

Transition from Inspiral to Plunge A Complete Near-Extremal Waveform

Ollie Burke,^{1,2,*} Jonathan Gair,^{1,2} and Joan Simón²

¹*Max Planck Institute for Gravitational Physics (Albert Einstein Institute),
Am Mühlenberg 1, Potsdam-Golm 14476, Germany*

²*School of Mathematics, University of Edinburgh, James Clerk Maxwell Building,
Peter Guthrie Tait Road, Edinburgh EH9 3FD, UK*

We extend the Ori and Thorne (OT) procedure to compute the transition from an adiabatic inspiral into a geodesic plunge *for any* spin. Our analysis revisits the validity of the approximations made in OT. In particular, we discuss possible effects coming from eccentricity and non-geodesic past-history of the orbital evolution. We find three different scaling regimes according to whether the mass ratio is much smaller, of the same order or much larger than the near extremal parameter describing how fast the primary black hole rotates. Eccentricity and non-geodesic past-history corrections are always sub-leading, indicating that the quasi-circular approximation applies throughout the transition regime. However, we show that the OT assumption that the energy and angular momentum evolve linearly with proper time must be modified in the near-extremal regime. Using our transition equations, we describe an algorithm to compute the full worldline in proper time for an extreme mass ratio inspiral (EMRI) and the full gravitational waveform in the high spin limit.

I. INTRODUCTION

The LIGO observation of the transient gravitational wave (GW) signal from the collision of two stellar mass black holes [1] in September 2015 spectacularly opened the new field of gravitational wave astronomy. By the end of the O2 observing run in August 2017, the LIGO/Virgo detectors had observed ten binary black hole mergers and a single binary neutron star inspiral [2]. This handful of observations has already had a profound impact on our understanding of the astrophysics of compact objects and ruled out a number of modified theories of gravity [3–7]. During the ongoing O3 observing run new events are being reported at the rate of one per week, so these constraints are rapidly improving. However, the masses of the objects being observed are all in the range of 1–100 M_{\odot} , which is determined by the frequency sensitivity of the instruments [8]. Black holes with much higher masses are expected to exist in the centres of most galaxies [9] and will be even stronger sources of GWs, but these waves will be at millihertz frequencies which are inaccessible to ground-based detectors due to the seismic noise background.

The launch of the Laser Interferometer Space Antennae (LISA) [10], scheduled for 2034, will open the millihertz band from 10^{-4} – 10^{-1} Hz for the first time. Expected sources in this frequency band include massive black hole binaries, cosmic strings and extreme mass ratio inspirals (EMRIs). Detection of these sources, and estimation of their parameters, will rely on the comparison of accurate theoretical models of the expected

gravitational waveforms to the observed data. Building these models for LISA is extremely challenging, in particular for EMRIs, which are expected to have a very rich structure and to be observed for hundreds of thousands of waveform cycles prior to merger with the central object [11]. In this paper we focus on modelling of a particular class of EMRIs, in which the central black hole has very large angular momentum (spin). All of the LIGO observations to date are consistent with zero or small spin [2], but the massive black holes that will be probed by LISA are a different population. These black holes are observed in high accretion states as quasars, and accretion tends to spin the black holes up. Semi-analytic models predict that the typical spins of these objects are $a \gtrsim 0.95$ [12].

The maximum spin of massive black holes is a quantity of fundamental interest for understanding the origin of black holes in the Universe. It was shown by Thorne [13] that the angular momentum of black holes being spun up through thin disc accretion saturates at a limit of $a = 0.998$ where an equilibrium is reached between spin up by accreted material and spin down by captured retrograde photons. Black holes with higher spin could in principle be formed directly in the early Universe and for sufficiently high mass these black holes can retain spins above the Thorne limit for a Hubble time [14]. A direct observation of a system with spin above the Thorne limit would thus have profound implications for our understanding of the origin and growth of black holes. It is therefore important to understand how well observations of EMRIs can constrain the spin of near-extremal black holes and to determine this we first need to build accurate representations of the gravitational waves emitted by such systems.

The near extremal limit is also relevant for more the-

* ollie.burke@aei.mpg.de

oretical considerations. Indeed, as the primary rotates faster, its Hawking’s temperature decreases because the distance between the inner (\tilde{r}_-) and outer (\tilde{r}_+) horizons in Boyer-Lindquist (BL) coordinates reduces according to

$$\tilde{r}_\pm = 1 + \sqrt{1 - a^2} = 1 + \epsilon, \quad (1)$$

where $\tilde{r} = r/M$. The existence of a double pole in the function determining the black hole horizons in this limit is responsible for an enhancement of symmetry in the near horizon geometry of the Kerr black hole [15], a feature that remains true for *any* extremal black hole [16]. This enhancement of symmetry from time translations to the conformal group has allowed several groups to analytically solve the master Teukolsky equation in the presence of the in spiraling probe particle leading to an analytic expression for the energy fluxes carried by the gravitational waves generated by this source [17–25]. This provides a very exciting opportunity where analytic tools developed in the high energy theoretical physics community can provide accurate predictions to generate gravitational waveform templates. Future observations using such templates will be directly testing these theoretical predictions.

It has already been shown that gravitational waveforms emitted by these sources contain unique qualitative features that provide a smoking gun for the existence of near-extremal systems [23]. The amplitude of an EMRI waveform (averaged over a suitable amount of orbits) typically increases linearly in time for moderate spin $a \approx 0.9$. It was shown in [23] that the amplitudes of these signals *dampen* in the high spin limit due to behaviour of the flux close to the horizon. There has been progress in modelling the inspiral from radial infinity to the innermost stable circular orbit (ISCO) [23], by integrating the geodesic equations in the near horizon geometry of the Kerr black hole [26] and exploiting the enhanced set of symmetries to compute the energy fluxes for more source trajectories [25]. However, no one has focused on providing a model which encapsulates the inspiral *and* plunge in the limit of high spins. This is precisely what this paper seeks to do.

In this work, we build such a model for an EMRI comprised of a small compact object of mass μ gravitationally bound to a supermassive Kerr black hole of mass M and study the transition from an adiabatic inspiral into a geodesic plunge *for any* spin of the primary black hole. This transition to plunge was originally discussed by Ori and Thorne (OT) [27] for moderate values of the spin. The technical reason why high spins require a separate discussion is because of the existence of a second independent small parameter competing with the mass ratio $\eta = \mu/M \ll 1$. This new parameter is the near-extremal parameter $\epsilon = \sqrt{1 - a^2}$ encoding the distance of the spin parameter a from its

upper/lower bound, since Kerr black holes have spin parameters $a \in [-1, +1]$. Since the dynamical equations describing the transition depend on the spin, the near extremal limit, i.e. $\epsilon \rightarrow 0$, modifies the original scaling discussed by OT. The transition to plunge for near-extremal EMRIs was previously considered in [28] and our work clarifies and extends those results in a number of ways. We point out the physical interpretation of the mathematical procedure used in that paper, identify a missing term in the near-extremal regime and incorporate recent analytic results for the near-extremal energy flux for the first time.

In this paper, we will first review the treatment of the transition regime given by OT in [27]. We analyse their methodology and approximations and carefully estimate the scaling of terms that are being omitted. In each of [27, 29, 30] the notion of eccentricities and non-circular motion was ignored. We discuss the potential growth of eccentricities before and during the transition regime and find that corrections to our equations due to eccentric motion are sub-leading *for any* spin. We identify three separate transition regimes, each with a slightly different equation of motion: $\eta \ll \epsilon$, $\epsilon \sim \eta$ and $\epsilon \ll \eta$. We then discuss a numerical algorithm to generate full inspiral trajectories in Boyer-Lindquist coordinates, alongside the corresponding evolution of the integrals of motion $E(\tau)$ and $L(\tau)$. Finally, we generate a gravitational waveform that represents a full inspiral of a compact object into a super-massive near-extremal black hole.

This paper is organised as follows. In section II, we review the properties of equatorial and circular orbits in the Kerr black hole and, in section II A, we review and compare the results describing gravitational fluxes emitted by circular EMRIs as a function of the spin. In section III we set-up the master transition equation of motion in general and, in subsection III B, we estimate corrections due to eccentricity and non-geodesic past-history of the orbital evolution. The transition equations of motion in the three different scaling regimes are described in subsections III C, III D and III E respectively. The numerical scheme to integrate our transition equations of motion for the $\epsilon \sim \eta$ regime is presented in section IV A. We describe how to generate a full near-extremal EMRI gravitational waveform encapsulating inspiral and plunge in subsections IV B and IV C. We finish with a summary of our main results in section V.

Notation: Any quantity carrying a tilde refers to a dimensionless quantity in units of the primary mass M , i.e. $\tilde{r} = r/M$, $\tilde{\tau} = \tau/M$, $\tilde{t} = t/M$, $\tilde{E} = E/\mu$ and $\tilde{L} = L/M\mu$. Dotted quantities (eg $\dot{\tilde{E}}$) denote proper time derivatives of that quantity. Finally, expressions $A \sim \mathcal{O}(B)$ or, for brevity, $A \sim B$ stress that both A and B scale in the same way with the small parameters under consideration.

Note added. In the final stages of this work, we became aware of overlapping results that were independently obtained in [31].

II. PRELIMINARIES

In Boyer-Lindquist (BL) coordinates $(\tilde{r}, \phi, \theta, \tilde{t})$, the motion of a point particle with mass μ in a Kerr black hole on the equatorial plane ($\theta = \pi/2$) is given by [32]

$$\left(\frac{d\tilde{r}}{d\tilde{\tau}}\right)^2 = \frac{[\tilde{E}(\tilde{r}^2 + a^2) - a\tilde{L}]^2 - \Delta[(\tilde{L} - a\tilde{E})^2 + \tilde{r}^2]}{\tilde{r}^4} = \tilde{E}^2 - V_{\text{eff}}(\tilde{r}, \tilde{E}, \tilde{L}, a) = G(\tilde{r}, \tilde{E}, \tilde{L}, a) \quad (2)$$

$$\frac{d\phi}{d\tilde{\tau}} = \frac{-\left(a\tilde{E} - \tilde{L}\right) + a(\tilde{E}[\tilde{r}^2 + a^2] - a\tilde{L})/\tilde{r}}{\tilde{r}^2} = \Phi(\tilde{r}, \tilde{E}, \tilde{L}, a) \quad (3)$$

$$\frac{d\tilde{t}}{d\tilde{\tau}} = \frac{-\Delta a(a\tilde{E} - \tilde{L}) + (\tilde{r}^2 + a^2)(\tilde{E}[\tilde{r}^2 + a^2] - a\tilde{L})}{\Delta\tilde{r}^2} = T(\tilde{r}, \tilde{E}, \tilde{L}, a), \quad (4)$$

where the largest root of $\Delta = \tilde{r}^2 - 2\tilde{r} + a^2$ corresponds to the outer horizon \tilde{r}_+

$$\tilde{r}_+ = 1 + \sqrt{1 - a^2},$$

$\tilde{\tau} = \tau/M$ denotes proper time in units of the Kerr black hole mass M and a is the *dimensionless* spin parameter $a \in [0, 1]$.

From hereon, we only consider particles on *prograde* circular orbits, i.e. orbits following the same direction as the rotation of the primary hole. In appendix B, we conduct an analysis on the transition from inspiral to plunge for retrograde orbits.

For an orbit to be circular, the BL radial coordinate \tilde{r} must be constant and to be stable, the latter must be at a minimum of the potential V_{eff} in (2) so that

$$G = \frac{\partial G}{\partial \tilde{r}} = 0, \text{ and } \frac{\partial^2 G}{\partial \tilde{r}^2} \geq 0.$$

These conditions determine the energy \tilde{E} and angular momentum \tilde{L} of these orbits to be [33]

$$\tilde{E} = \frac{1 - 2/\tilde{r} + a/\tilde{r}^{3/2}}{\sqrt{1 - 3/\tilde{r} + 2a/\tilde{r}^{3/2}}}, \quad (5)$$

$$\tilde{L} = \tilde{r}^{1/2} \frac{1 - 2a/\tilde{r}^{3/2} + a^2/\tilde{r}^2}{\sqrt{1 - 3/\tilde{r} + 2a/\tilde{r}^{3/2}}}. \quad (6)$$

Substituting (5)-(6) into (3)-(4) gives rise to

$$\frac{d\phi}{d\tilde{\tau}} = \frac{1}{\tilde{r}^{3/2} \sqrt{1 - 3/\tilde{r} + 2a/\tilde{r}^{3/2}}}, \quad (7)$$

$$\frac{d\tilde{t}}{d\tilde{\tau}} = \frac{1 + a/\tilde{r}^{3/2}}{\sqrt{1 - 3/\tilde{r} + 2a/\tilde{r}^{3/2}}}, \quad (8)$$

whose ratio defines the angular velocity $\tilde{\Omega}$ of the particle

$$\frac{d\phi}{d\tilde{t}} = \tilde{\Omega} = (\tilde{r}^{3/2} + a)^{-1}. \quad (9)$$

Equatorial circular orbits are also known to satisfy the identity [34]

$$\frac{\partial G}{\partial \tilde{E}}(\tilde{r})\tilde{\Omega}(\tilde{r}) + \frac{\partial G}{\partial \tilde{L}}(\tilde{r}) = 0, \quad (10)$$

where we stress the equality holds for any circular orbit labelled by $(\tilde{r}, \tilde{E}, \tilde{L})$. Differentiating (10) with respect to \tilde{r} , we can derive further equalities satisfied for any such orbits. The ones below

$$\frac{\partial^2 G}{\partial \tilde{r} \partial \tilde{E}} \tilde{\Omega} + \frac{\partial^2 G}{\partial \tilde{r} \partial \tilde{L}} = -\frac{\partial \tilde{\Omega}}{\partial \tilde{r}} \frac{\partial G}{\partial \tilde{E}}, \quad (11)$$

$$-\frac{1}{2} \left(\frac{\partial^3 G}{\partial \tilde{r}^2 \partial \tilde{E}} \tilde{\Omega} + \frac{\partial^3 G}{\partial \tilde{r}^2 \partial \tilde{L}} \right) = \frac{\partial \tilde{\Omega}}{\partial \tilde{r}} \frac{\partial^2 G}{\partial \tilde{r} \partial \tilde{E}} + \frac{1}{2} \frac{\partial^2 \tilde{\Omega}}{\partial \tilde{r}^2} \frac{\partial G}{\partial \tilde{E}} \quad (12)$$

will play a role in our analysis later on.

The *innermost stable circular orbit* (ISCO) is the marginal circular stable orbit satisfying

$$G|_{\text{isco}} = \frac{\partial G}{\partial \tilde{r}} \Big|_{\text{isco}} = \frac{\partial^2 G}{\partial \tilde{r}^2} \Big|_{\text{isco}} = 0. \quad (13)$$

The last equality, describing marginality, allows to solve for its radius as a function of the spin [35]

$$\tilde{r}_{\text{isco}} = 3 + Z_2 - [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2} \quad (14)$$

$$Z_1 = 1 + (1 - a^2)^{1/3} [(1 + a)^{1/3} + (1 - a)^{1/3}]$$

$$Z_2 = (3a^2 + Z_1^2)^{1/2}.$$

This is the last radii before plunging into the horizon occurs. In appendix A, we derive general formulas for (13) and higher order derivatives of $G(\tilde{r}, \tilde{E}, \tilde{L})$, which are valid for any spin a , when evaluated at ISCO that will be relevant in the rest of this work.

For *near-extremal* Kerr black holes, it is natural to introduce the near extremal parameter

$$\epsilon = \sqrt{1 - a^2} \quad \text{for } \epsilon \ll 1, \quad (15)$$

to mathematically capture the large spin limit $a \rightarrow 1$. For these black holes, the ISCO location can be expanded in ϵ

$$\tilde{r}_{\text{isco}} = 1 + 2^{1/3} \epsilon^{2/3} + \mathcal{O}(\epsilon^{4/3}) \quad (16)$$

and the physical parameters of this marginal orbit re-

duce to

$$\tilde{E}_{\text{isco}} \rightarrow \frac{1}{\sqrt{3}} \left(1 + 2^{1/3} \epsilon^{2/3} \right) + \mathcal{O}(\epsilon^{4/3}), \quad (17)$$

$$\tilde{L}_{\text{isco}} \rightarrow \frac{2}{\sqrt{3}} \left(1 + 2^{1/3} \epsilon^{2/3} \right) + \mathcal{O}(\epsilon^{4/3}), \quad (18)$$

$$\tilde{\Omega}_{\text{isco}} \rightarrow \frac{1}{2} \left(1 - \frac{3}{2^{5/3}} \epsilon^{2/3} \right) + \mathcal{O}(\epsilon^{4/3}). \quad (19)$$

A. Gravitational Wave Flux

The motion of the particle in the Kerr black hole generates gravitational waves carrying energy and angular momentum. Since the total energy and angular momentum are conserved, the rates of change $\dot{\tilde{E}}$ and $\dot{\tilde{L}}$ are entirely determined by the gravitational wave fluxes $\dot{\tilde{E}}_{\text{GW}}$ and $\dot{\tilde{L}}_{\text{GW}}$ through the balance law

$$\dot{\tilde{E}} = -\dot{\tilde{E}}_{\text{GW}}, \quad \dot{\tilde{L}} = -\dot{\tilde{L}}_{\text{GW}}. \quad (20)$$

The gravitational wave fluxes are determined by solving the Teukolsky equation in the presence of the particle source [36–41].

In [42], Finn and Thorne (F&T) parametrise the energy flux as the (Peters and Mathews [43]) leading order post newtonian correction with an extra general relativistic correction $\dot{\mathcal{E}}$ factor

$$\frac{d\tilde{E}_{\text{GW}}}{dt} = \frac{32}{5} \eta \tilde{\Omega}^{10/3} \dot{\mathcal{E}}(\tilde{r}). \quad (21)$$

These fluxes are spin dependent and are typically computed through numerical means. See the tables in [42] for some of the values of these relativistic corrections.

As we increase the spin of the black hole, the two roots $\tilde{r}_{\pm} = 1 \pm \epsilon$ of the function Δ determining the outer and inner horizons of the rotating black hole coincide in the extremal limit $\epsilon = 0$. In this limit, the geometry close to the horizon of the black hole, which can be isolated using the change of coordinates

$$\tilde{r} - \tilde{r}_+ = \lambda \rho, \quad \tilde{t} = \frac{T}{\lambda}, \quad \tilde{\phi} = \phi + \frac{\tilde{t}}{2\lambda} \quad (22)$$

has an enhancement of symmetry from $\mathbb{R} \times \text{U}(1)$, i.e. time translations and rotational symmetry, to $\text{SL}(2, \mathbb{R}) \times \text{U}(1)$. The resulting near horizon geometry is warped AdS_2 over a 2-sphere. The enhanced $\text{SL}(2, \mathbb{R})$, the isometry group of AdS_2 , includes the scaling symmetry $\rho \rightarrow c\rho$ and $T \rightarrow T/c$. This was already observed in the original work [15] and it is true for any extremal black hole [16].

Larger symmetry in physics implies larger kinematic constraints which can provide further analytic control over the given problem, in this case, the calculation of

a	$\dot{\tilde{E}}_{\text{Exact}}/\eta$	$\dot{\tilde{E}}_{\text{NHEK}}/\eta$	$ \dot{\tilde{E}}_{\text{NHEK}} - \dot{\tilde{E}}_{\text{Exact}} /\eta$
$1 - 10^{-5}$	0.0264197	0.0261523	0.0002674
$1 - 10^{-6}$	0.0129344	0.0125200	0.0004143
$1 - 10^{-7}$	0.0061516	0.0059484	0.0002031
$1 - 10^{-8}$	0.0028875	0.0028082	0.0000793
$1 - 10^{-9}$	0.0013472	0.0013193	0.0000280
$1 - 10^{-10}$	0.0006273	0.0006176	0.0000097
$1 - 10^{-11}$	0.0002915	0.0002883	0.0000031
$1 - 10^{-12}$	0.0001354	0.0001344	0.0000009

Table I: In this table we are comparing the NHEK flux (23) with exact flux data found in the BHPT. We fix the radial coordinate at $\tilde{r} = \tilde{r}_{\text{isco}}$ and change the spin parameter a .

the gravitational wave fluxes (20). It is precisely the emergence of this conformal group ($\text{SL}(2, \mathbb{R})$) and the use of asymptotic expansion matching methods that allowed to find analytic expressions for these energy and angular momentum fluxes for equatorial circular orbits close to the horizon [17–20, 22, 23, 25]. This body of work led to the simple relationship for the flux given in [23]

$$\frac{d\tilde{E}_{\text{GW}}}{dt} \approx \eta (\tilde{C}_H + \tilde{C}_\infty) \frac{\tilde{r} - \tilde{r}_+}{\tilde{r}_+}. \quad (23)$$

The quantities $\tilde{C}_H = 0.987$ and $\tilde{C}_\infty = -0.133$ are constants representing how much wave emission goes towards the horizon and infinity respectively. These constants are computed numerically in equations (76) and (77) in [20] where the first $l \leq m = 30$ modes are summed over.

The flux (23) is only reliable when working with near extremal black holes and interested in near horizon physics. This fact can be checked by comparing the exact fluxes (21), using exact results found in the black hole perturbation toolkit (BHPT), with the near extremal approximation (23). This comparison is shown in figure 1. Fixing the radial coordinate to $\tilde{r} = \tilde{r}_{\text{isco}}$ and varying the spin parameter a , we observe in Table (I) that as $a \rightarrow 1$, the NHEK flux (23) converges towards the exact value computed using the BHPT. Furthermore, fixing the spin parameter to $a = 1 - 10^{-9}$, as in figure (1), the NHEK flux (23) provides a nearly-perfect agreement up to a coordinate radii $\tilde{r} \approx 1.012$. The reason for the (extremely small) discrepancy at the ISCO is because Eq.(23) is only valid for $\epsilon \rightarrow 0$ and we consider $\epsilon \approx 10^{-5}$. Thus we can use (21) to build a trajectory throughout the adiabatic inspiral regime. Then, as we near the ISCO, we can use the powerful analytic result given by Eq.(23). Using Eq.(23) allows for a more analytic treatment of the analysis of the transition regime.

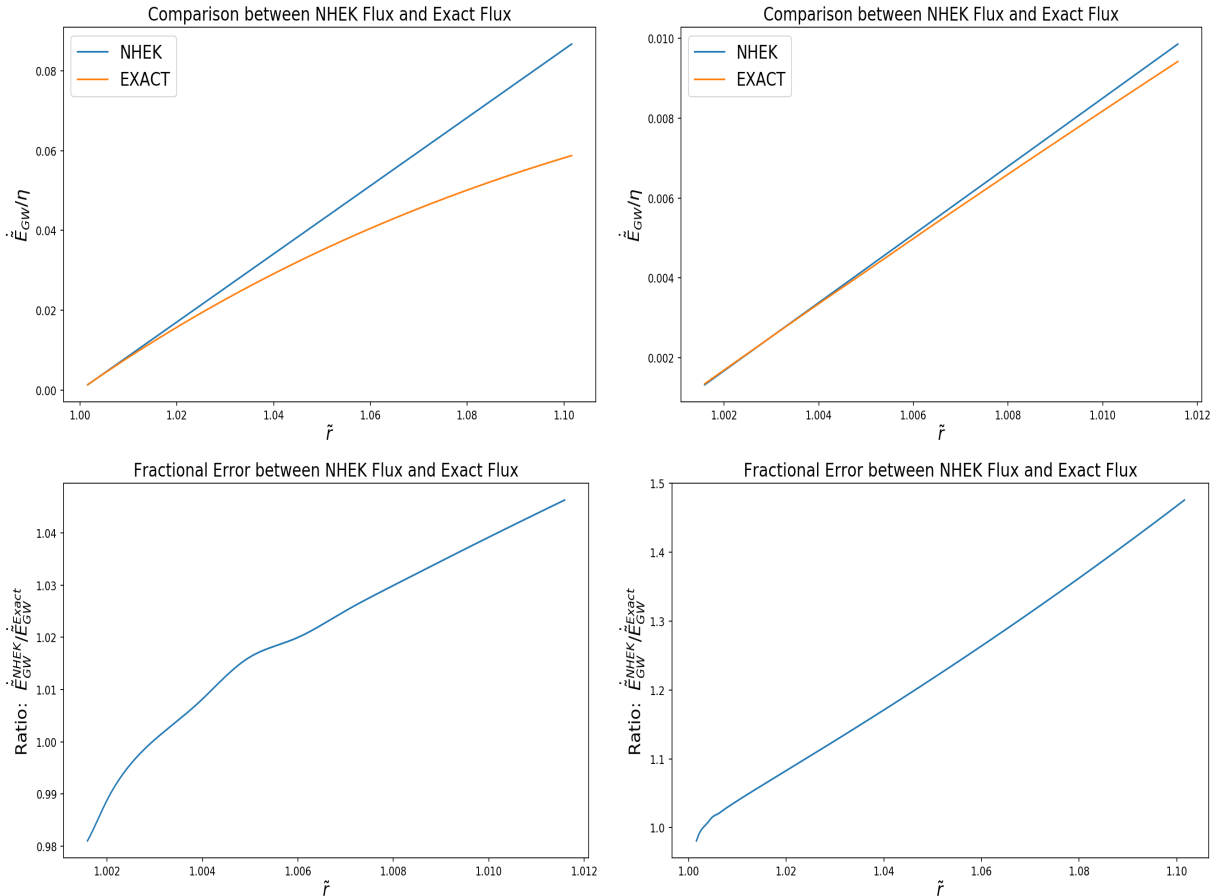


Figure 1: These plots show the deviation between using the exact results for the flux (21) and the near extremal approximation given in (23). Notice that, to keep the error $< 5\%$, we require $\tilde{r} \lesssim 1.01$. For each of these plots, we used a spin parameter $a = 1 - 10^{-9}$.

III. THE GENERAL MASTER EQUATION

In this section we revisit the earlier work by OT [27] describing how a small body following an initial equatorial circular orbit around the large black hole inspirals and eventually transitions into a plunging trajectory falling into the black hole. We do this for arbitrary black hole spins, paying special attention to near-extremal ones.

We want to understand the evolution of the orbit as it approaches the ISCO. To do this we will expand the equations of motion about this point. If we take the radial geodesic equation, Eq. (2), and differentiate it we obtain

$$\frac{d^2\tilde{r}}{d\tilde{\tau}^2} - \frac{1}{2} \frac{\partial G}{\partial \tilde{r}} = \frac{1}{2\dot{\tilde{r}}} \left(\dot{\tilde{E}} \frac{\partial G}{\partial \tilde{E}} + \dot{\tilde{L}} \frac{\partial G}{\partial \tilde{r}} \right). \quad (24)$$

The terms on the left hand side are the usual equations for geodesic motion. The terms on the right hand side are corrections arising due to the evolution of the orbit

under radiation reaction. The fluxes $\dot{\tilde{E}}$ and $\dot{\tilde{L}}$ are the temporal and azimuthal components of the radiation reaction self-force acting on the body. Due to conservation of the velocity norm, $u_\alpha u^\alpha = -1$, the self-force must be orthogonal to the motion, $u_\alpha f^\alpha = 0$, which demonstrates that the right hand side of the equation is in fact the radial component of the self-force, f^r . This equivalence is shown explicitly in Appendix C. For circular orbits, as considered by OT, the fluxes of energy and angular momentum are related by [34, 44]

$$\dot{\tilde{E}} = \tilde{\Omega}(\tilde{r}) \dot{\tilde{L}}. \quad (25)$$

Imposing this condition makes the term on the right hand side of Eq. (24) vanish at linear order in η . OT argued that corrections to this cancellation were negligible and set it to zero throughout their analysis. We will ultimately do the same, although we will use the expression above to deduce the scaling of such corrections and to carefully check that they are sub-dominant. We

note that to include these terms would require detailed knowledge of the self-force throughout the orbit, as the oscillatory components of the fluxes are required to ensure appropriate cancellation at turning points, not just the secular part.

To evolve the orbit, OT used the circular flux relationship and additionally assumed that the energy \tilde{E} and angular momenta \tilde{L} evolve linearly in proper time $\tilde{\tau}$ throughout the transition regime

$$\begin{aligned}\tilde{E} - \tilde{E}_{\text{isco}} &\approx \dot{\tilde{E}}_{\text{isco}}(\tilde{\tau} - \tilde{\tau}_{\text{isco}}), \\ \tilde{L} - \tilde{L}_{\text{isco}} &\approx \dot{\tilde{L}}_{\text{isco}}(\tilde{\tau} - \tilde{\tau}_{\text{isco}}).\end{aligned}\quad (26)$$

In our analysis of the transition, we will *not* assume (26). In other words, we will take into account corrections to (25) due to the non-geodesic past-history of the orbital evolution. Physically, this means we will keep track of the evolution of $\tilde{E} - \tilde{E}_{\text{isco}} - \tilde{\Omega}_{\text{isco}}(\tilde{L} - \tilde{L}_{\text{isco}})$, as also considered in [28].

A. Transition Equation - Generalities

OT proposed to analyse the transition to the plunging geodesic by expanding (24) around the ISCO trajectory

$(\tilde{r}_{\text{isco}}, \tilde{E}_{\text{isco}}, \tilde{L}_{\text{isco}})$, since the latter provides the natural starting point for the plunging trajectory for equatorial and circular orbits. It is physically natural to introduce the new variables

$$\begin{aligned}\tilde{E} - \tilde{E}_{\text{isco}} &= \tilde{\Omega}_{\text{isco}}\delta E \\ \tilde{L} - \tilde{L}_{\text{isco}} &= \delta L \\ \tilde{r} - \tilde{r}_{\text{isco}} &= R\end{aligned}\quad (27)$$

to study the inspiral evolution of the small body perturbatively around the primary. The presence of $\tilde{\Omega}_{\text{isco}}$ is for technical convenience.

Instead of expanding (24), we find it more convenient to expand (2). Our conclusions do not depend on this choice. The latter is given by

$$\begin{aligned}\left(\frac{d\tilde{r}}{d\tilde{\tau}}\right)^2 &= G(\tilde{r}_{\text{isco}}, \tilde{E}_{\text{isco}}, \tilde{L}_{\text{isco}}) + \sum_{i=1}^{\infty} \frac{1}{i!} \left. \frac{\partial^i G}{\partial \tilde{r}^i} \right|_{\text{isco}} (\tilde{r} - \tilde{r}_{\text{isco}})^i \\ &\quad + \sum_{i=0}^{\infty} \frac{1}{i!} \left(\left. \frac{\partial^{i+1} G}{\partial \tilde{r}^i \partial \tilde{E}} \right|_{\text{isco}} (\tilde{E} - \tilde{E}_{\text{isco}}) + \left. \frac{\partial^{i+1} G}{\partial \tilde{r}^i \partial \tilde{L}} \right|_{\text{isco}} (\tilde{L} - \tilde{L}_{\text{isco}}) \right) (\tilde{r} - \tilde{r}_{\text{isco}})^i \\ &\quad + \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{i!} \left(\left. \frac{\partial^{i+2} G}{\partial \tilde{r}^i \partial \tilde{E}^2} \right|_{\text{isco}} (\tilde{E} - \tilde{E}_{\text{isco}})^2 + 2 \left. \frac{\partial^{i+2} G}{\partial \tilde{r}^i \partial \tilde{L} \partial \tilde{E}} \right|_{\text{isco}} (\tilde{E} - \tilde{E}_{\text{isco}})(\tilde{L} - \tilde{L}_{\text{isco}}) + \left. \frac{\partial^{i+2} G}{\partial \tilde{r}^i \partial \tilde{L}^2} \right|_{\text{isco}} (\tilde{L} - \tilde{L}_{\text{isco}})^2 \right) (\tilde{r} - \tilde{r}_{\text{isco}})^i.\end{aligned}\quad (28)$$

Since $G(\tilde{r}, \tilde{E}, \tilde{L})$ is quadratic in \tilde{E} and \tilde{L} , we already ignored the derivatives

$$\frac{\partial^n G}{\partial \tilde{E}^n} = \frac{\partial^n G}{\partial \tilde{L}^{n-k} \partial \tilde{E}^k} = \frac{\partial^n G}{\partial \tilde{E}^{n-k} \partial \tilde{L}^k} = \frac{\partial^n G}{\partial \tilde{L}^n} = 0 \quad \text{for } n \geq 3 \text{ and } k < n. \quad (29)$$

Plugging the perturbative variables (27), using the definition of the coefficients (A8) and the results in (A9)-(A11), one can rewrite the general transition equation as

$$\left(\frac{dR}{d\tilde{\tau}}\right)^2 = \sum_{n=3}^{\infty} \frac{1}{n!} A_n R^n + \delta L \sum_{n=1}^{\infty} \frac{1}{n!} B_n R^n + \frac{\delta L^2}{2} \sum_{n=0}^{\infty} \frac{1}{n!} C_n R^n + \Gamma_{\odot}, \quad (30)$$

where Γ_{\odot} is defined by

$$\begin{aligned}\Gamma_{\odot} &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\Omega}_{\text{isco}} (\delta E - \delta L) \left(2 \left. \frac{\partial^{n+1} G}{\partial \tilde{r}^n \partial \tilde{E}} \right|_{\text{isco}} + 2 \left(\left. \frac{\partial^{n+2} G}{\partial \tilde{r}^n \partial \tilde{E} \partial \tilde{L}} \right|_{\text{isco}} + \tilde{\Omega}_{\text{isco}} \left. \frac{\partial^{n+2} G}{\partial \tilde{r}^n \partial \tilde{E}^2} \right|_{\text{isco}} \right) \delta L \right. \\ &\quad \left. + \tilde{\Omega}_{\text{isco}} \left. \frac{\partial^{n+2} G}{\partial \tilde{r}^n \partial \tilde{E}^2} \right|_{\text{isco}} (\delta E - \delta L) \right) R^n.\end{aligned}\quad (31)$$

Notice $\Gamma_{\odot} \propto \delta E - \delta L$. Hence, it encodes the deviations from the OT approximation (26).

The time evolution of $\delta E - \delta L$ near \tilde{r}_{isco} is controlled by the fluxes and the angular velocity. Throughout a quasi-circular inspiral far from ISCO, the compact object inspirals on a sequence of circular geodesics defined by the constants of motion $\tilde{E}(\tilde{r}_{\text{circ}}) = \tilde{E}_{\text{circ}}$ and $\tilde{L}(\tilde{r}_{\text{circ}}) = \tilde{L}_{\text{circ}}$, as given in Eq (5) and Eq.(6) respectively. The evolution of the constants of motion is linked through Eq. (25) above, which simply states that circular geodesics evolve into circular geodesics. It can be shown that solutions to the Teukolsky equation for circular orbits obey this condition [34, 45]. For circular evolutions we therefore see that

$$\begin{aligned} \frac{d}{d\tilde{\tau}}(\delta E - \delta L) &= \tilde{\Omega}_{\text{isco}}^{-1} \frac{d\tilde{E}}{d\tilde{\tau}} - \frac{d\tilde{L}}{d\tilde{\tau}} \\ &= (\tilde{\Omega}_{\text{isco}}^{-1} \tilde{\Omega}(\tilde{r}) - 1) \frac{d\tilde{L}}{d\tilde{\tau}} \\ &\approx - \left. \frac{\partial \log \tilde{\Omega}}{\partial \tilde{r}} \right|_{\text{isco}} \eta \kappa R \\ &\sim \eta R, \end{aligned} \quad (32)$$

where we expanded $\tilde{\Omega}(\tilde{r})$ to first order in R and approximated $\dot{\tilde{L}} \approx \dot{\tilde{L}}_0 = -\kappa$ for κ constant defined by

$$\kappa = \left(\tilde{\Omega}^{-1} \frac{d\tilde{t}}{d\tilde{\tau}} \frac{d\tilde{E}_{\text{GW}}}{d\tilde{t}} \right)_{\text{isco}} \sim \mathcal{O}(1) \quad \text{for } a \in [0, 1]. \quad (33)$$

We deduce that $\delta E - \delta L \sim \eta R \tilde{\tau}$ for circular inspirals. We shall see that these corrections are indeed sub-leading in the regime considered by OT [27]. However, they will not be negligible for near-extremal black holes.

Additional corrections arise from eccentricity and from higher-order corrections to the self-force sourced by the non-geodesic past-history of the evolution. The resulting corrections to the rate of change of Γ_{\odot} scale like ηe^2 (where e is the eccentricity) and $\eta \dot{\tilde{r}}$ respectively. We expect to be able to ignore eccentricity if $e < R$ and in that regime these corrections scale like ηR^2 and $\eta \dot{R} \eta R/T$, compared to the ηR of the leading term. The former corrections are sub-dominant since R is a small quantity, and the latter because T is a large quantity. These scalings will be discussed in more detail in the next section.

B. Corrections arising from deviations from adiabatic nearly-circular inspiral

Before we analyse the different transition regimes, we discuss possible physical effects giving rise to corrections to our set-up. In our analysis, we assume the orbit is nearly circular when it reaches the transition regime. For an exactly circular adiabatic inspiral, the

fluxes of energy and angular momentum are related via (25). Two possible corrections to this relationship come from *eccentricity* and from the *non-geodesic past-history of the orbital evolution*.

Eccentricity can lead to corrections to the master equation which we will discuss further below, but eccentricity corrections to the fluxes tend to be suppressed during the transition regime. This is because the transition corresponds to the orbit passing over the local maximum of the effective potential given by Eq.(2). The radial velocity throughout the transition regime is therefore always small, while the angular velocity remains $\mathcal{O}(1)$. Hence the orbit looks very much like a circular orbit, even if it is technically eccentric or even plunging. For nearly-circular transitions, the orbit is passing over a point of inflection of the effective potential and corrections to this approximately-circular assumption are even smaller.

Corrections from *non-geodesic past-history* enter because the self-force acting on the small object at a particular time is generated by the intersection of the particle world line with gravitational perturbations generated by the orbital motion in the immediate past [46]. The self-force acting on the orbit when it is at a particular radius will therefore have corrections that depend on how far, in radius, the orbit has moved over the relevant past-history. The latter is determined by the dominant, azimuthal, timescale, and is an $\mathcal{O}(1)$ quantity. The orbital radius therefore changes by an amount of $\mathcal{O}(\dot{\tilde{r}})$ over the relevant past-history and the non-geodesic past-history corrections thus scale like $\eta \dot{\tilde{r}}$. In the adiabatic inspiral phase, these corrections are $\mathcal{O}(\eta^2)$ and form part of the second-order component of the self-force. However, in the transition phase these corrections can be larger.

We have argued above that eccentricity corrections to the fluxes should be suppressed in the transition regime. We now make this more concrete. Eccentricity corrections to the fluxes enter as $\mathcal{O}(\eta e^2)$, since corrections to the orbit at linear order in eccentricity are oscillatory and average to zero over a complete orbit [34]. If these corrections are to be small relative to the non-geodesic past-history corrections, we need $e^2 < \dot{\tilde{r}}$. In the transition zone we will see that time scales like $\delta \tilde{r}^{-1/2}$, where $\delta \tilde{r}$ is the distance from the ISCO. The scaling of $\dot{\tilde{r}}$ is therefore the same as that of $\delta \tilde{r}^{3/2}$, and so the constraint we obtain on eccentricity is $e < \delta \tilde{r}^{3/4}$. However, there is also a geometric constraint, which is that the variation in the orbital radius due to eccentricity should be small compared to the variation due to radiation reaction through the transition zone. The latter is the scaling of $\delta \tilde{r}$, while the former is a quantity of $\mathcal{O}(e)$, so we deduce an additional constraint $e < \delta \tilde{r} < \delta \tilde{r}^{3/4}$, the latter inequality following from the fact that $\delta \tilde{r}$ is a small quantity throughout the transition. We deduce

that the geometrical constraint is always stronger than the flux-correction constraint, and so it is this that we must verify in deriving the transition equations of motion.

Eccentricity during the transition can arise either from the presence of residual eccentricity prior to the start of the transition zone, or due to the excitation of eccentricity during the transition. The latter manifests itself as additional terms in the master equation, the existence of which we will check for carefully in our analysis. To understand the former, we need to analyse the growth of eccentricity during the adiabatic inspiral. We will assume that at the beginning of the inspiral the orbit is nearly circular. It was shown in [34] that, for small eccentricity, the evolution of eccentricity under radiation reaction takes the form $\dot{e} = f(\tilde{r}_0)e$, where \tilde{r}_0 is the mean orbital radius and e is an eccentricity defined such that the orbital apoapse is at $\tilde{r} = \tilde{r}_0(1 + e)$. For large \tilde{r}_0 , $f(\tilde{r}_0) < 0$ and so the eccentricity decreases. In this regime any small eccentricity that is excited by small perturbations arising due to inspiral evolution or other effects is damped away and does not grow. However, for all spins $a < 1$, as the innermost stable circular orbit (or separatrix) is approached the sign of $f(\tilde{r}_0)$ changes and is greater than zero in the vicinity of the ISCO. This means that orbits near to the separatrix are unstable to eccentricity growth. We would therefore expect any eccentricity that is excited to begin to grow.

Denoting $\tilde{v}^2 = 1/\tilde{r}_0$, Kennefick [34] showed that the evolution of the orbital parameters, for small eccentricity, was governed by equations of the form

$$\frac{\dot{\tilde{r}}_0}{\tilde{r}_0} = -\frac{2(1 - 3\tilde{v}^2 + 2a\tilde{v}^3)^{3/2}}{\tilde{v}^2(1 - 6\tilde{v}^2 + 8a\tilde{v}^3 - 3a^2\tilde{v}^4)} \dot{E}_0 \quad (34)$$

$$\frac{\dot{e}}{e} = \frac{1}{e^2} \left(\dot{E}_0 - \Omega(\tilde{v})\dot{L}_0 \right) - j(\tilde{v}) \left[\Gamma - h(\tilde{v})\dot{E}_0 \right] \quad (35)$$

where

$$j(\tilde{v}) = \frac{(1 + a\tilde{v}^3)(1 - 2\tilde{v}^2 + a^2\tilde{v}^4)(1 - 3\tilde{v}^2 + 2a\tilde{v}^3)^{1/2}}{\tilde{v}^2(1 - 6\tilde{v}^2 + 8a\tilde{v}^3 - 3a^2\tilde{v}^4)}$$

$$h(\tilde{v}) = \frac{\mathcal{H}(\tilde{v})(1 + a\tilde{v}^3)^{-1}(1 - 2\tilde{v}^2 + a^2\tilde{v}^4)^{-2}}{2(1 - 6\tilde{v}^2 + 8a\tilde{v}^3 - 3a^2\tilde{v}^4)}$$

$$\begin{aligned} \mathcal{H}(\tilde{v}) = & 1 - 12\tilde{v}^2 + 66\tilde{v}^4 - 108\tilde{v}^6 + a\tilde{v}^3 + 8a^2\tilde{v}^4 \\ & - 72a\tilde{v}^5 - 20a^2\tilde{v}^6 + 204a\tilde{v}^7 + 38a^3\tilde{v}^7 - 42a^2\tilde{v}^8 \\ & - 9a^4\tilde{v}^8 - 144a^3\tilde{v}^9 + 116a^4\tilde{v}^{10} - 27a^5\tilde{v}^{11}. \end{aligned}$$

The quantities Γ and \dot{E}_0 are components of the self-force, which can be evaluated by solution of the Teukolsky equation. The quantity Γ is given explicitly in [34] and the quantity \dot{E}_0 is the energy flux given in (20). Numerical calculations show that these are finite quantities of $\mathcal{O}(1)$ throughout parameter space. The first term in the eccentricity evolution equation vanishes for evolu-

tion driven by gravitational radiation reaction, while the quantity $h(\tilde{v})$ is singular at the ISCO. Therefore, close to ISCO the eccentricity evolution takes the form

$$\begin{aligned} \frac{\dot{e}}{e} & \approx j(\tilde{v})h(\tilde{v})\dot{E}_0 \\ \Rightarrow r_0 \frac{d \ln e}{dr_0} & \approx -\frac{\tilde{v}^2(1 - 6\tilde{v}^2 + 8a\tilde{v}^3 - 3a^2\tilde{v}^4)j(\tilde{v})h(\tilde{v})}{2(1 - 3\tilde{v}^2 + 2a\tilde{v}^3)^{3/2}}. \end{aligned} \quad (36)$$

For non-extremal spin, both $j(\tilde{v})$ and $h(\tilde{v})$ have simple poles at $\tilde{r} = \tilde{r}_{\text{isco}}$ and there is a simple zero in the term $(1 - 6\tilde{v}^2 + 8a\tilde{v}^3 - 3a^2\tilde{v}^4)$ in the numerator. Therefore as ISCO is approached the eccentricity evolves as

$$\frac{d \ln e}{d\delta\tilde{r}} \approx -\frac{k(a)}{\delta\tilde{r}} \quad \Rightarrow \quad e = e_0 \left(\frac{\delta\tilde{r}_0}{\delta\tilde{r}} \right)^{k(a)} \quad (37)$$

in which $\delta\tilde{r} = \tilde{r} - \tilde{r}_{\text{isco}}$ as before, and e_0 denotes the eccentricity when $\delta\tilde{r} = \delta\tilde{r}_{\text{isco}}$. The exponent $k(a)$ is given by

$$\begin{aligned} k(a) & = \mathcal{H}(\tilde{v}_{\text{isco}})/\mathcal{D}(\tilde{v}_{\text{isco}}) \\ \text{where } \mathcal{D}(\tilde{v}) & = 2\tilde{v}^2(1 - 2\tilde{v}^2 + a\tilde{v}^4) \\ & \quad \times (12\tilde{v} - 24a\tilde{v}^2 + 12a^2\tilde{v}^3) \\ & \quad \times (1 - 3\tilde{v}^2 + 2a\tilde{v}^3) \end{aligned} \quad (38)$$

where $\tilde{v}_{\text{isco}}^2 = 1/\tilde{r}_{\text{isco}}$. This can be found to be constant and equal to 0.25 for $a < 1$. The behaviour for near-extremal black holes is slightly different, which we will discuss further below.

For extremal black holes the various factors in the expression for $d \ln e / d\tilde{r}_0$ have repeated roots at the ISCO. To understand the behaviour for near-extremal black holes we therefore need to do an expansion in both δr and ϵ . This takes the form

$$\frac{d \ln e}{d\tilde{r}_0} = \frac{a_0\epsilon^4 + a_1\epsilon^4\delta\tilde{r} + \sum_{i=2}^5 a_i\epsilon^{\frac{2(6-i)}{3}}\delta\tilde{r}^i + a_6\delta\tilde{r}^6 + \dots}{\sum_{i=1}^6 b_i\epsilon^{\frac{2(7-i)}{3}}\delta\tilde{r}^i + b_7\delta\tilde{r}^7 + \dots} \quad (39)$$

The terms omitted from both the numerator and denominator above are $\mathcal{O}(1)$ in ϵ . The ratio $a_0/b_1 = -1/4$, agreeing with the result for $k(a)$ found above. However, for $\epsilon \ll \delta r$, the behaviour is not dominated by this term, but by the terms from a_6 in the numerator and from b_7 in the denominator. The leading order behaviour in this regime is therefore

$$\frac{d \ln e}{d\tilde{r}_0} = \frac{a_6}{b_7} \frac{1}{\delta\tilde{r}}. \quad (40)$$

This is also exponential, but we find the ratio $a_6/b_7 = 3/2$, i.e., it is greater than zero and therefore the eccentricity *decreases* exponentially until we reach the regime $\delta\tilde{r} \sim \epsilon$. This is the statement that the critical curve,

where the sign of the eccentricity evolution changes, is in the near-horizon region, which is consistent with results in [40]. We conclude that for near-extremal black holes, eccentricity can only grow once the in spiralling object is already very close to the ISCO, which is typically already inside the transition zone.

To complete this discussion we need to determine the scaling of the initial eccentricity e_0 . If the orbit is truly circular then the eccentricity remains zero, so there must be some mechanism to excite an initial eccentricity which can then grow. Eccentricity can be excited by other physical processes, such as the presence of perturbing material, e.g., dust, or gravitational interactions with third bodies. Those processes are important, but in the pure-vacuum case eccentricity could still in principle be excited by the evolution under radiation reaction. We argued earlier that corrections to the fluxes scale like $\eta\dot{r}$ which is η^2 during the adiabatic inspiral. These corrections mean that the first term in Eq. (35) is no longer exactly zero. Setting that term to η^2 we find an evolution equation of the form $de^2/d\tilde{t} \sim \eta^2$. After a few orbits the eccentricity is then $\mathcal{O}(\eta)$. This eccentricity induced by second order corrections to the evolution is damped by the process described above, until we reach the critical curve where it grows, eventually exponentially near the ISCO. This suggests appropriate initial conditions are $e_0 \sim \eta$ and $\delta\tilde{r}_0 \sim \mathcal{O}(1)$ [47] We note that this mechanism could also excite eccentricity during the transition zone itself, but this would be of order $e^2 \sim \eta\dot{r}$ and hence no larger than the non-geodesic past-history corrections described above. If eccentricity grew coherently throughout the transition zone, the eccentricity induced by this process would be no larger than $e^2 \sim \eta\dot{r}T$, where T is the time elapsed through the transition zone, which is typically smaller than the eccentricity grown during adiabatic inspiral prior to the start of the transition zone.

To summarise, we expect corrections to the evolution equations that arise from higher-order terms in the flux to scale like $\eta\dot{r}$, and we expect corrections from eccentricity to scale like $e^2 \sim \eta^2\delta\tilde{r}^{-2k}$. These eccentricity corrections will be important when $e > \delta\tilde{r}$, which implies $\delta\tilde{r} < \eta^{1/(1+k)}$. In the analysis that follows we will evaluate the scaling of these terms and show that they are sub-dominant for inspirals into near-extremal black holes.

C. Ori and Thorne regime

Consider non-extremal black holes, i.e. rotating black holes where the extremality parameter ϵ is not close to zero so that $\eta \ll \epsilon$. In this regime of spins and according to the discussion below (A13)-(A18), all the coefficients controlling the general transition equation (30) and (31)

are $\mathcal{O}(1)$. This is the regime originally discussed in [27].

Omitting coefficients of order one, the dominant contributions to the transition equation are

$$\begin{aligned} \left(\frac{dR}{d\tilde{\tau}}\right)^2 &\sim R^3 + R\delta L + \Gamma_{\odot} \\ \Gamma_{\odot} &\sim \delta E - \delta L, \end{aligned} \quad (41)$$

where we also omitted any further terms from (30) and (31) since they are subleading. Looking for a scaling solution $R \sim \eta^p$ and $\tilde{\tau} \sim \eta^q$, it follows, using equation (26) that $\delta L \sim \eta^{1+q}$. Requiring all dominant terms to have the same scaling fixes $p = 2/5$ and $q = -1/5$, so that

$$R = \eta^{2/5}\mathcal{R}, \quad \tau = \eta^{-1/5}\mathcal{T}, \quad \delta L = \eta^{4/5}\delta\mathcal{L}. \quad (42)$$

Notice the overall scaling of the transition equation is $\dot{r}^2 \sim \eta^{6/5}$. The remaining question is whether the dominant term in $\Gamma_{\odot} \sim \delta E - \delta L$ is subleading or not. From (32), it follows $\delta E - \delta L \sim \eta^{6/5}$ in this regime, suggesting the change of variables

$$\Gamma_{\odot} = \eta^{6/5}\mathcal{Y}. \quad (43)$$

This allows to write the schematic transition equation as

$$\left(\frac{d\mathcal{R}}{d\mathcal{T}}\right)^2 \sim \mathcal{R}^3 + \mathcal{R}\delta\mathcal{L} + \mathcal{Y}. \quad (44)$$

Terms in Eqs. (30) and (31) that have been dropped can be seen to scale like the above terms multiplied by additional powers of \mathcal{R} or $\delta\mathcal{L}$. Since both \mathcal{R} and $\delta\mathcal{L}$ are small quantities in the transition zone, these terms are sub-leading and we can ignore them.

The above scaling analysis proves the dominant terms in (30) in the regime $\eta \ll \epsilon$ are captured by

$$\left(\frac{dR}{d\tilde{\tau}}\right)^2 \simeq -\frac{2}{3}\alpha R^3 + 2\beta\delta L R + \Gamma_{\odot} + \dots \quad (45)$$

where we neglected all subleading corrections, kept the same original notation as in OT [27] for the coefficients

$$\alpha = -\frac{1}{4}\frac{\partial^3 G}{\partial\tilde{r}^3}\Bigg|_{\text{isco}} \quad (46)$$

$$\beta = \frac{1}{2}\left(\frac{\partial^2 G}{\partial\tilde{r}\partial\tilde{E}}\tilde{\Omega} + \frac{\partial^2 G}{\partial\tilde{r}\partial\tilde{L}}\right)\Bigg|_{\text{isco}} \quad (47)$$

and the dominant contribution to (31) reduces to

$$\Gamma_{\odot} \simeq \tilde{\Omega}_{\text{isco}}(\delta E - \delta L)\frac{\partial G}{\partial\tilde{E}}\Bigg|_{\text{isco}} + \dots \quad (48)$$

Keeping all coefficients of order one, the natural scaled variables to introduce are

$$\begin{aligned} R &= \eta^{2/5} \alpha^{-3/5} (\beta \kappa)^{2/5} X \\ \tilde{\tau} - \tilde{\tau}_{\text{isco}} &= \eta^{-1/5} (\alpha \beta \kappa)^{-1/5} T \\ \delta E - \delta L &= \eta^{6/5} Y \\ \delta L &= -\eta^{4/5} (\alpha \beta)^{-1/5} \kappa^{4/5} T \end{aligned} \quad (49)$$

where

$$\kappa = \left(\frac{d\tilde{t}}{d\tau} \frac{d\tilde{L}}{d\tilde{t}} \right)_{\text{isco}}. \quad (50)$$

Plugging this into (45), one obtains

$$\left(\frac{dX}{dT} \right)^2 = -\frac{2}{3} X^3 - 2XT + C_0 \left(\tilde{\Omega} \frac{\partial G}{\partial \tilde{E}} \right)_{\text{isco}} Y + \mathcal{O}(\eta^{2/5})$$

where we defined $C_0 = \alpha^{4/5} (\kappa \beta)^{-6/5}$. From now on, we ignore the subleading corrections.

The analogue of the acceleration equation (24) reduces to

$$\begin{aligned} \frac{d^2 X}{dT^2} &= -X^2 - T \\ &- \frac{1}{2(dX/dT)} \left(2X - C_0 \left[\tilde{\Omega} \frac{\partial G}{\partial \tilde{E}} \right]_{\text{isco}} \frac{dY}{dT} \right) \end{aligned} \quad (51)$$

This depends on the time evolution of the circularity deviation parameter Y , whose dominant contribution is derived in (32). Inserting the re-scaled variables (49) in the latter

$$\frac{dY}{dT} = - \left. \frac{\partial \log \tilde{\Omega}}{\partial \tilde{r}} \right|_{\text{isco}} (\beta C_0)^{-1} X, \quad (52)$$

leads to a transition equation

$$\frac{d^2 X}{dT^2} = -X^2 - T - \frac{1}{2(dX/dT)} \left(2X + \beta^{-1} \left[\frac{\partial \tilde{\Omega}}{\partial \tilde{r}} \frac{\partial G}{\partial \tilde{E}} \right]_{\text{isco}} X \right). \quad \eta^{1-2k/5} < \eta^{2/5} \Rightarrow 3 - 2k > 0 \Rightarrow k < \frac{3}{2}.$$

Evaluating (11) at ISCO, we find the term in square brackets equals

$$\left(\frac{\partial \tilde{\Omega}}{\partial \tilde{r}} \frac{\partial G}{\partial \tilde{E}} \right)_{\text{isco}} = -2\beta \quad (53)$$

and so the last term vanishes. This was inevitable, since this term is precisely the term that arises from the radial self-force, as identified earlier. The leading order evolution of Y is driven by maintaining the circularity of the orbit and so with this condition we expect the radial self-force corrections to be sub-leading.

The resulting transition equation of motion in the regime of low spins $\eta \ll \epsilon$ is

$$\frac{d^2 X}{dT^2} = -X^2 - T \quad (54)$$

and Y is evolved through the ODE (52). We note that the transition equation does not depend on Y in this regime. Corrections to this equation arising from evolution of Y enter at an order $\eta^{2/5}$ higher than leading and so are sub-dominant. As discussed earlier the evolution of Y is related to deviations from the linear-in-proper-time evolution of energy and angular momentum and so the fact that these corrections do not enter the transition equation for $\eta \ll \epsilon$ demonstrate that the linear evolution assumed by OT is appropriate in this regime.

As discussed in Section III B, corrections to this equation arise from deviations from the flux balance law $\dot{\tilde{E}} = \tilde{\Omega}(\tilde{r})\dot{\tilde{L}}$ and from eccentricity. The former scale like $\eta\dot{\tilde{r}}$, which is $\mathcal{O}(\eta^{8/5})$. Corrections to the geodesic part of the master equation enter through corrections to $\delta\tilde{L}$ or $\delta\tilde{E}$, and these are a factor of $\eta^{3/5}$ smaller than the terms that have been retained and are hence sub-dominant. Corrections to the radial self-force term in the master equation enter divided by $\dot{\tilde{r}}$ and so contribute like η to the master equation. Other terms scale like $\eta^{4/5}$ so these corrections are sub-leading, albeit only by a factor of $\eta^{1/5}$. Corrections also arise from the approximation used to evolve the angular momentum, $\dot{\tilde{L}} = \text{constant}$. These corrections also scale like $\eta\dot{\tilde{r}}$ and so are sub-dominant.

Corrections arising from eccentricity are sub-leading provided $e < \tilde{r} - \tilde{r}_{\text{isco}}$, as discussed in Section III B. In the non-extremal case we therefore need $e < \eta^{2/5}$. The residual eccentricity in this regime come from Eq. (37), setting $\delta\tilde{r} = \tilde{r} - \tilde{r}_{\text{isco}}$ to $\mathcal{O}(\eta^{2/5})$. This yields the constraint

We saw previously that $k = 1/4$ for all spins $a < 1$, which satisfies this bound. We deduce that eccentricity corrections are sub-dominant in the non-near-extremal regime.

D. General Master Equation - Near-Extremal

Let us consider rapidly rotating black holes with spin parametrized by $a = \sqrt{1 - \epsilon^2}$ for $\epsilon \ll 1$, as in (15). The discussion below equations (A13)-(A18) allows to identify the a priori dominant contributions to the transition

equation (30) as

$$\begin{aligned} \left(\frac{dR}{d\tilde{\tau}}\right)^2 &\sim R^3 + R\delta L\epsilon^{2/3} + R^2\delta L + \delta L^2\epsilon^{4/3} + \Gamma_{\odot} \\ \Gamma_{\odot} &\sim (\delta E - \delta L)\left(\epsilon^{2/3} + R + \delta L\epsilon^{2/3}\right). \end{aligned} \quad (55)$$

Since the functional dependence of the above equation does not depend on η , we learn the η scaling should be the same as before if we keep the R^3 and $R\delta L$ terms. Hence, we are left to determine any possible ϵ scaling. Proceeding as before, we look for scalings of the form $R \sim \eta^{2/5}\epsilon^p$ and $\tau \sim \eta^{-1/5}\epsilon^q$. We learn from equation (26) that $\delta L \sim \eta^{4/5}\epsilon^q$. Requiring these dominant terms to scale in the same way determines $p = 4/15$ and $q = -2/15$, so that

$$\begin{aligned} R &= \eta^{2/5}\epsilon^{4/15}\mathcal{R}, \quad \tilde{\tau} = \eta^{-1/5}\epsilon^{-2/15}\mathcal{T}, \\ \delta L &= \eta^{4/5}\epsilon^{-2/15}\delta\mathcal{L}. \end{aligned} \quad (56)$$

Notice $R/\delta L \sim (\epsilon/\eta)^{2/5}$. Hence, if $\eta \sim \epsilon$, the term $R^2\delta L$ scales like the velocity squared $\dot{r}^2 \sim \eta^{6/5}\epsilon^{4/5} \sim \epsilon^2$ and must be kept in the transition equation, whereas the term $\delta L^2\epsilon^{4/3}$ is $\mathcal{O}(\epsilon^{2/3})$ smaller and, consequently, subdominant.

The only remaining question is whether Γ_{\odot} is relevant in this regime or not. Using (32) and the scalings (56), we infer $(\delta E - \delta L) \sim \eta^{6/5}\epsilon^{2/15}$. Since in the regime $\eta \sim \epsilon$, $R \sim \delta L \sim \epsilon^{2/3}$ we conclude $\Gamma_{\odot} \sim \eta^{6/5}\epsilon^{4/5} \sim \dot{r}^2$ and must be kept in the transition equation. Introducing the finite variable \mathcal{Y}

$$\Gamma_{\odot} = \eta^{6/5}\epsilon^{4/5}\mathcal{Y}, \quad (57)$$

the general transition equation in the $\eta \sim \epsilon$ regime reduces to

$$\left(\frac{d\mathcal{R}}{d\mathcal{T}}\right)^2 \sim \mathcal{R}^3 + \mathcal{R}\delta\mathcal{L} + \mathcal{R}^2\delta\mathcal{L} + \mathcal{Y}. \quad (58)$$

As a self-consistency check, we can write the radial geodesic equation using the change of variables (56) and (57)

$$\begin{aligned} \left(\frac{d\mathcal{R}}{d\mathcal{T}}\right)^2 &\sim \sum_{i=3}^{\infty} \eta^{2(i-3)/5} \epsilon^{4(i-3)/15} \mathcal{R}^i + \delta\mathcal{L}\mathcal{R} + \\ &\sum_{m=2}^{\infty} \left(\frac{\eta}{\epsilon}\right)^{2(m-1)/5} \epsilon^{2(m-2)/3} \mathcal{R}^m \delta\mathcal{L} + \eta^{2/5} \epsilon^{4/15} \delta\mathcal{L}^2 \\ &+ \sum_{n=1}^{\infty} \left(\frac{\eta}{\epsilon}\right)^{2(n+1)/5} \epsilon^{2(5n-1)/15} \delta\mathcal{L}^2 \mathcal{R}^n + \mathcal{Y}. \end{aligned} \quad (59)$$

It is apparent that the dominant terms are the $i = 3$ and $m = 2$ terms, all others being subleading.

The above scaling analysis proves the dominant terms in (30) in the regime $\eta \sim \epsilon$ are captured by

$$\left(\frac{dR}{d\tilde{\tau}}\right)^2 \simeq -\frac{2}{3}\alpha R^3 + 2\beta\delta L R + \gamma\delta L R^2 + \Gamma_{\odot} + \dots \quad (60)$$

where α and β are defined as in (46)-(47) and $\gamma = B_1$ in (30). As shown in appendix A, they are approximated by

$$\alpha \rightarrow 1, \quad \beta \rightarrow 2^{-2/3}\sqrt{3}\epsilon^{2/3} \equiv \hat{\beta}\epsilon^{2/3}, \quad \gamma \rightarrow \sqrt{3}. \quad (61)$$

Furthermore, the dominant contributions to Γ_{\odot} are

$$\Gamma_{\odot} = \tilde{\Omega}_{\text{isco}}(\delta E - \delta L) \left(\left. \frac{\partial G}{\partial \tilde{E}} \right|_{\text{isco}} + \left. \frac{\partial^2 G}{\partial \tilde{\tau} \partial \tilde{E}} \right|_{\text{isco}} R + \dots \right). \quad (62)$$

Keeping all coefficients of order one, the natural scaled variables to introduce are

$$\begin{aligned} R &= \eta^{2/5}\epsilon^{4/15}\alpha^{-3/5}(\hat{\beta}\kappa)^{2/5}X, \\ \tilde{\tau} - \tilde{\tau}_{\text{isco}} &= \eta^{-1/5}\epsilon^{-2/15}(\alpha\hat{\beta}\kappa)^{-1/5}T, \\ \delta E - \delta L &= \eta^{6/5}\epsilon^{2/15}Y \\ \delta L &= -\eta^{4/5}\epsilon^{-2/15}(\alpha\hat{\beta})^{-1/5}\kappa^{4/5}T. \end{aligned} \quad (63)$$

Since $\eta \sim \epsilon$, it follows $R \sim \epsilon^{2/3}$. Hence, the near ISCO expansion corresponds to the near horizon geometry of the primary black hole since, in Boyer-Lindquist coordinates, $|\tilde{r}_{\text{isco}} - \tilde{r}_+| \sim \epsilon^{2/3}$. As a result, we will be able to use the (leading order and analytic) expression for the energy flux due to gravitational radiation in (23). This allows to compute κ in (50) in this regime as

$$\kappa = \left(\tilde{\Omega}^{-1} \frac{d\tilde{t}}{d\tilde{\tau}} \frac{d\tilde{E}}{d\tilde{t}} \right)_{\text{isco}} \rightarrow \frac{8}{\sqrt{3}}(\tilde{C}_H + \tilde{C}_{\infty}). \quad (64)$$

Notice $\kappa \sim \mathcal{O}(1)$ since $\tilde{C}_H + \tilde{C}_{\infty} \sim \mathcal{O}(1)$.

Ignoring subleading terms, the general master equation (60) reduces to

$$\left(\frac{dX}{dT}\right)^2 = -\frac{2}{3}X^3 - 2XT - (\eta/\epsilon)^{2/5}C_1TX^2 + \tilde{\Gamma}_{\odot} \quad (65)$$

with

$$C_1 = \gamma(\alpha\hat{\beta}\kappa)^{-3/5}\kappa \quad (66)$$

$$\tilde{\Gamma}_{\odot} = \epsilon^{-4/5}\eta^{-6/5}\alpha^{4/5}(\hat{\beta}\kappa)^{-6/5}\Gamma_{\odot}. \quad (67)$$

Notice the appearance of the new term proportional to TX^2 , compared to the OT regime, is due to the regime $\eta \sim \epsilon$.

Taking a further $\tilde{\tau}$ derivative, we find the analogue of the acceleration equation (24) in this regime

$$\begin{aligned} \frac{d^2X}{dT^2} &= -X^2 - T - (\eta/\epsilon)^{2/5}C_1XT + \\ &\frac{1}{2(dX/dT)} \left(-2X - (\eta/\epsilon)^{2/5}C_1X^2 + \frac{d\tilde{\Gamma}_{\odot}}{dT} \right) \end{aligned} \quad (68)$$

The time evolution of Γ_{\odot} in (62) has two contributions : one proportional to dY/dT , which can be computed using (32) and a second one proportional to $Y\dot{X}$. Altogether yields

$$\begin{aligned} \frac{d\tilde{\Gamma}_{\odot}}{dT} = & 2X + (\eta/\epsilon)^{2/5}(\alpha\hat{\beta}\kappa)^{-3/5}\kappa\left.\frac{\partial\tilde{\Omega}}{\partial\tilde{r}}\right|_{\text{isco}}\left.\frac{\partial^2G}{\partial\tilde{r}\partial\tilde{E}}\right|_{\text{isco}}X^2 \\ & + \tilde{\Omega}_{\text{isco}}\left.\frac{\partial^2G}{\partial\tilde{r}\partial\tilde{E}}\right|_{\text{isco}}(\eta/\epsilon)^{2/5}\alpha^{1/5}(\hat{\beta}\kappa)^{-4/5}\dot{X} \end{aligned} \quad (69)$$

where we used (11) to simplify the first term. The latter cancels the $-2X$ term in (68). Using the dominant contribution to the identity (12) evaluated at ISCO, the second term cancels the C_1X^2 term in (68). Finally, the third term gives a non-trivial contribution to the acceleration equation

$$\frac{d^2X}{dT^2} = -X^2 - T - (\eta/\epsilon)^{2/5}(C_1XT - C_2Y) \quad (70)$$

with constant defined by

$$C_2 = \frac{1}{2}\alpha^{1/5}(\hat{\beta}\kappa)^{-4/5}\tilde{\Omega}_{\text{isco}}\left.\left(\frac{\partial^2G}{\partial\tilde{E}\partial\tilde{r}}\right)\right|_{\text{isco}}. \quad (71)$$

and evolution equation for Y such that

$$\frac{dY}{dT} = -\Lambda\left.\frac{\partial\log\tilde{\Omega}}{\partial\tilde{r}}\right|_{\text{isco}}X, \quad \text{with } \Lambda = \alpha^{-4/5}\kappa^{6/5}\hat{\beta}^{1/5}. \quad (72)$$

In our treatment of the OT regime (non near-extremal spins), the terms in Eq. (70) were neglected since they scaled with $\eta^{2/5}$ and were subdominant. In the near-extremal case, one can clearly see that the XT and Y term are comparable to the (rescaled) radial acceleration provided $\eta \sim \epsilon$. As such, they *must* be included in the analysis. Our final transition equation of motion differs from that in [28], which correctly included the Y term but missed the term XT , which is the same order as the terms being retained. Our analysis improves on [28] in two additional ways. Firstly, Y was introduced in [28] as a mathematical construct to ensure conservation of the four-velocity norm. The evolution equation for Y was derived by forcing the equation of motion obtained from differentiation of the kinetic energy equation, Eq. (2), to agree with that obtained by expansion of the left-hand-side of the acceleration equation, Eq. (24). This is equivalent to setting the radial self-force term to zero, which is equivalent to imposing the circular-to-circular condition. This physical interpretation of the procedure was not made clear in [28], nor the interpretation of Y as representing departures from the linear-in-proper-time evolution. Secondly, the scaling of the flux given in Eq. (23) was not known at that time and this was left as an unspecified power of ϵ . Now that we know this scaling we can do a more complete analysis of the near-extremal regime.

The quantities above can be computed in the near-extremal limit, $\epsilon \rightarrow 0$,

$$\begin{aligned} \Lambda & \rightarrow 2^{52/15}(\tilde{C}_H + \tilde{C}_{\infty})^{6/5}/\sqrt{3} \\ C_1 & \rightarrow 2^{8/5}(\tilde{C}_H + \tilde{C}_{\infty})^{2/5} \\ C_2 & \rightarrow 2^{-13/15} \cdot 3^{-1/2}(\tilde{C}_H + \tilde{C}_{\infty})^{-4/5}. \end{aligned}$$

Equations (70) and (72) are a coupled set of ODEs which will link the adiabatic inspiral to a plunging geodesic.

As in the previous section we now consider the size of corrections to the master equation. Corrections to the circular flux-balance law in the geodesic part of the master equation and corrections to the linear-in-time angular-momentum evolution enter through corrections to δE and δL and scale like \tilde{r} times terms that are being retained. These are therefore subdominant since $\tilde{r} \sim \eta^{3/5}\epsilon^{2/5} \ll 1$. These corrections also contribute terms of order $\eta \cdot \delta G/\partial\tilde{E}$ to the radial self-force part of the master equation. These terms are of order $\eta\epsilon^{2/3}$ and so are a factor of $(\eta/\epsilon)^{1/5}\epsilon^{1/3}$ smaller than the leading order terms in the transition equation and are therefore sub-dominant. Eccentricity corrections enter like e^2 but, as shown in Section III B, for near-extremal inspirals eccentricity can only grow once $\tilde{r} - \tilde{r}_{\text{isco}} \sim \mathcal{O}(\epsilon)$. In the transition zone $\tilde{r} - \tilde{r}_{\text{isco}} \sim (\eta/\epsilon)^{2/5}\epsilon^{2/3} \gg \epsilon$ and so the eccentricity has not started to grow when the transition zone is reached. Residual eccentricity from the adiabatic inspiral would be $\mathcal{O}(\eta)$ and eccentricity excited during the transition would be $\mathcal{O}(\eta^{4/5}\epsilon^{1/5})$ (or $\mathcal{O}(\eta^{7/10}\epsilon^{2/15})$) if it was coherently excited throughout the transition). These are sub-leading corrections.

E. General Master Equation - Very Near-Extremal

The final regime concerns very rapidly rotating black holes, where $\epsilon \ll \eta$. Using the results in appendix A, one can identify the a priori dominant contributions to the master equation (30) and (31) to be (ignoring coefficients of $\mathcal{O}(1)$)

$$\begin{aligned} \left(\frac{dR}{d\tilde{r}}\right)^2 & \sim R^3 + R\delta L\epsilon^{2/3} + R^2\delta L + \delta L^2\epsilon^{4/3} + \Gamma_{\odot} \\ \Gamma_{\odot} & \sim (\delta E - \delta L)\left(\epsilon^{2/3} + R + \epsilon^{2/3}\delta L\right). \end{aligned} \quad (73)$$

It is natural to expect that terms involving some explicit factors of ϵ should be sub-leading in this regime. Assuming a scaling solution of the form $R \sim \eta^{\alpha}$ and $\tilde{r} \sim \eta^{\beta}$, we learn using (26) that $\delta L \sim \eta^{\beta+1}$. Imposing the dominant terms R^3 and $R^2\delta L$ scale like \tilde{R}^2 yields the scaling solutions $\alpha = 2/3$ and $\beta = -1/3$, so that

$$R = \eta^{2/3}\mathcal{R}, \quad \tilde{r} = \eta^{-1/3}\mathcal{T}, \quad \delta L = \eta^{2/3}\delta\mathcal{L}. \quad (74)$$

As a consistency check, notice the terms $\epsilon^{2/3}R\delta L \sim \eta^2(\epsilon/\eta)^{2/3}$ and $\epsilon^{4/3}\delta L^2 \sim \eta^{8/3}(\epsilon/\eta)^{4/3}$ are subdominant compared to the leading scaling $\dot{r}^2 \sim \eta^2$.

The remaining question is whether Γ_\odot is negligible in this regime or not. Using the scalings (74) together with (32), we infer that $\delta E - \delta L \sim \eta^{4/3}$. It follows $\Gamma_\odot \sim \eta^2$ from the term linear in R in the second equation in (73). Introducing the finite variable \mathcal{Y}

$$\Gamma_\odot = \eta^2 \mathcal{Y} \quad (75)$$

leads to the transition equation of motion

$$\left(\frac{d\mathcal{R}}{dT}\right)^2 \sim \mathcal{R}^3 + \mathcal{R}^2\delta\mathcal{L} + \mathcal{Y}. \quad (76)$$

As a consistency check, we can substitute the scalings (74) and (75) into the general master equation (30)

$$\begin{aligned} \left(\frac{d\mathcal{R}}{dT}\right)^2 &\sim \sum_{i=3}^{\infty} \eta^{2(i-3)/3} \mathcal{R}^i + (\epsilon/\eta)^{2/3} \delta\mathcal{L}\mathcal{R} + \\ &\sum_{m=2}^{\infty} \eta^{2(m-2)/3} \mathcal{R}^m \delta\mathcal{L} + \epsilon^{2/3}(\epsilon/\eta)^{2/3} \delta\mathcal{L}^2 + \\ &\epsilon^{4/3} \delta\mathcal{L}^2 \mathcal{R} + \sum_{n=2}^{\infty} \eta^{2(n-1)/3} \delta\mathcal{L}^2 \mathcal{R}^n + \mathcal{Y}. \end{aligned} \quad (77)$$

Clearly the dominant terms occur when both $i = 3$ and $m = 2$ with the rest being subleading.

The above scaling analysis proves the dominant terms in (30) in the regime $\epsilon \ll \eta$ are captured by

$$\left(\frac{dR}{d\tilde{r}}\right)^2 \simeq -\frac{2}{3}\alpha R^3 + \gamma \delta L R^2 + \Gamma_\odot + \dots \quad (78)$$

where α and γ are given in Eq.(61) with

$$\Gamma_\odot \simeq \tilde{\Omega}_{\text{isco}}(\delta E - \delta L) \frac{\partial^2 G}{\partial \tilde{r} \partial \tilde{E}} \Big|_{\text{isco}} R + \dots \quad (79)$$

Keeping all coefficients of order one, the natural rescaled variables in this regime are

$$\begin{aligned} R &= \eta^{2/3} \alpha^{-3/5} \kappa^{2/5} X \\ \tau - \tilde{\tau}_{\text{isco}} &= \eta^{-1/3} (\alpha \kappa)^{-1/5} T \\ \delta E - \delta L &= \eta^{4/3} \mathcal{Y} \\ \delta L &= -\eta^{2/3} \alpha^{-1/5} \kappa^{4/5} T. \end{aligned} \quad (80)$$

In these variables, the radial velocity equation (78) can be expressed as

$$\left(\frac{dX}{dT}\right)^2 = -\frac{2}{3}X^3 - K_1 X^2 T + \tilde{\Gamma}_\odot \quad (81)$$

with

$$\begin{aligned} K_1 &= \gamma \alpha^{-3/5} \kappa^{2/5}, \\ \tilde{\Gamma}_\odot &= \eta^{-2} \alpha^{4/5} \kappa^{-6/5} \Gamma_\odot, \end{aligned} \quad (82)$$

and κ as in (50).

Taking a further derivative with respect to T yields the acceleration equation

$$\frac{d^2 X}{dT^2} = -X^2 - K_1 X T + \frac{1}{2(dX/dT)} \left(\frac{d\tilde{\Gamma}_\odot}{dT} - K_1 X^2 \right). \quad (83)$$

Using (32) together with (80), one finds that

$$\begin{aligned} \frac{d\tilde{\Gamma}_\odot}{dT} &= \alpha^{1/5} \kappa^{-4/5} \tilde{\Omega}_{\text{isco}} \frac{\partial^2 G}{\partial \tilde{r} \partial \tilde{E}} \Big|_{\text{isco}} \dot{X} Y \\ &- \alpha^{-3/5} \kappa^{2/5} \left(\frac{\partial^2 G}{\partial \tilde{r} \partial \tilde{E}} \frac{\partial \tilde{\Omega}}{\partial \tilde{r}} \right)_{\text{isco}} X^2. \end{aligned} \quad (84)$$

Plugging this back in (83) and using the dominant contribution to the identity (12), the $K_1 X^2$ term cancels and one is left with

$$\frac{d^2 X}{dT^2} = -X^2 - K_1 X T + K_2 Y \quad (85)$$

together with the evolution equation for $Y(T)$ given by

$$\frac{dY}{dT} = -\alpha^{-4/5} \kappa^{6/5} \frac{\partial \log \tilde{\Omega}}{\partial \tilde{r}} \Big|_{\text{isco}} X. \quad (86)$$

where

$$K_2 = \frac{1}{2} \alpha^{1/5} \kappa^{-4/5} \tilde{\Omega}_{\text{isco}} \frac{\partial^2 G}{\partial \tilde{r} \partial \tilde{E}} \Big|_{\text{isco}}.$$

In the limit $\epsilon \rightarrow 0$, the constants K_1 and K_2 approach the values

$$\begin{aligned} K_1 &\rightarrow 2^{6/5} 3^{3/10} (\tilde{C}_H + \tilde{C}_\infty)^{2/5} \\ K_2 &\rightarrow 2^{-7/5} 3^{-1/10} (\tilde{C}_H + \tilde{C}_\infty)^{-4/5}. \end{aligned}$$

As argued in previous sections, corrections to the circular flux-balance law contribute terms to the master equation which scale like $\dot{r} \sim \mathcal{O}(\eta)$ times terms that are being retained and like $\eta \epsilon^{2/3}$. Corrections to the linear-in-time angular momentum evolution enter with the same scaling as the former. The retained terms in the master equation scale like $\eta^{4/3}$ in the very near extremal regime and so these corrections are both subleading. Eccentricity corrections enter like e^2 but, as in the near-extremal case, eccentricity cannot grow until the transition zone has already been reached, and so these corrections are no larger than $\mathcal{O}(\eta^{5/3})$ and are also sub-leading.

We conclude this subsection by noting that the transition equation of motion (85) is perfectly well behaved in the limit $\epsilon \rightarrow 0$ and can therefore be used to compute an inspiral into a maximally spinning black hole with $a = 1$. In this case the horizon coincides with the ISCO and so we terminate integration of the ODE (85) at $\tilde{r} = \tilde{r}_+$. The presence of the horizon manifests itself in the transformation from proper time to coordinate time, which will be discussed, for non-extremal inspirals, in the next sub-section.

IV. RESULTS

A. Numerical Integration

We now seek to compute a full worldline $\tilde{r}(\tilde{\tau})$ for $\infty > \tilde{r} \geq \tilde{r}_+$. We restrict ourselves to the $\epsilon \sim \eta$ regime so we try to find the solution to

$$\begin{aligned} \frac{d^2 X}{d\tilde{T}^2} &= -X^2 - T - (\eta/\epsilon)^{2/5}(C_1 X T - C_2 Y) \\ \frac{dY}{d\tilde{T}} &= -\frac{3}{4}\Lambda X \end{aligned} \quad (87)$$

which deviates off the past adiabatic inspiral and evolves into a geodesic plunge. We can derive an equation for an adiabatic inspiral in proper time by using the quasi-circular approximation. Using our far-horizon expression for the energy flux defined by Eq.(21) with both equations (8) and (5), one derives

$$\frac{d\tilde{r}}{d\tilde{\tau}} = \eta^{-\frac{64}{5}} \tilde{\Omega}^{7/3} \frac{(2a - 3\tilde{r}^{1/2} + \tilde{r}^{3/2})\tilde{r}}{\tilde{r}^2 - 6\tilde{r} + 8a\tilde{r}^{1/2} - 3a^2} \dot{\mathcal{E}}(\tilde{r}). \quad (88)$$

This equation diverges at the ISCO which is a breakdown of the quasi-circular approximation. We shall use Eq.(87) to smoothly transition from the adiabatic inspiral Eq.(88) into a geodesic plunge to the horizon. We used a cubic spline to interpolate values for the relativistic correction $\dot{\mathcal{E}}(\tilde{r})$ using exact flux data found in the BHPT. We then numerically integrate Eq.(88) by stepping forwards in proper time until $\tilde{L}(\tilde{\tau}_i) - \tilde{L}_{\text{isco}} \sim \eta^{4/5} \epsilon^{-2/15}$. We feel this criteria is suitable for turning on the transition equation of motion since our model for the flux is well represented during the transition regime. When this criteria is met we can be sure that our model for flux evolution throughout the transition regime is correct to leading order. Once this is satisfied, we stop integrating our adiabatic inspiral solution and begin integrating our transition equation of motion (87).

Since we do not terminate our adiabatic inspiral solution at the ISCO, we do not know the precise proper time where the particle crosses the ISCO. As such, the variable T is not a good choice of variable to integrate on the right hand side of (87). Instead, we substitute

T for δL from Eq.(63) into our transition equation of motion, then

$$\begin{aligned} \frac{d^2 X}{d\tilde{T}^2} &= -X^2 + B_0 \delta L + (\eta/\epsilon)^{2/5}(C_1 B_0 \delta L + C_2 Y) \\ \frac{dY}{d\tilde{T}} &= -\frac{3}{4}\Lambda X \\ \frac{d\delta L}{d\tilde{T}} &= B_0^{-1}, \quad B_0 = -\eta^{-4/5} \epsilon^{2/15} (\alpha \hat{\beta})^{1/5} \kappa^{-4/5}. \end{aligned} \quad (89)$$

We use initial conditions determined by the end of the adiabatic inspiral Eq.(88) at some time τ_{init} .

$$\begin{aligned} X(T_{\text{init}}) &= \eta^{-2/5} \alpha^{-3/5} (\beta \kappa)^{-2/5} (\tilde{r} - \tilde{r}_{\text{isco}}) \\ \frac{dX}{d\tilde{T}} \Big|_{T_{\text{init}}} &= \eta^{-3/5} \alpha^{2/5} (\beta \kappa)^{-3/5} \frac{d\tilde{r}}{d\tilde{\tau}} \Big|_{\tilde{\tau}_{\text{init}}} \\ Y(T_{\text{init}}) &= \eta^{-6/5} \epsilon^{-2/15} (\tilde{\Omega}_{\text{isco}}^{-1} \delta E_{\text{init}} - \delta L_{\text{init}}) \\ \delta L(T_{\text{init}}) &= L_{\text{circ}}(\tilde{r}_{\text{init}}) - L_{\text{isco}}. \end{aligned} \quad (90)$$

Where $\tilde{L}_{\text{circ}}(r_{\text{init}})$ corresponds to the circular angular momenta evaluated at the end of the inspiral, r_{init} . Using this prescription, we are able to integrate the coupled ODEs Eq.(89) with initial conditions (90) to obtain Fig.(4). The transition solution smoothly deviates away from the adiabatic inspiral (blue curve), passes through the ISCO and reaches the horizon where the solution terminates. The plot on the right shows the full worldline in proper time $\tilde{r}(\tilde{\tau})$ where the inspiral starts at $\tilde{r} = 1.006$ and terminates at the horizon. This method ensures that $\tilde{r}(\tilde{\tau})$ is both continuous and once differentiable everywhere.

Also, by our choice of integrating (89) using the variable δL , we ensure continuity but *not* differentiability in \tilde{L} throughout the full inspiral. We note here that Apte and Hughes in [48] also found discontinuities in their evolution of both L and E and added corrections to ensure both (first order) differentiability and continuity at $\tilde{\tau}_{\text{init}}$. We consider a correction of the form

$$\tilde{L} = \Delta L_{\text{cor}} + \tilde{L}_{\text{isco}} + \dot{\tilde{L}}_{\text{isco}} (\tilde{\tau} - \tilde{\tau}_{\text{isco}}). \quad (91)$$

We have discussed previously that the leading order term in $\tilde{L}_{\tilde{\tau}_i} - \tilde{L}_{\text{isco}}$ scales proportionally to $\eta^{4/5} \epsilon^{-2/15}$. So we choose to add a constant offset $\Delta L_{\text{cor}} \sim \eta^{6/5} \epsilon^{2/15}$ to the angular momenta evolution to ensure continuity in the \tilde{L} evolution. Finally, to calculate the evolution in \tilde{E} , one has to evaluate

$$\tilde{E} = \Delta E_{\text{cor}} + \tilde{E}_{\text{isco}} + \int_{\tilde{\tau}_{\text{init}}}^{\tilde{\tau}_+} \tilde{\Omega}(\tilde{r}) \dot{\tilde{L}}_{\text{trans}} d\tilde{\tau} \quad (92)$$

from the flux balance law $\dot{\tilde{E}} = \tilde{\Omega}(\tilde{r}) \dot{\tilde{L}}$. The correction to ΔE_{cor} is chosen to ensure continuity with the end of the inspiral energy given by Eq.(6).

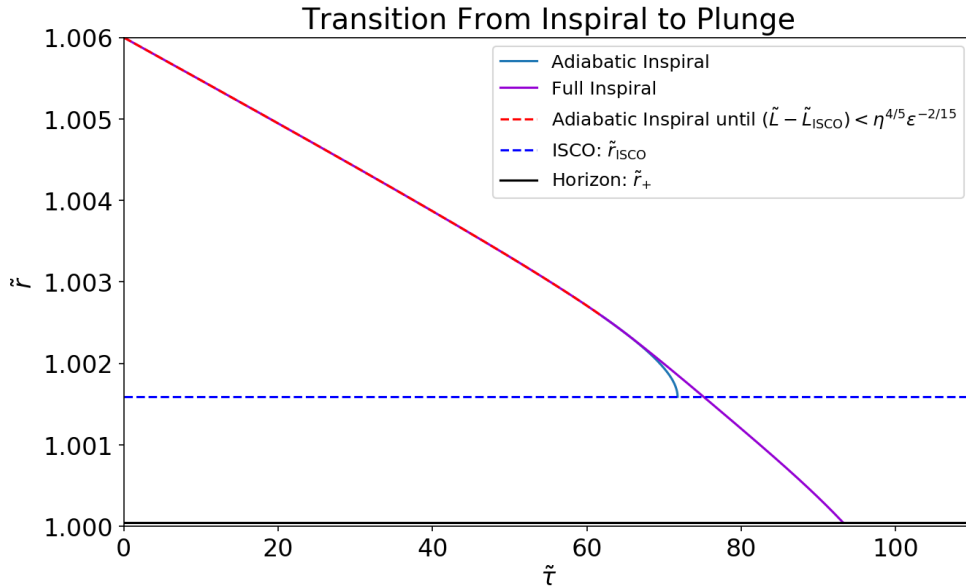


Figure 2: In both plots we consider mass ratio $\eta = 10^{-5}$ and spin $a = 1 - 10^{-9}$. The transition regime begins at $r_{init} \approx 1.0026$ at $\tau_{init} \approx 62.00$. The particle plunges into the horizon r_+ at $\tilde{\tau}_+ \approx 93.19$.

Notice here that this ensures that the energy obeys $\dot{\tilde{E}} = \tilde{\Omega}(\tilde{r})\dot{\tilde{L}}_{isco}$ and is thus *not* constant. This ensures that we are still granted a full cancellation of the radial self force piece in Eq.(24). This will yield a continuous evolution \tilde{E} at the matching point with a discontinuous first derivative. At this point we will have a full trajectory $\tilde{r}(\tilde{\tau})$ with (continuous) integrals of motion in proper time $\tilde{E}(\tilde{\tau})$ and $\tilde{L}(\tilde{\tau})$. In each of [27, 29, 48], the authors compute three separate worldlines in proper time; Adiabatic inspiral, transition, geodesic plunge. Apte *et al* in [48], provide an algorithm in which they freeze the constants of motion E and L when the extra terms in Eq.(54) exceed the leading order terms X^2 and T by 5%. As one would expect, as one ventures farther from the ISCO, the Taylor expansion method used to derive these transition equations of motion will break down. As such, it is very natural for each of the aforementioned authors to compute a geodesic plunge to complete their worldlines in proper time $\tilde{r}(\tilde{\tau})$. Simply because, for moderate spins (non near-extremal), $|\tilde{r}_+ - \tilde{r}_{isco}| \sim \mathcal{O}(1) \approx \eta^{2/5}$. For near-extreme black holes the ISCO is close to the horizon in Boyer Lindquist coordinates $|\tilde{r}_{isco} - \tilde{r}_+| \sim \epsilon^{2/3}$. The scaling of the near-extremal transition zone is also $\epsilon^{2/3}$ and so the horizon is reached while the object is still in the transition zone. We therefore do not expect to need to add a geodesic plunge to compute full near-extremal inspirals. To verify this we numerically calculate the extra terms in (89),

which are

$$C_3 X^3 \Rightarrow C_3 = \frac{1}{12} \left(\frac{\eta}{\epsilon}\right)^{2/5} \epsilon^{2/3} \frac{\partial^4 G}{\partial \tilde{r}^4} \Big|_{isco} \alpha^{-8/5} (\hat{\beta}\kappa)^{2/5}$$

$$C_4 XY \Rightarrow C_4 = \frac{1}{2} \left(\frac{\eta}{\epsilon}\right)^{4/5} \epsilon^{2/3} \tilde{\Omega}_{isco} \frac{\partial^3 G}{\partial \tilde{r}^2 \partial \tilde{E}} \Big|_{isco} (\alpha \hat{\beta}\kappa)^{-2/5}. \quad (93)$$

We compare the solution to (89) when these terms are omitted or included in Figure 3. The difference is at most 1% even at ISCO. We conclude that we can use the solution from (89) *throughout* the plunging regime, for $\tilde{r} \in [\tilde{r}_+, \tilde{r}_{isco}]$. It would be useful in the future to compare our results with the analytic geodesic plunges found in [25].

B. Worldline in Boyer-Lindquist Coordinates

In the previous section, we computed the full worldline comprised of inspiral, transition and plunge parametrized as $\tilde{r}(\tilde{\tau})$. We now intend to do the same but in coordinate time so that our worldline is in Boyer-Lindquist coordinates $(\tilde{t}, \tilde{r}(\tilde{t}), \theta = \pi/2, \phi(\tilde{t}))$. Loosely, this is the time measured from Earth (at radial infinity) so is extremely useful for observable purposes.

For the (quasi-circular) inspiral solution, we simply integrate the circular relation relating coordinate time to proper time via Eq.(8)

$$\tilde{t} = \int_0^{\tilde{\tau}_{init}} \frac{1 + a/\tilde{r}^{3/2}}{\sqrt{1 - 3/\tilde{r} + 2a/\tilde{r}^{3/2}}} d\tilde{\tau} \quad (94)$$

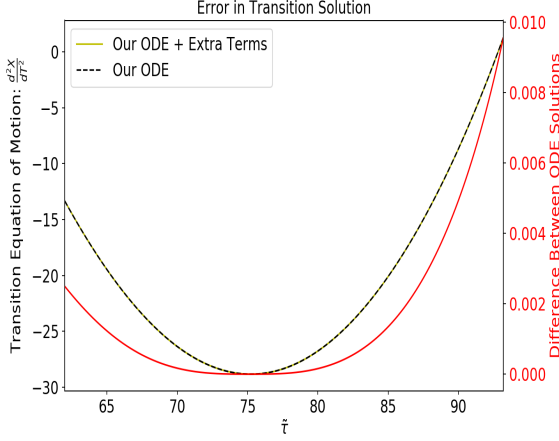


Figure 3: Solution to (89) (black dashed line) and difference in solution when including the higher-order corrections given in Eq. (93) (red solid line). The numerical difference is small throughout the transition regime reaching a maximum at plunge of $\sim 1\%$.

where $\tilde{r}(\tilde{\tau})$ is the worldline constructed by integrating Eq.(88) up to some suitable point to begin the transition solution, in our case, $\tilde{r}(\tilde{\tau}_{init}) = \tilde{r}_{init}$. To compute the trajectory in coordinate time $\tilde{r}(\tilde{t})$ throughout the transition regime, we must integrate

$$\tilde{t} = \tilde{t}_{insp} + \int_{\tilde{\tau}_{init}}^{\tilde{\tau}_+} T(\tilde{r}, \tilde{E}, \tilde{L}, a) d\tilde{\tau}. \quad (95)$$

where $T(\tilde{r}, \tilde{E}, \tilde{L}, a)$ is given by 4 and \tilde{t}_{insp} is defined through $\tilde{t}(\tilde{\tau}_{init})$. Throughout the transition regime, we use the model for both $\tilde{E}(\tilde{\tau})$ and $\tilde{L}(\tilde{\tau})$ given by Eq.(92) and Eq.(91). This will yield the $\tilde{r}(\tilde{t})$ throughout the transition regime. Combining these results yield a full trajectory from radial infinity to the horizon in coordinate time $\tilde{r}(\tilde{t})$.

To then calculate the orbital velocity $d\phi/d\tilde{t} = \tilde{\Omega}$ in coordinate time we substitute $\tilde{r}(\tilde{t})$ found previously into Eq.(9). This now gives $\tilde{\Omega}(\tilde{t})$ valid throughout the adiabatic adiabatic inspiral regime. Using our solutions for $\tilde{E}(\tilde{\tau})$ and $\tilde{L}(\tilde{\tau})$ defined through Eq.(91) and Eq.(92) and $\tilde{r}(\tilde{t})$ throughout the transition regime, we calculate

$$\tilde{\Omega} = \frac{d\phi}{d\tilde{t}} = \frac{2a\tilde{E}\tilde{r} - a^2\tilde{L} + \Delta\tilde{L}}{\tilde{E}(\tilde{r}^2 + a^2)^2 - 2a\tilde{L}\tilde{r} - \Delta a^2\tilde{E}}. \quad (96)$$

This algorithm will provide a worldline in coordinate time $\tilde{r}(\tilde{t})$ which will be used for our waveforms. We stress here that $\tilde{r}(\tilde{t})$ is continuous and (once) differentiable.

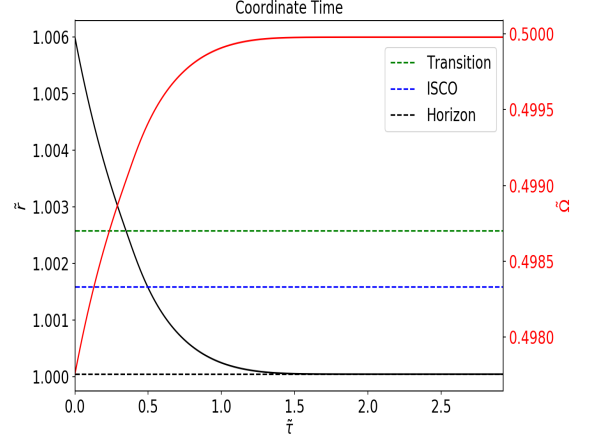


Figure 4: The red curve shows the orbital velocity $\tilde{\Omega}$ and the black curve shows the trajectory in coordinate time $\tilde{r}(\tilde{t})$. Notice the smooth evolution of both $\tilde{r}(\tilde{t})$ and $\tilde{\Omega}$ during the start of the transition (green dashed curve). This smooth evolution continues through the ISCO (blue dashed curve) and evolves towards the horizon (black dashed curve).

C. Near-Extremal Waveform

Following [42], the root mean square (rms) amplitude of gravitational waves emitted towards infinity at harmonic m is given by $h_{o,m} = \sqrt{\langle h_{+,m}^2 + h_{\times,m}^2 \rangle}$. The plus and cross each represent individual transverse-traceless polarisations of the gravitational wave strain h . The amplitudes are averaged $\langle \cdot \rangle$ over the direction and over the period of the waves. Furthermore, the rms amplitude is related to the outgoing radiation flux in harmonic m by

$$h_{o,m} = \frac{2M\sqrt{\eta\dot{\tilde{E}}_{\infty,m}}}{m\tilde{\Omega}D} \quad (97)$$

for distance D and outgoing fluxes defined by

$$\dot{\tilde{E}}_{\infty,m} = \eta\mathcal{A}_m\tilde{\Omega}^{2+2m/3}\dot{\mathcal{E}}_{\infty,m} \quad (98)$$

where the amplitude \mathcal{A}_m is defined by

$$\mathcal{A}_m = \frac{8(m+1)(m+2)(2m)!m^{2m-1}}{(m-1)[2^m m! (2m+1)!]^2}$$

for $m \geq 2$. In the above equation, $\dot{\mathcal{E}}_{\infty,m}$ represents the relativistic correction to $\dot{\tilde{E}}_{\infty,m}$ at each harmonic m . We highlight here that Eq.(97) is valid for a particle on a circular orbit on the equatorial plane in the small mass ratio limit $\eta \rightarrow 0$.

An EMRI signal a superposition of infinitely many harmonics of the fundamental frequency $\tilde{\Omega}$. In other

words, one can write

$$h = \sum_{m=2}^{\infty} h_{o,m} \sin(2\pi \tilde{f}_m \tilde{t} + \phi), \quad (99)$$

with frequencies defined by harmonics of the orbital frequency $\tilde{\Omega}$

$$\tilde{f}_m = m \cdot \frac{\tilde{\Omega}}{2\pi}. \quad (100)$$

Recall that the total emission of radiation through gravitational waves is related to the outgoing and ingoing flux by

$$\begin{aligned} \dot{\tilde{E}}_{GW} &= \dot{\tilde{E}}_{\infty} + \dot{\tilde{E}}_H \\ &= \sum_{m=2}^{\infty} (\dot{\tilde{E}}_{\infty,m} + \dot{\tilde{E}}_{H,m}). \end{aligned} \quad (101)$$

where $\dot{\tilde{E}}_{H,m}$ is the ingoing flux (towards the horizon) at each harmonic m . Using the exact results from the BHPT for a spin parameter of $a = 1 - 10^{-9}$, we constructed a cubic spline for each outgoing flux $\dot{\tilde{E}}_{\infty,m}$. Our results are plotted in 5. It should be clear that including the higher order modes become increasingly important as the spin parameter becomes unity. This has already been observed in [25]. Hence, for near-extremal systems, only using the $m = 2$ harmonic is not an accurate representation of the EMRI signal in general.

As seen in figure 5, we truncate at the eleventh harmonic in the outgoing flux (101) to model a full near-extremal waveform with suitable accuracy. We believe that a near-extremal waveform using the first 11 harmonics is accurate enough for parameter estimation studies [49]. We argue that this approximation is justified since the outgoing fluxes directly effect the amplitude of the GWs. It is the phase evolution that is determined by the total flux $\dot{\tilde{E}}_{GW}$ rather than the outgoing fluxes $\dot{\tilde{E}}_{\infty}$. So, to do interesting science, it is much more important to have an accurate representation of the phase evolution of the waveform. Hence forth we will only consider the first 11 modes for the remainder of our study and use the model

$$h \approx \sum_{m=2}^{11} h_{o,m} \sin(2\pi \tilde{f}_m \tilde{t} + \phi). \quad (102)$$

Once the ISCO is reached, we smoothly extrapolate each of the fluxes $\dot{\tilde{E}}_{\infty,m} \rightarrow 0$, as $\tilde{r} \rightarrow \tilde{r}_+$. This is a similar approach to that found in Taracchini *et al* in [50]. Using (102) and the results obtained in this paper, we plot a full near-extremal waveform, encapsulating transition from inspiral to plunge, in Fig.(6). We notice that the waveform in Figure 6 exhibits the usual dampening

before the ISCO is reached as seen by Gralla *et al* in [23]. This, qualitatively, is a unique feature to near-extreme EMRIs as a gravitational wave source.

V. CONCLUSION

This paper has presented a solution to the problem of the transition from inspiral to plunge, *for any* primary spin, for EMRIs on circular and equatorial orbits. This work has extended the treatment of Ori & Thorne [27] which was the first analysis of this problem but did not apply to systems with near-extremal spins. This work also extended the analysis of [28] which did consider near extremal spins, by providing a better physical interpretation of the procedure, identifying a missing term in the analysis and updating the treatment to use recent calculations of the near-extremal energy flux. We have also carefully identified the scaling of the various higher order terms arising from effects such as eccentricity and non-geodesic past-history to carefully demonstrate that these are all sub-dominant. Previous treatments have assumed that the quasi-circular assumption holds throughout the inspiral, but without rigorous justification. We have demonstrated that initial eccentricities excited during the adiabatic inspiral regime grow by the time the transition regime is reached, but are still sufficiently small to be sub-dominant. We have shown that corrections to the flux balance law (25) arising from eccentricity and from the non-geodesic past-history of the orbital evolution are also sub-dominant, if only marginally, but there are non-trivial deviations from the linear-in-proper-time evolution of energy and angular momentum in (26) that was assumed in OT. These deviations are encoded in the evolution of the parameter $\tilde{\Gamma}_{\odot}$ through the transition regime.

Based on these arguments, we have derived a transition equation for each of the three scaling regimes: $\eta \ll \epsilon$, $\eta \sim \epsilon$ and $\eta \gg \epsilon$ and described a numerical scheme to generate a full inspiral trajectory in coordinate time, from radial infinity to the horizon. For near-extremal black holes, we found that there was no need to attach a geodesic plunge onto the transition solution as the inspiraling object reaches the horizon while still within the transition regime. Finally, we used these inspiral trajectories to construct a full near-extremal waveform using results from the BHPT [51].

The OT procedure is straightforward, but with surprisingly rich phenomenology. Through semi-analytic means, one is able to derive an equation which describes the dynamics within the vicinity of the ISCO. However, *in practice*, the OT theory has several shortcomings. The point at which the transition solution is taken to start has a significant influence on the time it takes the particle to reach the horizon and so the OT procedure

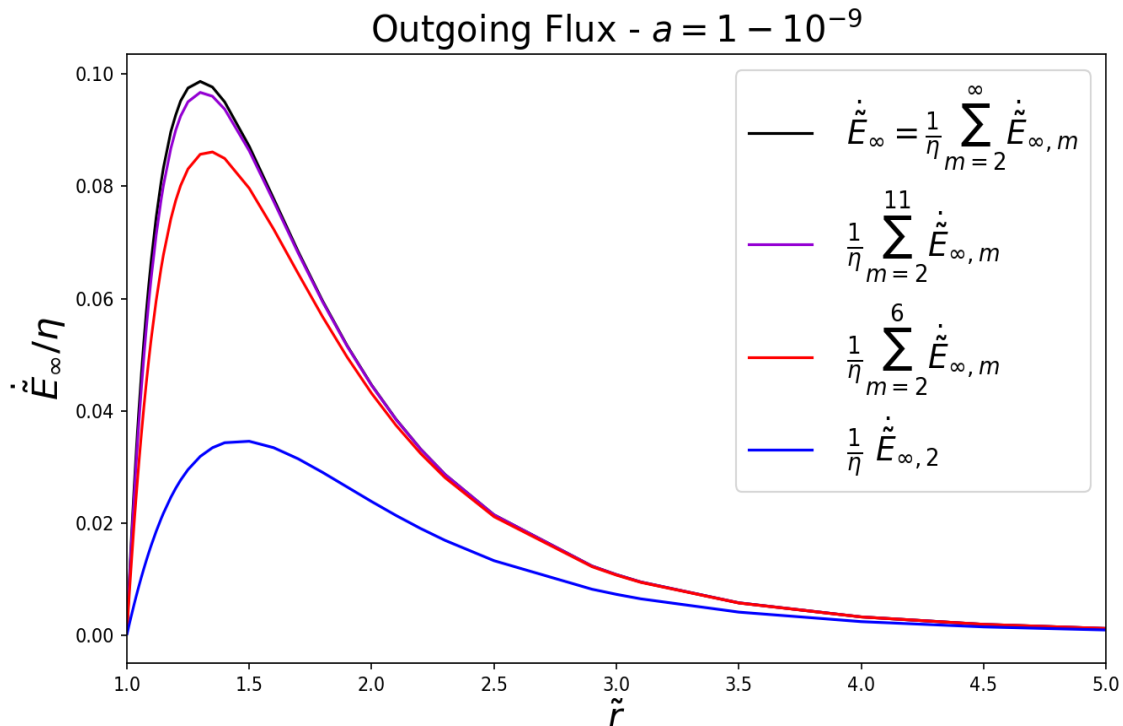


Figure 5: Comparison of the total energy flux at infinity (black curve) including different harmonic $\dot{E}_{\infty,m}$ contributions. Note that at $\tilde{r} \approx 1.3$, the $m = 2$ harmonic energy flux $\dot{E}_{\infty,2}$ contributes $\sim 32\%$ of the total energy flux, whereas including the first 11 harmonics (violet curve) contributes $\sim 98\%$ at the least.

does *not* define a unique worldline given a particular set of parameters for the source. This is clearly not physical behaviour. We argued in section IV A that if the switch from the adiabatic inspiral to the transition equation is made when the constraint $\delta L \sim \eta^{4/3} \epsilon^{-2/15}$ is satisfied, the solution will be almost unique. This was verified numerically and we found it leads to plunge times consistent to $\pm 0.5M$. This very same problem was found in [48] but they saw no effect in their waveform analysis. Another issue with the OT method is that it can lead to discontinuities in the constants of motion $\tilde{E}(\tau)$ and $\tilde{L}(\tau)$ if the OT equations are integrated backwards from the ISCO rather than forwards from the point of the switch from the adiabatic inspiral to transition regime. Discontinuities in the constants of motion lead to discontinuities in the coordinate time trajectories and in the waveforms which must be avoided if these waveforms are to give physically reasonable results in parameter estimation studies. Our solution, which was to integrate forward not backwards, yields continuous, but not first order differentiable, trajectories. The procedure described in [48] provides both. For parameter estimation studies we only require continuity of \tilde{E} and \tilde{L} and first order differentiability of $\tilde{r}(\tilde{t})$ and so our procedure should be

sufficient, although this should be examined more carefully.

There are natural extensions of this work. Most immediately the waveforms constructed in this paper can be used to carry out a parameter estimation study to understand how well the parameters of near-extremal EMRIs can be measured with observations by LISA. Of particular interest is how well the spin can be determined, since the identification of an object that definitely has spin above the Thorne limit would be of profound significance. It would also be of interest to extend this analysis to apply to inspirals that are not circular and equatorial. The extension to non-equatorial, but circular, orbits was presented in [48], who corrected the analysis of [29] to orbits of arbitrary inclination in the Kerr spacetime. No one, as of yet, has considered the transition from inspiral to plunge in the case of eccentric orbits, which are expected for EMRIs formed through standard astrophysical channels [11]. The extension to eccentric orbits will require more careful modelling of the self-force and the use of the (eccentricity-dependent) separatrix in place of the ISCO among other complications. A model of the transition for inspirals on *generic* orbits into black holes *arbitrary* spin will be invaluable

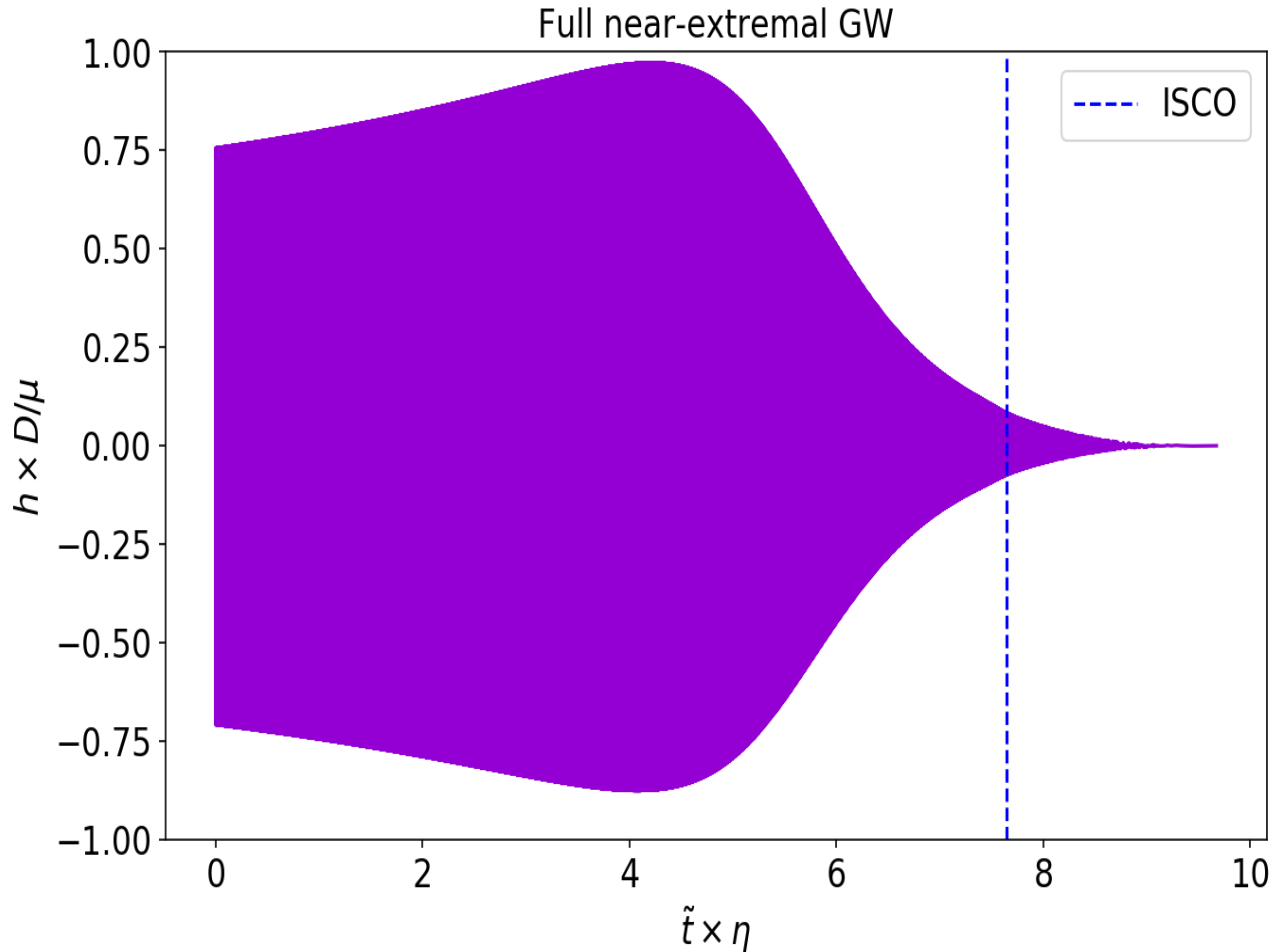


Figure 6: Here we plot both the root mean square gravitational waveform for both inspiral, transition and plunge using the first eleven harmonics. Notice the smooth evolution of $h(\tilde{t})$. We terminate evolution of the waveform close to the plunge $\tilde{r} = \tilde{r}_+ + \delta$ for suitably chosen $0 < \delta \ll 1$, otherwise the waveform will continue to decay for infinite coordinate time. This is obvious since the (point-like) particle (as observed from infinity) will never reach the horizon. In this example, we considered $a = 1 - 10^{-9}$ and $\eta = 10^{-5}$ so that we are in the $\epsilon \sim \eta$ regime.

for the analysis of future LISA EMRI observations and is an important future topic of study.

ACKNOWLEDGMENTS

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Appendix A: The Innermost Stable Circular Orbit

In this appendix we review the main properties of the function $G(\tilde{r}, \tilde{E}, \tilde{L})$ determining the radial geodesic (2)

$$G(\tilde{r}, \tilde{E}, \tilde{L}) = \tilde{E}^2 - 1 + \frac{a^2(\tilde{E}^2 - 1) - \tilde{L}^2}{\tilde{r}^2} + \frac{2(a\tilde{E} - \tilde{L})^2}{\tilde{r}^3} + \frac{2}{\tilde{r}}. \quad (\text{A1})$$

together with its derivatives when evaluated at the ISCO orbit \tilde{r}_{isco} . The spin dependence of these quantities will play a critical role in the identification of the different transition regimes discussed in section III.

Remember the ISCO radial coordinate \tilde{r}_{isco} is char-

acterised by marginal stability

$$G(\tilde{r}, \tilde{E}_{\text{isco}}, \tilde{L}_{\text{isco}}) = \left. \frac{\partial G}{\partial \tilde{r}} \right|_{\text{isco}} = \left. \frac{\partial^2 G}{\partial \tilde{r}^2} \right|_{\text{isco}} = 0. \quad (\text{A2})$$

Labelling the energy and angular momentum of the ISCO orbit by \tilde{E}_{isco} and \tilde{L}_{isco} , we can solve the second and third constraint equations by

$$\begin{aligned} \tilde{L}_{\text{isco}} &= \frac{\tilde{r}_{\text{isco}}^2 - 3a^2 + 6\tilde{r}_{\text{isco}}}{2\sqrt{3}\tilde{r}_{\text{isco}}}, \\ \tilde{E}_{\text{isco}} &= \frac{6\tilde{r}_{\text{isco}} - 3a^2 - \tilde{r}_{\text{isco}}^2}{2\sqrt{3}a\tilde{r}_{\text{isco}}}. \end{aligned} \quad (\text{A3})$$

Plugging these into $G(\tilde{r}_{\text{isco}}, \tilde{E}_{\text{isco}}, \tilde{L}_{\text{isco}}) = 0$, one derives the relation

$$\frac{2}{3\tilde{r}_{\text{isco}}} = 1 - \tilde{E}_{\text{isco}}^2, \quad (\text{A4})$$

which combined with (A3) yields

$$\tilde{r}_{\text{isco}}^2 - 6\tilde{r}_{\text{isco}} + 8a\sqrt{\tilde{r}_{\text{isco}}} - 3a^2 = 0, \quad (\text{A5})$$

whose solution $r_0(a)$ reproduces (14) [52]. This equality allows to simplify the energy (5) and angular momentum (6) of equatorial circular orbits when evaluated at ISCO to

$$\begin{aligned} \tilde{E}_{\text{isco}} &= \frac{1 - 2/\tilde{r}_{\text{isco}} + a/\tilde{r}_{\text{isco}}^{3/2}}{\sqrt{1 - 3/\tilde{r}_{\text{isco}} + 2a/\tilde{r}_{\text{isco}}^{3/2}}} = \frac{4\sqrt{\tilde{r}_{\text{isco}}} - 3a}{\sqrt{3}\tilde{r}_{\text{isco}}} \quad (\text{A6}) \\ \tilde{L}_{\text{isco}} &= \tilde{r}_{\text{isco}}^{1/2} \frac{1 - 2a/\tilde{r}_{\text{isco}}^{3/2} + a^2/\tilde{r}_{\text{isco}}^2}{\sqrt{1 - 3/\tilde{r}_{\text{isco}} + 2a/\tilde{r}_{\text{isco}}^{3/2}}} = 2\sqrt{3} - \frac{4a}{\sqrt{3}\tilde{r}_{\text{isco}}}. \end{aligned} \quad (\text{A7})$$

Armed with these identities, we move towards the evaluation of the derivatives controlling the expansions (28) relevant to the transition regime. First, we introduce some notation

$$\begin{aligned} A_n &= \left. \frac{\partial^n G}{\partial \tilde{r}^n} \right|_{\text{isco}}, \\ B_n &= \left(\frac{\partial^{n+1} G}{\partial \tilde{r}^n \partial \tilde{E}} \tilde{\Omega} + \frac{\partial^{n+1} G}{\partial \tilde{r}^n \partial \tilde{L}} \right)_{\text{isco}}, \\ C_n &= \left(\frac{\partial^{n+2} G}{\partial \tilde{r}^n \partial \tilde{E}^2} \tilde{\Omega}^2 + 2 \frac{\partial^{n+2} G}{\partial \tilde{r}^n \partial \tilde{L} \partial \tilde{E}} \tilde{\Omega} + \frac{\partial^{n+2} G}{\partial \tilde{r}^n \partial \tilde{L}^2} \right)_{\text{isco}} \end{aligned} \quad (\text{A8})$$

with $\tilde{\Omega}$ as in(9). Either by explicit calculation or by induction, one can prove for any integer $n \geq 0$

$$\begin{aligned} \frac{\partial^n G}{\partial \tilde{r}^n} &= (-1)^n \left(\frac{(n+2)!(a\tilde{E} - \tilde{L})^2}{\tilde{r}^{n+3}} + \frac{(n+1)!(a^2(\tilde{E}^2 - 1) - \tilde{L}^2)}{\tilde{r}^{n+2}} + \frac{2n!}{\tilde{r}^{n+1}} \right) - \delta_{n0}(1 - \tilde{E}^2), \\ \frac{\partial^{n+1} G}{\partial \tilde{r}^n \partial \tilde{E}} &= (-1)^n \left(\frac{2(n+2)!(a^2\tilde{E} - a\tilde{L})}{\tilde{r}^{n+3}} + \frac{2(n+1)!a^2\tilde{E}}{\tilde{r}^{n+2}} \right) + 2\delta_{n0}\tilde{E}, \\ \frac{\partial^{n+1} G}{\partial \tilde{r}^n \partial \tilde{L}} &= -(-1)^n \left(\frac{2(n+2)!(a\tilde{E} - \tilde{L})}{\tilde{r}^{n+3}} + \frac{2(n+1)\tilde{L}}{\tilde{r}^{n+2}} \right), \\ \frac{\partial^{n+2} G}{\partial \tilde{r}^n \partial \tilde{L}^2} &= (-1)^n \left(\frac{2(n+2)!}{\tilde{r}^{n+3}} - \frac{2(n+1)!}{\tilde{r}^{n+2}} \right), \\ \frac{\partial^{n+2} G}{\partial \tilde{r}^n \partial \tilde{L} \partial \tilde{E}} &= -(-1)^n \left(\frac{2a(n+2)!}{\tilde{r}^{n+3}} \right), \\ \frac{\partial^{n+2} G}{\partial \tilde{r}^n \partial \tilde{E}^2} &= (-1)^n \left(\frac{2a^2(n+2)!}{\tilde{r}^{n+3}} - \frac{2a^2(n+1)!}{\tilde{r}^{n+2}} \right) + 2\delta_{n0} \end{aligned}$$

where δ_{n0} stands for the Kronecker delta. Finally, evaluating these derivatives at $(\tilde{r}_{\text{isco}}, \tilde{E}_{\text{isco}}, \tilde{L}_{\text{isco}})$ and using the

properties (A2)-(A7), we can derive the exact results

$$A_n = (1 - \delta_{n0}) \frac{(-1)^n (n-1)(n-2)n!}{3\tilde{r}_{\text{isco}}^{1+n}}, \quad (\text{A9})$$

$$B_n = 2(1 - \delta_{n0})(-1)^n (n+1)! \frac{n(a - \sqrt{\tilde{r}_{\text{isco}}}) + a - 2\sqrt{\tilde{r}_{\text{isco}}} + \tilde{r}_{\text{isco}}^{3/2}}{\tilde{r}_{\text{isco}}^n \sqrt{3\tilde{r}_{\text{isco}}} (a - \sqrt{\tilde{r}_{\text{isco}}}) (a + \tilde{r}_{\text{isco}}^{3/2})}, \quad (\text{A10})$$

$$C_n = 2 \cdot \frac{\delta_{0n} - (-1)^n (2a + \sqrt{\tilde{r}_{\text{isco}}}[\tilde{r}_{\text{isco}} - 2 - n])(n+1)!}{\tilde{r}_{\text{isco}}^{(2n+1)/2} (a + \tilde{r}_{\text{isco}}^{3/2})^2}, \quad (\text{A11})$$

Notice equations (A9)-(A10) recover the familiar identities for circular orbits

$$\left(\frac{\partial G}{\partial \tilde{E}} \tilde{\Omega} + \frac{\partial G}{\partial \tilde{L}} \right)_{\text{isco}} = 0, \\ G_0 = \left. \frac{\partial G}{\partial \tilde{r}} \right|_{\text{isco}} = \left. \frac{\partial^2 G}{\partial \tilde{r}^2} \right|_{\text{isco}} = 0.$$

Let us study the behaviour of these derivatives for

near extremal black holes, i.e. in the limit $\epsilon \rightarrow 0$ as introduced in section II. Remember \tilde{r}_{isco} is given by

$$\tilde{r}_{\text{isco}} \rightarrow 1 + 2^{1/3} \epsilon^{2/3} + \frac{7}{4 \cdot 2^{1/3}} \epsilon^{4/3} + \mathcal{O}(\epsilon^2). \quad (\text{A12})$$

Using this expansion together with $a = \sqrt{1 - \epsilon^2}$, we can evaluate the leading terms of all previous derivatives to be

$$A_n \rightarrow (1 - \delta_{n0})(n-2)(n-1) \left(\frac{1}{3} (-1)^n \Gamma(n+1) + \mathcal{O}(\epsilon^{2/3}) \right), \quad (\text{A13})$$

$$B_n \rightarrow (1 - \delta_{n0}) \frac{(-1)^n \Gamma(n+2)}{\sqrt{3}} \left(n-1 - \frac{4n^2 + n + 1}{2^{5/3}} \epsilon^{2/3} \right) + \mathcal{O}(\epsilon^{4/3}), \quad (\text{A14})$$

$$C_n \rightarrow -\frac{1}{4} (-1)^n (n-1) \left(-2 + 2^{1/3} \epsilon^{2/3} [2n+3] \right) (n+1)! + \frac{(-1)^n (4n^2 - 3n - 3)(n+2)!}{2^{10/3}} \epsilon^{4/3} + p_{n0} \quad (\text{A15})$$

$$\left. \frac{\partial^{n+1} G}{\partial \tilde{r}^n \partial \tilde{E}} \right|_{\text{isco}} \rightarrow \frac{2}{\sqrt{3}} (-1)^{n+1} (n+1)! [(n+1) - 2^{1/3} (n^2 + 3n + 3) \epsilon^{2/3}] + \frac{2}{\sqrt{3}} (1 + 2^{1/3} \epsilon^{2/3}) \delta_{n0} + \mathcal{O}(\epsilon^{4/3}) \quad (\text{A16})$$

$$\left(\frac{\partial^{n+2} G}{\partial \tilde{r}^n \partial \tilde{E}} + \tilde{\Omega} \frac{\partial^{n+2} G}{\partial \tilde{r}^n \partial \tilde{E}^2} \right)_{\text{isco}} \rightarrow (\delta_{0n} - (-1)^n (n+1)^2 n!) + \frac{(-1)^n (7 + 13n + 4n^2)(n+1)! - 3\delta_{0n}}{2^{5/3}} \epsilon^{2/3} + \mathcal{O}(\epsilon^{4/3}) \quad (\text{A17})$$

$$\left. \frac{\partial^{n+2} G}{\partial \tilde{r}^n \partial \tilde{E}^2} \right|_{\text{isco}} \rightarrow 2((-1)^n (3+n)(n+1)! + \delta_{0,n}) - 2^{4/3} (-1)^n (4+n)(n+2)! \epsilon^{2/3} + \mathcal{O}(\epsilon^{4/3}) \quad (\text{A18})$$

where we defined

$$p_{n0} = \frac{2 - 3 \cdot 2^{1/3} \epsilon^{2/3}}{4} \delta_{0n} + \mathcal{O}(\epsilon^2). \quad (\text{A19})$$

What we learn is that $A_n \sim \mathcal{O}(1)$ for all $n \geq 3$, $B_1 \sim \epsilon^{2/3}$, $B_n \sim \mathcal{O}(1)$ for $n \geq 2$, $C_0 \sim C_1 \sim \epsilon^{4/3}$ and $C_n \sim \mathcal{O}(1)$ for $n > 1$. Furthermore, (A16) and (A17) are $\mathcal{O}(\epsilon^{2/3})$ for $n = 0$ and $\mathcal{O}(1)$ for $n \geq 1$, whereas (A18) is always $\mathcal{O}(1)$.

Appendix B: Retrograde Orbits

In this section, we will restrict our attention to retrograde orbits. That is, orbits opposing the direction with the primaries angular momenta. These orbits are of interest because the ISCO is much further away from the horizon, which implies that the radial distance travelled during plunge time is much longer. Due to frame-dragging, we expect the ISCO to be farther from the

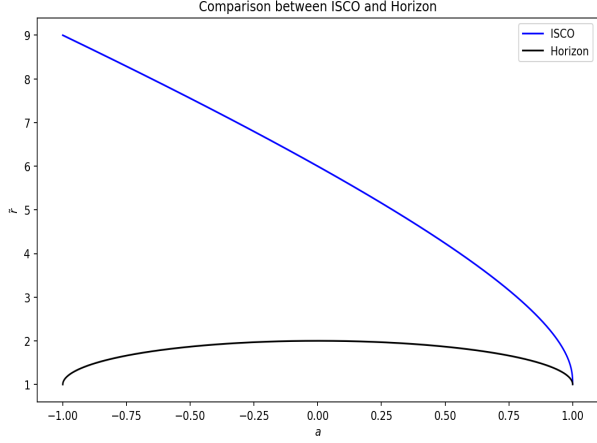


Figure 7: This plot shows the relationship between \tilde{r}_{isco} and \tilde{r}_+ with the spin parameter $a \in [-1, 1]$. For negative a , the compact object is on a retrograde orbit and positive otherwise (prograde). Notice that, for $a > 0$, in Boyer Lindquist coordinates the ISCO and horizon coincide. However, for $a < 0$, the horizons radial coordinate remains the same (this is obvious) and the ISCO grows in radial distance from the horizon.

hole since the space is dragged in the opposite direction to the compact objects orbital direction [see Fig.7]. In light of our previous discussion, we consider the case when the primary is near extremal with a compact object on a retrograde orbit. We characterise this using a spin parameter of negative parity $a \rightarrow -1$. We parametrise a by

$$a \rightarrow -\sqrt{1 - \epsilon^2}, \text{ where } \epsilon \ll 1. \quad (\text{B1})$$

notice that the horizon takes the same form as in the case of prograde orbits

$$\tilde{r}_+ = 1 + \sqrt{1 - a^2} = 1 + \epsilon$$

as to be expected. Using a spin parameter of negative parity, the expressions for $\tilde{E}, \tilde{L}, \tilde{\Omega}$ and \tilde{r}_{isco} remain the same. However, each quantity will be different at the ISCO of a retrograde orbit. By substituting Eq. (B1) into Eqs. (6,5, 14,9) and expanding for small $\epsilon \ll 1$

$$\tilde{r}_{\text{isco}} = 9 - \frac{45}{32}\epsilon^2 + \mathcal{O}(\epsilon^4) \quad (\text{B2})$$

$$\tilde{E}_{\text{isco}} = \frac{5}{3\sqrt{3}} - \frac{1}{96\sqrt{3}}\epsilon^2 + \mathcal{O}(\epsilon^4) \quad (\text{B3})$$

$$\tilde{L}_{\text{isco}} = \frac{22}{3\sqrt{3}} - \frac{3\sqrt{3}}{16}\epsilon^2 + \mathcal{O}(\epsilon^4) \quad (\text{B4})$$

$$\tilde{\Omega}_{\text{isco}} = \frac{1}{26} + \frac{373}{43264}\epsilon^2 + \mathcal{O}(\epsilon^4). \quad (\text{B5})$$

Notice here that the expansion in ϵ is no longer increasing in powers of $\epsilon^{2/3}$ and now in ϵ^2 . Also notice that $|\tilde{r}_{\text{isco}} - \tilde{r}_+| \sim \mathcal{O}(1)$ rather than of order $\epsilon^{2/3}$ like in the case of near-extremal prograde orbits. Like we have done previously, we consider the Kerr radial velocity expanded around the ISCO

$$\left(\frac{dR}{d\tilde{r}}\right)^2 \simeq -\frac{2}{3}\alpha R^3 + 2\beta\delta LR + \gamma\delta LR^2 + \Gamma_{\odot} + \dots \quad (\text{B6})$$

with small variables

$$\tilde{E} - \tilde{E}_{\text{isco}} = \tilde{\Omega}_{\text{isco}}\delta E \quad (\text{B7})$$

$$\tilde{L} - \tilde{L}_{\text{isco}} = \delta L \quad (\text{B8})$$

$$\tilde{r} - \tilde{r}_{\text{isco}} = R. \quad (\text{B9})$$

The coefficients in (B6) can be approximated for $\epsilon \rightarrow 0$ under the retrograde condition Eq.((B1))

$$\begin{aligned} \alpha &= -\frac{1}{4} \frac{\partial^3 G}{\partial \tilde{r}^3} \Big|_{\text{isco}} \rightarrow \frac{1}{6561} \\ \beta &= \frac{1}{2} \left(\frac{\partial^2 G}{\partial \tilde{r} \partial \tilde{L}} + \tilde{\Omega} \frac{\partial^2 G}{\partial \tilde{r} \partial \tilde{E}} \right)_{\text{isco}} \rightarrow \frac{4}{351\sqrt{3}} \\ \gamma &= \frac{1}{2} \left(\frac{\partial^3 G}{\partial \tilde{r}^2 \partial \tilde{L}} + \tilde{\Omega} \frac{\partial^3 G}{\partial \tilde{r}^2 \partial \tilde{E}} \right)_{\text{isco}} \rightarrow -\frac{1}{351\sqrt{3}}. \end{aligned}$$

and Γ_{\odot} in (B6) defined through equation (31). Notice that none of the coefficients in our transition equation of motion depend on the extremality parameter ϵ . This gives us no reason to introduce any scalings on \tilde{r}, \tilde{r} and δL . As such, let us introduce similar scalings to O&T

$$R = \eta^{2/5} \alpha^{-3/5} (\beta\kappa)^{2/5} X \quad (\text{B10})$$

$$\tau - \tilde{r}_{\text{isco}} = \eta^{-1/5} (\alpha\beta\kappa)^{-1/5} T \quad (\text{B11})$$

$$\delta E - \delta L = \eta^{6/5} Y. \quad (\text{B12})$$

$$\delta L = -\eta^{4/5} (\alpha\beta)^{-1/5} \kappa^{4/5} T \quad (\text{B13})$$

Substituting these results into Eq.(B6) we find that

$$\left(\frac{dX}{dT}\right)^2 = -\frac{2}{3}X^3 - 2XT + \alpha^{4/5} (\eta\beta\kappa)^{-6/5} \Gamma_{\odot}. \quad (\text{B14})$$

Since $R \sim \eta^{2/5}$, we only need the first term of Γ_{\odot}

$$\Gamma_{\odot} = \eta^{6/5} \tilde{\Omega}_{\text{isco}} Y \frac{\partial G}{\partial \tilde{E}} \Big|_{\text{isco}}. \quad (\text{B15})$$

and taking derivatives of Eq.(B14) and following an identical procedure to ,

$$\frac{d^2 X}{dT^2} = -X^2 - T \quad (\text{B16})$$

with evolution equation for Y

$$\frac{dY}{dT} = -\frac{\partial \log \tilde{\Omega}}{\partial \tilde{r}} \Big|_{\text{isco}} (C_1 K_0)^{-1} X \quad (\text{B17})$$

for $K_0 = \alpha^{4/5}(\beta\kappa)^{-6/5}$. Which is precisely the equation of motion for the transition regime derived by O&T in [27]. Although the constants present in the change of coordinates are different, the physics and ultimate end goal are the same. As a result, we stop our analysis of retrograde orbits here since we feel that this problem has already been solved by the community for smaller spin values $a \geq -0.999$. We conclude that, for near-extremal retrograde orbits, there is nothing new to learn about the transition regime. It can be solved in the matter of O&T in [27]. We do remark that the quantity κ can no longer be computed using the near-extremal formula defined by $\dot{E}_{GW} = (\tilde{C}_H + \tilde{C}_\infty)(\tilde{r} - \tilde{r}_+)/\tilde{r}_+$ (see Fig.1). Instead we have to use the (numerical) quantity

$$\begin{aligned} \kappa &= \left(\tilde{\Omega}^{-1} \frac{d\tilde{t}}{d\tilde{\tau}} \frac{d\tilde{E}}{d\tilde{\tau}} \right)_{\text{isco}} \\ &= \left(-\frac{32}{5} \tilde{\Omega}^{7/3} \frac{1 + a/\tilde{r}^{3/2}}{\sqrt{1 - 3/\tilde{r} + 2a/\tilde{r}^{3/2}}} \dot{\mathcal{E}} \right)_{\text{isco}}. \end{aligned}$$

Various results are tabulated (including retrograde orbits) in [42]. The downside of this equation is that it can only be evaluated numerically.

Appendix C: Osculating Elements Equations

The derivative of the radial geodesic equation (2) with respect to $\tilde{\tau}$ yields (24)

$$\frac{d^2\tilde{r}}{d\tilde{\tau}^2} - \frac{1}{2} \frac{\partial G}{\partial \tilde{r}} = \frac{1}{2\dot{\tilde{r}}} \left(\dot{\tilde{E}} \frac{\partial G}{\partial \tilde{E}} + \dot{\tilde{L}} \frac{\partial G}{\partial \tilde{r}} \right). \quad (\text{C1})$$

The purpose of this appendix is to review how this equation is equivalent to a forced geodesic equation

$$\frac{d^2\tilde{x}^\mu}{d\tilde{\tau}^2} + \Gamma_{\rho\sigma}^\mu \frac{d\tilde{x}^\rho}{d\tilde{\tau}} \frac{d\tilde{x}^\sigma}{d\tilde{\tau}} = \tilde{f}^\mu, \quad (\text{C2})$$

for $\tilde{x}^\mu = (\tilde{r}, \tilde{t}, \theta, \phi)$ where the force per unit mass \tilde{f}^μ is determined by the fluxes [53]

$$f^\alpha = \frac{\partial \dot{\tilde{x}}^\alpha}{\partial \tilde{E}} \dot{\tilde{E}} + \frac{\partial \dot{\tilde{x}}^\alpha}{\partial \tilde{L}} \dot{\tilde{L}}. \quad (\text{C3})$$

Since the four velocity \tilde{u}^α is normalised as $\tilde{u}^\alpha \tilde{u}_\alpha = -1$, differentiation with respect to proper time yields the identity

$$\tilde{f}^\alpha \tilde{u}_\alpha = 0. \quad (\text{C4})$$

Evaluating (C2) for the radial direction $\tilde{x}^r = \tilde{r}(\tilde{\tau})$ and solving (C4) for \tilde{f}^r yields

$$\frac{d^2\tilde{r}}{d\tilde{\tau}^2} + \Gamma_{\rho\sigma}^r \frac{d\tilde{x}^\rho}{d\tilde{\tau}} \frac{d\tilde{x}^\sigma}{d\tilde{\tau}} = -\frac{\tilde{f}^\phi \tilde{u}_\phi + \tilde{f}^t \tilde{u}_t}{\tilde{u}_r}. \quad (\text{C5})$$

For the left hand side, one simply needs to calculate the relevant Christoffel symbols and use the Kerr geodesic relations

$$\begin{aligned} \Gamma_{\rho\sigma}^{\tilde{r}} \frac{d\tilde{x}^\rho}{d\tilde{\tau}} \frac{d\tilde{x}^\sigma}{d\tilde{\tau}} &= \Gamma_{\tilde{r}\tilde{r}}^{\tilde{r}} \left(\frac{d\tilde{r}}{d\tilde{\tau}} \right)^2 + \Gamma_{\phi\phi}^{\tilde{r}} \left(\frac{d\phi}{d\tilde{\tau}} \right)^2 + \\ &\quad \Gamma_{\tilde{t}\tilde{t}}^{\tilde{r}} \left(\frac{d\tilde{t}}{d\tilde{\tau}} \right)^2 + 2\Gamma_{\tilde{t}\phi}^{\tilde{r}} \frac{d\tilde{t}}{d\tilde{\tau}} \frac{d\phi}{d\tilde{\tau}} \end{aligned} \quad (\text{C6})$$

so that

$$\begin{aligned} \Gamma_{\rho\sigma}^{\tilde{r}} \frac{d\tilde{x}^\rho}{d\tilde{\tau}} \frac{d\tilde{x}^\sigma}{d\tilde{\tau}} &= \frac{3(a\tilde{E} - \tilde{L})^2}{\tilde{r}^4} - \frac{a^2(\tilde{E}^2 - 1) - \tilde{L}^2}{\tilde{r}^3} + \frac{1}{\tilde{r}^2} \\ &= -\frac{1}{2} \frac{\partial G}{\partial \tilde{r}} \end{aligned}$$

For the right hand side, $\tilde{f}^{\tilde{r}}$, since the Kerr metric components $g_{\mu\nu}$ and functionally absent of the coordinates \tilde{t} and ϕ , there are two killing vectors, time-like $\xi^\mu = (1, 0, 0, 0)$ and space-like $\psi^\mu = (0, 0, 0, 1)$. Hence, due to the existence of these killing vectors, there exists the two conserved quantities

$$\tilde{E} = -\xi^\mu \dot{\tilde{x}}_\mu, \quad \tilde{L} = \psi^\mu \dot{\tilde{x}}_\mu. \quad (\text{C7})$$

Hence $\tilde{u}_\phi = \tilde{L}$ and $\tilde{u}_t = -\tilde{E}$. Finally, it's easy to show that

$$\tilde{u}_{\tilde{r}} = g_{\alpha\tilde{r}} \tilde{u}^{\tilde{r}} = g_{\tilde{r}\tilde{r}} \dot{\tilde{r}} = \frac{\tilde{r}^2}{\tilde{r}^2 - 2\tilde{r} + a^2} \dot{\tilde{r}} \quad (\text{C8})$$

Using these results, a little algebra shows that

$$\tilde{f}^{\tilde{r}} = \frac{\dot{\tilde{E}} g_1(\tilde{r}, \tilde{E}, \tilde{L}, a) + \dot{\tilde{L}} g_2(\tilde{r}, \tilde{E}, \tilde{L}, a)}{\dot{\tilde{r}}} \quad (\text{C9})$$

with

$$\begin{aligned} g_1(\tilde{r}, \tilde{E}, \tilde{L}, a) &= \left[\frac{\partial}{\partial \tilde{E}} \left(\frac{d\tilde{t}}{d\tilde{\tau}} \right) \tilde{E} - \frac{\partial}{\partial \tilde{E}} \left(\frac{d\phi}{d\tilde{\tau}} \right) \tilde{L} \right] \frac{\Delta}{\tilde{r}^2} \\ g_2(\tilde{r}, \tilde{E}, \tilde{L}, a) &= \left[\frac{\partial}{\partial \tilde{L}} \left(\frac{d\tilde{t}}{d\tilde{\tau}} \right) \tilde{E} - \frac{\partial}{\partial \tilde{L}} \left(\frac{d\phi}{d\tilde{\tau}} \right) \tilde{L} \right] \frac{\Delta}{\tilde{r}^2}. \end{aligned}$$

Substituting in the geodesic equations defined through equations (2) - (4), one can show that

$$\begin{aligned} g_1(r, \tilde{E}, \tilde{L}, a) &= \tilde{E} + \frac{2a(a\tilde{E} - \tilde{L})}{\tilde{r}^3} + \frac{a^2\tilde{E}}{\tilde{r}^2} = \frac{1}{2} \frac{\partial G}{\partial \tilde{E}} \\ g_2(\tilde{r}, \tilde{E}, \tilde{L}, a) &= \frac{2(\tilde{L} - a\tilde{E})}{\tilde{r}^3} - \frac{\tilde{L}}{\tilde{r}^2} = \frac{1}{2} \frac{\partial G}{\partial \tilde{L}} \end{aligned}$$

Hence

$$\tilde{f}^{\tilde{r}} = \frac{1}{2\dot{\tilde{r}}} \left(\dot{\tilde{E}} \frac{\partial G}{\partial \tilde{E}} + \dot{\tilde{L}} \frac{\partial G}{\partial \tilde{r}} \right). \quad (\text{C10})$$

as required.

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