A new kind of local symmetry without gauge bosons

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Abstract

In conventional gauge theories, any local gauge symmetry generator is accompanied by a corresponding gauge boson in order to compensate the transformation of the gauge-covariant derivation against the local gauge transformations as acting on the matter fields. Indeed, whenever the gauge group is purely compact, as in usual gauge theories, the invocation of corresponding gauge bosons as compensating fields is unavoidable. That is because of there exist no nontrivial forgetful homomorphisms onto some smaller Lie groups from the full gauge group. In this paper we show a mechanism that at the price of allowing some non-semisimple component of the gauge group besides the compact part, it is possible to construct such Lagrangians that the non-semisimple part of the local gauge group only acts on the matter fields, without invoking corresponding gauge bosons. It shall be shown that already the ordinary Dirac equation admits such a hidden symmetry related to the dilatation group, thus this mechanism cannot be called unphysical. Then, we give our more complicated example Lagrangian, in which the gauge group is an indecomposable Lie group built up of a nilpotent part and of a compact part. Since the nilpotent part does carry also Lorentz charges in our example, the first order symmetries of the pertinent theory give rise to a unified gauge–Poincaré group, bypassing Coleman–Mandula and related no-go theorems in a different way in comparison to SUSY. The existence of a local symmetry without a gauge boson is already mathematically very striking, but this new mechanism might even be useful to eventually try to substitute SUSY for a unification concept of symmetries.

Contents

1 Introduction

It is very well known in field theory that the first order symmetry group of a Lagrangian quite substantially restricts its possible structure. The ensemble of symmetries become the most restrictive whenever they belong to a non-direct product (unified) group. This simple principle motivated the gauge–gauge and gauge–spacetime symmetry unification strategies. A rather well-known set of no-go theorems by McGlinn [1], O'Raifeartaigh [2], Coleman and Mandula [3] strongly restricts the possibilities for the latter kind of unification. Actually, it turns out that the core of these restrictive arguments derive from the general structure theory of finite dimensional real Lie algebras [4], for any kind of symmetry unification attempts.

Detailed studies [5] of the proof of the pertinent no-go theorems [1, 2, 3] uncover that in order to obtain these prohibitive results, the assumption of the presence of a positive definite invariant scalar product on the Lie algebra of the internal symmetry group is very essential. This happens to be equivalent to the property that the group of internal symmetries can only be chosen to be purely compact. In a previous paper [4] it was demonstrated that whenever the assumption on this scalar product is somewhat weakened, namely e.g. merely positive semidefiniteness is required, then a loophole opens, and even a gauge–spacetime type symmetry unification can occur. Whenever indeed this is the case, it was shown that the internal symmetry group needs to be a semidirect product of a non-semisimple (solvable or nilpotent) and of a compact Lie group, i.e. it necessarily needs to be somewhat more general than being purely compact.

The requirement on the purely compactness of the internal symmetry group in conventional gauge theories has several motivations: (i) the classification of compact Lie groups is well understood, (ii) Standard Model with its gauge group U(1)×SU(2)×SU(3) satisfies the pertinent property, and (iii) Yang-Mills fields with compact gauge group admit strictly positive definite energy functional. So, if one allows for an internal Lie algebra with merely positive *semi*definite invariant scalar product, then it is implied that the non-compact gauge field directions shall have vanishing Yang-Mills kinetic Lagrangian and correspondingly shall have zero Yang-Mills kinetic energy term. This poses a clear question on the physical or unphysical nature of the pertinent non-compact gauge fields: what does it mean that in a gauge-theory-like setting if we have some gauge fields, which although do not disturb the non-negativity of the energy density of the usual compact part of Yang-Mills fields, but they do not possess a Yang-Mills kinetic term themselves? Surely these "exotic" (non-compact) part of the gauge fields shall not obey a conventional Yang-Mills equation-like Euler-Lagrange equation, since they do not have kinetic term. In this paper we shall show examples when actually the corresponding gauge fields do not even contribute to the Lagrangians of the matter fields, and thus can be completely transformed out of the theory. As such, in these kinds of theories one has an internal symmetry group acting faithfully on the matter fields, but only the compact part of this group has corresponding gauge fields. Thus, the question of physicality or unphysicality of such "exotic" (non-compact) gauge fields is naturally resolved: they do not contribute to the Lagrangian at all in such theories. The mathematical fact of the existence of a Lagrangian with some local symmetry without corresponding gauge field is already quite striking. Before one would think that such a theory must be very artificial, let us remark that e.g. the ordinary Dirac kinetic term if viewed in appropriate field variables, does admit an extremely simplified version of the above mechanism, related to the dilatation group, as shall be shown.

The structure of the paper is as follows. In Section 2 an interesting property of the ordinary Dirac kinetic Lagrangian is outlined: it shall be shown that it is invariant to the choice of a D(1) gauge connection, which is a bit stronger additional symmetry on top of its well-known conformal invariance. That shall serve as an oversimplified prototype example for our mechanism of elimination of non-compact gauge bosons. In Section 3 we recall some known results on the structure of generic (not necessarily semisimple) Lie groups and Lie algebras, and also SUSY shall be mentioned in the framework of these general structural theorems. In Section 4 we shall show an indecomposable (unified) finite dimensional real Lie group, containing the U(1) as compact part, the $\mathrm{SL}(2,\mathbb{C})$ encoding Lorentz symmetries, and a nilpotent part which makes the pertinent unification possible. Then, in Section 5 we begin to construct a Lagrangian admitting the above local symmetries acting faithfully and pointwise on the matter fields, but not having gauge fields corresponding to the "exotic" nilpotent part. Finally, in Section 6 we conclude. We already note here that the mathematical existence of the pertinent kind of toy models is a warning: it is not mathematically automatic that all kind of local continuous symmetries of the matter field sector manifest themselves by a corresponding gauge boson. This is only automatic for the compact or eventually for the semisimple part of the group of internal symmetries.

2 A hidden symmetry of the Dirac kinetic Lagrangian

The Lagrangian of the Dirac kinetic term can be viewed as a pointwise map taking a Clifford map γ^a, Dirac matter field Ψ, and the Dirac matter field gauge-covariant gradient $\nabla_a\Psi$ into a real volume form field over spacetime, according to the formula:

$$\mathbf{L}_{\mathrm{Dirac}}(\gamma,\Psi,\nabla\Psi) \qquad = \qquad \mathbf{v}_\gamma\;\mathrm{Re}\left(\overline{\Psi}\,\gamma^a\,\mathrm{i}\nabla_a\Psi\right). \tag{1}$$

Here, $\mathbf{v}_\gamma$ is the volume form uniquely associated to the spacetime metric subordinate to the Clifford map γ^a and to a chosen fixed spacetime orientation. The covariant derivation ∇_a is understood to be the sum of the natural metric spinorial covariant derivation associated to γ^a and of a U(1) gauge potential. In addition to this, one could assign D(1) gauge charges to the fields γ^a and Ψ in the following way.[1] Assign a D(1) gauge charge (physical dimension in terms of length powers) to γ^a of

[1]The D(1) is nothing but $\mathbb{R}^+$ with the real multiplication, i.e. the *dilatation group*.

-1 and to Ψ of $-\frac{3}{2}$. Then, the Dirac Lagrangian Eq.(1) turns out to be invariant to global D(1) gauge transformations

$$\begin{pmatrix} \Psi \\ \gamma^a \\ \nabla_b \end{pmatrix} \quad \overset{\Omega\in\mathbb{R}^+}{\longmapsto} \quad \begin{pmatrix} \Omega^{-\frac{3}{2}}\;\Psi \\ \Omega^{-1}\;\gamma^a \\ \nabla_b \end{pmatrix}, \tag{2}$$

i.e. to a scaling transformation with a constant Ω throughout spacetime. Moreover, the Dirac Lagrangian Eq.(1) turns out to be invariant to local D(1) gauge transformations

$$\begin{pmatrix} \Psi \\ \gamma^a \\ \nabla_b \end{pmatrix} \quad \overset{\Omega>0}{\longmapsto} \quad \begin{pmatrix} \Omega^{-\frac{3}{2}}\;\Psi \\ \Omega^{-1}\;\gamma^a \\ \Omega^{-\frac{3}{2}}\;\nabla_b\,\Omega^{\frac{3}{2}} \;=\nabla_b+\Omega^{-\frac{3}{2}}\;\mathrm{d}_b\,\Omega^{\frac{3}{2}} \end{pmatrix}, \tag{3}$$

in which situation the positive scaling field Ω is not required to be constant. Note, however, that this assumes also a D(1) gauge field to be implicitly understood within the gauge-covariant derivation ∇_a in order to absorb the compensating gradient term $\Omega^{-\frac{3}{2}}\;\mathrm{d}_a\,\Omega^{\frac{3}{2}}$. Whenever, this scenario is allowed, the Dirac Lagrangian is locally D(1) gauge invariant, meaning that it is invariant to the pointwise rescaling of measurement units, provided that this is compensated accordingly in the D(1) gauge connection understood within ∇_a. An interesting observation, not yet described in the literature, is that the Dirac Lagrangian Eq.(1) as understood in such variables, has a further symmetry: it is invariant to the choice of the D(1) gauge connection. Quite naturally, a change in the D(1) gauge connection is uniquely described by a shift transformation $\nabla_a \mapsto \nabla_a + C_a$ with C_a being a smooth real valued covector field over the spacetime. Direct evaluation shows that the Dirac Lagrangian Eq.(1) is invariant to such a shift transformation

$$\begin{pmatrix} \Psi \\ \gamma^a \\ \nabla_b \end{pmatrix} \quad \overset{C_d}{\longmapsto} \quad \begin{pmatrix} \Psi \\ \gamma^a \\ \nabla_b \;+\; C_b \end{pmatrix} \tag{4}$$

and therefore is indeed invariant to a choice of the D(1) gauge connection. Consequently, it is invariant to any local D(1) gauge transformation where it is only prescribed to act on the matter fields canonically and faithfully, but it is required to stay uncompensated for in the D(1) part of the gauge-covariant derivation:

$$\begin{pmatrix} \Psi \\ \gamma^a \\ \nabla_b \end{pmatrix} \quad \overset{\Omega>0}{\longmapsto} \quad \begin{pmatrix} \Omega^{-\frac{3}{2}}\;\Psi \\ \Omega^{-1}\;\gamma^a \\ \nabla_b \end{pmatrix}, \tag{5}$$

where Ω is a positive scaling field, not necessarily constant. In summary: due to this extra $\nabla_a \mapsto \nabla_a + C_a$ shift symmetry, one can disregard the presence of the corresponding D(1) gauge connection in ∇_a (it does not give any contribution to the theory), while the local D(1) gauge transformations still act faithfully on the matter field sector — which are encoded by γ^a and Ψ.

The physical meaning of the insensitiveness to the D(1) gauge connection is that the pertinent Lagrangian is insensitive to any kind of parallel transport rule of measurement units throughout spacetime. If one uses the appropriate dynamical variables, all the Standard Model kinetic terms can be seen to admit such a symmetry. This is related to their conformal invariance, but happens to be a slightly stronger symmetry property than that. The precise geometric meaning of the above property is formulated in [6] and reviewed in Appendix A using more formal differential geometry, for completeness.

It is seen that due to this $\nabla_a \mapsto \nabla_a + C_a$ shift symmetry of the Lagrangian, the original D(1)×U(1) internal symmetry group, acting locally and faithfully on the matter fields, only gives rise to gauge bosons in the compact direction, i.e. with U(1) degrees of freedom only. In our more elaborate example in this paper, we will show that such a forgetting mechanism can also be invoked for more complicated internal group structure, and even with non-direct product (unified) internal group. It follows, however, that the generators of the local symmetries whose gauge fields can be eliminated in such a manner, must sit in a normal sub-Lie algebra. Because of that and the general structural theorem on Lie algebras (Levi decomposition), those generators can only sit in the so-called solvable part of the Lie algebra, and can accompany the usual compact internal symmetry generators without revealing themselves through corresponding gauge bosons.

3 Structural theorems for generic Lie groups

In the followings we shall recall some general structural properties of Lie groups, only using some very elementary notions and notations.

Let G be a group. A *normal* subgroup of G is a subgroup which is Ad-invariant. That is: a subgroup N of G is called normal subgroup whenever for all elements g of the entire group G one has that $g\,N\,g^{-1} \subset N$. Whenever one has a normal subgroup N in a group G, the alternative notation $G = N\,\centerdot\,H$ is sometimes used where H is some complementing set to N within G, i.e. such a set that $N \cap H = \{\text{identity}\}$ and $G = N\,H$. This kind of structure, i.e. when a normal subgroup exists, can eventually be called *semi-semidirect product.*

Whenever one has the above semi-semidirect product situation, but one can find a complementing set H to N such that H closes as a subgroup, then we call G a *semidirect product* of N and H, and the usual notation is $G = N \rtimes H$.

Whenever one has the above semidirect product situation, but the complementing subgroup H can be chosen to be normal also, then we call G a *direct product* of N and H, and the usual notation is $G = N \times H$. In this case it follows that N and H do commute, i.e. they are completely independent, and G is called *decomposable.* The symmetry unification strategies try to avoid theories with decomposable symmetry groups.

Of course, on the infinitesimal level, i.e. at the level of the Lie algebra of a Lie group, corresponding notions exist as well: sub-Lie algebra, *normal* sub-Lie algebra (*semi-semidirect sum*), and the notions of *semidirect sum* and *direct sum* of two Lie algebras are defined accordingly.

3.1 On the structure of global topology

It is well-known that a finite dimensional real Lie group may be written in the following form

$$E \;=\; \Bigg(\underbrace{\tilde{E}_0}_{\text{universal covering group}} \;\centerdot\; \underbrace{\mathcal{I}}_{\text{some discrete symmetries}}\Bigg)\Bigg/\underbrace{\mathcal{J}}_{\text{some discrete symmetry group}} \tag{6}$$

Here, $\tilde{E}_0$ is the universal covering group of the unital connected component. $\mathcal{I}$ is a discrete set of outer automorphisms of $\tilde{E}_0$, such that $\tilde{E}_0\,\centerdot\,\mathcal{I}$ closes as a group, and it is responsible for enumerating the connected components of $\tilde{E}_0\,\centerdot\,\mathcal{I}$. Generally, $\mathcal{I}$ may or may not close as a standalone group. Finally, $\mathcal{J}$ is some discrete normal subgroup of $\tilde{E}_0\,\centerdot\,\mathcal{I}$, responsible for a possible non-simply connectedness of E. As it is well-known, by Ado's theorem, the structure of $\tilde{E}_0$ is uniquely determined by the Lie algebra of E. In this paper, we will not address general issues with discrete symmetries, and therefore when not otherwise mentioned, by Lie group we will only mean connected and simply connected Lie groups (universal covering groups), and eventually we will use the Lie group / algebra interchangeably when their distinction is not particularly relevant.

3.2 On the infinitesimal structure: general structure of Lie algebras

As it is well-known, modulo the global topology, finite dimensional real Lie groups are in one-to-one correspondence with their universal covering groups, and by Ado's theorem these are uniquely characterized by their Lie algebras. Any finite dimensional real Lie algebra, by construction, admits a natural real valued invariant symmetric bilinear form, the *Killing form*. For any element x of the Lie algebra introduce the notation $\mathrm{ad}_x := [x,\ ]$. Then, the Killing form is defined to be the mapping $(x,y) \mapsto \mathrm{Tr}\,(\mathrm{ad}_x\,\mathrm{ad}_y)$. Generally, the Killing form of a Lie algebra may be indefinite and even may be degenerate. The general structure of all possible Lie algebras is constrained by the Levi decomposition theorem [7, 8] using the properties of the Killing form. It states that the entire Lie algebra consists of a semidirect sum of the degenerate directions of the Killing form, called to be the *radical* or *solvable* part, and of the non degenerate directions of the Killing form, called to be the *Levi factor* or *semisimple* part. Due to the non-degeneracy of the Killing form inside the semisimple part, that can consist merely of direct sums of components which contain no nontrivial normal sub-Lie algebras, and are called *simple* components. In summary, for the general structure of Lie algebras, or equivalently, for connected and simply connected Lie groups, one can draw the following summary picture:

$$\underbrace{E}_{\substack{\text{any connected and}\\ \text{simply connected}\\ \text{Lie group}}} = \underbrace{R}_{\substack{\text{degenerate directions of Killing form}\\ (\textit{radical} \text{ or } \textit{solvable} \text{ part})}} \rtimes \underbrace{\overbrace{L_1}^{(\textit{simple})} \times \cdots \times \overbrace{L_n}^{(\textit{simple})}}_{\substack{\text{non-degenerate directions of Killing form}\\ (\textit{Levi factor} \text{ or } \textit{semisimple} \text{ part})}} \tag{7}$$

In conventional gauge theories usually only semisimple or simple groups are used, i.e. Lie groups with vanishing radical, like $\mathrm{SU}(N)$ or so. However, already e.g. the Poincaré group gives an example for a Lie group where a nonvanishing radical is present:

$$\underbrace{\mathcal{P}}_{\text{Poincaré group}} = \underbrace{\mathcal{T}}_{\substack{\text{translations}\\ (\textit{radical} \text{ or } \textit{solvable} \text{ part})}} \rtimes \underbrace{\overbrace{\mathcal{L}}^{(\textit{simple})}}_{\substack{\text{Lorentz group}\\ (\textit{Levi factor} \text{ or } \textit{semisimple} \text{ part})}} \tag{8}$$

As it was discussed in [4], the super-Poincaré group is an even less trivial example with a nonvanishing radical, in particular, with a nonabelian radical. For compact Lie groups the Levi decomposition is as follows:

$$\underbrace{G}_{\substack{\text{any connected and}\\ \text{simply connected}\\ \text{compact Lie group}}} = \underbrace{\mathrm{U}(1) \times \cdots \times \mathrm{U}(1)}_{\substack{\text{degenerate directions of Killing form}\\ (\textit{radical} \text{ or } \textit{solvable} \text{ part})}} \times \underbrace{\overbrace{G_1}^{(\textit{compact simple})} \times \cdots \times \overbrace{G_m}^{(\textit{compact simple})}}_{\substack{\text{non-degenerate directions of Killing form}\\ (\textit{Levi factor} \text{ or } \textit{semisimple} \text{ part})}} \tag{9}$$

Note that in the compact case the radical can merely consist of copies of $\mathrm{U}(1)$ (compact abelian part), and it can only be in direct product relation with its Levi factor (compact nonabelian, or equivalently, compact semisimple part). The Levi decomposition is useful when trying to construct Lie group embeddings, since that must be compatible with the Levi structure of the underlying Lie algebras.

Moreover, it also restricts nontrivial forgetful homomorphisms onto a smaller Lie algebra: the kernel of a homomorphism always must be normal, so in absence of normal sub-Lie algebras, no nontrivial forgetful homomorphism exists. As such, e.g. simple Lie algebras do not have nontrivial forgetful homomorphisms onto any smaller Lie algebra at all. It is seen from the general Levi decomposition Eq.(7), that for a non-direct product (unified) Lie algebra, one can only have nontrivial normal sub-Lie algebras within, whenever one allows for a nonvanishing radical. In conventional gauge theory, since usually the simpleness or semisimpleness of the internal symmetry group is assumed, nontrivial normal sub-Lie algebras are not present, and this whole issue with nontrivial normal sub-Lie algebras was not yet studied for a model building purpose.

As a closing remark to the Levi decomposition, we recall that the solvability of a subgroup, i.e. that the Killing form degenerates there, can be also tested without explicitly constructing the Killing form. It is well-known [7, 8] that a Lie group R is solvable if and only if its Lie algebra r has the following properties: with the definition $r^0 := r$, $r^1 := [r^0, r^0]$, $r^2 := [r^1, r^1]$, $\ldots$, $r^k := [r^{k-1}, r^{k-1}]$, $\ldots$, one has that $r^k = \{0\}$ for finite k. A special case is when the radical R is said to be *nilpotent*: there exists a finite k for which for all $x_1, \ldots, x_k \in r$ one has $\mathrm{ad}_{x_1} \ldots \mathrm{ad}_{x_k} = 0$, which also happens to be equivalent to the property that for all $x \in r$ the linear map ad_x is nilpotent, hence the name. An even more special case is when the radical R is *abelian*: for all $x \in r$, one has $\mathrm{ad}_x = 0$. For instance, the Poincaré group happens to have a nonvanishing but abelian radical. The super-Poincaré group happens to have a non-abelian, namely a two-step nilpotent radical. In conventional gauge theory, like theories involving $\mathrm{SU}(N)$, traditionally no nonvanishing radical is allowed, i.e. the gauge groups are purely semisimple.

3.3 Constraints on unification patterns by the Levi decomposition

If one studies the possible enlargements of Lie groups, the Levi decomposition theorem Eq.(7) gives important constraints. Namely, their Lie algebras must obey the following rule: the Levi factor of the smaller Lie algebra cannot be injected to the radical of the larger Lie algebra, since the Killing form degenerates there. Moreover, if the smaller Lie algebra has some direct product structure, its indecomposable components must end up in some indecomposable components of the larger Lie algebra. From this simple observation, O'Raifeartaigh developed its classification theorem [2] of the Lie algebra extensions of the Poincaré Lie algebra. The theorem states that if one injects the Poincaré Lie algebra Eq.(8) into a larger Lie algebra Eq.(7), then one can have the following disjoint cases:

(A) Trivial extension, i.e. $E = \mathcal{P} \times \{\text{some other symmetry group}\}$. (This case is the usual conclusion of the Coleman-Mandula-like no-go theorems.)

(B) Not (A), and the translation group $\mathcal{T}$ is embedded into the radical R of the enlarged group, whereas the Lorentz group $\mathcal{L}$ is embedded into one of the simple components of the Levi factor of the enlarged group. (For instance, super-Poincaré group [9, 10, 11], and our new example in this paper and also in [4, 12] falls into this case.)

(C) The entire Poincaré group $\mathcal{T} \rtimes \mathcal{L}$ is embedded into one of the simple components of the enlarged group. (For instance, conformal Poincaré group, being isomorphic to $\mathrm{SO}(2, 4)$, falls into this case, but there are also many other models of this kind, see a detailed review in [13]. One should note that for models built on this case, a heavy symmetry breaking mechanism needs to be invoked in order to identify the spacetime degrees of freedom.)

The above O'Raifeartaigh theorem on the classification of finite dimensional real Lie group extensions of the Poincaré group can be summarized in the following diagram:

case (A) and (B): case (C):

$$\begin{array}{ccccc} E & = & R & \rtimes & L_1 \times \ldots \times L_n \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{P} & = & \mathcal{T} & \rtimes & \mathcal{L} \end{array} \qquad\qquad \begin{array}{ccccc} E & = & R & \rtimes & L_1 \times \ldots \times L_n \\ \uparrow & & & \nearrow & \uparrow \\ \mathcal{P} & = & \mathcal{T} & \rtimes & \mathcal{L} \end{array} \qquad (10)$$

where the thick arrows indicate homomorphic injection. Similar enlargement theorem also applies to the internal symmetry group, which has not yet been discussed in the literature.

As a closing remark, it is useful to note that via the O'Raifeartaigh theorem it is easy to understand the principle of the Coleman-Mandula no-go theorem, without deeply invoking field theoretical notions and arguments. As a starting point, O'Raifeartaigh theorem states that all the finite dimensional real Lie algebras containing the Poincaré Lie algebra must fall into one of the cases (A) or (B) or (C). Then, Coleman-Mandula theorem has a number of explicit and implicit assumptions. For instance, it explicitly assumes that there exists a positive definite scalar product on the generators of the non-Poincaré part of the extended Lie algebra, which in finite dimensions implies that the extended part is purely compact. This, rules out case (B). Finally, Coleman-Mandula theorem also assumes that no symmetry breaking is present, which rules out case (C).

3.4 Conservative extensions of the Poincaré group

As mentioned in [4, 12], the super-Poincaré group is one of the possibilities for extending the Poincaré group with the mechanism of O'Raifeartaigh theorem case (B), i.e. via allowing for the extension of the radical. However, it was also discussed that the super-Poincaré group cannot be cast in the form of a vector bundle automorphism group over the four dimensional Lorentz spacetime: some of the non-Poincaré symmetries (the pure supertranslations) do not act spacetime pointwise, and therefore they cannot be put into the structure group of a vector bundle over the spacetime. This is a curious property of the super-Poincaré group: some of the strictly non-spacetime transformations do not preserve the spacetime points, and therefore they cannot really be considered as part of an internal group. In order to overcome this, but to still have a unified gauge–Poincaré group, in [4] a special kind of enlargement of the Poincaré group was proposed: the *conservative* extensions of the Poincaré group. These can be defined in three equivalent ways. Namely, we are looking for a non-direct product (unified) extension of the Poincaré Lie algebra which satisfies any of the below criteria:

(i) The non-Poincaré generators do not act on the spacetime, i.e. they are really *internal*.

(ii) There exists $\mathcal{P} \xrightarrow{i} E \xrightarrow{o} \mathcal{P}$ homomorphisms such that $o \circ i =$ identity.

(iii) E is part of a vector bundle automorphism group over spacetime.

The intuitive meaning of these equivalent conditions, most transparently seen from condition (iii), is that the pertinent extended group E is compatible with a gauge theory-like setting: no symmetry breaking is necessary in order to identify the spacetime degrees of freedom on which merely the Poincaré part acts. Instead, there exists a forgetful homomorphism from the extended group onto the Poincaré group. One could call such a mechanism *symmetry hiding*. Such mechanism is usually not employed in conventional field theories as mostly some kind of symmetry breaking is assumed. Note however, that the above mechanism can host the necessary "exotic" symmetries in a way that they can remain unbroken, while respecting the gauge theory-like structure (vector bundle of matter fields) of a model, while being indecomposable (unified) group. As was mentioned: due to the Levi decomposition theorem, such a mechanism is only possible if one leaves the realm of semisimple Lie algebras.

As was discussed in [4, 12], assuming that an energy non-negativity condition is required to hold for the gauge fields in a theory, the structure of conservative extensions of the Poincaré group is as follows:

(arrows: nonvanishing adjoint subgroup action)

$$\underbrace{\Big(\underbrace{\mathcal{T}}_{\text{translations}} \times \underbrace{\underbrace{\mathcal{N}}_{\text{solvable internal symmetries}}\Big) \rtimes \Big(\underbrace{\mathcal{G}_1 \times \ldots \times \mathcal{G}_m}_{\text{compact internal symmetries}}}_{\text{all internal (gauge) symmetries}} \times \underbrace{\mathcal{L}}_{\text{Lorentz group}}\Big)}_{\text{unified global symmetries of matter fields}} \tag{11}$$

The arrows here indicate that which of the subgroups has to have a nonvanishing adjoint action on which of the normal subgroups for achieving indecomposability (unifiedness). Since the Killing form degenerates on the noncompact (solvable) internal Lie algebra, in a theory with corresponding gauge fields, these "exotic" gauge field degrees of freedom shall have zero Yang-Mills kinetic Lagrangian and shall have vanishing Yang-Mills kinetic energy. Thus, physicswise it is a natural question to consider: what kind of Euler-Lagrange equations these "exotic" gauge field degrees of freedom will obey? In our example we shall show, that it is possible to build models, where these "exotic" gauge fields can be completely transformed out from the Lagrangian, and thus can be eliminated from all the observables in a way as introduced in Section 2. In such a construction, they only manifest themselves as acting faithfully on the matter field sector without being accompanied by corresponding gauge bosons.

The main idea behind the conservative unification pattern is that despite of the indecomposable (unified) group structure, there is a forgetful homomorphism back onto the usual product of the compact internal group $\times$ Poincaré group:

$$\underbrace{\underbrace{\mathcal{N}}_{\substack{\text{solvable}\\ \text{internal symmetries}}} \rtimes \Big(\underbrace{\mathcal{G}_1 \times \ldots \times \mathcal{G}_m}_{\substack{\text{compact}\\ \text{internal symmetries}}} \times \underbrace{\mathcal{P}}_{\substack{\text{Poincaré}\\ \text{symmetries}}}\Big)}_{\substack{\text{direct-indecomposable conservative extension of the Poincaré group,}\\ \text{acting on fundamental field degrees of freedom}}} \longrightarrow \underbrace{\underbrace{\mathcal{G}_1 \times \ldots \times \mathcal{G}_m}_{\substack{\text{compact}\\ \text{internal symmetries}}} \times \underbrace{\mathcal{P}}_{\substack{\text{Poincaré}\\ \text{symmetries}}}}_{\substack{\text{observed direct-decomposable symmetries,}\\ \text{acting on some derived field quantities}\\ \text{which are function of fundamental degrees of freedom}}} \tag{12}$$

and potentially can explain a Standard Model-like gauge theory setting from a direct-indecomposable fundamental symmetry, without symmetry breaking. As it was shown in [4, 12] this definition is not empty: there are nontrivial conservative extensions of the Poincaré group. In this paper we shall show that there exist Lagrangians which admit such a group as their symmetry.

Since in the presented arguments we were reflecting on the Coleman-Mandula theorem, the above arguments were given in the global symmetries limit: at special relativistic limit, and with globally acting gauge symmetries, constant throughout the spacetime. As a closing remark of this section we recall [4] that in order to make a theory with the local version of the above symmetry requirements, one merely needs to construct a vector bundle with its structure group being a conservative extension of the Lorentz group, i.e. like Eq.(11) but without the translations being counted into the structure group. In the remaining part of the paper, such a construction will be given.

4 The example structure group of the proposed toy model

The mathematically simplest, i.e. lowest dimensional nonabelian nilpotent Lie group is the so-called *Heisenberg Lie group* with 3 generators. Its name comes from the formal resemblance of its Lie algebra relations to the Heisenberg exchange relations: the Lie algebra of the lowest dimensional Heisenberg group is spanned by three elements q, p and e, the only nonvanishing bracket relation being $[p, q] = K\,e$ where K is some nonzero real number. Since for different values of K they are naturally isomorphic, one can fix the value of the constant K to an arbitrary preferred nonzero real number. Take the complexified 3 generator Heisenberg Lie group $\mathrm{H}_3(\mathbb{C})$. It is well known that the Lie algebra of its outer derivations is isomorphic to $\mathrm{gl}(2,\mathbb{C})$, which mixes the first two generators q and p, while it acts by a scaling with the trace on the third generator e. Thus, one can rightaway construct an indecomposable conservative Lorentz group extension with the smallest possible nilpotent part: $\mathrm{H}_3(\mathbb{C}) \rtimes \mathrm{GL}(2,\mathbb{C}) \equiv \mathrm{H}_3(\mathbb{C}) \rtimes \big(\mathrm{U}(1){\times}\mathrm{D}(1){\times}\mathrm{SL}(2,\mathbb{C})\big)$. Note that this group also contains a compact part, $\mathrm{U}(1)$. The key ingredient for the structure group of our toy model shall be the above group. In order to continue, we first show that the above is a matrix group, and will find an elegant defining representation, in order to see a possible field theoretical meaning of such symmetries.

In the followings, we shall use the ordinary two-spinor calculus [14, 15], and in particular its variant which is most wide spread in the general relativity (GR) literature. Take any two dimensional complex vector space S ("two-spinor space"), and take its corresponding dual, complex conjugate and complex conjugate dual vector spaces, denoted by S^*, $\bar{S}$ and $\bar{S}^*$, respectively. In the Penrose abstract index notation [14, 15], elements of S, S^*, $\bar{S}$, $\bar{S}^*$ are denoted by upper index, lower index, primed upper index and primed lower index spinors, respectively, with the spinor indices being based on upper case latin letters. The symbol T shall denote a four dimensional real vector space ("tangent space"), with T^* being its dual. As usually in the GR literature, Penrose abstract indices of elements of T and T^* shall be denoted by lower case latin letter upper and lower indices, respectively.

Take a Grassmann algebra $\mathcal{A}$ with 2 generators, i.e. an exterior algebra of a two-dimensional vector space without a fixed preferred $\mathbb{Z}$-grading. Whenever a preferred $\mathbb{Z}$-grading is fixed, then $\mathcal{A}$ may be identified as $\mathcal{A} \equiv \Lambda(S^*)$, i.e. a spinorial representation of it can be given. $\mathcal{A}$, being a four dimensional complex unital associative algebra, can act on itself via left multiplication by its invertible elements.[2] Clearly, this defines a Lie group action on $\mathcal{A} \equiv \Lambda(S^*)$, and the acting Lie group can be seen to be isomorphic to $\mathbb{C}^{\times} \times \mathrm{H}_3(\mathbb{C})$, here $\mathbb{C}^{\times}$ denoting the group of complex numbers without the zero element and with the complex multiplication. To put it short: $\exp L_{\mathcal{A}} \equiv \mathbb{C}^{\times} \times \mathrm{H}_3(\mathbb{C})$, where $L_{(\cdot)}$ denotes left multiplication within $\mathcal{A}$. Moreover, it is not difficult to see that $\exp L_{M(\mathcal{A})} \equiv \mathrm{H}_3(\mathbb{C})$, where $M(\mathcal{A}) \subset \mathcal{A}$ denotes the maximal ideal (invariant subalgebra of at-least-1-forms) within $\mathcal{A}$. Thus, one can describe $\mathrm{H}_3(\mathbb{C})$ with a handy defining representation on $\mathcal{A}$, using ordinary two-spinors. Since $\mathrm{GL}(S^*) \equiv \mathrm{GL}(\Lambda_1) \equiv \mathrm{GL}(M(\mathcal{A})/M^2(\mathcal{A}))$ describes the $\mathbb{Z}$-grading preserving algebra automorphisms of $\mathcal{A}$ [16], the group $\big(\mathbb{C}^{\times} \times \mathrm{H}_3(\mathbb{C})\big) \rtimes \big(\mathrm{U}(1){\times}\mathrm{D}(1){\times}\mathrm{SL}(2,\mathbb{C})\big) \equiv \exp(L_{\mathcal{A}}) \rtimes \mathrm{GL}(\Lambda_1)$ indeed has a natural faithful defining complex-linear representation on $\mathcal{A}$. The structure of the algebra $\mathcal{A}$ along with the natural action of the group $\exp(L_{\mathcal{A}}) \rtimes \mathrm{GL}(\Lambda_1)$ on it is illustrated in Figure 1. A convenient basis of the algebra $\mathcal{A}$ is spanned by the unit element $\mathbb{1} \in \Lambda_0$, two linearly independent elements (generators) $a_1, a_2 \in \Lambda_1$, and a corresponding two-form $a_1 a_2 \in \Lambda_2$. Using this, the Lie algebra of $\exp L_{\mathcal{A}} \equiv \mathbb{C}^{\times} \times \mathrm{H}_3(\mathbb{C})$ can be conveniently parameterized by the basis $\{L_{\mathbb{1}}, L_{a_1}, L_{a_2}, L_{a_1 a_2}\}$, and indeed one may evaluate that the Lie algebra spanned by $L_{a_1}, L_{a_2}, L_{a_1 a_2}$ is isomorphic to the Lie algebra of $\mathrm{H}_3(\mathbb{C})$, and of course the Lie algebra spanned by $L_{\mathbb{1}}$ alone is isomorphic to the Lie algebra of $\mathbb{C}^{\times}$. Clearly, the defining representation over $\mathcal{A}$ induces a canonical faithful representation on the complex

[2]It is well known that the invertible elements of Grassmann algebra are those which have nonzero scalar (zero-form) component. To put it differently: invertible elements are those which are exponentials of any elements.

conjugate space $\bar{\mathcal{A}}$ via the requirement of being compatible with the $\mathcal{A} \to \bar{\mathcal{A}}$ complex conjugation map.

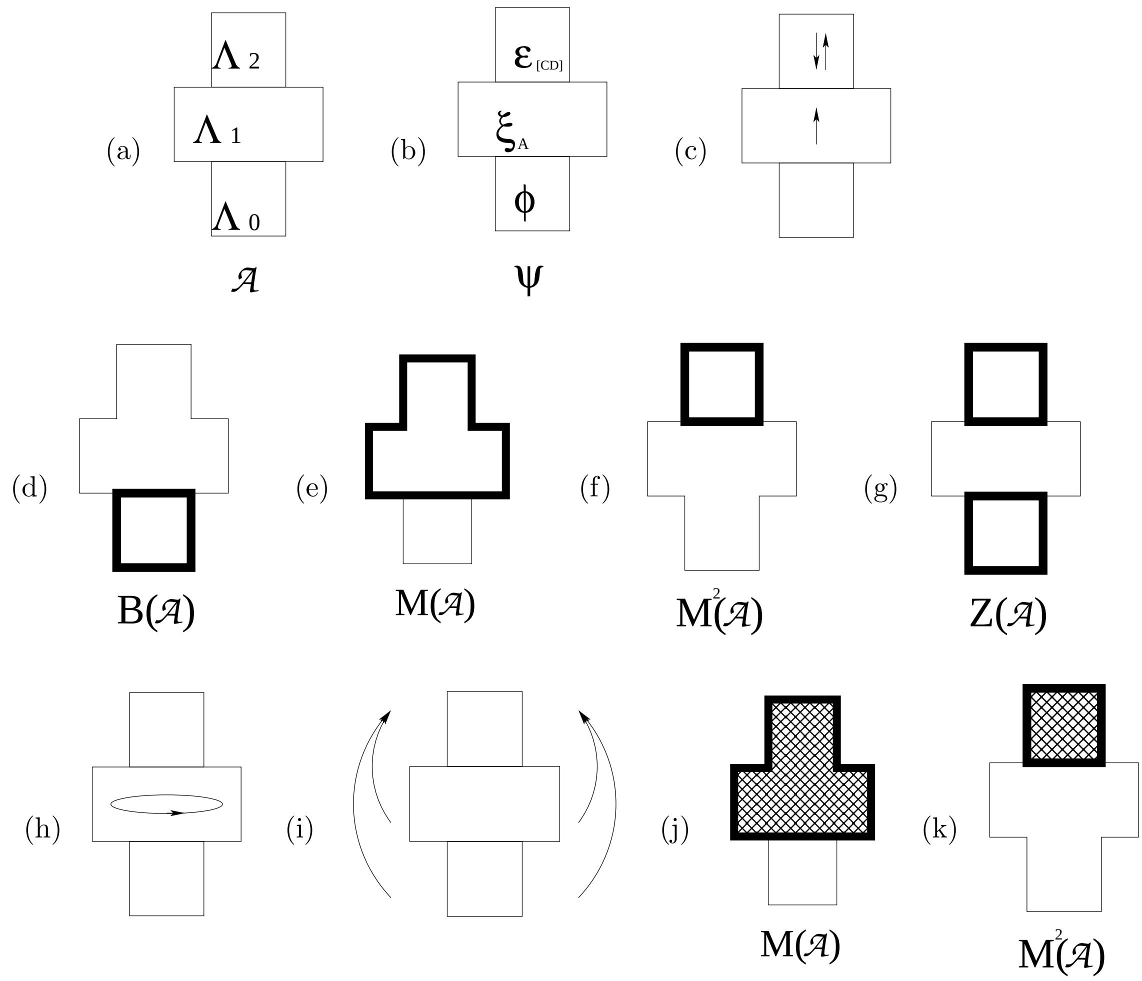

Figure 1: Illustration of the structure of the complex unital associative algebra $\mathcal{A} \equiv \Lambda(S^*)$ and the natural group action of the conservative Lorentz group extension $\exp(L_{\mathcal{A}}) \rtimes \mathrm{GL}(\Lambda_1)$ over it. Panel (a): the algebra $\mathcal{A}$ with a fixed $\mathbb{Z}$-grading ($\Lambda_k \equiv \overset{k}{\wedge} S^*$). Panel (b): whenever a fixed $\mathbb{Z}$-grading is taken, an element ψ of $\mathcal{A}$ can be represented by a tuple of spinors. Panel (c): heuristically speaking, the algebra $\mathcal{A}$ can be considered as a creation operator algebra of fermions with 2 degrees of freedom. Panels (d)–(e)–(f)–(g): important subspaces of the algebra $\mathcal{A}$, namely the scalar sector $B(\mathcal{A})$, the maximal ideal $M(\mathcal{A})$, and its second power $M^2(\mathcal{A})$, moreover its center $Z(\mathcal{A})$. Panels (h)–(i): illustration of the group action of the grading preserving part ($\mathrm{GL}(\Lambda_1)$) and of the grading non-preserving part ($\exp L_{\mathcal{A}}$) of the full symmetry group $\exp(L_{\mathcal{A}}) \rtimes \mathrm{GL}(\Lambda_1)$. The grading preserving part, by definition conserves the k-form subspaces, whereas the grading non-preserving part mixes higher forms to lower forms. Panels (j)–(k): list of all the invariant subspaces, which are invariant to the group action of the full symmetry group $\exp(L_{\mathcal{A}}) \rtimes \mathrm{GL}(\Lambda_1)$. It is seen that none of the invariant subspaces possess an invariant complementing subspace, and thus the defining representation on $\mathcal{A}$ is not totally reducible. In other words: the pertinent group action puts $\mathcal{A}$ into a single multiplet. Note that in the representation space of a non-semisimple Lie group an invariant subspace might not have invariant complement, i.e. a reducible representation might still be an indecomposable (non-direct sum) multiplet.

Our actual representation space shall be $A := \bar{\mathcal{A}} \otimes \mathcal{A}$, where $\bar{\mathcal{A}}$ denotes the complex conjugate vector space of $\mathcal{A}$ and $\otimes$ denotes ordinary, i.e. vector space sense tensor product (and not a graded tensor product, for instance). Clearly, $\mathcal{A}$ and $\bar{\mathcal{A}}$ may be naturally embedded into A as the subspaces $\mathbb{1} \otimes \mathcal{A}$ and $\bar{\mathcal{A}} \otimes \mathbb{1}$, respectively. The algebra A is a kind of doubled exterior algebra, which we named *spin algebra*, being a 16 dimensional complex unital associative algebra, along with a natural conjugate-linear $\overline{(\cdot)} : A \to A$ involution on it, which we call *charge conjugation*, and which has the property $\overline{x\,y} = \overline{x}\,\overline{y}$. The charge conjugation map is simply defined by the composition of the complex conjugation as a $\bar{\mathcal{A}} \otimes \mathcal{A} \to \mathcal{A} \otimes \bar{\mathcal{A}}$ map and of the tensor product swapping as a $\mathcal{A} \otimes \bar{\mathcal{A}} \to \bar{\mathcal{A}} \otimes \mathcal{A}$ map, hence giving rise to a $\bar{\mathcal{A}} \otimes \mathcal{A} \to \bar{\mathcal{A}} \otimes \mathcal{A}$ conjugate-linear involution on A. Clearly, our pertinent conservative Lorentz group extension has a canonical action on $A = \bar{\mathcal{A}} \otimes \mathcal{A}$, inherited from the defining action on $\mathcal{A}$. It is easy to observe that by construction, the complex phase of the group action of the $\exp L_{B(\mathcal{A})} \equiv \mathbb{C}^{\times}$ part will only act trivially on the representation space A, but the other parts act faithfully there. That is, the invariant subgroup $\exp L_{B(\mathcal{A})}$ shall act merely as $\exp L_{\mathrm{Re}(B(\mathcal{A}))} \equiv \mathrm{D}(1)$ on A, while the other parts act faithfully. The matter fields in our model will take their values in A, and our Lorentz group extension

$$\begin{aligned} \mathcal{G}_0 \quad &:= \quad \exp\left(L_{\mathrm{Re}(B(\mathcal{A}))+M(\mathcal{A})}\right) \rtimes \mathrm{GL}(\Lambda_1) \\ &\equiv \quad \left(\mathrm{D}(1)_{\mathrm{Z}} \times \mathrm{H}_3(\mathbb{C})\right) \rtimes \left(\mathrm{U}(1)\times\mathrm{D}(1)\times\mathrm{SL}(2,\mathbb{C})\right) \\ &\equiv \quad \mathrm{D}(1)_{\mathrm{Z}} \times \Big(\mathrm{H}_3(\mathbb{C}) \rtimes \left(\mathrm{U}(1)\times\mathrm{D}(1)\times\mathrm{SL}(2,\mathbb{C})\right)\Big) \end{aligned} \tag{13}$$

acts faithfully and real-linearly on it, while also preserving the charge conjugation map $\overline{(\cdot)}$. It is seen from Eq.(13) that one has two copies of $\mathrm{D}(1)$ within $\mathcal{G}_0$: one of them originates from the subgroup $\exp L_{\mathrm{Re}(B(\mathcal{A}))}$, and is direct decomposable from the other parts of $\mathcal{G}_0$, and we structurally denoted it by $\mathrm{D}(1)_{\mathrm{Z}}$. The other $\mathrm{D}(1)$ part sits in the remaining, direct indecomposable part of $\mathcal{G}_0$. A further observation is that, by construction, the conjugate-linear involution $\overline{(\cdot)}$ of charge conjugation leaves $\mathcal{G}_0$ invariant, thus, one can form the semidirect product group $\mathcal{G} := \mathcal{G}_0 \rtimes \{I, \overline{(\cdot)}\}$, which is $\mathcal{G}_0$ together with the charge conjugation operation. The structure of the spin algebra A along with the natural action of the group $\mathcal{G}$ on it is illustrated in Figure 2. It is seen that although $\mathcal{G}$-invariant subspaces within A do exist, but none of them has an invariant complement, and thus the representation space of A is direct indecomposable. One should note that due to the presence of the nilpotent part of $\mathcal{G}$ the usual behavior of semisimple group representations does not apply: the presence of an invariant subspace does not imply the presence of an invariant complement, i.e. it does not imply total reducibility.

Before we continue, we briefly mention the heuristic meaning of the representation space A and the group action of $\mathcal{G}$ on it. Since $A = \bar{\mathcal{A}} \otimes \mathcal{A}$, the algebra A can be heuristically considered as a creation operator algebra of two kinds of fermions, each having 2 degrees of freedom, and the two kinds being related to each-other via the charge conjugation operation $\overline{(\cdot)}$. The finite dimensional real Lie group $\mathcal{G}$, defined above, acts naturally on A, and the meaning of grading preserving transformations of $\mathcal{G}$ is clear: they generate $\mathrm{U}(1)$ and $\mathrm{D}(1)\times\mathrm{SL}(2,\mathbb{C})$ transformations on the generating sector $\Lambda_{\bar{0}1}$ and corresponding natural action on all of the sectors $\Lambda_{\bar{p}q}$, and thus on the entire $A \equiv \bigoplus_{p,q=0}^{2} \Lambda_{\bar{p}q}$. The grading non-preserving transformations mix higher forms to lower forms, deforming the original $\mathbb{Z}\times\mathbb{Z}$-grading of A to an other equivalent one. In the heuristic picture of creation operator algebras, the corresponding grading non-preserving group action of $\mathcal{G}$ on an element $\Psi \in A$ would mean insertion of equal amount of particles and corresponding charge conjugate particles into Ψ. In the following part we investigate important $\mathcal{G}$-invariant functions on the representations $\mathcal{A}$ and A in order to cast further light on the meaning of these grading deforming nilpotent transformations.

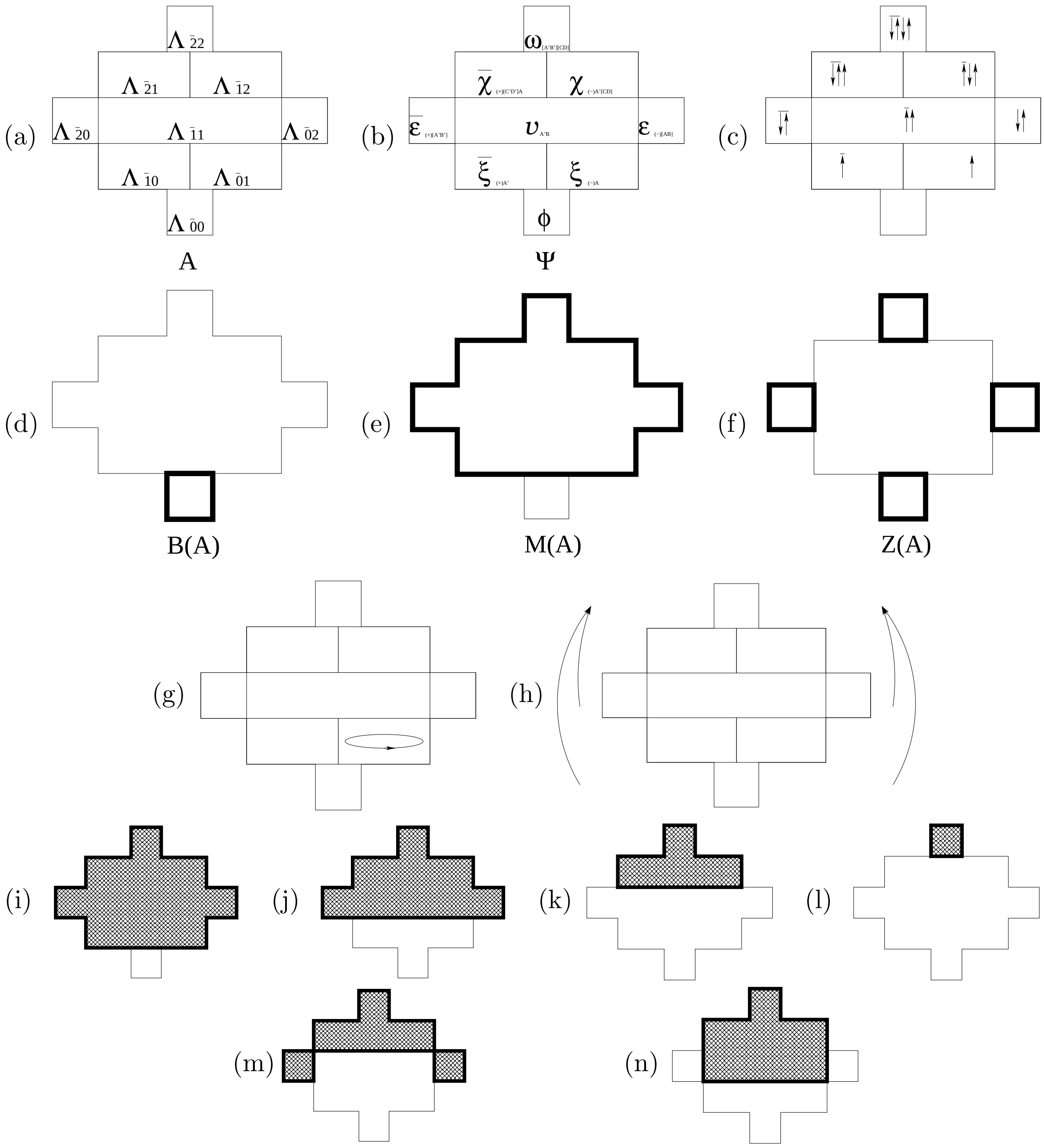

Figure 2: Illustration of the structure of the *spin algebra* $A \equiv \Lambda(\bar{S}^*) \otimes \Lambda(S^*)$ and the natural group action of the conservative Lorentz group extension $\mathcal{G}$ from over it. Panel (a): the algebra A with a fixed $\mathbb{Z}\times\mathbb{Z}$-grading ($\Lambda_{\bar{p}q} \equiv \overset{p}{\wedge}\bar{S}^* \otimes \overset{q}{\wedge} S^*$). Panel (b): whenever a fixed $\mathbb{Z}\times\mathbb{Z}$-grading is taken, an element Ψ of A can be represented by a tuple of spinors. Panel (c): heuristically speaking, the algebra A can be considered as a creation operator algebra of two distinct kind of fermions with 2 degrees of freedom each, and the two kinds being charge conjugate to each-other. Panels (d)–(e)–(f): important subspaces of the algebra A, namely the scalar sector $B(A)$, the maximal ideal $M(A)$, moreover its center $Z(A)$. Panels (g)–(h): illustration of the group action of the grading preserving part and of the grading non-preserving part of the symmetry group $\mathcal{G}$. Panels (i)–(n): list of all the subspaces of A, which are invariant under the group action of the symmetry group $\mathcal{G}$. It is seen that no invariant complementing subspaces exist, i.e. A is an indecomposable multiplet.

4.1 Important invariant functions on representations of the example group

For convenience, introduce an important subgroup $S\mathcal{G}_0$ of $\mathcal{G}_0$ which acts trivially on the scalars $A/M(A)$ and on the maximal forms $M^4(A)$ of the spin algebra A. It is nothing but $\mathcal{G}_0$ without the D(1) and $\mathrm{D}(1)_{\mathrm{Z}}$ parts, i.e. it is the "special" subgroup of $\mathcal{G}_0$, and is seen to be isomorphic to $\mathrm{H}_3(\mathbb{C}) \rtimes \big(\mathrm{U}(1)\times\mathrm{SL}(2,\mathbb{C})\big)$, being the smallest dimensional conservative unification of a compact Lie group and of the Lorentz symmetries. The invariant theory of $\mathcal{G}$ is most easily studied via the invariant theory of its special subgroup $S\mathcal{G}_0$, and subsequent study of the action of the dilatation groups D(1) and $\mathrm{D}(1)_{\mathrm{Z}}$. We will need to introduce one even smaller subgroup $\overset{\times}{S}\mathcal{G}_0$ which is the subgroup of $S\mathcal{G}_0$ acting also trivially on $M^2(\mathcal{A})$, i.e. on the maximal forms of $\mathcal{A}$. That is, $\overset{\times}{S}\mathcal{G}_0$ is $S\mathcal{G}_0$ without the U(1) component, and thus it is isomorphic to $\mathrm{H}_3(\mathbb{C}) \rtimes \mathrm{SL}(2,\mathbb{C})$, being the smallest dimensional conservative extension of the Lorentz symmetries.

Using the `LieAlgebras` Maple package [17], one can search for invariant functions of the pertinent group. For instance, one can show that there is a single functionally independent $\mathcal{A} \to \mathbb{C}$ map, which is invariant to the group action of $\overset{\times}{S}\mathcal{G}_0$, and is nothing but the scalar component function $b : \mathcal{A} \to \mathbb{C}, \psi \mapsto b\psi$, where b picks out the scalar component (zero-form) of an element of $\mathcal{A}$. In a two-spinor representation $\psi \equiv (\phi,\, \xi_A,\, \varepsilon_{[BC]})$ of an element $\psi \in \mathcal{A}$, one has that $b\psi = \phi$. Similarly, one can search for $\mathcal{A} \times \mathcal{A} \to \mathbb{C}$ functions, invariant to $\overset{\times}{S}\mathcal{G}_0$, and these turn out to be functional combinations of these three invariants:

$$\begin{aligned}
(\psi, \psi') &\mapsto b\psi, \\
(\psi, \psi') &\mapsto b\psi', \\
(\psi, \psi') &\mapsto \lambda(\psi,\psi') := (b\partial_1\psi)(b\partial_2\psi') - (b\partial_2\psi)(b\partial_1\psi') \\
&\qquad\qquad -(b\psi)(b\partial_2\partial_1\psi') + (b\partial_2\partial_1\psi)(b\psi')
\end{aligned} \tag{14}$$

where ∂_1, ∂_2 are stepping down operators associated to some arbitrarily chosen generators $a_1, a_2 \in \Lambda_1$. The invariance of the $\mathcal{A} \times \mathcal{A} \to \mathbb{C}$ bilinear function λ is most easily understood via verifying the identity $\lambda(\psi,\psi') = (b\psi)^2\,\big(b\partial_2\partial_1(\psi^{-1}\psi')\big)$ for any element $\psi' \in \mathcal{A}$ and any invertible element $\psi \in \mathcal{A}$. It is clear that the function $b : \mathcal{A} \to \mathbb{C}$ is invariant, moreover, by construction of $\overset{\times}{S}\mathcal{G}_0$, the map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}, (\psi,\psi') \to \psi^{-1}\psi'$ is invariant, thus the bilinear form λ indeed has to be invariant. In two-spinor representation by setting $\psi \equiv (\phi,\, \xi_A,\, \varepsilon_{[BC]})$ and $\psi' \equiv (\phi',\, \xi'_A,\, \varepsilon'_{[BC]})$ one has that $\lambda(\psi,\psi') = -\frac{1}{2}\boldsymbol{\epsilon}^{[AB]}\left(\xi_A\xi'_B - \xi'_A\xi_B + \phi\,\varepsilon'_{[AB]} - \phi'\,\varepsilon_{[AB]}\right)$, where $\boldsymbol{\epsilon}_{[AB]} \in \overset{2}{\wedge} S^* \equiv M^2(\mathcal{A})$ is an arbitrary but fixed nonzero maximal form in $\mathcal{A}$, and $\boldsymbol{\epsilon}^{[AB]}$ is its corresponding inverse maximal form satisfying $\boldsymbol{\epsilon}_{[AB]}\,\boldsymbol{\epsilon}^{[BC]} = \delta_A{}^C$. It is seen that λ is a nondegenerate symplectic form, and that its choice is unique up to a complex multiplier, i.e. up to the choice of $\boldsymbol{\epsilon}_{[AB]}$. With this, we have also shown that $\overset{\times}{S}\mathcal{G}_0$ is a subgroup of Sp(4). One could say that the symplectic form λ is a natural generalization of the spinor symplectic form $\boldsymbol{\epsilon}^{[AB]}$, from the lower index two-spinor space S^* to the exterior algebra $\mathcal{A} \equiv \Lambda(S^*)$ of that. The ambiguous complex normalization of λ can be made more explicit by introducing the notation $\lambda_{\frac{1}{\boldsymbol{\epsilon}}}$, where one requires $\lambda_{\frac{1}{\boldsymbol{\epsilon}}}(\mathbb{1},\boldsymbol{\epsilon}) = 1$ to hold for a fixed nonzero maximal form $\boldsymbol{\epsilon} \in M^2(\mathcal{A})$. Here, $\frac{1}{\boldsymbol{\epsilon}}$ symbolically denotes the unique dual maximal form in $M^{2^*}(\mathcal{A})$ which satisfies $(\frac{1}{\boldsymbol{\epsilon}}|\boldsymbol{\epsilon}) = 1$, the symbol $(\cdot|\cdot)$ denoting duality pairing form.

Using again the `LieAlgebras` Maple package [17], one can search for $S\mathcal{G}_0$-invariant functions of A. For instance, one can show that there is a single functionally independent invariant $A \to \mathbb{C}$ function, namely $\bar{b}\otimes b$, picking out the scalar component (zero-form) of an element in A. In the followings we shall use the abbreviation b for $\bar{b}\otimes b$, since their distinction is not relevant. Similarly, one can search for $A \times A \to \mathbb{C}$ functions, invariant to $S\mathcal{G}_0$, and these turn out to be functional combinations of these three invariants:

$$\begin{array}{lll}(\Psi,\Psi') & \mapsto & b\Psi,\\ (\Psi,\Psi') & \mapsto & b\Psi',\\ (\Psi,\Psi') & \mapsto & L(\Psi,\Psi') := \big(\bar{\lambda}\otimes\lambda\big)\circ\big(I_{\bar{\mathcal{A}}}\otimes J\otimes I_{\mathcal{A}}\big)\,(\Psi\otimes\Psi')\end{array} \tag{15}$$

where J denotes the $\mathcal{A}\otimes\bar{\mathcal{A}}\to\bar{\mathcal{A}}\otimes\mathcal{A}$ swapping map, whereas $I_{\bar{\mathcal{A}}}$ and $I_{\mathcal{A}}$ denote the identity map of $\bar{\mathcal{A}}$ and $\mathcal{A}$, respectively. If a preferred $\mathbb{Z}\times\mathbb{Z}$-grading is taken along with generators $a_1, a_2\in\Lambda_{\bar{0}1}$, and corresponding stepping down operators ∂_1,∂_2, then the concrete expression

$$L(\Psi,\Psi') = b\bar{\partial}_2\bar{\partial}_1\partial_2\partial_1\big((\Psi_{\bar{0}0}-\Psi_{\bar{1}0}-\Psi_{\bar{0}1}+\Psi_{\bar{1}1}-\Psi_{\bar{2}0}-\Psi_{\bar{0}2}+\Psi_{\bar{2}1}+\Psi_{\bar{1}2}+\Psi_{\bar{2}2})\,\Psi'\big)$$

holds for all $\Psi,\Psi'\in A$. By construction, L is a nondegenerate symmetric bilinear form with alternating signature $(+1,-1,+1,-1,\dots)$. The invariant bilinear map L shall play a key role in the construction of $\mathcal{G}$-invariant Lagrangians. When expressed in terms of two-spinor representation $A\equiv\Lambda(\bar{S}^*)\otimes\Lambda(S^*)$, then for two elements

$$\Psi\equiv\Big(\phi,\,\bar{\xi}_{(+)A'},\,\xi_{(-)A},\,\bar{\varepsilon}_{(+)[A'B']},\,v_{[A'B]},\,\varepsilon_{(-)[AB]},\,\bar{\chi}_{(+)[A'B']C},\,\chi_{(-)C'[AB]},\,\omega_{[A'B'][AB]}\Big)$$

and

$$\Psi'\equiv\Big(\phi',\,\bar{\xi}'_{(+)A'},\,\xi'_{(-)A},\,\bar{\varepsilon}'_{(+)[A'B']},\,v'_{[A'B]},\,\varepsilon'_{(-)[AB]},\,\bar{\chi}'_{(+)[A'B']C},\,\chi'_{(-)C'[AB]},\,\omega'_{[A'B'][AB]}\Big)$$

one has the identity

$$\begin{aligned}L(\Psi,\Psi') = \frac{1}{4}\boldsymbol{\omega}^{[A'B'][CD]}\Big(&\\ \phi\,\omega'_{[A'B'][CD]} - 2\bar{\xi}_{(+)[A'}\,\chi'_{(-)B'][CD]} - 2\xi_{(-)[C}\,\bar{\chi}'_{(+)[A'B']D]} + 4v_{[A'[C}\,v'_{B']D]}&\\ -\bar{\varepsilon}_{(+)[A'B']}\,\varepsilon'_{(-)[CD]} - \varepsilon_{(-)[CD]}\,\bar{\varepsilon}'_{(+)[A'B']}&\\ +2\bar{\chi}_{(+)[A'B'][C}\,\xi'_{(-)D]} + 2\chi_{(-)[A'[CD]}\,\bar{\xi}'_{(+)B']} + \omega_{[A'B'][CD]}\,\phi'\Big)&,\end{aligned}$$

where $\boldsymbol{\omega}_{[A'B'][CD]}\in\overset{2}{\wedge}\bar{S}^*\otimes\overset{2}{\wedge}S^*\equiv M^4(A)$ is an arbitrary but fixed nonzero positive maximal form of A, and $\boldsymbol{\omega}^{[A'B'][CD]}$ is its inverse maximal form with the normalization convention $\boldsymbol{\omega}_{[A'B'][DE]}\,\boldsymbol{\omega}^{[B'C'][EF]} = \bar{\delta}_{A'}{}^{C'}\,\delta_D{}^F$. Due to the invariance of the nondegenerate symmetric bilinear form L, the nondegenerate hermitian sesquilinear form $L(\overline{(\cdot)},\cdot)$ is also invariant, and shall be shown to be a generalization of the sesquilinear form defined by the Dirac adjoint operation, in conventional Dirac theory.[3] The ambiguous real normalization of L can be made more explicit by introducing the notation $L_{\frac{1}{\boldsymbol{\omega}}}$, where one requires $L_{\frac{1}{\boldsymbol{\omega}}}(\mathbb{1},\boldsymbol{\omega})=1$ to hold for a fixed nonzero real maximal form $\boldsymbol{\omega}\in\mathrm{Re}(M^4(A))$. Here, $\frac{1}{\boldsymbol{\omega}}$ symbolically denotes the unique dual maximal form in $M^{4*}(A)$ which satisfies $(\frac{1}{\boldsymbol{\omega}}|\boldsymbol{\omega})=1$, the symbol $(\cdot|\cdot)$ denoting duality pairing form.

Before we can go on to the formulation of $\mathcal{G}$-invariant theories, invocation of a further invariant object needs to be introduced. As it is well-known [14, 15], the ordinary two-spinor formalism is based on the fact that the vector space $\mathrm{Re}\big(\bar{S}\otimes S\big)$ is four real dimensional, moreover for any nonzero maximal form $\boldsymbol{\epsilon}_{[AB]}\in\overset{2}{\wedge}S^*$ one has that the form $\boldsymbol{\omega}_{[A'B'][AB]} := \bar{\boldsymbol{\epsilon}}_{[A'B']}\otimes\boldsymbol{\epsilon}_{[AB]}$ defines a nondegenerate, symmetric, Lorentz signature (+,-,-,-) real-bilinear form on $\mathrm{Re}\big(\bar{S}\otimes S\big)$. Moreover, if

[3] Also, let us also remark that the formal resemblance of the structure of the spin algebra $A\equiv\Lambda(\bar{S}^*)\otimes\Lambda(S^*)$ to the algebra of superfields in SUSY theories is only superficial: when represented by two-spinors, the superfields take their values in $\Lambda(\bar{S}^*\oplus S^*)$ which has very different algebraic structure in comparison to $\Lambda(\bar{S}^*)\otimes\Lambda(S^*)$.

T is any four dimensional real vector space ("tangent space"), then one may fix a linear injection $\sigma_a^{\;\;A'A} : T \to \mathrm{Re}\big(\bar{S}\otimes S\big)$ (called *soldering form* or *Pauli map* or *Infeld–Van der Waerden symbol*), and with that one induces a Lorentz metric $g(\sigma,\boldsymbol{\omega})_{ab} := \sigma_a^{\;\;A'A}\,\sigma_b^{\;\;B'B}\,\boldsymbol{\omega}_{[A'B'][AB]}$ over T. This philosophy naturally generalizes to the spin algebra case: the soldering form can be taken instead to be a linear injection $\sigma_a^{\;\;A'A} : T \to \mathrm{Re}\Big(\big(M(\bar{\mathcal{A}})/M^2(\bar{\mathcal{A}})\big)^* \otimes \big(M(\mathcal{A})/M^2(\mathcal{A})\big)^*\Big)$, since the natural identification $M(\mathcal{A})/M^2(\mathcal{A}) \equiv S^*$ holds in any spinorial realization. Clearly, this induces a natural real-linear representation of $\mathcal{G}$ on T, by requiring the soldering form $\sigma_a^{\;\;A'A}$ to be constant under the simultaneous group action on A and T. This linear action on T is nothing but the Weyl group: the Lorentz group together with the metric rescalings. Having this, one can construct a further equivalent realization of the soldering form. Using again the `LieAlgebras` Maple package [17], one can show that the subspace of elements of $\mathrm{Lin}(A)$ which are invariant to the Heisenberg (nilpotent) group action of $\exp\, L_{M(\mathcal{A})}$, is nothing but R_A, i.e. the image of the spin algebra A in $\mathrm{Lin}(A)$ by the right multiplication. Of course, the Heisenberg-invariance of the elements of $R_A \subset \mathrm{Lin}(A)$ follows from the construction of $\mathcal{G}$, but the pertinent check is needed in order to prove that the subspace of Heisenberg-invariant operators is not larger. After verifying this, it follows that, up to a real multiplier, the only $\mathcal{G}$-invariant $T^* \to \mathrm{Lin}(A)$ real-linear injective map is $\sigma^a_{\;\;A'A}$ where the lower spinorial indices here are to be understood as right multiplication operations in A, in any spinorial representation of A. In order to avoid confusion, from now on we denote that map by $\boldsymbol{\sigma}^a$, when suppressing the (lower) spinor indices, i.e. when not referring to an explicit spinorial representation of A. Several identities inherit, by construction, from ordinary two-spinor calculus, for instance $\boldsymbol{\sigma}^a\boldsymbol{\sigma}^b$ is a $\mathrm{Re}_+(M^4(A))$ valued Lorentz signature metric on T^*, or equivalently, it is a real valued Lorentz signature metric conformal equivalence class on T^*. As in ordinary two-spinor calculus, $\sigma^a_{\;\;A'A}$ simply denotes the inverse of the linear map $\sigma_a^{\;\;A'A} : T \to \mathrm{Re}(\bar{S}\otimes S)$, i.e. it is uniquely determined via the relation $\sigma^a_{\;\;A'A}\,\sigma_b^{\;\;A'A} = \delta^a{}_b$. Due to the Heisenberg-invariance of $R_{\mathcal{A}}$ and R_A, one has that the natural injection of $S \to R_{\mathcal{A}}$ is $\mathcal{G}$-invariant, which shall be denoted by the symbol $R_{\boldsymbol{\delta}^A}$, and that takes an element of S into a corresponding right multiplication by a one-form. Using this notation, one has the expression $\boldsymbol{\sigma}^a = \sigma^a_{\;\;A'A}\, R_{\bar{\boldsymbol{\delta}}^{A'}}\otimes R_{\boldsymbol{\delta}^A}$, which will be needed later.

As mentioned in the above paragraph, given a soldering form $\sigma_a^{\;\;A'A}$ and a real maximal form $\boldsymbol{\omega} \in \mathrm{Re}(M^4(A))$, the Lorentz metric $g(\sigma,\boldsymbol{\omega})_{ab}$ is naturally defined, and therefore also its corresponding inverse metric. The formula for the inverse metric, as usually, can be expressed as $\sigma^a_{\;\;A'A}\,\sigma^b_{\;\;B'B}\,\boldsymbol{\omega}^{[A'B'][AB]}$ in the two-spinor calculus, and will be symbolically denoted by $g(\sigma,\frac{1}{\boldsymbol{\omega}})^{ab}$, emphasizing the inverse behavior with the scaling of its $\boldsymbol{\omega}$ argument. Associated to the metric $g(\sigma,\boldsymbol{\omega})_{ab}$, also a unique volume form in $\overset{4}{\wedge}T^*$ exists (up to orientation sign), and that can be expressed in the form

$$\begin{aligned}\mathbf{v}(o,\sigma,\boldsymbol{\omega})_{[abcd]} \quad := \quad & o\,\big(\mathrm{i}\,\sigma_a^{\;\;E'E}\,\sigma_b^{\;\;F'F}\,\sigma_c^{\;\;B'A}\,\sigma_d^{\;\;A'B}\,\boldsymbol{\omega}_{[E'A'][EA]}\,\boldsymbol{\omega}_{[F'B'][FB]} \\ & -\mathrm{i}\,\sigma_a^{\;\;E'E}\,\sigma_b^{\;\;F'F}\,\sigma_d^{\;\;B'A}\,\sigma_c^{\;\;A'B}\,\boldsymbol{\omega}_{[E'A'][EA]}\,\boldsymbol{\omega}_{[F'B'][FB]}\,\big)\end{aligned}$$

[14, 15], where $o = \pm 1$ describes the chosen orientation sign. The *spin tensor* is a further function of $\sigma_a^{\;\;A'A}$ according to the definition

$$\Sigma(\sigma)_a{}^b{}_C{}^D := \mathrm{i}\,\sigma_a^{\;\;A'D}\,\sigma^b_{\;\;A'C} - \mathrm{i}\,g(\sigma,\tfrac{1}{\boldsymbol{\omega}})^{cb}\,g(\sigma,\boldsymbol{\omega})_{ad}\,\sigma_c^{\;\;A'D}\,\sigma^d_{\;\;A'C}$$

which is a tensor of $T^*\otimes T\otimes S^*\otimes S$, with S^* denoting the factor space $M(\mathcal{A})\big/M^2(\mathcal{A})$ in the analogy of the dual two-spinor space. This is all as usual in the ordinary two-spinor calculus [14, 15], with the slight generalization of providing representation space for the nilpotent $\mathrm{H}_3(\mathbb{C})$ Lie group component of our symmetry group $\mathcal{G}$.

5 The example Lagrangian

In order to define our Lagrangian, we assume that our matter fields are sections of an A-valued vector bundle over a four dimensional spacetime, as illustrated in Figure 3. A distantly similar construction was tried by Anco and Wald [18], but the algebra they employed was too small in order to accommodate representation space for any symmetries larger than the conventional ones. In our construction, we take a four dimensional real manifold $\mathcal{M}$, and a spin algebra valued vector bundle $A(\mathcal{M})$ over it, being of the form $A(\mathcal{M}) = \bar{\mathcal{A}}(\mathcal{M})\otimes\mathcal{A}(\mathcal{M})$ with $\mathcal{A}(\mathcal{M})$ being a two generator Grassmann algebra bundle over $\mathcal{M}$. The vector bundle $A(\mathcal{M})$ is considered to be associated to the principal bundle $\mathcal{G}(\mathcal{M})$ with a structure group $\mathcal{G}$. In the analogy of ordinary two-spinor calculus [14, 15], we assume a $\sigma_a^{\;A'A}$ pointwise $T(\mathcal{M}) \to \mathrm{Re}\,\big(\bar{S}(\mathcal{M})\otimes S(\mathcal{M})\big)$ soldering form to be present, where $S^*(\mathcal{M}) := M(\mathcal{A})(\mathcal{M})\big/M^2(\mathcal{A})(\mathcal{M})$. Taking into account the findings in the previous section, one has a $\boldsymbol{\sigma}^a$ alternative realization of the soldering, being a pointwise intertwining map $T^*(\mathcal{M}) \to \mathrm{Lin}(A)(\mathcal{M})$. The fields $\sigma_a^{\;A'A}$ and $\boldsymbol{\sigma}^a$ are in one-to-one correspondence as mentioned in the previous section. If one takes a nowhere zero maximal form field $\boldsymbol{\omega}$ being a section of $M^4(A)(\mathcal{M})$, then one has the Lorentz metric field $g(\sigma,\boldsymbol{\omega})_{ab}$ as mentioned in the previous section, with its corresponding inverse metric tensor field $g(\sigma,\frac{1}{\boldsymbol{\omega}})^{ab}$, corresponding volume form field $\mathbf{v}(o,\sigma,\boldsymbol{\omega})_{[abcd]}$ (with $o = \pm 1$ being the chosen spacetime orientation sign), and corresponding field of spin tensor $\Sigma(\sigma)_a{}^b{}_C{}^D$. These will be the auxiliary quantities with which it is most transparent to express our invariant Lagrangian.

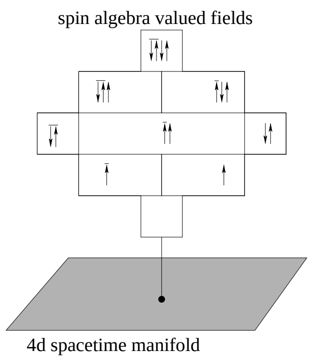

Figure 3: Illustration of the concept of spin algebra valued fields. The structure group of such a theory can be set to be a conservative unification $\mathcal{G}$ of the Lorentz and of the compact U(1) symmetries.

Our action principle shall be Palatini-like, i.e. the metric will not be a distinguished field. In fact, it will be a function of other fundamental fields (the soldering form $\sigma_a^{\;A'A}$ etc). The matter field sector of the theory will consist of the soldering form $\sigma_a^{\;A'A}$ and of a section Ψ of the spin algebra bundle $A(\mathcal{M})$. Moreover, as in the Palatini formalism, the $\mathcal{G}$-gauge-covariant derivation ∇_a is independently varied from the matter field sector. The actual Lagrangian shall be a real volume form valued pointwise vector bundle mapping

$$(o,\,\sigma_a^{\;A'A},\,\Psi,\,\nabla_a\Psi,\,P(\nabla)_{ab}) \qquad \longmapsto \qquad \mathbf{L}(o,\,\sigma_a^{\;A'A},\,\Psi,\,\nabla_a\Psi,\,P(\nabla)_{ab}) \tag{16}$$

with the requirement of being $\mathcal{G}$-gauge-covariant in the internal and tangent indices, and being diffeomorphism covariant in the tangent indices. Here $P(\nabla)_{ab}$ denotes the curvature tensor of a $\mathcal{G}$-gauge-covariant derivation ∇_a. The action functional is, as usually, defined as local integrals of the pertinent volume form over compact regions of the spacetime $\mathcal{M}$. Besides the $\mathcal{G}$-gauge-covariance and diffeomorphism covariance, we require further symmetry properties. Namely, we require that the theory is chiral, i.e. that the Lagrangian changes sign when changing the orientation o to opposite. Moreover we require the theory to be CPT covariant in the sense that a pointwise charge conjugation $\overline{(\cdot)}$ on $A(\mathcal{M})$ shall have a representation $(\Psi, \sigma_a^{A'A}) \mapsto (\overline{\Psi}, -\sigma_a^{A'A})$, i.e. a sign reversal of spacetime vectors. We also require that the Lagrangian does not depend on an overall complex phase of Ψ. In addition, we require that the Lagrangian is invariant to a shift transformation of the gauge-covariant derivation according to $\nabla_a \mapsto \nabla_a + C_a$ in the manner of Section 2, where now C_a denotes a smooth covector field taking its values in the normal sub-Lie algebra of $\mathcal{G}$, corresponding to the $\mathrm{D}(1)_{\mathrm{Z}} \times \big(\mathrm{H}_3(\mathbb{C}) \rtimes \mathrm{D}(1)\big)$ part, i.e. to the noncompact part of the radical of $\mathcal{G}$. The search for all such invariant volume form valued expressions in principle can be addressed by the `LieAlgebras` Maple package [17]. However, due to the relative large dimension of the total pointwise degrees of freedom, the pertinent library was not able to answer this question in its full generality. We were able to find, though, several invariant terms, and there is strong evidence that these are all. The pertinent invariant terms assuming fixed polynomial order in either $P(\nabla)_{ab}$ or in $\nabla_a\Psi$ are enumerated in the followings.

Yang–Mills-like term. Clearly, the tensor field $\mathbf{v}(o,\sigma,\boldsymbol{\omega})\, g(\sigma,\frac{1}{\boldsymbol{\omega}})^{ab}\, g(\sigma,\frac{1}{\boldsymbol{\omega}})^{cd}$ only depends on the orientation o and the soldering form σ, in particular it does not depend on $\boldsymbol{\omega}$. It therefore does not come as a surprise that the only invariant Lagrangian bilinear in the curvature $P(\nabla)_{ab}$ and satisfying positive energy density condition for gauge fields is:

$$\mathbf{L}_{\mathrm{YM}}(o,\, \sigma_a^{A'A},\, \Psi,\, \nabla_a\Psi,\, P(\nabla)_{ab}) \;=\; \mathbf{v}(o,\sigma,\boldsymbol{\omega})\, g(\sigma,\tfrac{1}{\boldsymbol{\omega}})^{ac}\, g(\sigma,\tfrac{1}{\boldsymbol{\omega}})^{bd}\, \mathrm{Im}\big(\,\mathrm{Tr}|_{\Lambda_{\bar{0}1}}\, P(\nabla)_{ab}\big)\, \mathrm{Im}\big(\,\mathrm{Tr}|_{\Lambda_{\bar{0}1}}\, P(\nabla)_{cd}\big). \tag{17}$$

This is nothing but literally the Maxwell Lagrangian, as expressed in our field variables. It is remarkable that only the U(1) part of the connection gives contribution, while being $\mathcal{G}$-covariant.

Einstein–Hilbert-like term. The tensor field $\mathbf{v}(o,\sigma,\boldsymbol{\omega})\, g(\sigma,\frac{1}{\boldsymbol{\omega}})^{ab}\, L_{\frac{1}{\boldsymbol{\omega}}}(\overline{\Psi},\Psi)$ does not depend on the maximal form $\boldsymbol{\omega}$. Thus, it is not surprising that the only invariant Lagrangian linear in the curvature $P(\nabla)_{ab}$ is:

$$\mathbf{L}_{\mathrm{EH}}(o,\, \sigma_a^{A'A},\, \Psi,\, \nabla_a\Psi,\, P(\nabla)_{ab}) \;=\; \mathbf{v}(o,\sigma,\boldsymbol{\omega})\, g(\sigma,\tfrac{1}{\boldsymbol{\omega}})^{ab}\, L_{\frac{1}{\boldsymbol{\omega}}}(\overline{\Psi},\Psi)\, \mathrm{Re}\big(\,\mathrm{Tr}|_{\Lambda_{\bar{0}1}}\, \big(\mathrm{i}\Sigma(\sigma)_a{}^c\, P(\nabla)_{cb}\big)\big). \tag{18}$$

This is nothing but a rather straightforward generalization of the Einstein–Hilbert Lagrangian, as expressed in spinorial variables. The only difference is that the prefactor of the scalar curvature is the field $L_{\frac{1}{\boldsymbol{\omega}}}(\overline{\Psi},\Psi)$ instead of the constant $1/(\text{Planck length})^2$. It is remarkable that only the $\mathrm{SL}(2,\mathbb{C})$ part of the connection gives contribution while the full expression being $\mathcal{G}$-covariant.

Klein–Gordon-like term is not allowed. That is because although the form field field $\mathbf{v}(o,\sigma,\boldsymbol{\omega})\, g(\sigma,\frac{1}{\boldsymbol{\omega}})^{ab}\, L_{\frac{1}{\boldsymbol{\omega}}}(\overline{(\cdot)},\cdot)$ does not depend on $\boldsymbol{\omega}$, and therefore the expression

$$\mathbf{L}_{\mathrm{KG}}(o,\, \sigma_a^{A'A},\, \Psi,\, \nabla_a\Psi,\, P(\nabla)_{ab}) \;=\; \mathbf{v}(o,\sigma,\boldsymbol{\omega})\, g(\sigma,\tfrac{1}{\boldsymbol{\omega}})^{ab}\, L_{\frac{1}{\boldsymbol{\omega}}}\big(\overline{\mathrm{i}\nabla_a(\Psi)},\, \mathrm{i}\nabla_b(\Psi)\big) \tag{19}$$

is an invariant of the aimed kind, except that it is not invariant to the shift symmetry $\nabla_a \mapsto \nabla_a + C_a$ with C_a being smooth covector field taking its values in the normal sub-Lie algebra of $\mathcal{G}$, corresponding to the $\mathrm{D}(1)_{\mathrm{Z}} \times \big(\mathrm{H}_3(\mathbb{C}) \rtimes \mathrm{D}(1)\big)$. Thus, a Klein–Gordon-like second order term in $\nabla_a\Psi$ is disallowed by the pertinent shift symmetry.

Dirac-like term. Here the calculations have to rely more intensively on the symbolic Maple calculation. It turns out that the $\mathcal{G}$-gauge-covariance, the diffeomorphism covariance, along with the CPT covariance, and the independence from the phase of Ψ singles out 13 linearly independent Lagrangians, which are first order in $\nabla_a\Psi$. However, the connection shift invariance singles out 1 unique invariant, resembling to a generalization of a Dirac term. It reads:

$$\mathbf{L}_{\mathrm{D}}(o,\, \sigma_a^{\;A'A},\, \Psi,\, \nabla_a\Psi,\, P(\nabla)_{ab}) \;=$$

$$\mathbf{v}(o,\sigma,\boldsymbol{\omega})\,\tfrac{1}{\sqrt{2}}\;\mathrm{Re}\Big(\,L_{\frac{1}{\boldsymbol{\omega}}}\big(\overline{\Psi},\;\gamma(\overline{\Psi},\Psi,\tfrac{1}{\boldsymbol{\omega}})^a\;\mathrm{i}\nabla_a(\Psi)\big)\,\Big) \tag{20}$$

where $\gamma(\overline{\Psi},\Psi,\frac{1}{\boldsymbol{\omega}})^a$ is a $T^*(\mathcal{M})\to\mathrm{Lin}(A)(\mathcal{M})$ pointwise linear map, defined as

$$\gamma(\overline{\Psi},\Psi',\tfrac{1}{\boldsymbol{\omega}})^a(\cdot)\;:=$$

$$\tfrac{1}{\sqrt{2}}\,\sigma^a_{\;\;A'A}\;\Big(\,(R_{\boldsymbol{\delta}^A}\,\overline{\Psi}\,)\;L_{\frac{1}{\boldsymbol{\omega}}}\big(R_{\bar{\boldsymbol{\delta}}^{A'}}\,\Psi',\,\cdot\,\big)\;+\;(R_{\bar{\boldsymbol{\delta}}^{A'}}\,\overline{\Psi}\,)\;L_{\frac{1}{\boldsymbol{\omega}}}\big(R_{\boldsymbol{\delta}^A}\,\Psi',\,\cdot\,\big)\,\Big).$$

Here, the notation $R_{\boldsymbol{\delta}^A}$ and $R_{\bar{\boldsymbol{\delta}}^{A'}}$ denote the pointwise injections $S^*\to R_{\mathcal{A}}$ and $\bar{S}^*\to R_{\bar{\mathcal{A}}}$, which are well defined (one has the identity $\boldsymbol{\sigma}^a\;=\;\sigma^a_{\;\;A'A}\;R_{\bar{\boldsymbol{\delta}}^{A'}}\otimes R_{\boldsymbol{\delta}^A}$). This Lagrangian is a kind of generalization of the Dirac kinetic term in the following sense. Introduce a fixed $\mathbb{Z}\times\mathbb{Z}$-grading of A, then one can realize that when restricted to the ±1 U(1) charge subspaces $D_+:=\Lambda_{\bar{1}0}\oplus\Lambda_{\bar{2}1}$ and $D_-:=\Lambda_{\bar{0}1}\oplus\Lambda_{\bar{1}2}$, then $\gamma(\overline{\Psi}_0,\Psi_0,\frac{1}{\boldsymbol{\omega}})^a$ admits the Clifford property against a metric proportional to $g(\sigma,\frac{1}{\boldsymbol{\omega}})^{ab}$, whenever the background field Ψ_0 takes its value in the spin-free subspace, i.e. in the center $Z(A)$ of the spin algebra A. Moreover, also when the sesquilinear map $L_{\frac{1}{\boldsymbol{\omega}}}\big(\overline{(\cdot)},\cdot\big)$ is restricted to D_+ or D_-, it corresponds to the one generated by the Dirac adjoint in ordinary Dirac bispinor formalism. This is illustrated in Figure 4.

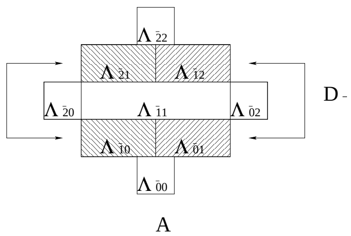

Figure 4: Illustration of the fact that whenever a fixed $\mathbb{Z}\times\mathbb{Z}$-grading of the spin algebra A is taken, then the ±1 U(1) charge subspaces $D_+:=\Lambda_{\bar{1}0}\oplus\Lambda_{\bar{2}1}$ and $D_-:=\Lambda_{\bar{0}1}\oplus\Lambda_{\bar{1}2}$ can be considered as embedded Dirac bispinor spaces in A.

Fourth order self-interaction potential. The volume form valued fourth order form field $\mathbf{v}(o,\sigma,\boldsymbol{\omega})\,L_{\frac{1}{\boldsymbol{\omega}}}(\cdot,\cdot)\,L_{\frac{1}{\boldsymbol{\omega}}}(\cdot,\cdot)$ does not depend on $\boldsymbol{\omega}$, thus one expects invariant Lagrangians based on it. Relying on the symbolic Maple calculation it turns out that there are 5 independent self-interaction terms, merely dependent on Ψ and $\sigma_a^{\;A'A}$. These all happen to be all fourth order in Ψ. Moreover, only 2 of these satisfy the condition of having definite sign (non-negativity condition). The most general form of the self-interaction potential is thus:

$$\mathbf{L}_{\mathrm{V}}(o,\, \sigma_a^{\;A'A},\, \Psi,\, \nabla_a\Psi,\, P(\nabla)_{ab}) \;=$$

$$\mathbf{v}(o,\sigma,\boldsymbol{\omega})\,\Big(\,\alpha_1\,L_{\frac{1}{\boldsymbol{\omega}}}\big(\overline{\Psi},\Psi\big)\,L_{\frac{1}{\boldsymbol{\omega}}}\big(\overline{\Psi},\Psi\big)\;+\;\alpha_2\,L_{\frac{1}{\boldsymbol{\omega}}}\big(\overline{\Psi},\overline{\Psi}\big)\,L_{\frac{1}{\boldsymbol{\omega}}}\big(\Psi,\Psi\big)\,\Big) \tag{21}$$

with real constants α_1 and α_2.

As mentioned, due to the high dimensionality of the problem we were not able to formally prove that the above exhaust the set of linearly independent invariant Lagrangians, but there is strong evidence that these are all. In any case, their linear combination

$$\mathbf{L}_{A_{\mathrm{YM}},\,A_{\mathrm{EH}},\,A_{\mathrm{D}},\,A_{\mathrm{V}}} \;:=\; A_{\mathrm{YM}}\,\mathbf{L}_{\mathrm{YM}} \;+\; A_{\mathrm{EH}}\,\mathbf{L}_{\mathrm{EH}} \;+\; A_{\mathrm{D}}\,\mathbf{L}_{\mathrm{D}} \;+\; A_{\mathrm{V}}\,\mathbf{L}_{\mathrm{V}} \tag{22}$$

also satisfy the pertinent invariance properties, with real coupling constants $A_{\mathrm{YM}}, A_{\mathrm{EH}}, A_{\mathrm{D}}, A_{\mathrm{V}}$. The question naturally arises: to what degree the behavior of the theory depends on these coupling constants. In order to address this question, the equivalence of two instances of such theory needs to be defined first. An instance

$$\Big(\mathcal{M},\, A(\mathcal{M}),\, \mathcal{G}(\mathcal{M}),\, (A_{\mathrm{YM}}, A_{\mathrm{EH}}, A_{\mathrm{D}}, A_{\mathrm{V}}),\, \mathbf{L}_{A_{\mathrm{YM}},\,A_{\mathrm{EH}},\,A_{\mathrm{D}},\,A_{\mathrm{V}}}\Big)$$

of the theory is said to be equivalent to an other instance

$$\Big(\mathcal{M}',\, A'(\mathcal{M}'),\, \mathcal{G}'(\mathcal{M}'),\, (A'_{\mathrm{YM}}, A'_{\mathrm{EH}}, A'_{\mathrm{D}}, A'_{\mathrm{V}}),\, \mathbf{L}'_{A'_{\mathrm{YM}},\,A'_{\mathrm{EH}},\,A'_{\mathrm{D}},\,A'_{\mathrm{V}}}\Big)$$

whenever there exists a principal bundle isomorphism $\mathcal{G}(\mathcal{M}) \to \mathcal{G}'(\mathcal{M}')$ with underlying vector bundle isomorphism $A(\mathcal{M}) \to A'(\mathcal{M}')$ and underlying diffeomorphism $\mathcal{M} \to \mathcal{M}'$, such that $\mathbf{L}_{A_{\mathrm{YM}},A_{\mathrm{EH}},A_{\mathrm{D}},A_{\mathrm{V}}}$ is mapped to $\mathbf{L}'_{A'_{\mathrm{YM}},A'_{\mathrm{EH}},A'_{\mathrm{D}},A'_{\mathrm{V}}}$, up to a nonzero real multiplier. The overall normalization can be disregarded for a classical field theory, since the Euler-Lagrange equations do not depend on the absolute normalization of the Lagrange form. Assume that we have one instance of the theory with all the coupling constants being nonzero. Then, all such theories are equivalent to an instance with coupling factors $1, A_{\mathrm{EH}}, A_{\mathrm{D}}, A_{\mathrm{V}}$, i.e. when the Yang–Mills coupling factor is fixed to 1, by convention. Moreover, due to the different homogeneity degree of the terms $\mathbf{L}_{\mathrm{YM}}$, $\mathbf{L}_{\mathrm{EH}}$, $\mathbf{L}_{\mathrm{D}}$, $\mathbf{L}_{\mathrm{V}}$, with global rescaling of the fields $\sigma_a^{\;A'A}$ and Ψ, one can find equivalent instances of the theory with coupling factors $1, 1, 1, A_{\mathrm{V}}$. The remaining coupling factor A_{V} can then be merged with the coupling factors α_1, α_2 within $\mathbf{L}_{\mathrm{V}}$. In the end, after such equivalence factorization, thus the theory has two coupling factors α_1 and α_2.

The heuristical meaning of the theory can be deduced from the picture outlined in Figure 3. According to that, such a model describes the field equations of a classical field, which spacetime pointwise has degrees of freedom similar to a second quantized fermionic theory, i.e. with pointwise degrees of freedom obeying Pauli principle. As such, it may be a kind of semiclassical limit of a QFT-like model. In this QFT heuristic picture, besides the usual compact gauge, Lorentz and dilatation symmetries, the theory is symmetric to the transformation when equal amount of fermions and antifermions are injected into a configuration spacetime pointwise, and this happens to be isomorphic to a pointwise $\mathrm{H}_3(\mathbb{C})$ Heisenberg group action. It also turns out that the "exotic" gauge fields can be completely transformed out from the theory due to an extra affine shift symmetry on the connection, which is similar to that of Section 2.

6 Concluding remarks

In this paper we showed an example toy model which exhibits a curious behavior: not all its local symmetry generators are accompanied by corresponding gauge bosons. As an introductory example we showed that already the Dirac kinetic Lagrangian shows an extremely simplified version for such behavior: the gauge boson fields corresponding to the dilatation symmetry do not give contribution to the theory. In other words: the Lagrangian has a hidden affine symmetry, namely it is invariant to an affine shift of the dilatation gauge connection. We showed that such behavior can eventually be also exhibited by more complicated internal symmetry groups, and even by indecomposable (unified) ones. The necessary condition, however, is that these "exotic" symmetry generators, whose gauge bosons can be transformed out, do span a normal sub-Lie algebra of the internal symmetries. Due to the general structural theorem of Lie algebras (Levi decomposition) this implies that only theories having nilpotent and compact gauge symmetry generators can eventually show such behavior. We have shown a constructive example Lagrangian for such a case, with indecomposable (unified) internal symmetry group.

Acknowledgments

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A Geometric picture behind the D(1) invariance

The notion of D(1) gauge charge mentioned in Section 2 can be reformulated in a geometrically even more elegant setting. The key idea is motivated by a work of Matolcsi [19] and of Janyška, Modugno, Vitolo [20], in which they proposed a simple mathematical framework for formal mathematical handling of physical units. In their concept, the mathematical model of special relativistic spacetime is considered to be a triplet $(\mathcal{M}, L, \eta)$, where $\mathcal{M}$ is a four dimensional real affine space (modeling the flat spacetime), L is a one dimensional oriented vector space (modeling the one dimensional vector space of length values), and $\eta : \overset{2}{\vee} T \to \overset{2}{\otimes} L$ is the flat Lorentz signature metric (constant throughout the spacetime), where T is the underlying vector space of $\mathcal{M}$ ("tangent space"). The important idea in that construction is: the field quantities, such as the metric tensor η, are not simply real valued, but they take their values in the tensor powers of the *measure line* L.[4] Due to the one-dimensionality of L, it can be shown that all rational tensor powers of it makes sense as a distinct vector space.[5] Such a setting formalizes the physical expectation that quantities actually have physical dimensions (the metric carries length-square dimension in this case), and that quantities with different physical dimensions cannot be added since they reside in different vector spaces. It is seen that the technique of measure lines is nothing but the precise mathematical formulation of ordinary dimensional analysis in physics.

Such mathematically precise formulation of dimensional analysis, although may seem to be a relatively innocent, almost tautological idea at a first glance, it becomes quite a powerful tool when carried over to a general relativistic framework. Namely, let our spacetime manifold $\mathcal{M}$ be some four dimensional real manifold, and let $L(\mathcal{M})$ be a real oriented vector bundle over $\mathcal{M}$, with one dimensional fiber. The fiber of $L(\mathcal{M})$ over each point of $\mathcal{M}$ shall model the oriented vector space of length values, and the pertinent line bundle shall be called the *measure line bundle*, or *line bundle of lengths*. We do not assume anything more about the line bundle $L(\mathcal{M})$, and in particular, we do not assume that a preferred trivialization is given. Just like proposed in [19, 20], the field quantities shall carry certain tensor powers of $L(\mathcal{M})$. For instance, reflecting on the example of Section 2, we assume that a Dirac field Ψ is a section of the vector bundle $L^{-\frac{3}{2}}(\mathcal{M}) \otimes D(\mathcal{M})$, where $D(\mathcal{M})$ is an ordinary (dimension-free) Dirac bispinor vector bundle. Similarly, one can assume that the spacetime metric g_{ab} is a section of the vector bundle $L^2(\mathcal{M}) \otimes \vee^2 T^*(\mathcal{M})$, and that the Clifford map γ^a is a section of

[4] The term *measure line* was introduced by [19], whereas the same concept is called *scale space* by [20]. Apparently, these two group of authors discovered the pertinent rather useful notion independently.

[5] Indeed, L^* denoting the dual vector space of L, for any non-negative integer n one can set $L^n := \overset{n}{\otimes} L$ and $L^{-n} := \overset{n}{\otimes} L^*$ in order to make sense of any signed integer tensor powers of L. Moreover, due to the one-dimensionality of L, the n-th tensorial root $\sqrt[n]{L}$ of L also can be shown to make sense uniquely [19, 20], via requiring the defining property $\overset{n}{\otimes} \left(\sqrt[n]{L} \right) = L$. As such, all rational tensor powers of a one dimensional oriented vector space L makes sense, and that is a key ingredient for encoding dimensional analysis.

the vector bundle $L^{-1}(\mathcal{M})\otimes T(\mathcal{M})\otimes D(\mathcal{M})\otimes D^{*}(\mathcal{M})$, accordingly. All this differential geometrical formalism encodes the physical idea that the quantities are tagged by physical dimensions, and that the units of measurements can only be a priori defined spacetime pointwise: in order to transport the unit length to different spacetime points, a connection on $L(\mathcal{M})$ must be specified. As such, in order to make sense of the field gauge-covariant gradient $\nabla_a\Psi$, besides the usual gauge-covariant derivation on $D(\mathcal{M})$, a covariant derivation on the line bundle of lengths $L(\mathcal{M})$ must be implicitly assumed within ∇_a, and these two joined canonically on the tensor product space via the usual Leibniz rule. When constructing the Lagrangian as a volume form valued mapping, one should keep in mind that it must be dimension-free (carrying zero tensor powers of $L(\mathcal{M})$), since only pure volume forms may be integrated over a manifold without any further assumptions, so that the action functional can be defined. As such, with the above assignment of dimensions, our example Lagrangian for the Dirac kinetic term Eq.(1) indeed takes its values purely as section of $\overset{\dim(\mathcal{M})}{\wedge} T^{*}(\mathcal{M})$, i.e. as a pure volume form.

First of all, on the above example of the Dirac kinetic term, one may see that an $L(\mathcal{M})\to L(\mathcal{M})$ pointwise vector bundle automorphism is uniquely described by a smooth positive real valued field Ω over the spacetime manifold $\mathcal{M}$, i.e. via a local D(1) gauge transformation. As trivially seen, Eq.(1) is invariant to these, when it acts canonically on all the fields Ψ, γ^a and ∇_a. This means that the Lagrangian is invariant to the pointwise rescaling of the measurement unit of lengths. Moreover, it is also seen in this formulation, that Eq.(1) is in fact invariant to the choice of the connection over $L(\mathcal{M})$. This latter fact means that the Lagrangian is invariant to the choice of any parallel transport rule of measurement units throughout spacetime. This is a slightly stronger symmetry on top of the previous rescaling invariance property. It can be shown, that all the Standard Model kinetic terms, when viewed in such variables, admit these symmetries.

References

[1] W. D. McGlinn, *Problem of combining interaction symmetries and relativistic invariance*, *Phys. Rev. Lett.* **12** (1964) 467.

[2] L. O'Raifeartaigh, *Lorentz invariance and internal symmetry*, *Phys. Rev.* **139** (1965) B1052.

[3] S. Coleman and J. Mandula, *All possible symmetries of the S matrix*, *Phys. Rev.* **159** (1967) 1251.

[4] A. László, *Unification mechanism for gauge and spacetime symmetries*, *J. Phys.* **A50** (2017) 115401.

[5] S. Weinberg, *The quantum theory of fields III.* Cambridge University Press, 2000.

[6] A. László, *A hidden symmetry of conformally invariant Lagrangians*, 1406.5888.

[7] A. L. Onishchik and E. B. Vinberg, *Lie Groups and Algebraic Groups.* Springer, Berlin, 1990.

[8] M. Ise and M. Takeuchi, *Lie groups I-II.* American Mathematical Society, 1991.

[9] S. Ferrara, B. Zumino and J. Wess, *Supergauge multiplets and superfields*, *Phys. Lett.* **B51** (1974) 239.

[10] A. Salam and J. Strathdee, *Super-gauge transformations*, *Nucl. Phys.* **B76** (1974) 477.

[11] S. Ferrara, *Supersymmetry 1–2*, vol. 1, ch. 2, pp. 31–34. Elsevier Science, World Scientific, 1987.

[12] A. László, *Alternative mechanism to SUSY: conservative extensions of the Poincaré group*, in *proceedings of Quantum Theory and Symmetries 10 conference* (V. Dobrev, ed.), Springer, 2018. 1801.03463.

[13] K. Krasnov and R. Percacci, *Gravity and unification: a review*, *Class. Quant. Grav.* **35** (2018) 143001.

[14] R. Penrose and W. Rindler, *Spinors and spacetime 1-2.* Cambridge University Press, 1984.

[15] R. Wald, *General relativity.* University of chicago Press, 1984.

[16] D. Z. Djokovic, *Derivations and automorphisms of exterior algebras*, *Can. J. Math.* **30** (1978) 1336.

[17] I. M. Anderson and C. G. Torre, *The DifferentialGeometry Package (Maple package).* Utah State University Digital Commons, 2016.

[18] S. Anco and R. M. Wald, *Does there exist a sensible quantum theory of an algebra-valued scalar field?*, *Phys. Rev.* **D39** (1989) 2297.

[19] T. Matolcsi, *Spacetime without reference frames.* Hungarian Academy of Sciences Press, 1993.

[20] J. Janyška, M. Modugno and R. Vitolo, *An algebraic approach to physical scales*, *Acta Applicandae Mathematicae* **110** (2010) 1249.