'(\eth + \tau - 2\bar{\tau}') - \tau(\tau' - 4\bar{\tau})\}h_{n\bar{m}} \\
& + \tfrac{1}{2}\{-(\eth' - \tau')\sigma' - \sigma'\eth'\}h_{mm} \\
& + \tfrac{1}{2}\{-(\text{\th}' - \bar{\rho}')(\eth - \tau + \bar{\tau}') + 2\bar{\tau}'\bar{\rho}' - \bar{\kappa}'(\text{\th} - 2\rho + 2\bar{\rho}) + \eth'(\bar{\sigma}') - \bar{\tau}\bar{\sigma}'\}h_{\bar{m}\bar{m}} \\
& + \tfrac{1}{2}\{(\text{\th}' + \rho' - \bar{\rho}')(\eth' - \tau' + \bar{\tau}) + 2\bar{\tau}(\text{\th}' - 2\rho') - (\eth' - \tau' - \bar{\tau})\bar{\rho}' + 2\rho'\tau' \\
& \quad + (\eth - \tau - \bar{\tau}')\sigma' + \sigma'\eth - \kappa'\text{\th} - \Psi_3\}h_{m\bar{m}},
\end{aligned}
\tag{A.5}
$$

$$
\begin{aligned}
(\mathcal{E}h)_{mm} = {} & \{(\text{\th}' - 2\rho')\bar{\sigma}' + \bar{\kappa}'(\eth + \tau - \bar{\tau}')\}h_{ll} \\
& + \{-\eth(\eth - \tau - \bar{\tau}') - 2\tau\bar{\tau}' + \bar{\sigma}'(\rho - \bar{\rho})\}h_{ln} \\
& + \{(\text{\th}' - \rho')(\eth - \bar{\tau}') - (\eth - \tau - \bar{\tau}')\rho' + \tau(\text{\th}' + \rho' - \bar{\rho}') - (\text{\th} - 2\bar{\rho})\bar{\kappa}' \\
& \quad - \bar{\tau}'(\text{\th} + \bar{\rho}') + \bar{\tau}\bar{\sigma}' - \bar{\Psi}_3\}h_{lm} \\
& + \{-(\eth - \tau - \bar{\tau}')\bar{\sigma}' - \bar{\sigma}'(\eth - \tau)\}h_{l\bar{m}} \\
& + \{(\text{\th} - \bar{\rho})(\eth - \tau) - (\eth - \tau - \bar{\tau}')\bar{\rho} - \tau(\text{\th} + \rho) + \bar{\tau}'(\text{\th} - \rho + \bar{\rho})\}h_{nm} \\
& + \{-(\text{\th}' - \rho')(\text{\th} - \bar{\rho}) + (\eth - \tau)\tau' - \tau(\eth' + \tau' - \bar{\tau}) + \Psi_2\}h_{mm} \\
& + \{(\text{\th} - 2\bar{\rho})\bar{\sigma}' + (\tau + \bar{\tau}')\eth + (\tau - \bar{\tau}')^2\}h_{m\bar{m}},
\end{aligned}
\tag{A.6}
$$

$$
\begin{aligned}
(\mathcal{E}h)_{m\bar{m}} = \ &\frac{1}{2}\{\text{\th}'(\text{\th}' - \rho' - \bar{\rho}') + 2\rho'\bar{\rho}' + \kappa'(\tau - \bar{\tau}') - \bar{\kappa}'(\bar{\tau} - \tau') + 2\sigma'\bar{\sigma}'\}h_{ll} \\
&+ \frac{1}{2}\{\text{\th}(\text{\th} - \rho - \bar{\rho}) + 2\rho\bar{\rho}\}h_{nn} \\
&+ \frac{1}{2}\{-(\text{\th}' + \rho' - \bar{\rho}')(\text{\th} - \rho + \bar{\rho}) - \text{\th}'(\text{\th} + \rho) + \rho(\text{\th}' + \rho' - \bar{\rho}') - \bar{\Psi}_2 \\
&\quad + (\eth' - \bar{\tau})(\eth - \tau - \bar{\tau}') + \eth'\eth - (\eth - 2\bar{\tau}')\tau' - \bar{\tau}(2\eth + \bar{\tau}') \\
&\quad - 2\tau(\eth' - \bar{\tau}) + 2\tau'\bar{\tau}' + \bar{\rho}\bar{\rho}'\}h_{ln} \\
&+ \frac{1}{2}\{-(\text{\th}' - 2\rho')(\eth' - 2\bar{\tau}) + \bar{\tau}(\text{\th}' + 2\rho' - 2\bar{\rho}') + 2(\eth - \bar{\tau}')\sigma' - \sigma'\eth \\
&\quad - 2\tau'\bar{\rho}' - 2\kappa'(\rho - \bar{\rho}) - \Psi_3\}h_{lm} \\
&+ \frac{1}{2}\{-(\text{\th}' - 2\bar{\rho}')(\eth - 2\tau) + \tau(\text{\th}' + 2\bar{\rho}' - 2\rho') + 2(\eth' - \tau')\bar{\sigma}' - \bar{\sigma}'\eth' \\
&\quad - 2\bar{\tau}'\rho' - 2\bar{\kappa}'(\bar{\rho} - \rho) - \bar{\Psi}_3\}h_{l\bar{m}} \\
&+ \frac{1}{2}\{-(\text{\th} - 2\bar{\rho})(\eth' - 2\tau') + \tau'(\text{\th} - 2\rho - 2\bar{\rho}) - 2\rho\bar{\tau} + 4\tau'\bar{\rho}\}h_{nm} \\
&+ \frac{1}{2}\{-(\text{\th} - 2\rho)(\eth - 2\bar{\tau}') + \bar{\tau}'(\text{\th} - 2\bar{\rho} - 2\rho) - 2\bar{\rho}\tau + 4\bar{\tau}'\rho\}h_{n\bar{m}} \\
&+ \frac{1}{2}\{-\bar{\tau}(\eth' - \bar{\tau}) - \tau'(\eth' - \tau') - (\text{\th} - 2\bar{\rho})\sigma'\}h_{mm} \\
&+ \frac{1}{2}\{-\tau(\eth - \tau) - \bar{\tau}'(\eth - \bar{\tau}') - (\text{\th} - 2\rho)\bar{\sigma}'\}h_{\bar{m}\bar{m}} \\
&+ \frac{1}{2}\{2\text{\th}'\text{\th} - (\text{\th}' - \bar{\rho}')\bar{\rho} - (\text{\th} - \rho)\rho' - \rho(\text{\th}' - \rho' + \bar{\rho}') - \bar{\rho}'(\text{\th} + \rho - \bar{\rho}) \\
&\quad - (\eth' - 2\tau')\bar{\tau}' + \tau(\eth' + 2\bar{\tau}) - \tau'(\eth - \bar{\tau}') + \bar{\tau}(\eth + \tau) - \eth'(\tau) \\
&\quad - \Psi_2 - \bar{\Psi}_2\}h_{m\bar{m}}.
\end{aligned}
\tag{A.7}
$$

Appendix B. GHP and Kerr quantities

In this work we use a Newman–Penrose tetrad $(l^a, n^a, m^a, \bar{m}^a)$ wherein

$$
g_{ab} = 2l_{(a}n_{b)} - 2m_{(a}\bar{m}_{b)}. \tag{B.1}
$$

We choose the normalization $n_a l^a = 1$ and $m_a \bar{m}^a = -1$, corresponding to the '-2' signature. The Kerr metric may be defined by the Kinnersley tetrad, which is convenient in explicit computations. The covariant Boyer–Lindquist coordinate components in the order (t, r, θ, ϕ) are

$$
l_a = \left(-1, \Sigma/\Delta, 0, a\sin^2\theta\right), \tag{B.2a}
$$

$$
n_a = \frac{1}{2}\left(-\Delta/\Sigma, -1, 0, a\sin^2\theta\Delta/\Sigma\right), \tag{B.2b}
$$

$$
m_a = \frac{1}{\sqrt{2}(r + \mathrm{i}a\cos\theta)}\left(-\mathrm{i}a\sin\theta, 0, \Sigma, \mathrm{i}(r^2 + a^2)\sin\theta\right), \tag{B.2c}
$$

where

$$
\Delta = r^2 + a^2 - 2Mr, \qquad \Sigma = r^2 + a^2\cos^2\theta. \tag{B.3}
$$

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The Kinnersley tetrad is regular in the exterior region $\mathscr{M}$ of Kerr, with the exception of $\mathscr{I}^-, \mathscr{H}^+$, and the north and south poles. Both l^a, n^a are repeated principal null directions, so this tetrad satisfies (n2). A corresponding tetrad satisfying (n1) may be defined by the null rotation described in (n1). With

$$u = t - r_* \equiv t - r - \frac{r_+^2 + a^2}{r_+ - r_-} \ln\left(\frac{r - r_+}{r_+}\right) + \frac{r_-^2 + a^2}{r_+ - r_-} \ln\left(\frac{r - r_-}{r_+}\right), \tag{B.4a}$$

$$\varphi_* = \varphi - \frac{a}{r_+ - r_-} \ln \frac{r - r_+}{r - r_-} \tag{B.4b}$$

the outgoing Kerr–Newman coordinates, the contravariant components in the coordinate system $(u, r, \theta, \varphi_*)$ are

$$l^a = (0, 1, 0, 0), \tag{B.5a}$$

$$n^a = (r^2 + a^2, -\Delta/2, 0, a)/\Sigma, \tag{B.5b}$$

$$m^a = \frac{1}{\sqrt{2}(r + ia\cos\theta)} \left(ia\sin\theta, 0, 1, i\csc\theta\right). \tag{B.5c}$$

These forms are useful to write out the various transport equations along the orbits of l^a because $\text{þ} = \partial/\partial r$ is rather simple, and because the coordinate system $(u, r, \theta, \varphi_*)$ is regular at the past horizon $\mathscr{H}^-$ (i.e. $r = r_+$).

The complex spin coefficients with definite GHP weights are given by

$$\kappa = m^a l^b \nabla_b l_a \overset{\circ}{=} (3, 1), \tag{B.6a}$$

$$\tau = m^a n^b \nabla_b l_a \overset{\circ}{=} (1, -1), \tag{B.6b}$$

$$\sigma = m^a m^b \nabla_b l_a \overset{\circ}{=} (3, -1), \tag{B.6c}$$

$$\rho = m^a \bar{m}^b \nabla_b l_a \overset{\circ}{=} (1, 1), \tag{B.6d}$$

together with their primed counterparts $\kappa', \tau', \sigma', \rho'$ defined by exchanging $l^a \leftrightarrow n^a, m^a \leftrightarrow \bar{m}^a$. The remaining four complex spin coefficients $\epsilon, \epsilon', \beta, \beta'$ may be read off from (15). They do not have definite GHP weight (i.e. they are not GHP scalars), but in effect form part of the definition of the GHP covariant derivative Θ (15), or equivalently, of $\text{þ}, \text{þ}', \eth, \eth'$. The explicit values of all spin coefficients in the Kinnersley tetrad may be found, e.g. in [5].

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