Portugal. Math. (N.S.)
Vol. 76, Fasc. 1, 2019, 57–100
DOI 10.4171/PM/2027

Portugaliae Mathematica

Courant-sharp Robin eigenvalues for the square and other planar domains

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(Communicated by Pedro Freitas)

Abstract. This paper is devoted to the determination of the cases where there is equality in Courant's nodal domain theorem in the case of a Robin boundary condition. For the square, we partially extend the results that were obtained by Pleijel, Bérard–Helffer, Helffer–Persson Sundqvist for the Dirichlet and Neumann problems. After proving some general results that hold for any value of the Robin parameter h, we focus on the case where h is large. We consider the case where h is small in a second paper. We also obtain some semi-stability results for the number of nodal domains of a Robin eigenfunction of a domain with piecewise $C^{2,\alpha}$ boundary ($\alpha > 0$) as h large varies.

Mathematics Subject Classification: 35P99, 58J50, 58J37

Keywords: Courant-sharp, Robin eigenvalues, square, planar domains

Contents

1. Introduction

Let $\Omega \subset \mathbb{R}^m$, $m \geq 2$, be a bounded, connected, open set with Lipschitz boundary and let $h \in \mathbb{R}$, $h \geq 0$. The case when $h < 0$ is mathematically interesting but less motivated by Physics. The Robin eigenvalues of the Laplacian on Ω with parameter h are $\lambda_{k,h}(\Omega) \in \mathbb{R}$, $k \in \mathbb{N}$, $k \geq 1$, such that there exists a function $u_k \in H^1(\Omega)$ which satisfies

$$
\begin{aligned}
-\Delta u_k(x) &= \lambda_{k,h}(\Omega) u_k(x), \qquad x \in \Omega,\\
\frac{\partial}{\partial \nu} u_k(x) + h u_k(x) &= 0, \qquad x \in \partial\Omega,
\end{aligned}
$$

where ν is the outward-pointing unit normal to $\partial\Omega$.

We recall that by the minimax principle, the Robin problem is associated with the quadratic form:

$$
H^1(\Omega) \ni u \mapsto \int_\Omega |\nabla u|^2 \, dx + h \int_{\partial\Omega} |u_{\partial\Omega}|^2 \, d\sigma,
$$

where $u_{\partial\Omega}$ is the trace of u. So the spectrum is monotonically increasing with respect to h for $h \in [0, +\infty)$. That is, the Robin eigenvalues with $h > 0$ interpolate between the Neumann eigenvalues $(h = 0)$ and the Dirichlet eigenvalues $(h = +\infty)$.

The Robin eigenvalues satisfy the celebrated Courant nodal domain theorem [11] stating that any eigenfunction corresponding to $\lambda_{k,h}(\Omega)$ has at most k nodal domains. We consider the Courant-sharp Robin eigenvalues of Ω. We call a Robin eigenvalue $\lambda_{k,h}(\Omega)$ Courant-sharp if it has a corresponding eigenfunction that has exactly k nodal domains.

The study of the Courant-sharp eigenvalues can be motivated by the fact that the nodal partition of a corresponding eigenfunction is a spectral minimal partition. This was shown for the eigenfunctions of a Schrödinger operator with Dirichlet boundary condition in [26]. The case of the Neumann Laplacian was described in [27] (the case of the Robin Laplacian with positive parameter h holds analogously to the latter).

Typical questions about Courant-sharp eigenvalues of the Laplacian on a given domain are: How many are there, and how large are they? For the Robin case, these questions have recently received some attention in the literature for domains with sufficiently smooth boundary, see [22], [34]. As for the Dirichlet and Neumann eigenvalues, $\lambda_{1,h}(\Omega)$ and $\lambda_{2,h}(\Omega)$ are Courant-sharp for all $h \geq 0$.

A further interesting question is whether it is possible to follow the Courant-sharp (Neumann) eigenvalues with $h = 0$ to Courant-sharp (Dirichlet) eigenvalues as $h \to +\infty$, or whether there are some critical values $h^*(k, \Omega)$ after which the Robin eigenvalues $\lambda_{k,h}(\Omega)$, $h \geq h^*(k, \Omega)$ become Courant-sharp or are no longer Courant-sharp.

We consider the particular example where Ω is a square S in $\mathbb{R}^2$ of side-length π and the main question is: *Is it possible to determine the Courant-sharp Robin eigenvalues of this square?*

As $\lambda_{2,h}(S) = \lambda_{3,h}(S)$ by a symmetry argument, it follows immediately that $\lambda_{3,h}(S)$ is not Courant-sharp for any $h \geq 0$. In addition, $\lambda_{4,h}(S)$ is Courant-sharp for all $h \geq 0$, see Subsection 2.2.

It was asserted by Pleijel in [36] that the only Courant-sharp Dirichlet eigenvalues of the square are for $k = 1, 2, 4$. This was shown rigorously in [3]. The only Courant-sharp Neumann eigenvalues of the square are for $k = 1, 2, 4, 5, 9$, as shown in [29].

The first step to obtain the results of [3], [29] is to reduce the number of potential Courant-sharp eigenvalues by invoking an argument which was inspired by the founding paper of Pleijel [36]. We employ a similar argument in Section 3 to reduce the possible cases that may give rise to Courant-sharp Robin eigenvalues. We have the following theorem.

Theorem 1.1. *Let $h \geq 0$. If $\lambda_{k,h}(S)$ is an eigenvalue of the Robin Laplacian on S with parameter h and $k \geq 520$, then it is not Courant-sharp.*

We note that in the case of a Dirichlet boundary condition, the equivalent statement in [36] gives $k \geq 34$ and in the case of a Neumann boundary condition, [29], $k \geq 209$. The strategies of [3], [29] are then either to re-implement the Faber–Krahn inequality, or to use symmetry properties of the corresponding eigenfunctions to further eliminate potential Courant-sharp eigenvalues. One is then reduced to the analysis of the nodal structure of very few families of eigenfunctions that belong to two-dimensional eigenspaces.

We show that the Robin eigenfunctions satisfy analogous symmetry properties. We note that it is possible that a Robin eigenvalue has multiplicity larger than 2 and the corresponding eigenfunctions have no common symmetries (see [20]).

In addition, for a Robin eigenvalue $\lambda_{k,h}(S)$, we do not know how to take the relationship between k and h into account in an efficient way. Indeed, to prove Theorem 1.1 our arguments are independent of h as they rely on the monotonicity of the Robin eigenvalues and comparison to the corresponding Dirichlet and Neumann eigenvalues.

We also treat the problem asymptotically as $h \to +\infty$. We show that for h large enough the only Courant-sharp Robin eigenvalues are for $k = 1, 2, 4$.

Theorem 1.2. *There exists $h_1 > 0$ such that for $h \geq h_1$, the Courant-sharp cases for the Robin problem on S are the same as those for $h = +\infty$ (i.e. the Dirichlet case).*

In order to prove this theorem, we follow the strategy due to Pleijel, [36]. It is therefore necessary to estimate the number of nodal domains whose boundaries intersect the boundary of the square in at least a non-trivial interval. For such nodal domains, we cannot use the Faber–Krahn inequality for the Dirichlet problem. Nevertheless, there is a Faber–Krahn inequality for the Robin problem when $h > 0$ (see [7], [8], [9], [12]). We will see how this can be used for h sufficiently large in Subsection 3.3 and Section 4.

In Section 5, we analyse the number of nodal domains of Robin eigenfunctions in the general context of a planar domain with piecewise $C^{2,\alpha}$ boundary ($\alpha > 0$). We obtain some semi-stability results for the number of nodal domains as the Robin parameter (h large) varies. In particular, we show that if we start with the nodal partition of a Dirichlet eigenfunction with corresponding eigenvalue $\lambda_{k,+\infty}$ which is not Courant-sharp, and we take a small perturbation of h large, then $\lambda_{k,h}$ is not Courant-sharp.

For the square, the results of Section 5 allow us to deal with the remaining case $k = 5$ which is not covered by Pleijel's strategy or by symmetry arguments. In Section 6, we describe explicitly the situations where the eigenfunction corresponding to the fifth Robin eigenvalue has 2, 3, 4 nodal domains respectively (for $h > 0$ sufficiently large).

In a second paper, [21], we consider the situation where the Robin parameter h tends to 0 and prove the following theorem.

Theorem 1.3. *There exists $h_0 > 0$ such that for $0 < h \leq h_0$, the Courant-sharp cases for the Robin problem on S are the same, except the fifth one, as those for $h = 0$ (i.e. the Neumann case).*

In light of the results of [3], [29], [36] and of the previous asymptotic results, a key question is to what extent is it possible to follow the Courant-sharp (Neumann) eigenvalues with $h = 0$ to Courant-sharp Robin eigenvalues as $h \to +\infty$?

In Section 7, we prove a first general result concerning the possible crossings between curves corresponding to Robin eigenvalues of the square. We then focus on the case $k = 9$ where we investigate if there exist critical values $\bar{h}_9^*(S)$, respectively $\underline{h}_9^*(S)$, after which the Robin eigenvalue $\lambda_{9,h}(S)$ is not Courant-sharp, respectively before which it is Courant-sharp. We show that $\underline{h}_9^* = \bar{h}_9^*$.

We note that in [20], we consider $\lambda_{25,h}$ and we observe that for $h = 20$ the nodal partitions of an associated eigenfunction do not satisfy the same symmetry properties that are satisfied by the corresponding Dirichlet eigenfunction. In addition, we present an example of an eigenvalue $\lambda_{k,h}(S)$ that is given by more than two distinct curves as h varies.

Finally, we remark that in order to use the results of Section 5 to determine which of the eigenvalues of the Robin Laplacian on a planar rectangle with parameter h large enough are not Courant-sharp, one would first need to know which of the Dirichlet eigenvalues of this rectangle are not Courant-sharp. These eigenvalues have been identified in certain cases, in particular where the square of the ratio of side-lengths is irrational see [28], but there are still some remaining cases to be dealt with even for the Dirichlet problem.

2. Formulas for the eigenvalues and eigenfunctions of the Robin Laplacian for a rectangle

2.1. Main formulas. In this subsection, we show that an orthogonal basis of eigenfunctions for the Robin realisation of the Laplacian on the square $S := \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^2$ with parameter $h > 0$ is given by

$$u_{p,q,h}(x, y) = u_{p,h}(x)u_{q,h}(y), \tag{2.1}$$

where, for $p, q \in \mathbb{N}$ (where $\mathbb{N}$ is the set of the non-negative integers)

$$u_{p,h}(x) = \frac{1}{\sin\frac{\alpha_p}{2}} \cos\left(\frac{\alpha_p x}{\pi}\right), \tag{2.2}$$

when p is even, and

$$u_{p,h}(x) = \frac{1}{\cos\frac{\alpha_p}{2}} \sin\left(\frac{\alpha_p x}{\pi}\right), \tag{2.3}$$

when p is odd, and where $\alpha_p = \alpha_p(h)$ is the non-zero solution in $[p\pi, (p+1)\pi)$ of

$$\frac{2\alpha_p}{h\pi}\cos\alpha_p + \left(1 - \frac{(\alpha_p)^2}{h^2\pi^2}\right)\sin\alpha_p = 0. \tag{2.4}$$

Here we follow the description given in [23] and we specialise to 2 dimensions. For rectangles $\Omega = (0, \ell_1) \times (0, \ell_2) \subset \mathbb{R}^2$ and $(x, y) \in \Omega$, an orthogonal basis for the Robin problem is given by (2.1) where, for $p, q \in \mathbb{N}$, $u_{p,h}$ is the $(p+1)$-st eigenfunction of the Robin problem in $(0, \ell_1)$:

$$u_{p,h}(x) = \sin\big(\alpha_p(h)x/\ell_1\big) + \frac{\alpha_p(h)}{h\ell_1}\cos\big(\alpha_p(h)x/\ell_1\big).$$

One should assume $\alpha_p(h) \neq 0$ which holds for $h \neq 0$. For $h = 0$, the solution is trivial, hence not the right one! Here $\alpha_p = \alpha_p(h)$ is the solution in $[p\pi, (p+1)\pi)$ of

$$\frac{2\alpha_p}{h\ell_1}\cos\alpha_p + \left(1 - \frac{(\alpha_p)^2}{h^2\ell_1^2}\right)\sin\alpha_p = 0. \tag{2.5}$$

We note that $u_{q,h}(y)$ with $y \in (0, \ell_2)$ and $\alpha_q(h)$ are defined analogously. The Robin eigenvalues are then given by

$$\left(\frac{\alpha_p}{\ell_1}\right)^2 + \left(\frac{\alpha_q}{\ell_2}\right)^2. \tag{2.6}$$

So in 2 dimensions, the Robin eigenvalues correspond to pairs of non-negative integers (p, q).

We analyse the one-dimensional situation in more detail and delete the reference to p, q, h. We note that the condition (2.5) reads (for $h \neq 0$ and $\alpha \neq 0$),

$$\frac{\alpha}{h\ell} = \pm\left(\sin\alpha + \frac{\alpha}{h\ell}\cos\alpha\right).$$

In this way one understands the symmetry properties of the eigenfunctions better (see Lemma 2.1).

One also obtains the localisation of the eigenvalues in the following way. If we consider the symmetric case, $\frac{\alpha}{h\ell} = \left(\sin\alpha + \frac{\alpha}{h\ell}\cos\alpha\right)$, we get

$$\frac{2\alpha}{h\ell}\sin^2\left(\frac{\alpha}{2}\right) = \sin\alpha,$$

which leads to

$$\alpha \tan\left(\frac{\alpha}{2}\right) = h\ell. \tag{2.7}$$

Similarly, if we consider the antisymmetric case, $\frac{\alpha}{h\ell} = -\left(\sin\alpha + \frac{\alpha}{h\ell}\cos\alpha\right)$, we get

$$\frac{2\alpha}{h\ell}\cos^2\left(\frac{\alpha}{2}\right) = -\sin\alpha,$$

which leads to

$$\frac{\alpha}{h\ell} = -\tan\left(\frac{\alpha}{2}\right). \tag{2.8}$$

With these formulas in mind, we get simpler expressions for the eigenfunctions. In the first case, we observe that

$$\begin{aligned}
u(x) &= \sin(\alpha x/\ell) + \frac{\alpha}{h\ell}\cos(\alpha x/\ell) \\
&= \sin(\alpha x/\ell) + \operatorname{cotan}\left(\frac{\alpha}{2}\right)\cos(\alpha x/\ell) \\
&= \frac{1}{\sin\frac{\alpha}{2}}\cos\left(\frac{\alpha x}{\ell} - \frac{\alpha}{2}\right).
\end{aligned}$$

In the second case, we observe that

$$\begin{aligned}
u(x) &= \sin(\alpha x/\ell) + \frac{\alpha}{h\ell}\cos(\alpha x/\ell) \\
&= \sin(\alpha x/\ell) - \tan\left(\frac{\alpha}{2}\right)\cos(\alpha x/\ell) \\
&= \frac{1}{\cos\frac{\alpha}{2}}\sin\left(\frac{\alpha x}{\ell} - \frac{\alpha}{2}\right).
\end{aligned}$$

In this way, we clearly see the symmetry properties of the eigenfunctions and we are closer to the Neumann case by considering $x \mapsto \cos\left(\frac{\alpha x}{\ell} - \frac{\alpha}{2}\right)$ or $x \mapsto \sin\left(\frac{\alpha x}{\ell} - \frac{\alpha}{2}\right)$ as eigenfunctions.

The first case corresponds to p even. When $h = 0$, we have $\alpha = p\pi$ and $\cos\left(\frac{\alpha x}{\ell} - \frac{\alpha}{2}\right) = (-1)^{p/2}\cos\left(\frac{p\pi x}{\ell}\right)$.

The second case corresponds to $p = 2n + 1$ odd ($n \in \mathbb{N}$). When $h = 0$, we have $\alpha = p\pi$ and $\sin\left(\frac{\alpha x}{\ell} - \frac{\alpha}{2}\right) = \pm\cos\left(\frac{p\pi x}{\ell}\right)$.

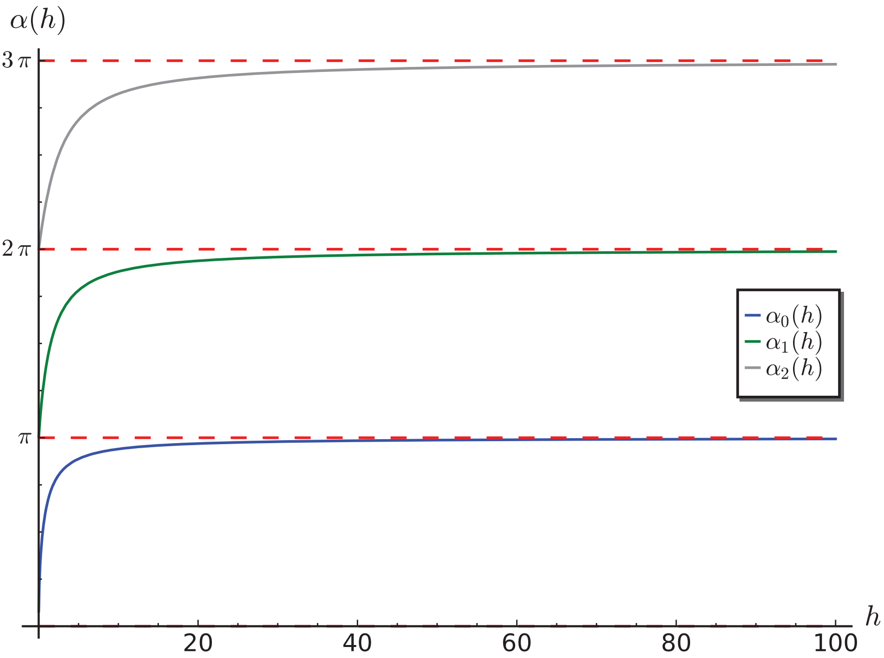


Figure 1. Solutions $\alpha_0(h)$, $\alpha_1(h)$, $\alpha_2(h)$ for the square of side-length π with $h \leq 100$.

By setting $\ell = \pi$ and then translating $x \mapsto x + \frac{\pi}{2}$, we obtain (2.2), (2.3) respectively. In Figure 1, we plot $\alpha_0(h)$, $\alpha_1(h)$, $\alpha_2(h)$ for S with $h \leq 100$.

2.2. Particular cases $k = 1, 2, 3, 4$. We recall from the introduction that $\lambda_{1,h}$ and $\lambda_{2,h}$ (which for eigenfunctions of the form $u_{p,q}(x, y)$ correspond to $(p, q) = (0, 0), (1, 0)$ respectively) are Courant-sharp via Courant's nodal domain theorem and orthogonality of eigenfunctions. We note that $\lambda_{3,h}(S)$ is not Courant-sharp since it corresponds to the case where $(p, q) = (0, 1)$ so $\lambda_{3,h}(S) = \lambda_{2,h}(S)$.

Consider $\lambda_{4,h}(S)$ with $h > 0$. Then $p = q = 1$ and the corresponding eigenfunction is

$$u_{1,1}(x, y) = \frac{1}{\cos^2 \frac{\alpha_1}{2}} \sin\left(\frac{\alpha_1 x}{\pi}\right) \sin\left(\frac{\alpha_1 y}{\pi}\right),$$

for $(x, y) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^2$.

We see that $x = 0$ and $y = 0$ are nodal lines of $u_{1,1}(x, y)$ which partition S into 4 nodal domains. There cannot be any further nodal lines of $u_{1,1}(x, y)$ as these would give rise to additional nodal domains so we would get a contradiction to Courant's nodal domain theorem. Thus $\lambda_{4,h}(S)$ with $h \geq 0$ is Courant-sharp.

Hence, from this point onwards, we are only interested in the remaining eigenvalues, i.e. in the eigenvalues $\lambda_{n,h}(S)$ with $n \geq 5$. Note that, due to the monotonicity of the Robin eigenvalues with respect to h, we have for $n \geq 5$,

$$\lambda_{n,h}(S) \geq \lambda_{4,h}(S) \geq \lambda_{4,0}(S) = 2. \tag{2.9}$$

2.3. Symmetry properties. For the case of the square with a Neumann boundary condition, [29], the symmetry properties of the eigenfunctions were quite powerful in reducing the number of potential Courant-sharp eigenvalues. In particular, via an argument due to Leydold, [35], a Courant nodal domain theorem was deduced from these symmetry properties.

The goal of this subsection is to show that this invariance by symmetry is satisfied by all the Robin problems on the interval and the square. In addition, the number of nodal domains inherits some particular properties from these symmetries.

2.3.1. Symmetry of Robin eigenfunctions in 1D. We recall that $h = 0$ corresponds to the Neumann case and $h = +\infty$ corresponds to the Dirichlet case. The Robin condition for $\left(-\frac{\ell}{2}, \frac{\ell}{2}\right)$ reads

$$\frac{du}{dx}(-\ell/2) = hu(-\ell/2), \qquad \frac{du}{dx}(\ell/2) = -hu(\ell/2).$$

We also observe the following invariance by symmetry.

Lemma 2.1. *If u is an eigenfunction of the 1D-Robin problem, the function $\tilde{u}(x) = u(-x)$ is also an eigenfunction of the same problem.*

Hence, we necessarily have (using the conservation of the norm) $u(-x) = \pm u(x)$. Moreover, if $u(0) \neq 0$, we have $u(-x) = u(x)$ and if $u'(0) \neq 0$ we get $u(-x) = -u(x)$. Therefore, the eigenfunctions u_p (see (2.2) and (2.3)) are alternately symmetric and antisymmetric:

$$u_p(-x) = (-1)^p u_p(x), \tag{2.10}$$

like in the Dirichlet or Neumann case. We note that one can obtain the symmetry property (2.10) immediately from (2.2), (2.3).

2.3.2. Symmetry of Robin eigenfunctions in 2D. In 2D, we now consider the possible symmetries of a general eigenfunction associated with the eigenvalues $\lambda_{n,h}$ of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^2$ which reads,

$$u(x, y) = \sum_{i,j:\lambda_{n,h}(S)=\pi^{-2}(\alpha_i^2+\alpha_j^2)} a_{ij} u_i(x) u_j(y), \tag{2.11}$$

where u_p (or $u_{p,h}$ if we want to mention the reference to the Robin parameter) is the $(p+1)$-*st* eigenfunction of the h-Robin problem in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

By considering the transformation $(x, y) \mapsto (-x, -y)$, we obtain

$$u(-x, -y) = \sum_{i,j:\lambda_{n,h}(S)=\pi^{-2}(\alpha_i^2+\alpha_j^2)} a_{ij}(-1)^{i+j} u_i(x) u_j(y). \tag{2.12}$$

Remark 2.2. We note that if $(i+j)$ is odd for any pair (i, j) such that $\lambda_{n,h}(S) = \pi^{-2}(\alpha_i^2 + \alpha_j^2)$, then we get by (2.12), $u(-x, -y) = -u(x, y)$ and as a consequence u has an even number of nodal domains. As we shall see in [21], other symmetries related to the finite group generated by the identity and the symmetries $(x, y) \mapsto (-x, y)$ and $(x, y) \mapsto (x, -y)$ can be considered.

In what follows, we obtain an upper bound for the number of Courant-sharp Robin eigenvalues of S via arguments that do not depend on the parameter h.

3. Upper bound for the number of Courant-sharp Robin eigenvalues of a square

In this section, we prove h-independent bounds for the number of Courant-sharp Robin eigenvalues. This was indeed the first step proposed by Pleijel [36] in the Dirichlet case to reduce the analysis of the Courant-sharp cases to the analysis of finitely many eigenvalues. His proof was a combination of the Faber–Krahn inequality and the Weyl formula. In the Neumann case considered in [29], a new difficulty arises as it is not possible to apply the Faber–Krahn inequality to the elements of the nodal partition whose boundaries touch the boundary of the square at more than isolated points. In this section, we extend the analysis to the Robin case.

3.1. Lower bound for the Robin counting function. Recall that for $\lambda > 0$, the Robin counting function for the corresponding eigenvalues of Ω is defined as

$$N_\Omega^{R,h}(\lambda) := \#\{k \in \mathbb{N} : k \geq 1,\ \lambda_{k,h}(\Omega) < \lambda\}. \tag{3.1}$$

Similarly we have the Dirichlet counting function

$$N_\Omega^{D}(\lambda) := \#\{k \in \mathbb{N} : k \geq 1,\ \lambda_{k,+\infty}(\Omega) < \lambda\}, \tag{3.2}$$

and the Neumann counting function

$$N_\Omega^{Ne}(\lambda) := \#\{k \in \mathbb{N} : k \geq 1,\ \lambda_{k,0}(\Omega) < \lambda\}. \tag{3.3}$$

Due to the monotonicity of the Robin eigenvalues with respect to $h \in [0, +\infty)$, we have the following upper and lower bounds for $N_\Omega^{R,h}(\lambda)$.

$$N_\Omega^{Ne}(\lambda) = N_\Omega^{R,0}(\lambda) \geq N_\Omega^{R,h}(\lambda) \geq N_\Omega^{R,+\infty}(\lambda) = N_\Omega^{D}(\lambda).$$

For the Neumann counting function of S, we have

$$\frac{\pi}{4}\lambda + 2\lfloor\sqrt{\lambda}\rfloor + 1 \geq N_S^{Ne}(\lambda) > \frac{\pi}{4}\lambda, \tag{3.4}$$

(see, for example, [17]) and for the Dirichlet counting function of S, if $\lambda \geq 2$, we have by [36], that

$$N_S^D(\lambda) > \frac{\pi}{4}\lambda - 2\sqrt{\lambda} + 1. \tag{3.5}$$

Assume that $\lambda \geq 2$ (this is true for $n \geq 4$ by (2.9)). Then, by (3.5) and monotonicity of the Robin eigenvalues with respect to h,

$$N_S^{R,h}(\lambda) \geq N_S^D(\lambda) > \frac{\pi}{4}\lambda - 2\sqrt{\lambda} + 1. \tag{3.6}$$

With $\lambda = \lambda_{n,h} > \lambda_{n-1,h}$ and Ψ an associated eigenfunction, (3.6) becomes

$$n > \frac{\pi}{4}\lambda_{n,h} - 2\sqrt{\lambda_{n,h}} + 2. \tag{3.7}$$

We now work analogously to the proof of Proposition 2.1 in [29]. Denote by Ω^{inn} the union of nodal domains of Ψ whose boundaries do not touch the boundary of Ω (except at isolated points), and $\mu^{\mathrm{inn}}(\Psi)$ the number of nodal domains of Ψ in Ω^{inn}. We call a nodal domain in Ω^{inn} an interior nodal domain. Similarly denote by Ω^{out} the nodal domains in $\Omega\backslash\Omega^{\mathrm{inn}}$, and $\mu^{\mathrm{out}}(\Psi)$ the number of nodal domains of Ψ in Ω^{out}. We call a nodal domain in $\Omega\backslash\Omega^{\mathrm{inn}}$ a boundary nodal domain. We note that the closure of a boundary nodal domain intersects $\partial\Omega$ in at least a non-trivial arc with non-empty interior. We have that

$$\mu^{\mathrm{inn}}(\Psi) = \mu(\Psi) - \mu^{\mathrm{out}}(\Psi)$$

and we require an upper bound for $\mu^{\mathrm{out}}(\Psi)$.

3.2. Counting the number of nodal domains touching the boundary for the Robin problem. We give a proof which holds for all the Robin problems in the square, except the Dirichlet case. We make use of the following theorem that is due to Sturm, (see [4] and references therein).

Theorem 3.1 (Sturm, 1836). *Let $u = a_m u_m + \cdots + a_n u_n$ be a non-trivial linear combination of eigenfunctions of the one-dimensional Robin problem in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, with $1 \leq m \leq n$, and $\{a_j, m \leq j \leq n\}$ real constants such that $a_m^2 + \cdots + a_n^2 \neq 0$. Then, the function u has at least $(m-1)$, and at most $(n-1)$ zeros in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.*

As observed originally by Pleijel [36], the analysis of the zeros of linear combinations of eigenfunctions appear in the following context. We observe that if an eigenfunction associated with $\lambda_{n,h}$ (see (2.11)) satisfies the Robin condition on the square, then its restriction to one side satisfies the Robin condition relative to the interval and is not zero (except of course in the Dirichlet case). In general, when the multiplicity is not one, this is no longer an eigenfunction but a linear combination of eigenfunctions on the segment $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

For example, the restriction to one side of the square, say $x = \frac{\pi}{2}$, is a linear combination of eigenfunctions on the segment $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$:

$$u(\pi/2, y) = \sum_{i,j:\lambda_{n,h}(S)=\pi^{-2}(\alpha_i^2+\alpha_j^2)} a_{ij} u_i(\pi/2) u_j(y).$$

We can then use Theorem 3.1 which gives a lower bound on the number of zeros of $u(\pi/2, y)$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ by

$$i_n(h) := \min\left(i : \lambda_{n,h}(S) = \pi^{-2}(\alpha_i^2 + \alpha_j^2) \text{ for some } j\right),$$

and an upper bound by

$$j_n(h) := \max\left(j : \lambda_{n,h}(S) = \pi^{-2}(\alpha_i^2 + \alpha_j^2) \text{ for some } i\right). \tag{3.8}$$

Recall that by (2.6) we have

$$\lambda_{n,h}(S) = (\alpha_{i_n(h)}^2 + \alpha_{j_n(h)}^2)/\pi^2 \geq i_n(h)^2 + j_n(h)^2 \geq j_n(h)^2,$$

which gives that

$$j_n(h) \leq \sqrt{\lambda_{n,h}(S)}.$$

We can argue in the same way for the other sides of the square. Therefore, the number of zeros of $u(x, y)$ on the boundary of S is bounded from above by $4\sqrt{\lambda_{n,h}(S)}$. Coming back to the number of boundary nodal domains, we have the following lemma.

Lemma 3.2. *Let λ be an eigenvalue of the Robin Laplacian on S with parameter $0 < h < +\infty$. If Ψ is a Robin eigenfunction associated to λ, then*

$$\mu^{out}(\Psi) \leq 4\sqrt{\lambda}. \tag{3.9}$$

Remark 3.3. There are other proofs given in Pleijel [36] and [29], but the one given above is much more general and not restricted to two-dimensional eigenspaces (and also not based on an explicit knowledge of the eigenfunctions). On the other hand, the claim in [29] is much more involved. It says that taking the whole boundary into consideration, the number of points on the boundary in the nodal set of an eigenfunction $\cos\theta\, u_i(x)u_j(y) + \sin\theta\, u_j(x)u_i(y)$ $(i \neq j)$ is comparable with $i + j$ (See Section 5 of [29]). The proof[1] is restricted to eigenfunctions whose corresponding eigenvalues have multiplicity 2. It would be interesting to prove the same result for the Robin case for $h < +\infty$.

3.3. Upper bound for Courant-sharp Robin eigenvalues of a square. By Lemma 3.2, we have

$$\mu^{\mathrm{inn}}(\Psi) \geq \mu(\Psi) - 4\sqrt{\lambda_{n,h}}. \tag{3.10}$$

Now, $\Omega^{\mathrm{inn}} = \bigcup_i \omega_i^{\mathrm{inn}}$ is a finite union of nodal domains of Ψ. Assuming that Ω^{inn} is not empty, we get, on each ω_i^{inn}, by Faber–Krahn (see [36]), that

$$\frac{A(\omega_i^{\mathrm{inn}})}{\pi \mathbf{j}^2} \geq \frac{1}{\lambda_{n,h}}, \tag{3.11}$$

where $A(\omega_i^{\mathrm{inn}})$ denotes the area of ω_i^{inn} and $\mathbf{j}$ denotes the first positive zero of the Bessel function J_0. Adding, and invoking (3.10), we find

$$\frac{\pi}{\mathbf{j}^2} = \frac{A(S)}{\pi \mathbf{j}^2} > \frac{A(\Omega^{\mathrm{inn}})}{\pi \mathbf{j}^2} \geq \frac{\mu^{\mathrm{inn}}(\Psi)}{\lambda_{n,h}} \geq \frac{\mu(\Psi) - 4\sqrt{\lambda_{n,h}}}{\lambda_{n,h}},$$

from which we obtain

$$\frac{\pi}{\mathbf{j}^2} > \frac{\mu(\Psi) - 4\sqrt{\lambda_{n,h}}}{\lambda_{n,h}}. \tag{3.12}$$

Due to (3.10), this inequality is still true if Ω^{inn} is empty.

If we are in the Courant-sharp situation, then $\mu(\Psi) = n$. Combining (3.7) and (3.12), we find that

$$0.543229 \approx \frac{\pi}{\mathbf{j}^2} > \frac{n - 4\sqrt{\lambda_{n,h}}}{\lambda_{n,h}} > \frac{\pi}{4} + \frac{2}{\lambda_{n,h}} - \frac{6}{\sqrt{\lambda_{n,h}}}. \tag{3.13}$$

The mapping

$$\lambda \mapsto f(\lambda) = \frac{2}{\lambda} - \frac{6}{\sqrt{\lambda}} + \frac{\pi}{4} - \frac{\pi}{\mathbf{j}^2}$$

[1] There is a small gap in the proof which can be repaired using Theorem 3.1 due to Sturm.

is increasing for $\lambda \geq 4/9$. Moreover, $f(597) < 0$ and $f(598) > 0$. Thus, if $\lambda_{n,h} \geq 598$, we violate inequality (3.13), and we are not in the Courant-sharp situation. So, similarly to [36] and [29], Proposition 2.1, we obtain the following proposition.

Proposition 3.4. *If $\lambda_{n,h} \geq 598$ is an eigenvalue of the Robin Laplacian on S with parameter $h > 0$, then it is not Courant-sharp. Equivalently, any Courant-sharp Robin eigenvalue satisfies $\lambda_{n,h}(S) < 598$.*

3.4. Proof of Theorem 1.1. By invoking the upper bound of (3.4), we obtain an upper bound for n such that $\lambda_{n,h}(S) < 598$. Indeed, suppose $\lambda_{n,h}(S) < 598$, then

$$\begin{aligned} n-1 = N_S^{R,h}\big(\lambda_{n,h}(S)\big) &= \#\{k \in \mathbb{N} : k \geq 1,\, \lambda_{k,h}(S) < \lambda_{n,h}(S)\} \\ &\leq \frac{\pi}{4}\lambda_{n,h}(S) + 2\lfloor\sqrt{\lambda_{n,h}(S)}\rfloor + 1 < 518.67. \end{aligned} \tag{3.14}$$

Hence we have shown Theorem 1.1.

We remark that the above arguments do not depend on the Robin parameter h. In the sections that follow, we consider the case where h is large and improve the result.

4. Analysis as $h \to +\infty$

In this section we show that for h sufficiently large, the Courant-sharp Robin eigenvalues of the square are the same as those in the Dirichlet case, [3], [36], that is the first, second and fourth, except possibly the fifth which we deal with in Section 5. We first briefly revisit the strategy that was used by Pleijel for the Dirichlet problem.

4.1. Pleijel's approach for Dirichlet. In this subsection, λ_n denotes the eigenvalues of the Dirichlet Laplacian on S. We recap Pleijel's proof that the only Courant-sharp Dirichlet eigenvalues of S are λ_1, λ_2, λ_4.

We recall from (3.5) that if $\lambda_n \geq 2$ is Courant-sharp, then

$$n > \frac{\pi}{4}\lambda_n - 2\sqrt{\lambda_n} + 2. \tag{4.1}$$

On the other hand, if λ_n is Courant-sharp, the Faber–Krahn inequality gives the necessary condition

$$\frac{n}{\lambda_n} \leq \pi \mathbf{j}^{-2} < 0.54323. \tag{4.2}$$

Recall that $\mathbf{j}$ is the smallest positive zero of J_0 the Bessel function of order 0, and that $\pi\mathbf{j}^2$ is the ground state energy of the disc of area 1. Combining (4.1) and (4.2), leads to the inequality

$$\pi\mathbf{j}^{-2} > \frac{\pi}{4} - 2\lambda_n^{-1/2} + 2\lambda_n^{-1}, \tag{4.3}$$

and to

$$\lambda_n \leq 50. \tag{4.4}$$

Then the proof that λ_1, λ_2, λ_4 are the only Courant-sharp Dirichlet eigenvalues of S is achieved in the following steps (see [3] for the full details).

- By a direct computation of the quotient $\frac{n}{\lambda_n}$, it is possible to eliminate all the eigenvalues except for $n = 1, 2, 4, 5, 7$ and 9.
- The eigenvalues for $n = 7$ and $n = 9$ are eliminated by symmetry arguments (analogously to Remark 2.2).
- The final step is to analyse the fifth eigenfunction for which a specific analysis of the nodal structure can be done (see [3]).

In the subsections that follow, we work through these steps and investigate the extent to which they still work for h large.

4.2. Faber–Krahn for the Robin case. We recall the result of Bossel–Daners [7], [12], which asserts that the Robin eigenvalues of the Laplacian satisfy the following Faber–Krahn inequality. For a Lipschitz domain $\omega \subset \mathbb{R}^2$ and $h > 0$,

$$\lambda_{1,h}(\omega) \geq \lambda_{1,h}(D_\omega), \tag{4.5}$$

where $D_\omega \subset \mathbb{R}^2$ is a disc such that $A(D_\omega) = A(\omega)$. We will refer to inequality (4.5) as the h-Faber–Krahn inequality in what follows.

For the interior nodal domains, the approach via the standard Faber–Krahn inequality still applies (see Subsection 3.3).

We observe that an eigenfunction u of the Robin Laplacian on S can be extended to all of $\mathbb{R}^2$ as a solution $\tilde{u}$ of $-\Delta\tilde{u} = \lambda\tilde{u}$ (we have an explicit expression as a trigonometric polynomial). Hence the nodal sets of $\tilde{u}$ have a nice local structure (see P. Bérard [2] for a survey) and have the same properties as in the Dirichlet case. In particular, these nodal sets are locally Lipschitz domains (actually with piecewise analytic boundary). If we observe that a nodal set of u is the intersection of a nodal set of $\tilde{u}$ with the square S, we immediately deduce that the interior nodal domains, ω_i^{inn}, are Lipschitz domains.

We will apply the h-Faber–Krahn inequality to a boundary nodal domain of a Robin eigenfunction $u = u_{n,h}$ associated with $\lambda_{n,h}$. However we do not know whether the boundary nodal domains have Lipschitz boundary or not. By Lemma 3.2, the nodal set intersects the boundary finitely many times, so $\partial\omega_j^{\mathrm{out}}$ consists of a finite number of arcs belonging either to S or to ∂S. So we can apply Theorem 4.1 of [9]. Alternatively, we can use the strategy given in Section 3 of [31] to obtain (4.5) for these domains (see also [32], p. 3620). We will discuss the regularity of the nodal domains further in Section 5.

For a boundary nodal domain ω_j^{out}, $u|_{\omega_j^{\mathrm{out}}}$ satisfies a mixed Robin–Dirichlet condition on its boundary, but we can use the monotonicity of the eigenvalues with respect to the Robin parameter which leads to

$$\lambda_{n,h} \geq \lambda_{1,h}(\omega_j^{\mathrm{out}}). \tag{4.6}$$

The h-Faber–Krahn inequality can then be applied.

In order to follow Pleijel's strategy, we now wish to rescale the discs so that they each have area 1. Consider a scaling of the domain ω by $t > 0$, $t\omega := \{tx \in \mathbb{R}^2 : x \in \Omega\}$. It is well known that the Robin eigenvalues satisfy the following scaling property.

$$\lambda_{n,h}(\omega) = t^2 \lambda_{n,h/t}(t\omega). \tag{4.7}$$

A serious issue here is that the scaling also affects the Robin parameter. So, in particular, replacing D_ω by D_1, the disc of area 1, we have

$$\lambda_{1,h}(D_\omega) = \lambda_{1,hA(\omega)^{1/2}}(D_1)/A(\omega). \tag{4.8}$$

When $h = +\infty$, the reference is $\lambda_{1,+\infty}(D_1)$. In the Robin case, if we start from h large, we will not necessarily have $hA(\omega)^{1/2}$ large if we use inequality (4.8) with ω a boundary nodal domain. Hence we have to be careful in the application of the Faber–Krahn argument. This is actually the main difficulty.

In the following proposition, we recall the asymptotic behaviour of the first Robin eigenvalue as the Robin parameter tends to $+\infty$ or to 0 (see, for example, [16], [23]).

Proposition 4.1. *Let D_1 denote the disc of unit area. Then*

(i) *$\lambda_{1,h}(D_1)$ tends to $\lambda_{1,+\infty}(D_1) = \pi \mathbf{j}^2$ as $h \to +\infty$,*

(ii) *there exists $c > 0$ such that, as $h \to +\infty$,*

$$\lambda_{1,h}(D_1) = \lambda_{1,+\infty}(D_1) - \frac{c}{h} + \mathcal{O}\left(\frac{1}{h^2}\right), \tag{4.9}$$

(iii) *there exists* $d > 0$ *such that as* $h \to 0$,

$$\lambda_{1,h}(D_1) = dh + \mathcal{O}(h^2). \tag{4.10}$$

We give the proof for completeness.

Proof. To determine the first eigenvalue of the Robin Laplacian on the disc of area 1 and radius $\pi^{-1/2}$ with parameter h, one looks for an eigenfunction of the form $J_0(\alpha\pi^{1/2}r)$ where the corresponding eigenvalue is $\pi\alpha^2$. The Robin condition[2] reads

$$\alpha\pi^{1/2}J_0'(\alpha) + hJ_0(\alpha) = 0.$$

For the asymptotic behaviour near $h = 0$ or $h = +\infty$, we use the Taylor expansion of J_0 or J_0' at $\alpha = 0$ and $\alpha = \mathbf{j}$. We recall that $J_0'(0) = 0$ and $J_0''(0) < 0$. For $h \geq 0$, for the first solution, we get $\alpha^2\pi^{1/2}J_0''(0) \sim -hJ_0(0)$. Hence the corresponding eigenvalue satisfies as $h \to 0$,

$$\lambda_{1,h}(D_1) = -\big(\pi^{1/2}J_0(0)\big)/\big(J_0''(0)\big)h + \mathcal{O}(h^2).$$

We also have $J_0(\mathbf{j}) = 0$ and $J_0'(\mathbf{j}) \neq 0$. With $\tau = \frac{1}{h}$, we write

$$\tau\alpha\pi^{1/2}J_0'(\alpha) + J_0(\alpha) = 0,$$

and expanding at $\alpha = \mathbf{j}$, we obtain:

$$\alpha = \mathbf{j} - \pi^{1/2}\mathbf{j}\tau + \mathcal{O}(\tau^2),$$

and

$$\pi\alpha^2 = \pi\mathbf{j}^2 - 2\pi^{3/2}\mathbf{j}^2\tau + \mathcal{O}(\tau^2). \qquad \square$$

The proof gives explicit values for the constants c and d in (4.9) and (4.10).

4.3. Pleijel's approach as $h \to +\infty$. In light of what was recalled in Subsection 4.1 for $h = +\infty$, we now consider the different steps in the limit $h \to +\infty$. We first show that for h sufficiently large, the eigenvalues $\lambda_{n,h}$ with $n \geq 10$ are not Courant-sharp.

[2]Note that there is a misprint in [23] after formula (3.9) for the Robin eigenvalue which is corrected here.

We recall that the eigenvalues depend continuously on h until $+\infty$, in particular

$$\forall n \in \mathbb{N}, \qquad \lim_{h \to +\infty} \lambda_{n,h} = \lambda_n. \tag{4.11}$$

We keep the notation of the previous section. If we are in the Courant-sharp situation, then $\mu(u) = n$, where u is an eigenfunction associated with $\lambda_{n,h}$.

If there exists ω_i^{inn} such that $A(\omega_i^{\text{inn}}) \leq A(S)/n$, we are done like in the Dirichlet case. Indeed, we combine the latter inequality with inequality (3.11) to obtain (4.2). Together with (4.1), this gives $\lambda_{n,h} \leq 50$. In particular, for these eigenvalues n is finite and using (4.11) we get that for h sufficiently large, (4.2) is not satisfied for $n \geq 10$.

If not, the situation is more delicate, but we can assume that there exists ω_j^{out} such that

$$A(\omega_j^{\text{out}}) \leq A(S)/n, \tag{4.12}$$

and we take one of smallest area with this property.

Combining (4.1), (4.6), (4.5), (4.12) and (4.8), we find that

$$\frac{A(S)}{\lambda_{1,hA(\omega_j^{\text{out}})^{1/2}}(D_1)} > \frac{\pi}{4} - \frac{2}{\sqrt{\lambda_{n,h}}} + \frac{2}{\lambda_{n,h}}. \tag{4.13}$$

Here, comparing with (4.3), we need to have $\tilde{h} := hA(\omega_j^{\text{out}})^{1/2}$ large enough if we want to arrive at the same conclusion as for the Dirichlet case (namely $\lambda_{n,h} \leq 50$). So we have to find a lower bound for $A(\omega_j^{\text{out}})^{1/2}$. This seems difficult, at least with explicit lower bounds. We will use our initial h-independent upper bound for $\lambda_{n,h}$ from the previous section. Hence, we can assume in this Courant-sharp situation, that

$$n \leq 520. \tag{4.14}$$

Under these assumptions, we will now show that there exists a constant $c > 0$ such that $A(\omega_j^{\text{out}}) \geq c$. According to (4.10), there exist constants $c_1 > 0$ and $h_1 > 0$ such that

$$\lambda_{1,\tilde{h}}(D_1) \geq c_1 \tilde{h} \quad \text{ if } 0 \leq \tilde{h} \leq h_1. \tag{4.15}$$

By monotonicity of the Robin eigenvalues with respect to h and the h-Faber–Krahn inequality, we have

$$\lambda_{520,+\infty} \geq \lambda_{n,h} \geq A(\omega_j^{\text{out}})^{-1} \lambda_{1,hA(\omega_j^{\text{out}})^{1/2}}(D_1). \tag{4.16}$$

If $hA(\omega_j^{\text{out}})^{1/2} \leq h_1$, then $c_1 h^2 \leq h_1 \lambda_{520,+\infty}$. Indeed, if $hA(\omega_j^{\text{out}})^{1/2} \leq h_1$, then by (4.15),

$$\lambda_{1,hA(\omega_j^{\text{out}})^{1/2}}(D_1) \geq c_1 hA(\omega_j^{\text{out}})^{1/2},$$

which implies by (4.16) that

$$\lambda_{520,+\infty} \geq c_1 hA(\omega_j^{\text{out}})^{-1/2},$$

hence

$$c_1 h^2 \leq hA(\omega_j^{\text{out}})^{1/2} \lambda_{520,+\infty} \leq h_1 \lambda_{520,+\infty}.$$

Consequently, if $h > h_1^{1/2} c_1^{-1/2} \lambda_{520,+\infty}^{1/2}$, then $c_1 h^2 > h_1 \lambda_{520,+\infty}$ which implies that $hA(\omega_j^{\text{out}})^{1/2} > h_1$. Therefore by (4.16) we have

$$A(\omega_j^{\text{out}}) \geq \lambda_{1,h_1}(D_1)/\lambda_{520,+\infty}.$$

This gives the existence of $c > 0$ such that $A(\omega_j^{\text{out}}) \geq c$ (see also Proposition 5.3). Using the latter inequality and (4.13), we have

$$\frac{\pi^2}{\lambda_{1,c^{1/2}h}(D_1)} > \frac{\pi}{4} - \frac{2}{\sqrt{\lambda_{n,h}}} + \frac{2}{\lambda_{n,h}}. \tag{4.17}$$

Hence for h large enough, we also get in this case that $\lambda_{n,h} \leq 50$ (compare with inequality (4.3)).

We can now follow the proof of Pleijel for the Dirichlet case (which was outlined in Subsection 4.1). By the above and the continuity of the eigenvalues with respect to h as $h \to +\infty$, (4.12), for h large enough, it remains to consider the cases $\lambda_{5,h}$, $\lambda_{7,h}$, $\lambda_{9,h}$ as left by Pleijel in the Dirichlet case. The next step is to rule out the cases $\lambda_{7,h}(S)$ and, for h sufficiently large, $\lambda_{9,h}(S)$. Here the symmetry argument due to Leydold, [35], holds in the same way as for the Dirichlet case [3] for the two cases corresponding to the seventh and the ninth Robin eigenvalues. We briefly recall the relevant particular case of the argument due to Leydold.

Lemma 4.2. *Let $0 \leq h < +\infty$. Suppose that $\lambda_{n,h}(S)$ is an eigenvalue of the Robin Laplacian on S with parameter h and with corresponding eigenfunction defined in* (2.11). *Suppose that n is odd and that the conditions of Remark* 2.2 *are satisfied. Then $\lambda_{n,h}(S)$ is not Courant-sharp.*

We know indeed by the standard Courant nodal domain theorem that the number of nodal domains is not larger than n and by Remark 2.2 that it is even. Hence the number is less than n.

As an application, we observe that any eigenfunction corresponding to the seventh Robin eigenvalue is a linear combination of $u_{2,1}(x,y)$ and $u_{1,2}(x,y)$ and that $1+2$ is odd. So $\lambda_{7,h}(S)$ is not Courant-sharp for any $h \geq 0$.

Similarly, for h large enough, any eigenfunction corresponding to the ninth Robin eigenvalue is a linear combination of $u_{3,0}(x,y)$ and $u_{0,3}(x,y)$ (see Subsection 7.2) and $0+3$ is odd.

Hence at this stage, we have proved the following proposition.

Proposition 4.3. *There exists $h_1 > 0$ such that for $h \geq h_1$, the Courant-sharp cases for the Robin problem are the same, except possibly for $k=5$, as those for $h=+\infty$.*

With what was done for the Dirichlet case [3] in mind, in order to prove Theorem 1.2 for h large enough, it remains to count the number of nodal domains of any eigenfunction corresponding to the fifth eigenvalue. This will be analysed in Section 6 as a direct consequence of Section 5.

5. A general perturbation argument

5.1. Preliminary discussion. We analyse a θ-dependent family $\Phi_{h,\theta}$ of eigenfunctions, more explicitly

$$\Phi_{h,\theta,p,q}(x,y) = \cos\theta\, u_{p,h}(x)u_{q,h}(y) + \sin\theta\, u_{p,h}(y)u_{q,h}(x), \tag{5.1}$$

for $(x,y) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^2$.

For most of the arguments in this section, we will not use the explicit expression of the eigenfunction, but only the property that $\Phi_{h,\theta}$ is a very smooth family of eigenfunctions (with respect to h and θ) where, for $h \in (0,+\infty]$, $\Phi_{h,\theta}$ is an eigenfunction of the Robin Laplacian with parameter h associated with a smooth eigenvalue $\lambda(h)$. Nevertheless, except the cases where h is fixed (and so we consider a smooth family inside a fixed eigenspace) or where we are in a product situation, e.g. a rectangle, it is not easy to give examples of such families as introduced above. The parameter θ, which above belongs to $\mathbb{R}/(2\pi\mathbb{Z})$, could also be thought of as belonging to some open neighbourhood of some point θ_0 in $\mathbb{R}$.

In addition, most of the arguments in this section extend to more general domains. We consider the case of bounded, planar domains with piecewise $C^{2,\alpha}$ ($\alpha > 0$) boundary.

For $h=+\infty$ (or $h=h_0>0$) and $\theta=\theta_0$, we assume that the number of nodal domains is known (for example, that the corresponding eigenvalue is not Courant-sharp). The aim of this section is to prove that by perturbation (i.e. for

$\left|\frac{1}{h} - \frac{1}{h_0}\right| + |\theta - \theta_0|$ small enough) the number of nodal domains cannot increase (see Proposition 5.7). The proof involves various general statements which are interesting in a more general context, hence not restricted to the case of the square.

5.2. Robin Faber–Krahn inequality revisited and applications. In this section, $C^{2,+}$ means $C^{2,\alpha}$ for some $\alpha > 0$. The subsequent proposition follows from [8].

Proposition 5.1. *Let Ω be a connected, bounded set with piecewise $C^{2,+}$ boundary, and, for some $h > 0$, let Φ_h be an eigenfunction of the h-Robin realisation of the Laplacian in Ω. Then each nodal domain of Φ_h satisfies the h-Faber–Krahn inequality* (4.5).

Proof. In [8], inequality (4.5) is proven when Ω has $C^{2,+}$ boundary. Hence it holds for an interior nodal domain and a boundary nodal domain whose boundary intersects $\partial\Omega$ away from a corner (see also Theorem A.1). The case of a boundary nodal domain whose boundary intersects $\partial\Omega$ at a corner is also covered by the results of [8]. Indeed, according to [8], the h-Faber–Krahn inequality holds for any open set with finite area. In this general case, the first eigenvalue is defined as in Definition 4.2 of [8]. It is also proven in [8] that with this choice of definition, this eigenvalue is not larger than any other definition given in a more regular situation. □

Remark 5.2. In the case of the square $\Omega = S$ there is a more direct proof. As in Subsection 4.2, we indeed observe that $\Phi_{h,\theta}$ admits an extension $\tilde{\Phi}_{h,\theta}$ to $\mathbb{R}^2$ such that $-\Delta\tilde{\Phi}_{h,\theta} = \lambda(h)\,\tilde{\Phi}_{h,\theta}$. This gives more information about the local nodal structure of $\Phi_{h,\theta}$ up to the boundary (actually in a neighbourhood of $\overline{S}$).

Proposition 5.3. *Let Ω be a connected, bounded set with piecewise $C^{2,+}$ boundary. Let $h_0 > 0$ and $M > 0$. For $h \in I \subset [h_0, +\infty)$ and $\theta \in [0, \pi)$, let $\Phi_{h,\theta}$ denote a smooth family of eigenfunctions for the h-Robin realisation of the Laplacian on Ω associated with $\lambda_h(\Omega) \le M$. Then, there exists $\varepsilon_0 > 0$ such that no nodal domain of $\Phi_{h,\theta}$ can have area less than ε_0. (This includes the Dirichlet case).*

Proof. This follows directly from the h-Faber–Krahn inequality. If ω is a nodal domain of $\Phi_{h,\theta}$ satisfying the assumptions of the lemma, we have

$$M \ge \lambda(h) \ge \lambda(h_0) \ge \lambda_{1,h_0}(D_\omega) = \lambda_{1,h_0 A(\omega)^{1/2}}(D_1)/A(\omega) \sim d h_0/A(\omega)^{1/2}. \tag{5.2}$$

This shows that as soon as we avoid the Neumann situation, the ground state energy in a domain ω tends to $+\infty$ as the area of the domain tends to 0. □

5.3. On the nodal set at the boundary.

Proposition 5.4. *Under the assumptions of Proposition* 5.3*, there exists* $C > 0$ *such that, for any* $h \in I$ *and any* θ*, the number of zeros of* $\Phi_{h,\theta}$ *at the boundary is less than* C*.*

Remark 5.5. In the case of the square, Proposition 5.4 follows from Sturm's theorem.

Proof. We will use the Euler formula with boundary. The conditions for its application are satisfied by using Theorem A.1 and it reads as follows (see, for example, [30]).

Proposition 5.6. *Let* Ω *be an open set in* $\mathbb{R}^2$ *with* $C^{2,+}$ *boundary,* u *an eigenfunction of the h-Robin realisation of the Laplacian on* Ω *with* k *nodal domains,* $N(u)$ *its zero-set. Let* b_0 *be the number of components of* $\partial\Omega$ *and* b_1 *be the number of components of* $N(u) \cup \partial\Omega$*. Denote by* $\nu(\mathbf{x}_i)$ *and* $\rho(\mathbf{y}_i)$ *the numbers of curves ending at a critical point* $\mathbf{x}_i \in N(u)$*, respectively* $\mathbf{y}_i \in N(u) \cap \partial\Omega$*. Then*

$$k = 1 + b_1 - b_0 + \sum_{\mathbf{x}_i} \left(\frac{\nu(\mathbf{x}_i)}{2} - 1 \right) + \frac{1}{2} \sum_{\mathbf{y}_i} \rho(\mathbf{y}_i). \tag{5.3}$$

In our application, we immediately obtain that the number $\rho(u)$ of boundary points (counted with multiplicity) in the nodal set of $u = \Phi_{h,\theta}$ satisfies

$$\rho(u) \leq 2k - 2.$$

To achieve the proof of Proposition 5.4, we observe that by Courant's nodal domain theorem, k is less than the minimal labelling of $\lambda(h)$ and that this labelling is uniformly bounded if $\lambda(h)$ is uniformly bounded. By monotonicity, this labelling is indeed bounded by the maximal labelling of an eigenvalue $\lambda_j(h_1)$ satisfying $\lambda_j(h_1) \leq M$.

It remains to treat what is going on in the neighbourhood of a corner x_c. We first show that there cannot exist an infinite sequence of zeros of $\Phi_{h,\theta}$ in the boundary (outside the corner) tending to the corner x_c. Indeed, by Proposition 5.1, similarly to the proof of Proposition 5.3, there exists some sufficiently small $\varepsilon > 0$ such that any line starting from one of these zeros (which necessarily belongs to the boundary of one nodal domain) should cross $\partial D(x_c, \varepsilon) \cap \Omega$ transversally and only once. Hence the number of points is finite, and moreover not greater than the cardinality of $N(u) \cap D(x_c, \varepsilon) \cap \Omega$. Observing that, by Proposition 5.3, the number of nodal domains of u in Ω is the same as the number of nodal domains of u in $\Omega\backslash D(x_c,\varepsilon)$, we can apply the Euler Formula in $\Omega\backslash D(x_c,\varepsilon)$ and get the same bound. □

5.4. On the variation of the cardinality of the nodal domains by perturbation. Our main result is the following proposition.

Proposition 5.7. *Under the assumptions of Proposition* 5.3, *let* $\rho(h,\theta)$ *denote the cardinality of the nodal domains of* $\Phi_{h,\theta}$. *For any* θ_0, $h_0 \in (0,+\infty]$, *there exists* $\eta_0 > 0$ *such that if* $\left|\frac{1}{h}-\frac{1}{h_0}\right| + |\theta-\theta_0| < \eta_0$, *then*

$$\rho(h,\theta) \le \rho(h_0,\theta_0).$$

We prove this proposition in the following subsections by analysing the structure of the zero set in a neighbourhood of the interior critical points and the boundary points.

5.4.1. Analysis in a neighbourhood of an interior point. We treat what is going on at an interior point z_0. We assume that z_0 is a critical point of Φ_{h_0,θ_0} associated with an eigenvalue $\lambda(h_0)$. We choose $\varepsilon_1 > 0$ small enough such that

- Proposition 5.3 applies for $\left(\frac{1}{h},\theta\right)$ close to $\left(\frac{1}{h_0},\theta_0\right)$;
- $D(z_0,\varepsilon_1) \subset \Omega$;
- $\pi\varepsilon_1^2 < \varepsilon_0$;
- the circle $\mathscr{C}(z_0,\varepsilon_1)$ crosses the 2ℓ half-lines emanating from z_0 transversally at 2ℓ points $z_j(h_0,\theta_0)$ $(j=1,\dots,2\ell)$.

Here we have used the general results on the local structure of an eigenfunction of the Laplacian (see [2] and Appendix A).

Lemma 5.8. *With the previous notation and the assumptions of Proposition* 5.3, *we have that there exists* $\eta_0 > 0$ *such that if* $\left|\frac{1}{h}-\frac{1}{h_0}\right| + |\theta-\theta_0| < \eta_0$, *then the number of nodal domains of* $\Phi_{h,\theta}$ *intersecting the disc* $D(z_0,\varepsilon_1)$ *cannot increase.*

Proof. If we look at the nodal structure inside $D(z_0,\varepsilon_1)$, we have 2ℓ local nodal domains.

By local nodal domain of an eigenfunction $\Phi_{h,\theta}$, we mean the nodal domains of the restriction of $\Phi_{h,\theta}$ to $D(z_0,\varepsilon_1)$. We note that any local nodal domain belongs to a global nodal domain but that two distinct local nodal domains can be included in the same global nodal domain.

In the second case, there exists a path γ in Ω joining these two local nodal domains on which $\Phi_{h,\theta}$ is positive (or negative), which necessarily will not be included in $D(z_0,\varepsilon_1)$.

Starting from (h_0, θ_0) we now look at a small perturbation. By considering the restriction of $\Phi_{h,\theta}$ to the circle $\partial D(z_0, \varepsilon_1)$, we see that the 2ℓ zeros of $\Phi_{h,\theta}$ on $\partial D(z_0, \varepsilon_1)$ move very smoothly, we denote them by $z_j(h, \theta)$.

We indeed observe that the tangential derivative to $\partial D(z_0, \varepsilon_1)$ of Φ_{h_0,θ_0} at each point $z_j(h_0, \theta_0)$ is not zero (again we use the general results for eigenfunctions, in particular the transversal property, see Appendix A). By perturbation, this condition is still true if we choose η_0 small enough. Hence the restriction of $\Phi_{h,\theta}$ changes sign at each point $z_j(h, \theta)$. Moreover, there are 2ℓ local domains $\omega_j(h, \theta)$ of $\Phi_{h,\theta}$ with the property that $\partial\omega_j(h, \theta)$ intersects $\partial D(z_0, \varepsilon_1)$ along the arc $\big(z_j(h, \theta), z_{j+1}(h, \theta)\big)$ (with the convention that $j + 1$ is 1 for $j = 2\ell$).

In addition, we have the following property:

If $\omega_j(h_0, \theta_0)$ and $\omega_{j'}(h_0, \theta_0)$ belong to the same nodal domain $(j \neq j')$, the property remains true for $\omega_j(h, \theta)$ and $\omega_{j'}(h, \theta)$ with (h, θ) sufficiently close to (h_0, θ_0) (i.e. for η_0 in the lemma sufficiently small).

Indeed let $x_{j,0} \in \omega_j(h_0, \theta_0)$, $x_{j',0} \in \omega_{j'}(h_0, \theta_0)$ and γ_0 be a path joining $x_{j,0}$ and $x_{j',0}$ inside the nodal domain. Since Φ_{h_0,θ_0} does not vanish on γ_0 and by continuity, $\Phi_{h,\theta}$ does not vanish on γ_0 for (h, θ) sufficiently close to (h_0, θ_0).

If, for (θ_0, h_0), $\omega_j(h_0, \theta_0)$ and $\omega_{j'}(h_0, \theta_0)$ do not belong to the same nodal domain, then there are two cases

- either the situation is unchanged by perturbation;
- or, after perturbation, they belong to the same nodal domain via a new path in $D(z_0, \varepsilon_1)$.

In the second case, the number of nodal domains touching $\partial D(z_0, \varepsilon_1)$ is decreasing (see Figure 2).

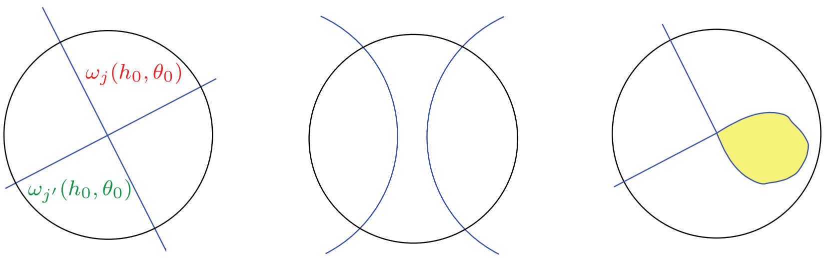


Figure 2. In the leftmost figure, we begin with $\omega_j(h_0, \theta_0)$ and $\omega_{j'}(h_0, \theta_0)$ in different nodal domains. After perturbation, they may belong to the same nodal domain as in the middle figure. The rightmost figure cannot occur as the area of the nodal domain that has been created inside the disc is too small.

On the other hand, by Proposition 5.3 and our choice of ε_1, any nodal domain that intersects $D(z_0, \varepsilon_1)$ crosses $\partial D(z_0, \varepsilon_1)$. If not, it would be contained in $D(z_0, \varepsilon_1)$ whose area is too small (see Proposition 5.3 and Figure 2). This achieves the proof. □

Remark 5.9. If $\ell = 2$, Φ_{h_0,θ_0} is a Morse function whose Hessian has two non-zero eigenvalues of opposite sign. For ε_0 small enough, $\Phi_{h,\theta}$ remains a Morse function for η_0 small enough and admits a unique critical point $z_{h,\theta}$ in $D(z_0, \varepsilon_1)$. Then there are four local nodal domains if $\Phi_{h,\theta}(z_{h,\theta}) = 0$ and three local nodal domains if $\Phi_{h,\theta}(z_{h,\theta}) \neq 0$ (see Subsection 6.3.1 for a detailed proof).

5.4.2. Analysis in a neighbourhood of a boundary point. It remains to control what is going on at the boundary. We consider a point $z_0 \in \partial\Omega$ such that z_0 is a zero of Φ_{h_0,θ_0} which in addition is assumed to be critical when $h_0 = +\infty$.

We first assume that we avoid the corners and successively consider three cases:

- $h_0 = +\infty$, perturbation only in θ.
- $0 < h_0 < +\infty$, general perturbation.
- $h_0 = +\infty$, general perturbation.

In the first case, the proof follows the same argument as that used in the proof of Lemma 5.8 and uses the local structure of a Dirichlet eigenfunction at the boundary (see [2] and Appendix A).

For the second case, considering the proof of Lemma 5.8 once again, we choose $\varepsilon_1 > 0$ sufficiently small such that z_0 is the only boundary point in the nodal set. Then the proof goes in the same way.

In the third case, the situation is more delicate due to the complete vanishing of $\Phi_{+\infty,\theta_0}$ on the boundary, which should not be the case for Φ_{h_0,θ_0} with $h_0 < +\infty$. To deal with this, we need the following lemma.

Lemma 5.10. *Let $\theta = \theta_0$ and Z^{bnd} denote the intersection of the nodal set of $\Phi_{+\infty,\theta_0}$ with the boundary. Then for any $\varepsilon > 0$ there exists h^*_ε such that the set $\{z : d(z, \partial\Omega) < \varepsilon\} \cap \{z : d(z, Z^{bnd}) > \varepsilon\}$ does not meet the zero set of $\Phi_{h,\theta}$ for any $h^*_\varepsilon \leq h < +\infty$ and any θ such that $|\theta - \theta_0| < \frac{1}{h^*_\varepsilon}$.*

In other words we have some nodal stability up to the boundary as $h \to +\infty$.

Proof. We consider the following two cases.

At a regular point of the boundary. We consider a point z_0 of the boundary (or a closed interval I in the boundary) which is not a critical point for $\Phi_{+\infty,\theta_0}$. By perturbation, this is still true for $|\theta - \theta_0|$ small. In this case, the normal derivative of $\Phi_{+\infty,\theta}$ for $z_0 \in I$ does not vanish, and to fix the ideas we can assume that

$$\partial_\nu \Phi_{+\infty,\theta}(z_0,\theta) > c > 0$$

(the other case would be treated similarly). By continuity, replacing c by $\frac{c}{2}$, this is still true for $\Phi_{h,\theta}$, with z in a h-independent neighbourhood of I and $\frac{1}{h}$ small enough.

On the other hand, we know that $\Phi_{h,\theta}$ satisfies the Robin condition:

$$\partial_\nu \Phi_{h,\theta}(z_0,\theta) + h\,\Phi_{h,\theta}(z_0,\theta) = 0.$$

Hence

$$\Phi_{h,\theta}(z_0,\theta) = -\frac{1}{h}\partial_\nu \Phi_{h,\theta}(z_0,\theta) < 0.$$

This implies that there exists a neighbourhood of I and $\eta > 0$ such that, for $\frac{1}{h} + |\theta - \theta_0| < \eta$, $\Phi_{h,\theta}$ is negative (actually $< -\frac{c}{2h}$).

At a corner. After translation, we assume that the corner is at $(0,0)$. We also assume that $(0,0)$ does not belong to the nodal set of $\Phi_{+\infty,\theta_0}$ and that $\Phi_{+\infty,\theta_0} < 0$ in Ω near the corner.

We now use the previous argument outside of $(0,0)$. For $\varepsilon_0 > 0$ small enough we can take $\eta > 0$ small enough such that, for $\frac{1}{h} + |\theta - \theta_0| < \eta$, $\Phi_{h,\theta}(x,y) < 0$ for $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = \varepsilon_0^2\} \cap \Omega$.

Suppose now that $\Phi_{h,\theta}(x,y) > 0$ for some $(x,y) \in D\big((0,0),\varepsilon_0\big)$. Then there is a nodal domain inside $D\big((0,0),\varepsilon_0\big)$ and this is excluded by Proposition 5.3 provided that we have chosen ε_0 sufficiently small. $\square$

Remark 5.11. We have not proven in full generality that $\Phi_{h,\theta}$ is negative at the boundary near the corner but this is not required. We do not know what occurs if the corner belongs to the zero set.

If the corner is not in the zero set of the Dirichlet eigenfunction for some θ_0, we can prove by the previous argument that this is still the case for h large enough and θ close to θ_0. In the case of the square, we get immediately that

$$\partial^2_{x,y}\Phi_{+\infty,\theta_0}(0,0) < 0. \tag{5.4}$$

We now estimate $\Phi_{h,\theta}(0,0)$. Using the Robin condition on the two sides, we obtain the formulas

$$\Phi_{h,\theta}(x,0) = \frac{1}{h}\partial_y \Phi_{h,\theta}(x,0), \qquad \Phi_{h,\theta}(0,y) = \frac{1}{h}\partial_x \Phi_{h,\theta}(0,y)$$

and

$$\partial_x \Phi_{h,\theta}(x,0) = \frac{1}{h} \partial_x \partial_y \Phi_{h,\theta}(x,0),$$

which imply that

$$\Phi_{h,\theta}(0,0) = h^{-2} \partial^2_{x,y} \Phi_{h,\theta}(0,0).$$

By perturbation of (5.4), we also have,

$$\partial^2_{x,y} \Phi_{h,\theta}(0,0) < 0,$$

which implies

$$\Phi_{h,\theta}(0,0) < 0.$$

This leads to the following result when $z_0 \in \partial\Omega$. We assume that z_0 is a critical point of $\Phi_{+\infty,\theta_0}$ associated with an eigenvalue $\lambda(\infty)$. We choose ε_1 small enough such that

- Proposition 5.3 applies;
- $\mathscr{C}(z_0, \varepsilon_1) \cap \Omega$ crosses the ℓ half-lines emanating from z_0 transversally at ℓ points $z_j(h_0, \theta_0)$ $(j = 1, \dots, \ell)$.

Here we have used the general results for the local structure of an eigenfunction of the Dirichlet Laplacian (see [2], see also [28] for the case with corners).

Lemma 5.12. *With the previous notation and the assumptions of Proposition* 5.3, *we have that there exists* $\eta_0 > 0$ *such that if* $|\frac{1}{h} - \frac{1}{h_0}| + |\theta - \theta_0| < \eta_0$, *then the number of nodal domains of* $\Phi_{h,\theta}$ *intersecting the disc* $D(z_0, \varepsilon_1)$ *cannot increase. If* $\ell = 1$, *the number of nodal domains equals two and remains fixed.*

5.5. Application to the square. We come back to the case of the square and prove Theorem 1.2. To this end, with Proposition 4.3 in mind, it is sufficient to show the following.

Proposition 5.13. *There exists* $h_0 > 0$ *such that for any* $h > h_0$, *any eigenfunction corresponding to* $\frac{1}{\pi^2}\left(\alpha_0(h)^2 + \alpha_2(h)\right)^2$ *has* 2, 3, *or* 4 *nodal domains* (*as in the Dirichlet case*). *Hence for* $h > h_0$, $\lambda_{5,h}$ *is not Courant-sharp.*

Proof. The property is indeed true for $h = +\infty$ and, by the results of the preceding sections, the number of nodal domains cannot increase and is necessarily > 1. □

In the next section, we carry out a deeper analysis for the eigenfunction associated with the fifth eigenvalue, where we count the nodal domains case by case. For some cases, the proof will use the explicit properties of the eigenfunctions $\Phi_{h,\theta}$ (see below).

In relation to Proposition 5.13, we note that by choosing non-critical values of θ we can obtain that 2, 3 and 4 nodal domains are attained for h_0 large enough.

6. Particular case $k = 5$

6.1. Main statement. Looking at the fifth eigenvalue corresponding to the pair $(0, 2)$, which is Courant-sharp for Neumann and not Courant-sharp for Dirichlet, we consider the family of eigenfunctions in $\left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)^2$ with $\theta \in (-\pi, \pi]$:

$$\begin{aligned}\Phi_{h,\theta,0,2}(x,y) &:= \cos\theta \cos\big(\alpha_0(h)x/\pi\big)\cos\big(\alpha_2(h)y/\pi\big)\\ &\quad + \sin\theta\cos\big(\alpha_2(h)x/\pi\big)\cos\big(\alpha_0(h)y/\pi\big).\end{aligned} \tag{6.1}$$

Up to changing the sign of the eigenfunction, it is sufficient to consider $\theta \in [0, \pi)$. We prove the following proposition.

Proposition 6.1. *There exists $h_0 > 0$ such that for any $h > h_0$, any eigenfunction corresponding to $\frac{1}{\pi^2}\left(\alpha_0(h)^2 + \alpha_2(h)\right)^2$ has 2, 3, or 4 nodal domains (as in the Dirichlet case). More precisely, there are three critical values $\theta_j^*(h) \in [0, \pi)$ $(j = 1, 2, 3)$ such that*

$$\theta_1^*(h) = \arctan\left(-\frac{1}{q_2(h)}\right), \qquad \theta_2^*(h) = \frac{\pi}{2} - \theta_1^*(h), \qquad \theta_3^* = \frac{3\pi}{4},$$

where

$$q_2(h) = \frac{\cos\left(\frac{\alpha_2}{2}\right)}{\cos\left(\frac{\alpha_0}{2}\right)},$$

and such that $\Phi_{h,\theta}$ has:

- 3 *nodal domains for $\theta \in [0, \theta_1^*(h)]$;*
- 2 *nodal domains for $\theta \in \left(\theta_1^*(h), \theta_2^*(h)\right)$;*
- 3 *nodal domains for $\theta \in [\theta_2^*(h), \theta_3^*)$;*
- 4 *nodal domains for $\theta = \theta_3^*$;*
- 3 *nodal domains for $\theta \in (\theta_3^*, \pi)$.*

Note that for the whole family of eigenfunctions, we have symmetry with respect to the two axes. In addition, the corresponding eigenvalue $\frac{1}{\pi^2}\left(\alpha_0(h)^2+\alpha_2(h)^2\right)$ is the fifth eigenvalue for any $h \in [0,+\infty]$. For $h=0$, we have $\alpha_0(0)=0$ and $\alpha_2(0)=2\pi$.

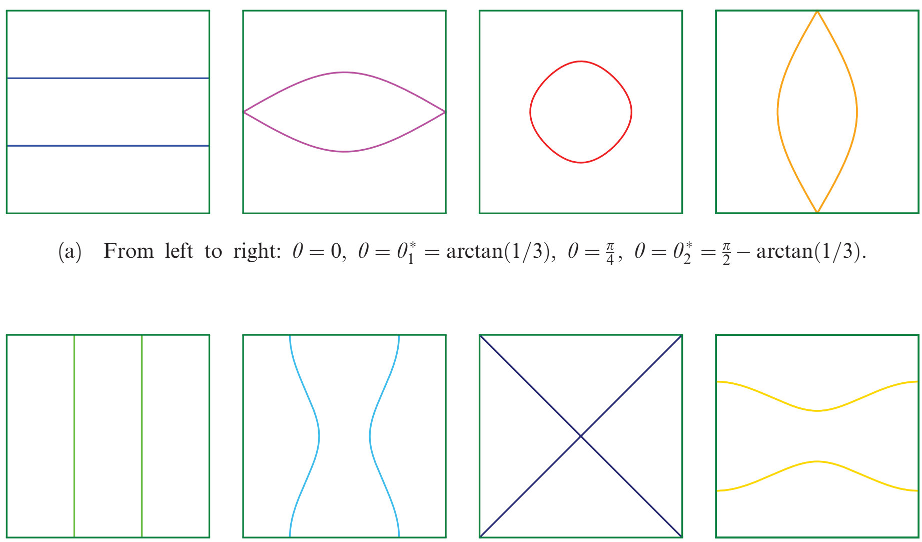
6.2. The Dirichlet case. For $h=+\infty$, i.e. in the Dirichlet case, we have $\alpha_0(+\infty) = \pi$ and $\alpha_2(+\infty)=3\pi$. The figures of Pockels, [37], give the various possibilities as a function of θ. We refer to [3] for a more rigorous mathematical analysis but note that Pockels gives all the possible topologies. He also gives the pictures for the θ corresponding to transitions between these topologies. In Figure 3, we plot the fifth Dirichlet eigenfunction

$$\Phi_{+\infty,\theta,0,2}(x,y)=\cos\theta\cos(x)\cos(3y)+\sin\theta\cos(3x)\cos(y)$$

for $(x,y)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)^2$ and various values of θ.

The critical values of θ corresponding to a change in the number of interior critical points or the number of boundary critical points in the nodal set are $\theta_1^*=\arctan(1/3)$, $\theta_2^*=\frac{\pi}{2}-\arctan(1/3)$, and $\theta_3^*=\frac{3\pi}{4}$.

As was proven in [3] and can be seen in Figure 3, the fifth Dirichlet eigenfunction has either 2, 3 or 4 nodal domains. More precisely, we have for $\theta \in [0,\pi)$:



(a) From left to right: $\theta=0$, $\theta=\theta_1^*=\arctan(1/3)$, $\theta=\frac{\pi}{4}$, $\theta=\theta_2^*=\frac{\pi}{2}-\arctan(1/3)$.

(b) From left to right: $\theta=\frac{\pi}{2}$, $\theta=\frac{5\pi}{8}$, $\theta=\frac{3\pi}{4}$, $\theta=\frac{7\pi}{8}$.

Figure 3. The Dirichlet eigenfunction $\Phi_{+\infty,\theta,0,2}$ for various values of θ.

- 3 nodal domains for $\theta \in [0, \theta_1^*]$;
- 2 nodal domains for $\theta \in (\theta_1^*, \theta_2^*)$;
- 3 nodal domains for $\theta \in [\theta_2^*, \theta_3^*)$;
- 4 nodal domains for $\theta = \theta_3^*$;
- 3 nodal domains for $\theta \in (\theta_3^*, \pi)$.

In what follows, we prove that this holds for h sufficiently large.

6.3. Application of Section 5. For h large enough, we analyse

$$\begin{aligned}\psi(\theta, x, y) &= \Phi_{h,\theta,0,2}(x, y)\\ &= \cos\theta\cos\left(\frac{\alpha_0 x}{\pi}\right)\cos\left(\frac{\alpha_2 y}{\pi}\right) + \sin\theta\cos\left(\frac{\alpha_2 x}{\pi}\right)\cos\left(\frac{\alpha_0 y}{\pi}\right).\end{aligned}$$

This solution has a double symmetry with respect to $x \mapsto -x$ and $y \mapsto -y$.

6.3.1. Interior critical points. We can look at the critical points of ψ as a function of θ. In the case of Dirichlet, the only possible critical point is for $x = y = 0$ and can only occur for $\cos\theta + \sin\theta = 0$ (we assume $\theta \notin \mathbb{Z}\frac{\pi}{2}$).

For $\cos\theta + \sin\theta = 0$, $x = \pm y$ belong to the zero set of ψ. We show that the zero set is exactly given by $x = \pm y$. We observe that the Hessian of ψ at $(x, y) = (0, 0)$ is

$$H_{(x,y)=(0,0)} = \frac{\cos\theta}{\pi^2}\begin{pmatrix} \alpha_2^2 - \alpha_0^2 & 0 \\ 0 & \alpha_0^2 - \alpha_2^2 \end{pmatrix},$$

which has negative determinant so $(x, y) = (0, 0)$ is a non-degenerate critical point of ψ. We see that $H_{(x,y)=(0,0)}$ has one positive eigenvalue and one negative eigenvalue, so the Morse index of the critical point $(0, 0)$ is 1. By the Morse Lemma, in a neighbourhood U of $(0, 0)$, there is a diffeomorphism $\phi = (u, v) : U \mapsto V \subset \mathbb{R}^2$ with $\phi(0, 0) = (0, 0)$ such that $\tilde{\psi} := \psi \circ \phi^{-1}$ has the form

$$\tilde{\psi}(u, v) = \tilde{\psi}(0, 0) - u^2 + v^2 = \cos\theta + \sin\theta - u^2 + v^2.$$

So we see immediately that the critical point $(0, 0)$ is isolated. With the condition that $\cos\theta + \sin\theta = 0$, the zero set is given by $u = \pm v$. Since ϕ is a bijection and $x = \pm y$ is contained in the zero set of ψ, the zero set of ψ is given by $x = \pm y$.

More generally, the same proof gives that the zero set of $\psi(\theta, \cdot) - (\cos\theta + \sin\theta)$ is given near $(0, 0)$ by $x = \pm y$. We remark that in this case there are 4 nodal domains.

6.3.2. Boundary edge. Considering the boundary edge $x = \frac{\pi}{2}$, we have that either $y = \pm\frac{\pi}{2}$ is in the nodal set, in which case there are 4 nodal domains by symmetry, or $y = \pm\frac{\pi}{2}$ is not in the nodal set. In the latter case, Theorem 3.1 gives that there are at most two points on the boundary edge $x = \frac{\pi}{2}$ that are in the nodal set. If there are exactly two such points in the nodal set, then this corresponds to 3 nodal domains. If there are no boundary points in the nodal set, then this corresponds to 2 nodal domains. For example, see Figure 3.

6.3.3. Double point on the boundary. We now analyse what is going on at the double point on the boundary. This occurs for Dirichlet when $\tan\theta = \frac{1}{3}$ and for $y = 0$. Here the situation is simple (see [37]). We observe that $y = 0$ is a double point for $\tan\theta = -\frac{1}{q_2(h)}$. From $\psi(\theta, \frac{\pi}{2}, y) = 0$, we have

$$\cos\left(\frac{\alpha_2 y}{\pi}\right) + t\frac{\cos\left(\frac{\alpha_2}{2}\right)}{\cos\left(\frac{\alpha_0}{2}\right)}\cos\left(\frac{\alpha_0 y}{\pi}\right) = 0.$$

The critical $t = \tan\theta$ is defined by $t = -1/q_2(h)$ with

$$q_2(h) = \frac{\cos\left(\frac{\alpha_2}{2}\right)}{\cos\left(\frac{\alpha_0}{2}\right)}.$$

Hence $t = \frac{1}{3} + \mathcal{O}\left(\frac{1}{h}\right)$, and we have near $y = 0$,

$$y^2 = \left(c + \mathcal{O}\left(\frac{1}{h}\right)\right)\left(t + \frac{1}{q_2(h)}\right).$$

Again, this is the perturbation of a Morse function depending on the parameters h and θ with the particularity that when $\psi = 0$ and $y = 0$, the critical point is always $\left(\frac{\pi}{2}, 0\right)$. We remark that in this case there are 3 nodal domains.

6.4. Interior critical points for any $h > 0$. In this subsection, we show that there are no other critical points than $(0, 0)$ without any restriction on $h > 0$. It is immediate that $(0, 0)$ is a critical point and we get the same condition as in the Dirichlet case. Writing $\psi = 0$ and $\nabla\psi = 0$, we get as a necessary condition that

$$\alpha_2 \tan\left(\frac{\alpha_2 x}{\pi}\right) = \alpha_0 \tan\left(\frac{\alpha_0 x}{\pi}\right), \qquad \alpha_2 \tan\left(\frac{\alpha_2 y}{\pi}\right) = \alpha_0 \tan\left(\frac{\alpha_0 y}{\pi}\right). \tag{6.2}$$

Lemma 6.2. *Let α_0 and α_2 satisfy* (2.7). *For $x \in \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)$, $\alpha_0 \tan(\alpha_0 x/\pi) = \alpha_2 \tan(\alpha_2 x/\pi)$ if and only if $x = 0$.*

Proof. Let us look at the function

$$\left[0,\frac{\pi}{2}\right] \ni x \mapsto W(x) = \alpha_0 \sin(\alpha_0 x)\cos(\alpha_2 x) - \alpha_2 \sin(\alpha_2 x)\cos(\alpha_0 x).$$

Up to some multiplicative renormalisation of the eigenfunctions, we recognise the Wronskian of the eigenfunctions u_0 and u_2. But for the Wronskian, we have

$$W'(x) = (\lambda_0 - \lambda_2)u_0(x)u_2(x).$$

Now we observe that $W(0) = 0$ and that by (2.7), $W\left(\frac{\pi}{2}\right) = 0$. Moreover W has a unique critical point in $\left(0,\frac{\pi}{2}\right)$ at the first zero of u_2. Hence $W(x)$ cannot vanish except at $x = 0$ and $\frac{\pi}{2}$. □

It is clear that this implies that $(0,0)$ is the only possible critical point in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)^2$. The condition that $\psi(0,0) = 0$ implies $\cos\theta + \sin\theta = 0$.

7. Analysis of crossings

In this section, we analyse the possible crossings of two curves $h \mapsto \lambda_{p,q,h}(S)$ and $h \mapsto \lambda_{p',q',h}(S)$ defined in an interval of $[0,+\infty)$. This is indeed quite important as we want to follow the labelling of these eigenvalues when h varies. We then consider the eigenvalue $\lambda_{9,h}(S)$ which is Courant-sharp when $h = 0$ but not when $h = +\infty$.

7.1. A general result.

Proposition 7.1. *For distinct pairs (p,q) and (p',q'), with $p \le q$ and $p' \le q'$, there is at most one value of h in $[0,+\infty)$ such that $\lambda_{p,q,h}(S) = \lambda_{p',q',h}(S)$.*

Proof. Suppose that $\lambda_{p,q,h}(S) = \lambda_{p',q',h}(S)$. Without loss of generality, suppose $p < p' \le q' < q$. Consider the variation of

$$(0,+\infty) \ni h \mapsto \sigma(h) := \frac{1}{\pi^2}\left(\alpha_p(h)^2 + \alpha_q(h)^2 - \alpha_{p'}(h)^2 - \alpha_{q'}(h)^2\right).$$

The zeros of σ correspond to the values of h for which the curves corresponding to (p,q), (p',q') intersect. To analyse its variation, we note that

$$\sigma'(h) = \frac{2}{\pi^2}\left(\alpha_p(h)\alpha_p'(h) + \alpha_q(h)\alpha_q'(h) - \alpha_{p'}(h)\alpha_{p'}'(h) - \alpha_{q'}(h)\alpha_{q'}'(h)\right).$$

Now, we deduce from (2.7) and (2.8), that $h \mapsto \alpha_k(h)$ satisfies the differential equation

$$\frac{\alpha_k'}{\alpha_k}\left(h\pi + \frac{\alpha_k^2}{2} + \frac{h^2\pi^2}{2}\right) = \pi, \tag{7.1}$$

which implies

$$\alpha_k'\alpha_k\left(h\pi + \frac{\alpha_k^2}{2} + \frac{h^2\pi^2}{2}\right) = \pi\alpha_k^2. \tag{7.2}$$

We introduce for $h > 0$ and $k \in \mathbb{N}$,

$$a_k(h) = h\pi + \frac{\alpha_k^2}{2} + \frac{h^2\pi^2}{2} > 0.$$

We deduce

$$\sigma'(h) = \frac{2}{\pi}\left(\frac{\alpha_p^2}{a_p} + \frac{\alpha_q^2}{a_q} - \frac{\alpha_{p'}^2}{a_{p'}} - \frac{\alpha_{q'}^2}{a_{q'}}\right) = -\frac{4}{\pi}\left(h\pi + \frac{h^2\pi^2}{2}\right)\left(\frac{1}{a_p} + \frac{1}{a_q} - \frac{1}{a_{p'}} - \frac{1}{a_{q'}}\right).$$

We now assume that $\sigma(h) = 0$, which implies

$$a_p + a_q = a_{p'} + a_{q'}.$$

This gives

$$\sigma'(h) = -\frac{4}{\pi}\left(h\pi + \frac{h^2\pi^2}{2}\right)\left(\frac{(a_p + a_q)(a_{p'}a_{q'} - a_pa_q)}{(a_pa_qa_{p'}a_{q'})}\right).$$

So the sign of $\sigma'(h)$ is the sign of $a_pa_q - a_{p'}a_{q'}$. For $\varepsilon > 0$, we can now write $a_p = a_{p'} - \varepsilon$ and $a_q = a_{q'} + \varepsilon$, and compute

$$a_pa_q - a_{p'}a_{q'} = (a_{p'} - \varepsilon)(a_{q'} + \varepsilon) - a_{p'}a_{q'} = (a_{p'} - a_{q'})\varepsilon - \varepsilon^2 < 0.$$

Since the derivative of σ has constant sign, there can be at most one point of intersection.

Remark 7.2. The proof of Proposition 7.1 shows that if $p < p' \leq q' < q$ and $\lambda_{p,q,h^*} = \lambda_{p',q',h^*}$ for some $h^* \geq 0$, then the map

$$h \mapsto \pi^{-2}\left(\alpha_{p'}(h)^2 + \alpha_{q'}(h)^2 - \alpha_p(h)^2 - \alpha_q(h)^2\right)$$

is increasing for $h > h^*$. Hence the curve $\pi^{-2}\big(\alpha_p(h)^2 + \alpha_q(h)^2\big)$ is below the curve $\pi^{-2}\big(\alpha_{p'}(h)^2 + \alpha_{q'}(h)^2\big)$ for $h > h^*$.

7.2. The eigenvalue $\lambda_{9,h}(S)$. The ninth eigenvalue of the Neumann Laplacian for the square is Courant-sharp, [29], and corresponds to the eigenvalue $2^2 + 2^2 = 8$. This eigenvalue is simple and corresponds to the labelling $(2,2)$. The eigenfunction reads

$$\Phi_{0,\theta,2,2}(x,y) = \cos 2x \cos 2y, \qquad \text{for } (x,y) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^2.$$

It is easy to see that the Courant-sharp property is still true for h small enough. By deformation, the eigenfunction is

$$\Phi_{h,\theta,2,2}(x,y) = \cos\big(\alpha_2(h)x/\pi\big)\cos\big(\alpha_2(h)y/\pi\big),$$

with corresponding eigenvalue $\frac{2}{\pi^2}\big(\alpha_2(h)\big)^2$. The nodal structure is given by

$$\frac{\alpha_2(h)x}{\pi} = -\frac{\pi}{2}, \qquad \frac{\alpha_2(h)x}{\pi} = \frac{\pi}{2}, \qquad \frac{\alpha_2(h)y}{\pi} = -\frac{\pi}{2}, \qquad \frac{\alpha_2(h)y}{\pi} = \frac{\pi}{2},$$

hence for this eigenfunction and for $h \in [0,+\infty)$, there are nine nodal domains as long as $\frac{2}{\pi^2}\big(\alpha_2(h)\big)^2$ is the ninth eigenvalue.

The issue is to follow the labelling of $\frac{2}{\pi^2}\big(\alpha_2(h)\big)^2$ and we observe that when $h = +\infty$ this eigenvalue is 18 which has minimal labelling 11. Because $9 < 11$, this eigenfunction is NOT Courant-sharp for h sufficiently large.

On the other hand the eigenvalue $\frac{1}{\pi^2}\big(\alpha_0(h)^2 + \alpha_3(h)^2\big)$ which has minimal labelling 10 for $h = 0$ arrives with labelling 9 at $h = +\infty$. Hence some transition occurs for at least one $h_9^* > 0$ which satisfies

$$\alpha_0(h)^2 + \alpha_3(h)^2 = 2\alpha_2(h)^2.$$

By Proposition 7.1, there is at most one point of intersection between the curves corresponding to $(2,2)$ and $(3,0)$.

We recall that $\alpha_0(0) = 0$, $\alpha_1(0) = \pi$, $\alpha_2(0) = 2\pi$, $\alpha_3(0) = 3\pi$ and that $\alpha_0(+\infty) = \pi$, $\alpha_1(+\infty) = 2\pi$, $\alpha_2(+\infty) = 3\pi$, $\alpha_3(+\infty) = 4\pi$, so $\alpha_0(h)^2 + \alpha_3(h)^2$ is increasing from $9\pi^2$ to $17\pi^2$ when $2\alpha_2(h)^2$ goes from $8\pi^2$ to $18\pi^2$.

We first show that the curves corresponding to the pairs $(2,2)$, $(3,0)$ do not intersect the curves corresponding to the other pairs.

From above, we see that the eigenvalues corresponding to the pairs $(3,3)$, $(4,2)$, $(2,4)$ and so on are all larger than or equal to 18. So we need to consider the eigenvalues corresponding to the pairs $(3,1)$, $(3,2)$, $(4,0)$, $(4,1)$ and show that they do not correspond to the ninth, tenth or eleventh eigenvalues for any $h > 0$.

Numerically we find that,

$$
\begin{aligned}
\lambda_{3,1,h}(S) = \lambda_{1,3,h}(S) &\geq 18 \qquad \text{for } h > 11.4225, \\
\lambda_{3,2,h}(S) = \lambda_{2,3,h}(S) &\geq 18 \qquad \text{for } h > 2.6288, \\
\lambda_{4,0,h}(S) = \lambda_{0,4,h}(S) &\geq 18 \qquad \text{for } h > 1.2668, \\
\lambda_{4,1,h}(S) = \lambda_{1,4,h}(S) &\geq 18 \qquad \text{for } h > 0.4208.
\end{aligned}
$$

So we are left to consider $h \leq 11.4225$.

From below, we see that the eigenvalues corresponding to the pairs $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$ are smaller than or equal to 8 for all $0 < h < +\infty$. So we need to consider the eigenvalues corresponding to the pairs $(2, 0)$, $(2, 1)$ and show that they do not correspond to the ninth, tenth or eleventh eigenvalues for any $h > 0$. Numerically we find that,

$$
\begin{aligned}
\lambda_{2,2,h}(S) &\geq 13 \qquad \text{for } h > 2.9804, \\
\lambda_{3,0,h}(S) = \lambda_{0,3,h}(S) &\geq 13 \qquad \text{for } h > 3.5468.
\end{aligned}
$$

So we are left to consider $h \leq 3.5468 < 11.4225$.

In Figure 4, we plot the Robin eigenvalues of the square $\left(\alpha_m(h)^2 + \alpha_n(h)^2\right)/\pi^2$ for $h \leq 12$ corresponding to the pairs $(0, 0)$, $(1, 0)$, $(1, 1)$, $(2, 0)$, $(2, 1)$, $(2, 2)$, $(3, 0)$, $(3, 1)$, $(3, 2)$, $(4, 0)$, $(4, 1)$.

From Figure 4, we see that for $h \leq 12$ the curves corresponding to the pairs $(2, 2)$, $(3, 0)$ do not intersect the curves corresponding to the other pairs. By Proposition 7.1, the curves corresponding to $(2, 2)$ and $(3, 0)$ intersect for a unique value of $h = h_9^* > 0$.

Since $u_{2,2}(x, y)$ is an eigenfunction corresponding to $\lambda_{9,h_9^*}(S)$ that has 9 nodal domains, we have proved the following proposition.

Proposition 7.3. *There exists $h_9^* > 0$ such that $\lambda_{9,h}$ is Courant-sharp for $0 \leq h \leq h_9^*$ and not Courant-sharp for $h > h_9^*$.*

By the bisection method, we compute h_9^* numerically and find that $h_9^* \sim 1.6967$.

By the above, $\lambda_{9,h}$ is given by the pair $(2, 2)$ for $h \leq h_9^*$ and the pair $(3, 0)$ for $h > h_9^*$. Also, $\lambda_{10,h}$ is given by the pair $(0, 3)$ and $\lambda_{11,h}$ is given by the pair $(3, 0)$ for $h \leq h_9^*$ and the pair $(2, 2)$ for $h > h_9^*$.

This shows that whether the eigenfunction corresponding to a Robin eigenvalue of the square is an odd function or an even function depends on h (in the case where there are crossings).

For example, for $\lambda_{9,h}$ with $h \leq h_9^*$, we have that $u_{2,2}(-x, -y) = u_{2,2}(x, y)$. On the other hand, for $h > h_9^*$,

$$
u_{3,0}(-x, -y) = -u_{3,0}(x, y) \qquad \text{and} \qquad u_{0,3}(-x, -y) = -u_{0,3}(x, y).
$$

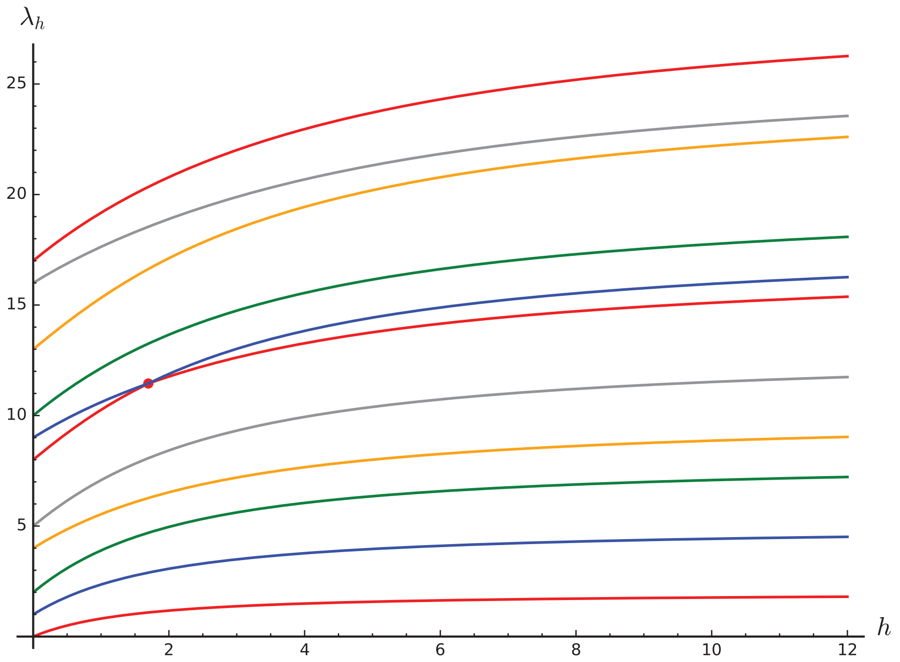


Figure 4. The Robin eigenvalues of the square $\left(\alpha_m(h)^2 + \alpha_n(h)^2\right)/\pi^2$ for $h \leq 12$ corresponding to the pairs $(0,0)$, $(1,0)$, $(1,1)$, $(2,0)$, $(2,1)$, $(2,2)$, $(3,0)$, $(3,1)$, $(3,2)$, $(4,0)$, $(4,1)$. The intersection between the curves corresponding to $(2,2)$ and $(3,0)$ occurs at $(1.6970, 11.4498)$.

So any linear combination of $u_{3,0}(x,y)$ and $u_{0,3}(x,y)$ is antisymmetric with respect to the transformation $(x,y) \mapsto (-x,-y)$. Hence $\lambda_{9,h}$ is not Courant-sharp for $h > h_9^*$ (via Lemma 4.2).

For $h = h_9^*$, any eigenfunction corresponding to $\lambda_{9,h_9^*}(S)$ is a linear combination of $u_{2,2}(x,y)$, $u_{3,0}(x,y)$ and $u_{0,3}(x,y)$, so in general it is neither symmetric nor antisymmetric with respect to the transformation $(x,y) \mapsto (-x,-y)$.

Appendix. On the local structure of the nodal set

In this appendix, we prove some well-known results for the nodal set of an eigenfunction of the Neumann problem and extend them to the Robin problem. Although used in various contributions, for example [26], no detailed proofs seem to be published for the Neumann problem. For the Dirichlet problem, see [30] and [28] where the case with corners or cracks is also considered. In addition, we require these results under weaker regularity assumptions on the boundary.

A.1. Main statement. Our main result describes the local structure of an eigenfunction of the Laplacian with a Robin boundary condition around an interior critical point or at the boundary.

Theorem A.1. *Let Ω be an open set in $\mathbb{R}^2$ with $C^{2,+}$ boundary. Let $h \in [0, +\infty)$ and let u be a real-valued eigenfunction of the Laplacian with h-Robin boundary condition. Then $u \in C^2(\overline{\Omega})$. Furthermore, u has the following properties:*

(1) *If u and ∇u vanish at a point $x_0 \in \Omega$ then there exists $\ell > 1$, $\varepsilon > 0$ and a real-valued, non-zero, harmonic, homogeneous polynomial of degree ℓ such that:*

$$u(x) = p_\ell(x - x_0) + \mathcal{O}(|x - x_0|^{\ell+\varepsilon}). \tag{A.1}$$

(2) *If u vanishes at $x_0 \in \partial\Omega$, then* (A.1) *holds for some $\ell > 0$ and*

$$u(x) = a\, r^\ell \cos \ell\omega + \mathcal{O}(r^{\ell+\varepsilon}) \tag{A.2}$$

for some non-zero $a \in \mathbb{R}$, where (r, ω) are polar coordinates of x around x_0. The angle ω is chosen so that the tangent to the boundary at x_0 is given by the equation $\sin \omega = 0$.

(3) *The nodal set $N(u)$ is the union of finitely many, C^2-immersed circles in Ω, and C^1-immersed lines which connect points of $\partial\Omega$. Each of these immersions is called a nodal line. Note that self-intersections are allowed. The connected components of $\Omega \backslash N(u)$ are called nodal domains.*

(4) *If u has a zero of order ℓ at a point $x_0 \in \Omega$ then exactly ℓ segments of nodal lines pass through x_0. The tangents to the nodal lines at x_0 dissect the full circle of radius $B(x_0, \alpha)$ (for $\alpha > 0$ small enough) into 2ℓ equal angles.*

(5) *If u has a zero of order ℓ at a point $x_0 \in \partial\Omega$ then exactly ℓ segments of nodal lines meet the boundary at x_0. The tangents to the nodal lines at x_0 are given by the equation* $\cos \ell\omega = 0$, *where ω is chosen as in* (A.2).

A.2. Proof of the theorem. The C^2-regularity of u up to the boundary is a consequence of standard Schauder estimates (see [19]). The proof now is in four steps.

A.2.1. Reduction to the Neumann case. The first step is to reduce the problem from the Robin case to the Neumann case. This is done through a change of functions. Setting $u = \exp \phi_h\, v$, we can choose ϕ_h such that $v \in C^2(\overline{\Omega})$ satisfies the Neumann condition. Indeed, this ϕ_h should be in $C^2(\overline{\Omega})$ and satisfy $\partial_\nu \phi_h = -h$ on the boundary of Ω (take $h\, \mathrm{dist}(x, \partial\Omega)$ near $\partial\Omega$ and then use a cut-off function). We obtain a Neumann problem where the Laplacian is replaced by $\exp -\phi_h \circ$

$(-\Delta) \circ \exp \phi_h$, that is the Laplacian with an additional term of degree 1 with $C^1(\overline{\Omega})$ coefficients and an additional term of degree 0 in $C^0(\overline{\Omega})$.

From this point onwards, we consider the Neumann case.

A.2.2. Double manifold. The second step is to use the double manifold as suggested in Donnelly–Feffermann, [13], [14], [15]. As we only wish to prove a local result, by a diffeomorphism we can reduce to the case when the boundary is given by $x_1 = 0$. In these new coordinates, the operator reads

$$H := \sum_{ij} g_{ij}(x_1, x_2)\partial_{x_i}\partial_{x_j} + \sum_i a_i(x_1, x_2)\partial_{x_i} + c(x).$$

In addition, this diffeomorphism can be chosen as a conformal map (see [15]), so more precisely, we have

$$H := -\rho(x)\Delta + \sum_i a_i(x)\partial_{x_i} + c(x).$$

Note that if we had started with the Neumann case ($h = 0$), then there would be no terms of degree one. This would make the proof easier and would permit weaker assumptions.

Starting from u as in Subsection A.2.1, after all these transformations, we get a local solution in $C^2(\overline{\mathbb{R}^2_+})$ of $Hw = \lambda w$.

We define $\tilde{w}$ by

$$\tilde{w}(x_1, x_2) = \begin{cases} w(x_1, x_2) & \text{for } x_1 > 0, \\ w(-x_1, x_2) & \text{for } x_1 < 0. \end{cases}$$

We can then define the extension of the operator as $\tilde{H}$

$$\tilde{H} := -\tilde{\rho}(x)\Delta + \sum_{i=1}^{2} \tilde{a}_i\partial_{x_i} + \tilde{c}(x).$$

where $\tilde{\rho}$, $\tilde{a}_2$ and $\tilde{c}$ are the extensions of ρ, a_2 and c by reflection and $\tilde{a}_1$ is defined by odd reflection.

We observe for later that $\tilde{\rho}$ is Lipschitz and that the other coefficients are bounded.

With this definition, we verify that $\tilde{w}$ is an even function (with respect to x_1) that satisfies the Neumann condition, and a solution of

$$\tilde{H}\tilde{w} = \lambda\tilde{w}.$$

By analysing the way that we obtain $\tilde{w}$ from u (starting from the first line of the proof), it is clear that $\tilde{w} \in C^2(\overline{\mathbb{R}_-} \times \mathbb{R}) \cap C^2(\overline{\mathbb{R}_+} \times \mathbb{R})$. Also, $\tilde{w}$ is clearly in $C^{1,1}(\mathbb{R}^2)$ (as it's an even function with respect to x_1).

We note that from $\tilde{w}(x_1, x_2) = \tilde{w}(-x_1, x_2)$, we get $\partial^2_{x_1,x_2}\tilde{w}(0, x_2) = 0$. The other second derivatives match on $x_1 = 0$. Moreover $\tilde{w}$ is actually locally in $C^2(\mathbb{R}^2)$.

A.2.3. Nodal structure for solutions of a second-order elliptic operator with coefficients with less regularity. The third step is to determine whether the local nodal structure that holds for the Laplacian still holds for this second-order elliptic operator which has coefficients with less regularity. This problem is analysed by Hardt–Simon in [25] (at least in a weaker sense) and more precisely in [24] (see Theorem 1.5 and Theorem 3.1). The following theorem is Theorem 3.1 of [24] applied to $\tilde{w}$ and

$$L := \tilde{H} - \lambda$$

in the neighbourhood of a point in the zero set on the boundary, which is assumed to be $(0, 0)$. In [24], the author proves the result for a second-order elliptic operator of the form

$$L = \sum_{ij} a_{ij}\partial_{x_i}\partial_{x_j} + \sum_i b_i(x)\partial_{x_i} + c$$

where the a_{ij} are Hölder and the other coefficients are bounded.

From this point onwards, we omit the tildes.

Theorem A.2. *Suppose that $Lu = 0$ and that u is not flat at $(0, 0)$, that is, u has finite vanishing order at $(0, 0)$. Then there exists a homogeneous harmonic polynomial P of some degree $d \geq 0$ and, for any $p > 1$, an $\varepsilon > 0$ such that $\psi := u - P$ satisfies:*

$$\psi(x) = \mathcal{O}(|x|^{d+\varepsilon}),$$

and

$$r^2\left(\int_{B(0,r)} |\nabla^2\psi(x)|^p \, dx\right)^{1/p} + r\left(\int_{B(0,r)} |\nabla\psi(x)|^p \, dx\right)^{1/p} \leq Cr^{d+\varepsilon+2/p}, \qquad r > 0.$$

In other words u is locally like the harmonic polynomial P in the pointwise sense, and the first and second derivatives of u are locally like the corresponding derivatives of P in the integral sense.

Remark A.3. Note that to apply Theorem A.2, we need to know that u is not flat. According to [24] (p. 985, lines 7–9), this is the case under our assumptions and the reference is [18].

This theorem gives a good indication of the nodal structure: it should be close to the zero set of the harmonic polynomial P whose structure is well known.

A.2.4. Cheng–Kuo's argument. Hence the last step is to verify if Cheng's argument [10] applies (a former reference is [6]). We can apply the following lemma attributed by Cheng [10] to Kuo [33].

Lemma A.4. *Suppose that u and ϕ are smooth functions in $\mathbb{R}^2$ such that, with $\psi = u - \phi$, we have for some $d \geq 1$ and $\varepsilon > 0$,*

(i) $\psi(x) = \mathcal{O}(|x|^{d+\varepsilon})$,

(ii) $\nabla\psi(x) = \mathcal{O}(|x|^{d-1+\varepsilon})$,

(iii) *ϕ vanishes with order d at* 0,

(iv) $|\nabla\phi(x)| \geq \frac{1}{C}|x|^{d-1}$.

Then there exists a local C^1 diffeomorphism Θ fixing the origin such that

$$u(x) = \phi\big(\Theta(x)\big).$$

In [10], Cheng applies the lemma to C^∞ functions, but the regularity of u and ϕ is not discussed there. The proof clearly holds for C^2 functions and this assumption is satisfied in our case.

To apply this lemma to the present situation, we observe that a homogeneous harmonic polynomial of degree d in dimension 2 satisfies (iii) and (iv) above. It has indeed, for some $\gamma \in \mathbb{C}$, the form $\Re(\gamma z^d)$ with $z = (x_1 + ix_2)$. We note that (i) holds by Theorem A.2.

It remains to verify that (ii) holds. We compare this condition with the property established in the previous theorem. By Theorem A.2, we get a control of $\nabla\psi$ in $W^{1,p}$ in any ball $B(0,r)$ hence by Sobolev's embedding theorem we have, as soon as $p > 2$, the control of $\nabla\psi$ in $L^\infty\big(B(0,r)\big)$ (see, for example, Part II Case C′ of Theorem 5.4 in [1]). It remains to control the constants appearing in the continuity of this injection. To do this, for $r > 0$, we introduce a cut-off $\chi(x/r)$ where $\chi = 1$ on $B(0,1)$ and $\operatorname{supp}\chi \subset B(0,2)$, and apply the standard Sobolev embedding theorem to $\chi(x/r)\partial_{x_i}\psi$ and use the two estimates from Theorem A.2. We get

$$\sup_{x\in B(0,r)} |\nabla\psi(x)| \leq C_p r^{-2+d+\varepsilon+2/p}, \qquad \text{for } p > 2.$$

For $p > 2$ sufficiently close to 2 (for example $-1 + \frac{2}{p} = -\frac{\varepsilon}{2}$), we get

$$\sup_{x \in B(0,r)} |\nabla \psi(x)| \leq C_p r^{-1+d+\varepsilon/2}, \qquad \text{for } p > 2.$$

This is sufficient to apply the lemma.

Remark A.5. There is a gap in Cheng's paper [10] when applied to a dimension larger than 2. The reason for this is that a harmonic homogeneous polynomial does not always satisfy item (iv) when the dimension is larger than 2 (see Appendix E in [5] for a complete discussion). Here we only use the statement in dimension 2.

A.3. Remarks. We note that all the proofs are local and the results can be obtained locally if we have the corresponding local regularity property.

In the Dirichlet case, we do not require the argument from Subsection A.2.1. We begin with the doubling argument as in Subsection A.2.2 (see [14], [15]). We then apply a conformal diffeomorphism as in [13], [15] and, as we work in dimension 2, the corresponding Laplacian has no terms of degree 1 (see, for example, equation (2.8) of [13]). Similarly to Subsection A.2.2, we obtain a local solution of $Hu = \lambda u$ in $C^2(\overline{\mathbb{R}^2_+})$. Instead of the reflection argument, in order to construct $\tilde{w}$, we can introduce an extension via odd reflection:

$$\tilde{w}(x_1, x_2) = \begin{cases} u(x_1, x_2) & \text{for } x_1 > 0, \\ -u(-x_1, x_2) & \text{for } x_1 < 0. \end{cases}$$

Analogously to the above, if u is an eigenfunction locally in $C^2(\overline{\Omega})$ satisfying the Dirichlet condition (note that $\partial^2_{x_1 x_1} u(0, x_2) = 0$), one can verify that $\tilde{w}$ is locally in $C^2(\mathbb{R}^2)$.

Theorem A.6. *Let Ω be an open set in $\mathbb{R}^2$ with $C^{2,+}$ boundary and let u be a real-valued eigenfunction of the Laplacian with Dirichlet boundary conditions. Then $u \in C^2(\overline{\Omega})$. Furthermore, u has the following properties:*

(1) *If u and ∇u vanish at a point $x_0 \in \overline{\Omega}$ then there exists $\ell > 1$, $\varepsilon > 0$ and a real-valued, non-zero, harmonic, homogeneous polynomial of degree ℓ such that:*

$$u(x) = p_\ell(x - x_0) + \mathcal{O}(|x - x_0|^{\ell+\varepsilon}). \tag{A.3}$$

(2) *If moreover $x_0 \in \partial\Omega$, then*

$$u(x) = ar^\ell \sin \ell\omega + \mathcal{O}(r^{\ell+\varepsilon}) \tag{A.4}$$

for some non-zero $a \in \mathbb{R}$, *where* (r, ω) *are polar coordinates of* x *around* x_0. *The angle* ω *is chosen so that the tangent to the boundary at* x_0 *is given by the equation* $\omega = 0$.

(3) *The nodal set* $N(u)$ *is the union of finitely many,* C^2*-immersed circles in* Ω*, and* C^1*-immersed lines which connect points of* $\partial\Omega$.

(4) *If* u *has a zero of order* ℓ *at a point* $x_0 \in \Omega$*, then exactly* ℓ *segments of nodal lines pass through* x_0. *The tangents to the nodal lines at* x_0 *dissect the full circle of radius* $B(x_0, \alpha)$ *(for* $\alpha > 0$ *small enough) into* 2ℓ *equal angles.*

(5) *If* u *has a zero of order* ℓ *at a point* $x_0 \in \partial\Omega$ *then exactly* $\ell - 1$ *segments of nodal lines meet the boundary at* x_0. *The tangents to the nodal lines at* x_0 *are given by the equation* $\sin \ell\omega = 0$, $\omega \neq 0, \pi$.

We can, for example, refer to [28] for the Dirichlet case which gives the results (except C^2 regularity) under the weaker assumption that the boundary is piecewise $C^{1,+}$.

Acknowledgments. We are very grateful to Pierre Bérard, Dorin Bucur and Thomas Hoffmann–Ostenhof for useful discussions. We thank D. Bucur for his enlightening explanation of the results of [8] and [9]. We thank T. Hoffmann–Ostenhof for the useful suggestion to establish and use Proposition 5.3. We are also very grateful to Alexander Weisse for introducing us to the mathematics software system "SageMath" and helping us to produce some graphs of the Robin eigenvalues of the square. KG acknowledges support from the Max Planck Institute for Mathematics, Bonn, from October 2017 to July 2018. Both authors wish to thank the referees for their helpful suggestions.

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Received February 8, 2019; revision received July 5, 2019

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