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ON DIMENSIONS OF THE REAL NERVE OF THE MODULI SPACE OF RIEMANN SURFACES OF ODD GENUS

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Abstract. In the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus $g \geq 2$ there is important, so-called, real locus $\mathcal{R}_g$, consisting of points representing Riemann surfaces having symmetries, by which we understand antiholomorphic involutions. $\mathcal{R}_g$ itself is covered by the strata $\mathcal{R}_g^k$, each being formed by the points corresponding to surfaces having a symmetry of given topological type k. These strata are known to be real analytic varieties of dimension $3(g-1)$. Also, their topological structure is pretty well known; Goulden-Jackson-Harer and Harer-Zagier have found their Euler characteristic, expressing them through the Riemann zeta function. However, topological properties of the whole real locus $\mathcal{R}_g$ were less studied. The most known fact is its connectivity, proved independently by Buser-Seppälä-Silhol, Catanese-Frediani and Costa-Izquierdo. This paper can be seen as a further contribution to the study of topology of $\mathcal{R}_g$, which was possible through the notion of the nerve $\mathcal{N}_g$, associated to $\mathcal{R}_g$ and called the *real nerve*. We find upper bounds for its geometrical and homological dimensions and we show their sharpness for infinitely many values of odd g. Precise values of these dimensions for even g have been found by the authors in an earlier paper.

1. Introduction

By a symmetry of a Riemann surface of genus g we understand its antiholomorphic involution σ. The *topological type* of the symmetry σ is described by an integer k, whose absolute value is the number of connected components of the set $F = \mathrm{Fix}(\sigma)$ of points fixed by σ, and which is positive if σ is separating, i.e. $X - F$ is disconnected, and negative or 0 otherwise. Each of the components of F is homeomorphic to a circle and called an *oval* of σ in the Hilbert's nineteenth century terminology. The possible types of individual symmetries are known from the classification of Harnack [14] and Weichold [19]. In such a way, the real locus $\mathcal{R}_g$ of the moduli space $\mathcal{M}_g$ of compact Riemann surfaces of given genus g can be covered by $[(3g+4)/2]$ real analytic varieties $\mathcal{R}_g^k$ of dimension $3(g-1)$, consisting of the points represented by surfaces having a symmetry of the type k [7, 17, 18]. Also a topological structure of these varieties is well known. In [13, 8] the authors found their Euler characteristic expressing them through the Riemann zeta function.

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However, topological properties of the whole real locus $\mathcal{R}_g$ were less studied. The most known fact is its connectivity, proved independently by Buser-Seppälä-Silhol, Catanese-Frediani and Costa-Izquierdo. This paper can be seen as a further contribution to the study of topology of $\mathcal{R}_g$, which was possible through the using of the notion of a nerve $\mathcal{N}_g$, associated to the covering of $\mathcal{R}_g$, by the loci $\mathcal{R}_g^k$, and called the *real nerve*.

By the definition, $(k_0, k_1, \ldots, k_n)$ constitutes an n-simplex of our nerve $\mathcal{N}_g$ if and only if

$$\mathcal{R}_g^{k_0} \cap \mathcal{R}_g^{k_1} \cap \ldots \cap \mathcal{R}_g^{k_n} \neq \emptyset,$$

which in turn means that there exists a Riemann surface having simultaneously symmetries of distinct types $k_0, k_1, \ldots, k_n$. The most challenging task here would be to compute the Euler characteristic of $\mathcal{N}_g$. This seems, however, to be rather difficult, though calculation of higher-dimensional Betti numbers seems to be more tractable. Here we shall deal with geometrical and homological dimensions of $\mathcal{N}_g$. We find upper bounds for them and we show their sharpness for infinitely many values of odd g. Precise values of these dimensions for even g have been found by the authors in an earlier paper [12].

By the mentioned results of of Harnack and Weichold (c.f. [5]), $\mathcal{N}_g$ has $[(3g+4)/2]$ points. Moreover, $\mathcal{N}_g$ is connected by the results of Buser-Seppälä-Silhol [4], Catanese-Frediani and it was also shown by Costa and Izquierdo in [6], that given arbitrary type k of a symmetry of a Riemann surface of genus g, a surface can be chosen in such a way, to also have a symmetry of the type -1. This in fact means that -1 is a *spine* of $\mathcal{N}_g$ for any g.

Due to functorial equivalence between compact, connected Riemann surfaces and projective, irreducible, smooth, complex algebraic curves, we can also translate our results to the language of complex curves and their real forms. Under this equivalence, a Riemann surface X admits a symmetry σ if and only if the corresponding curve $\mathcal{C}_X$ has a real form $\mathcal{C}_X(\sigma)$. Moreover, two symmetries σ and τ define real forms $\mathcal{C}_X(\sigma)$ and $\mathcal{C}_X(\tau)$ isomorphic over the reals $\mathbb{R}$ if and only if they are conjugate in $\mathrm{Aut}^{\pm}(X)$. Finally, the set $\mathrm{Fix}(\sigma)$ is homeomorphic to a smooth projective model of the corresponding real form $\mathcal{C}_X(\sigma)$. The image $\mathcal{M}_g^{\mathbb{R}}$, resulting from mapping the moduli space of real algebraic curves into the moduli space $\mathcal{M}_g$ of complex algebraic curves of genus g, is called the *real locus* and it is covered by the strata $\mathcal{M}_{g,k}^{\mathbb{R}}$, consisting of the points of $\mathcal{M}_g$ representing complex algebraic curves having a real form, whose smooth projective model has $|k|$ connected components and the set of its $\mathbb{R}$-rational points leaves, when removed, its complexification connected or not according to k being negative (or 0) or positive respectively.

2. Preliminaries

We obtain our results by using the combinatorial group theory, i.e. theory of non-euclidean crystallographic groups (*NEC groups* in short), which are the discrete and cocompact subgroups of the group $\mathcal{G}$ of all isometries of the hyperbolic plane $\mathcal{H}$. The algebraic structure of such a group Λ is determined by the signature:

$$s(\Lambda) = (h; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}), \tag{1}$$

where the brackets $(n_{i1}, \ldots, n_{is_i})$ are called the *period cycles*, the integers n_{ij} are the *link periods*, m_i *proper periods* and h is the *orbit genus* of Λ.

A group Λ with signature (1) has the presentation with the following generators, called *canonical generators* :

$x_1, \ldots, x_r, e_i,\ c_{ij}, 1 \leq i \leq k, 0 \leq j \leq s_i$ and $a_1, b_1, \ldots, a_h, b_h$ if the sign is $+$ or $d_1, \ldots, d_h$ otherwise,

and relators:

$x_i^{m_i}, i = 1, \ldots, r,\ c_{ij-1}^2, c_{ij}^2, (c_{ij-1}c_{ij})^{n_{ij}}, c_{i0}e_i^{-1}c_{is_i}e_i,\ i = 1, \ldots, k, j = 1, \ldots, s_i$ and

$$x_1 \ldots x_r e_1 \ldots e_k a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_h b_h a_h^{-1} b_h^{-1} \text{ or } x_1 \ldots x_r e_1 \ldots e_k d_1^2 \ldots d_h^2,$$

according to whether the sign is $+$ or $-$. The elements x_i are elliptic transformations, a_i, b_i hyperbolic translations, d_i glide reflections and c_{ij} hyperbolic reflections. We shall call the reflections c_{ij-1}, c_{ij} *consecutive.*

Now an abstract group with such a presentation can be realized as an NEC group Λ if and only if the value

$$\eta h + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right),$$

is positive, where $\eta = 2$ or 1 according to the sign being $+$ or $-$. This value corresponds to the hyperbolic area $\mu(\Lambda)$ of any fundamental region for Λ and we have the Hurwitz-Riemann formula

$$[\Lambda : \Lambda'] = \frac{\mu(\Lambda')}{\mu(\Lambda)},$$

where Λ' is a subgroup of finite index in an NEC group Λ.

NEC groups having no orientation reversing elements are just the classical Fuchsian groups and among them particularly important are the *Fuchsian surface groups*, which are just torsion free Fuchsian groups. A Fuchsian surface group Γ has signature of the type $(g; -)$ and in such a case $\mathcal{H}/\Gamma$ is a compact Riemann surface of genus g. Conversely, every compact Riemann surface X can be represented as such an orbit space for some Fuchsian surface group Γ and a finite group G is a group of automorphisms of X if and only if $G = \Lambda/\Gamma$ for some NEC group Λ. The following result from [9, 10] is a main tool that we use in this paper.

Theorem 2.1. *Let $X = \mathcal{H}/\Gamma$ be a Riemann surface with the group G of all automorphisms of X, let $G = \Lambda/\Gamma$ for some NEC group Λ and let $\theta : \Lambda \to G$ be the canonical projection. Then the number of ovals of a symmetry σ of X equals*

$$\sum [\mathrm{C}(G, \theta(c_i)) \,:\, \theta(\mathrm{C}(\Lambda, c_i))],$$

where C *denotes the centralizer and the sum is taken over a set of representatives of all the conjugacy classes of canonical reflections, whose images under θ are conjugate to σ.*□

The index $w_i = [\mathrm{C}(G, \theta(c_i)) \,:\, \theta(\mathrm{C}(\Lambda, c_i))]$ will be called a *contribution* of c_i to $\|\sigma\|$. The next result of Singerman allows us to compute the centralizer of a canonical reflection in an NEC group

Proposition 2.2 ([15, 16]). *Let $c_0, c_1, \ldots, c_s, e$ be the canonical generators corresponding to a period cycle $(n_1, \ldots, n_s)$ of an NEC group Λ with signature (1). If all n_i are even, then the centralizer $C(\Lambda, c_i)$ of c_i in Λ is*

$$\begin{array}{ll} \langle c_i\rangle \oplus \big(\langle (c_{i-1}c_i)^{n_i/2}\rangle * \langle (c_ic_{i+1})^{n_{i+1}/2}\rangle\big) = \mathrm{Z}_2 \oplus (\mathrm{Z}_2 * \mathrm{Z}_2) & \text{for } i \neq 0,\\ \langle c_0\rangle \oplus \big(\langle (c_0c_1)^{n_1/2}\rangle * \langle e^{-1}(c_{s-1}c_s)^{n_s/2}e\rangle\big) = \mathrm{Z}_2 \oplus (\mathrm{Z}_2 * \mathrm{Z}_2) & \text{for } i = 0,\\ \langle c_0\rangle \oplus \langle e\rangle = \mathrm{Z}_2 \oplus \mathrm{Z} & \text{for } s = 0. \end{array}$$

A group G is said to be *abstractly oriented* if there is an epimorphism $\alpha : G \to \mathrm{Z}_2 = \{\pm 1\}$, called an abstract orientation. An element g of abstractly oriented group G with an abstract orientation ε is said to be *orientation preserving* (respectively *orientation reversing*) if $\alpha(g) = +1$ (respectively $\alpha(g) = -1$). Observe that the abstract orientations of G correspond to the subgroups of index 2 in G. Moreover, by the Sylow theorem, in our studies of symmetries we may assume that G is a 2-group. Indeed, for $\sigma_1, \sigma_2, \ldots, \sigma_k$ being the representatives of conjugacy classes of symmetries we know that all Sylow 2-groups are conjugate and so we can assume that these symmetries generate a 2-group G. In [2] the authors proved the following two theorems:

Theorem 2.3. *A 2-group G containing a dihedral group D_n as a subgroup of index 2^r has at most $2^{r+2} - 1$ conjugacy classes of elements of order 2. Furthermore, if G is abstractly oriented and the generators x_0, y_0 of D_n reverse the orientation, then G has at most 2^{r+1} conjugacy classes of orientation reversing elements of order 2.*

Theorem 2.4. *Let X be a Riemann surface of genus $g = 2^{r-1}u + 1$ with u odd. Let G be a 2-group of automorphisms of X of order 2^t and assume that $t \geq r + 1$. Then G contains a cyclic or a dihedral subgroup of index 2^r.*

obtaining as a corollary, the sharp upper bound on a number of nonconjugate symmetries with fixed points is given in [2]:

Corollary 2.5. *Let X be a Riemann surface of genus $g = 2^{r-1}u + 1$ with u odd. Then the maximal number of nonconjugate symmetries with fixed points that X may admit is 2^{r+1}. Furthermore, this bound is attained if and only if $u \geq 2^{r+1} - 3$.*

Remark 2.6. Here we shall mention, that one of the possible types of symmetries is 0, which corresponds to a fixed point free (and hence nonseparating) symmetry. As in our studies of the geometrical and homological dimension of the nerve $\mathcal{N}_g$ we will look for symmetries with distinct types, we have to take a fixed point free symmetry into account as one (and only one!) of the possibilities. It was shown in [3] that the bound from corollary 2.5 is also true, if we allow fixed point free symmetries.

For the sake of completeness, before we move to the main sections of the paper, let us cite results concerning even values of g, which were obtained in [12].

Theorem 2.7. *The following conditions hold:*
(1) *for any even* $g \geq 2$, $\dim_G(\mathcal{N}_g) = 3$;
(2) *for any even* $g \geq 6$, $\dim_H(\mathcal{N}_g) = 3$, *while* $\dim_H(\mathcal{N}_2) = 0$ *and* $\dim_H(\mathcal{N}_4) = 1$. □

3. Geometrical dimension of $\mathcal{N}_g$

Since symmetries of a Riemann surface X having distinct topological types are nonconjugate in $\mathrm{Aut}^{\pm}(X)$, the quantitative results concerning symmetries from the corollary 2.5 give us upper bounds for $\dim_G(\mathcal{N}_g)$ and $\dim_H(\mathcal{N}_g)$. Observe, however, that to study the attainment, one needs qualitative results allowing topological type of single symmetry σ of X to be found in terms of $\mathrm{Aut}^{\pm}(X)$ and topological type of the action. This role in our paper will be played by theorem 2.1. Here we shall prove the following

Theorem 3.1. *Let* $g = 2^{r-1}u + 1$ *where* $r \geq 2$ *and* u *is odd. The geometrical dimension of* $\mathcal{N}_g$ does not exceed $2^{r+1} - 1$ *and this bound is attained if* $u \geq 2^r(2^r + 1) - 5$.

Example 3.2. Before we present the proof of the above theorem, let us consider an example, which shall give us an idea and intuition for the upcoming proof. The example can be viewed as the proof for the specific case $r = 2$ in the theorem above.

Let $g = 2u + 1$ for some odd integer $u \geq 15$. It is obvious, by corollary 2.5, that the geometrical dimension of $\mathcal{N}_g$ cannot be greater than 7. Indeed, there are at most 8 nonconjugate symmetries, hence at most 8 symmetries of distinct types and hence the dimension of a simplex in $\mathcal{N}_g$ cannot be greater than 7. We shall show that in fact it is maximal and equals 7 in such a case. Take $G = \mathbb{Z}_2^4$ to be an abstractly oriented group having 8 orientation reversing involutions $\sigma_0, \sigma_1, \ldots, \sigma_7$. Consider an NEC group with signature

$$(0; +; [2, \overset{(u-15)/2}{\ldots}, 2]; \{(2, \overset{19}{\ldots}, 2)\}),$$

and define and epimorphism $\theta : \Lambda \to \mathbb{Z}_2^4$ on a sequence of consecutive canonical reflections in the following way:

$$\underbrace{\sigma_7, \sigma_6, \sigma_7, \ldots, \sigma_7}_{7}\underbrace{\sigma_5, \sigma_4, \sigma_5, \sigma_4, \sigma_5}_{5}\underbrace{\sigma_3, \sigma_2, \sigma_3}_{3}\sigma_1, \sigma_7, \sigma_5, \sigma_3.$$

Moreover, we define $\theta(x_l) = \sigma_0\sigma_1$ for all values of l and $\theta(e_1) = 1$ if $(u-15)/2$ is even and $\theta(e_1) = \sigma_0\sigma_1$ otherwise. Here we consider our reflections as situated on a circle, so we unify the first and the last reflection. It is easy to see that symmetry σ_i has $2i$ ovals. Indeed, whenever a symmetry appears with the same neighbors, the corresponding reflection contributes to it with 4 ovals, by theorem 2.1 and proposition 2.2. Now if it appears with distinct neighbors we have a contribution of 2 ovals to the respective symmetry. It follows easily, that we constructed a Riemann surface $\mathcal{H}/\ker\theta$, having 8 nonconjugate symmetries, each with different number of ovals, hence of distinct topological types. Therefore we constructed a 7-dimensional simplex and the geometrical dimension here is maximal and equals 7.

Proof. The upper bound on $\dim_G \mathcal{N}_g$ is obvious by corollary 2.5. Again, as there are at most 2^{r+1} nonconjugate symmetries, then there are at most 2^{r+1} distinct types. It follows immediately that the dimension of a simplex in $\mathcal{N}_g$ is bounded by $2^{r+1}-1$ and so $\dim_G(\mathcal{N}_g) \le 2^{r+1}-1$. Therefore for the proof it is enough to construct, for any g as in the theorem, a Riemann surface of genus g, having 2^{r+1} symmetries of distinct topological types. Let $u \ge 2^r(2^r+1)-5$ and consider an NEC group Λ with signature

$$(0; +; [2, \overset{m}{\ldots}, 2]; \{(2, \overset{s}{\ldots}, 2)\}), \tag{2}$$

where $s = 2^r(2^r+1)-1$, $m = (u-2^r(2^r+1)+5)/2$. We shall construct an epimorphism $\theta : \Lambda \to G = \mathbb{Z}_2^{r+2}$, where G is an abstractly oriented group. Denote by σ_i, $0 \le i \le 2^{r+1}-1$, all the orientation reversing involutions in G. Let $\theta(x_l) = \sigma_0\sigma_1$ for $1 \le l \le m$ and $\theta(e_1) = 1$ or $\sigma_0\sigma_1$ depending on the parity of m, that is $\theta(e_1) = 1$ for m being even and $\theta(e_1) = \sigma_0\sigma_1$ otherwise. Let us divide the sequence of canonical reflections $c_0, \ldots, c_{s-1}$ in the following way: first we have 2^r segments, where the n-th segment has length $2^{r+1}-(2n-1)$, and at the end we have the remaining 2^r-1 reflections; we shall call these a *tail* of the sequence. Now we define θ to map the reflections of n-th segment alternatively to $\sigma_{2^{r+1}-(2n-1)}$ and $\sigma_{2^{r+1}-2n}$, starting and finishing with $\sigma_{2^{r+1}-(2n-1)}$. The last 2^r-th segment has length 1 and its reflection is mapped to σ_1. Next, we map the remaining consecutive reflections respectively to

$$\sigma_{2^{r+1}-1},\ \sigma_{2^{r+1}-3},\ \ldots,\ \sigma_3.$$

As before, we unify the first and the last reflection, hence we can view the canonical reflections as situated on a circle and treat the first one and the last one as the same. Clearly θ is an epimorphism and it is also easy to determine the number of ovals of all the symmetries. Obviously σ_0 is a fixed point free symmetry. The centralizer of any symmetry σ_i in G has order 2^{r+2}. By proposition 2.2, an image of the centralizer of a reflection c in Λ is generated by $\theta(c) = \sigma_i$ and the images of its neighboring reflections on a circle mentioned above. Therefore, if these images are distinct, then the reflection c contributes, by theorem 2.1, with $2^{r+2}/8 = 2^{r-1}$ ovals to symmetry σ_i. If these images are the same, then we have $2^{r+2}/4 = 2^r$ ovals.

Let us now consider symmetries of the n-th segment, $\sigma_{2^{r+1}-(2n-1)}$ and $\sigma_{2^{r+1}-2n}$ for some $1 \le n \le 2^r$. The segment has odd length and its first and last reflections contribute with 2^{r-1} ovals to symmetry $\sigma_{2^{r+1}-(2n-1)}$. Indeed, recall that for the first and last reflection in the segment, the images of neighboring reflections are distinct, so the image of this centralizer has order 8. Now by the theorem 2.1, the last and the first reflection in the segment contribute $|G|/8 = 2^{r-1}$ ovals each. Observe also, that all the remaining reflections of this segment contribute with $|G|/4 = 2^r$ ovals to respective symmetries. This is so, because now the image of the centralizer of a reflection has order 4, as the neighboring reflections have the same image. Summing up, the n-th segment gives $(2^r - n) \cdot 2^r$ ovals to symmetries $\sigma_{2^{r+1}-(2n-1)}$ and $\sigma_{2^{r+1}-2n}$ each. Furthermore, symmetry $\sigma_{2^{r+1}-(2n-1)}$ appears once more, as it is the image of one of the reflections at the tail of the cycle (not connected to any segment). This reflection again contributes with 2^{r-1} ovals. Therefore, symmetry $\sigma_{2^{r+1}-(2n-1)}$ has $(2^{r+1} - 2n + 1) \cdot 2^{r-1}$ ovals and $\sigma_{2^{r+1}-2n}$ has $(2^{r+1} - 2n) \cdot 2^{r-1}$ ovals. Observe also that symmetries connected with distinct segments have distinct numbers of ovals, as the lengths of the segments differ. Hence we arrived to the configuration of 2^{r+1} symmetries σ_i, where the symmetry σ_i has $i \cdot 2^{r-1}$ ovals, for some $0 \le i \le 2^{r+1} - 1$. This shows that we have a $(2^{r+1} - 1)$-simplex in $\mathcal{N}_g$ and so $\dim_G \mathcal{N}_g = 2^{r+1} - 1$, the proof is finished □

4. Homological dimension of $\mathcal{N}_g$

Now we shall deal with the problem of the homological dimension $\dim_H \mathcal{N}_g$. Obviously the homological dimension of the nerve cannot be greater than the geometrical dimension of $\mathcal{N}_g$. The next result shows, that in fact the bound again is attained for infinitely many values of g.

Theorem 4.1. *Let $g = 2^{r-1}u + 1$ for some $r \ge 2$ and u odd. Then the homological dimension of $\mathcal{N}_g$ does not exceed $2^{r+1} - 1$ and equals $2^{r+1} - 1$ if $u \ge 2^{r+1}(2^{r-1} + 1) - 5$.*

Example 4.2. The first part of the statement is clearly true, as $\dim_H \mathcal{N}_g \le \dim_G \mathcal{N}_g \le 2^{r+1} - 1$. As before, we shall start with a specific case of $r = 2$, which shall help us to go through the general proof. Let us assume that $g = 2u + 1$ for some odd $u \ge 19$. We shall construct 9 Riemann surfaces $X_0, \ldots, X_8$ in such a way, that a surface X_j has 8 commuting symmetries σ_i with $2i$ ovals each, where $0 \le i \le 8$ and $i \ne j$. Let us take an NEC group Λ with signature (2). Now we shall consider 9 cases defining m, s and respective epimorphisms θ_j onto Z_2^4 such that $X_j = \mathcal{H}/\ker \theta_j$ is the surface we looked for. Throughout the proof we assume σ_i to be all the orientation reversing involutions in Z_2^4. We also assume that the canonical reflections in an NEC group Λ are situated on a circle so that the last and first one are unified and treated as one.

Case 0: Take m $= \frac{u-19}{2}, s = 23$ and define θ_0 by mapping the consecutive canonical reflections respectively to

$$\underbrace{\sigma_8, \sigma_7, \sigma_8, \ldots, \sigma_7}_{8} \underbrace{\sigma_6, \sigma_5, \ldots, \sigma_5}_{6} \underbrace{\sigma_4, \sigma_3, \sigma_4, \sigma_3}_{4} \sigma_1, \sigma_4, \sigma_6, \sigma_8, \sigma_2$$

and all the elliptic generators to $\sigma_1\sigma_2$. The connecting generator e_1 is mapped to $\sigma_1\sigma_2$ or 1 for m odd or even respectively. By theorem 2.1 and proposition 2.2, in the same way as we did in the proof of theorem 3.1, it is easy to see that σ_i has $2i$ ovals for $i = 1, \ldots, 8$.

Case 1: Take m $= \frac{u-19}{2}, s = 23$ and define θ_1 by mapping the connecting and elliptic generators similarly as above, with the image being $\sigma_0\sigma_2$ or 1, and consecutive canonical reflections respectively to

$$\underbrace{\sigma_8, \sigma_7, \sigma_8, \ldots, \sigma_7}_{8} \underbrace{\sigma_6, \sigma_5, \ldots, \sigma_5}_{6} \underbrace{\sigma_4, \sigma_3, \sigma_4, \sigma_3}_{4} \sigma_2, \sigma_8, \sigma_6, \sigma_4, \sigma_2.$$

Here again σ_i has $2i$ ovals for $i = 2, \ldots, 7$ and $i = 0$. Observe also that in this case we have in addition a fixed point free symmetry σ_0. This also holds true for all the remaining cases.

Case 2: Take m $= \frac{u-19}{2}, s = 23$ and define θ_2 by mapping the consecutive canonical reflections respectively to

$$\underbrace{\sigma_8, \sigma_7, \sigma_8, \ldots, \sigma_7}_{8} \underbrace{\sigma_6, \sigma_5, \ldots, \sigma_5}_{6} \sigma_4, \sigma_3, \sigma_4, \sigma_1, \sigma_3, \sigma_4, \sigma_8, \sigma_6, \sigma_4$$

and connecting and elliptic generators similarly as above, with the nontrivial image being $\sigma_0\sigma_1$. Here again σ_i has $2i$ ovals for $0 \leq i \leq 8$ and $i \neq 2$.

Case 3: Take m $= \frac{u-17}{2}, s = 21$ and define θ_3 by mapping the consecutive canonical reflections respectively to

$$\underbrace{\sigma_8, \sigma_7, \sigma_8, \ldots, \sigma_7}_{8} \underbrace{\sigma_6, \sigma_5, \ldots, \sigma_5}_{6} \sigma_4, \sigma_2, \sigma_4, \sigma_1, \sigma_6, \sigma_8, \sigma_4.$$

The elliptic and connecting generators are mapped as in the previous case.

Case 4: Take m $= \frac{u-17}{2}, s = 21$ and define θ_4 by mapping the consecutive canonical reflections respectively to

$$\underbrace{\sigma_8, \sigma_7, \sigma_8, \ldots, \sigma_7}_{8} \underbrace{\sigma_6, \sigma_5, \ldots, \sigma_5}_{6} \sigma_3, \sigma_2, \sigma_3, \sigma_8, \sigma_6, \sigma_3, \sigma_1.$$

The elliptic and connecting generators are mapped as in the previous case.

Case 5: Take m $= \frac{u-15}{2}, s = 19$ and define θ_5 by mapping the consecutive canonical reflections respectively to

$$\underbrace{\sigma_8, \sigma_7, \sigma_8, \ldots, \sigma_7}_{8} \sigma_6, \sigma_4, \sigma_6, \sigma_4, \sigma_6, \sigma_3, \sigma_6, \sigma_3, \sigma_1, \sigma_8, \sigma_2.$$

The elliptic and connecting generators are mapped as in the previous case.

Case 6: Take m $= \frac{u-17}{2}, s = 21$ and define θ_6 by mapping the consecutive canonical reflections respectively to

$$\underbrace{\sigma_8, \sigma_7, \sigma_8, \ldots, \sigma_7}_{8} \sigma_5, \sigma_4, \sigma_5, \sigma_4, \sigma_5, \sigma_3, \sigma_1, \sigma_2, \sigma_3, \sigma_8, \sigma_5, \sigma_3, \sigma_2.$$

The elliptic and connecting generators are mapped as in the previous case.

Case 7: Take m $= \frac{u-15}{2}, s = 19$ and define θ_7 by mapping the consecutive canonical reflections respectively to

$$\sigma_8, \sigma_4, \sigma_8, \sigma_4, \sigma_8, \sigma_3, \sigma_8, \sigma_3, \sigma_6, \sigma_5, \sigma_6, \sigma_5, \sigma_6, \sigma_5, \sigma_1, \sigma_2, \sigma_8, \sigma_2, \sigma_6.$$

The elliptic and connecting generators are mapped as in the previous case.

Case 8: Take m $= \frac{u-15}{2}, s = 19$ and define θ_8 by mapping the consecutive canonical reflections as in the first Example concerning theorem 3.1. It is not hard to see, that in the j-th Case we obtained a surface X_j with the configuration of symmetries announced before.

Proof. Let g be as in the theorem with $u \geq 2^{r+1}(2^{r-1}+1) - 5$. Our aim will be to construct $2^{r+1}+1$ Riemann surfaces X_j, $j = 0, 1, \ldots, 2^{r+1}$ of genus g, each having 2^{r+1} symmetries with $0 \cdot 2^{r-1}, 1 \cdot 2^{r-1}, \ldots, \widehat{j \cdot 2^{r-1}}, \ldots, 2^{r+1} \cdot 2^{r-1}$ ovals, where the symbol $\widehat{\ }$ means that the corresponding value is removed. We shall divide our considerations into a few cases, depending on the 4-adic structure of $j-1$. Throughout this part of the proof we shall again denote $G = \mathbb{Z}_2^{r+2}$. We also employ the following convention: during the construction of X_j assume that all the orientation reversing involutions in G are σ_i for $i \neq j$ and $0 \leq i \leq 2^{r+1}$. These will also become symmetries in question and the convention will allow us to link the number of the symmetry with its number of ovals, which for symmetry σ_i will be equal to $i \cdot 2^{r-1}$. In addition, during constructions of the epimorphisms $\theta_j, j \neq 0, j \neq 1$ for which $X_j = \mathcal{H}/\ker\theta_j$, we assume that all the m elliptic generators are mapped to $\sigma_0\sigma_1$ and $\theta_j(e_1) = \sigma_0\sigma_1$ for m odd and $\theta_j(e_1) = 1$ otherwise. For $j = 0$ we replace $\sigma_0\sigma_1$ in the above definition with $\sigma_1\sigma_2$, and for $j = 1$ we take it to be $\sigma_0\sigma_2$.

First of all we shall construct X_0, which will be a Riemann surface having 2^{r+1} symmetries respectively with $i \cdot 2^{r-1}$ ovals, where $1 \leq i \leq 2^{r+1}$. Consider an NEC group Λ with signature (2), where $m = (u - 2^{r+1}(2^{r-1}+1)+5)/2$ and $s = 2^{r+1}(2^{r-1}+1) - 1$. Let $\sigma_1, \ldots, \sigma_{2^{r+1}}$ denote all the symmetries in G. We define $\theta_0 : \Lambda \to G$ in the following way: we divide our cycle into pieces such that first we have $2^r - 1$ segments, where the n-th segment has length $2^{r+1} - 2(n-1)$. The consecutive canonical reflections corresponding to the n-th segment are sent alternatively to $\sigma_{2^{r+1}-2(n-1)}$ and $\sigma_{2^{r+1}-2n+1}$, starting with the former and finishing with the latter. The last of these segments has length 4 and its reflections are sent to $\sigma_4, \sigma_3, \sigma_4, \sigma_3$. Now the next reflection is mapped to σ_1 and the remaining reflections respectively to $\sigma_4, \sigma_6, \ldots, \sigma_{2^{r+1}}, \sigma_2$, as shown below:

$$\underbrace{\sigma_{2^{r+1}},\ \sigma_{2^{r+1}-1},\ \sigma_{2^{r+1}},\ \ldots\ \sigma_{2^{r+1}-1}}_{2^{r+1}}\underbrace{\sigma_{2^{r+1}-2},\ \sigma_{2^{r+1}-3},\ \ldots\ \sigma_{2^{r+1}-3}}_{2^{r+1}-2}\cdots$$

$$\cdots\underbrace{\sigma_{2^{r+1}-2(n-1)},\ \sigma_{2^{r+1}-2n+1},\ \ldots\ \sigma_{2^{r+1}-2n+1}}_{2^{r+1}-2(n-1)}\cdots\underbrace{\sigma_4, \sigma_3, \sigma_4, \sigma_3}_{4}\sigma_1, \sigma_4, \sigma_6, \ldots, \sigma_{2^{r+1}}, \sigma_2.$$

In the same way as in the proof of theorem 3.1, we see that the reflections of the n-th segment are contributed with $(2^{r+1} - 2n + 1) \cdot 2^{r-1}$ ovals each. Furthermore, symmetry $\sigma_{2^{r+1}-2(n-1)}$ appears once again as the image of one of the reflections at the end of the cycle. Therefore it has $(2^{r+1} - 2n + 2) \cdot 2^{r-1}$ ovals. In addition, symmetry σ_1 has 2^{r-1} ovals and symmetry σ_2 has 2^r ovals. Hence the epimorphism θ_0 leads to the configuration of 2^{r+1} symmetries σ_i with $i \cdot 2^{r-1}$ ovals each, where $1 \leq i \leq 2^{r+1}$.

We construct the surface X_2 in the similar way. Consider an epimorphism $\theta_2 : \Lambda \to G$, which is defined in the same way as θ_0 on all the canonical reflections except the last segment and the tail of the cycle. On these canonical reflections we define θ_2 as

$$\underbrace{\sigma_4, \sigma_3, \sigma_4, \sigma_1}_{4}\sigma_3, \sigma_4, \sigma_{2^{r+1}}, \sigma_{2^{r+1}-2}, \ldots, \sigma_8, \sigma_6, \sigma_4.$$

Compared to θ_0, we do not have the symmetry with $2 \cdot 2^{r-1}$ ovals, which was our aim. This one is replaced by fixed point free symmetry σ_0. The numbers of ovals of all the other symmetries did not change. Note, that we replaced the symmetry σ_2 at the end of the cycle with symmetry σ_4 but we also changed the sequence of the symmetries at the end of the cycle and switched σ_3 with σ_1 in the last segment. Therefore also now σ_4 has $4 \cdot 2^{r-1}$ ovals, although it appears once more in the cycle.

Now we shall construct surfaces X_{2n} for $1 < n \leq 2^r$. Consider an NEC group Λ with signature (2) where $m = (u - 2^{r+1}(2^{r-1}+1)+5)/2 + \lfloor n/2 \rfloor$ and $s = 2^{r+1}(2^{r-1}+1) - 1 - 2\lfloor n/2 \rfloor$. We define $\theta_{2n} : \Lambda \to G$ by dividing our cycle into pieces, as before, such that first we have $2^r - 1$ segments and after these segments we have the tail of the cycle. Now the segments with numbers from 1 to $2^r - n$ are the same as in the case of X_0 and we define θ_{2n} in the same way as θ_0 on the canonical reflections corresponding to these segments. Now we shall modify the latter part, depending on the parity of n.

Let first n be even. The next segments (with numbers from $2^r - n + 1$ up to $2^r - 1$) are shortened and their lengths are diminished by 1, which means they have lengths $2n - 1, 2n - 3, \ldots, 3$ respectively. The consecutive canonical reflections corresponding to these shortened segments are mapped as follows:

$$\underbrace{\sigma_{2n-1}, \sigma_{2n-2}, \ldots \sigma_{2n-1}}_{2n-1}\underbrace{\sigma_{2n-3}, \sigma_{2n-2} \ldots \sigma_{2n-3}}_{2n-3}\cdots\underbrace{\sigma_3, \sigma_2, \sigma_3}_{3}.$$

Hence for the a-th segment (with $a \geq 2^r - n + 1$) the reflections are sent alternatively to $\sigma_{2^{r+1}-2a+1}$ and $\sigma_{2^{r+1}-2a}$, starting and finishing with the former. These segments are exactly the same as in the proof of theorem 3.1. Now the consecutive reflections from the tail of the cycle, not belonging to the segments, are mapped respectively to $\sigma_{2^{r+1}}, \sigma_{2^{r+1}-2}, \ldots, \sigma_{2n+2}, \sigma_{2n-1}, \sigma_{2n-3}, \ldots, \sigma_1$. Obviously the numbers of ovals of symmetries with numbers distinct from $2n$ have not changed compared to θ_0. But here we do not have symmetry with $2n \cdot 2^{r-1}$ ovals and in fact this one has been replaced by fixed point free symmetry σ_0, which is the configuration we looked for.

Let now n be odd. The only difference we make, compared to the case of n being even, is on the last segment and the latter part of the cycle. Recall that the last segment had length 3 and its reflections were mapped to $\sigma_3, \sigma_2, \sigma_3$. We take this segment to have length 4 and map the corresponding reflections to $\sigma_3, \sigma_1, \sigma_2, \sigma_3$. Observe that this operation causes the symmetry σ_2 to loose 2^{r-1} ovals. But we also modify the epimorphism on the last part of the cycle, by taking the reflections to be mapped to $\sigma_{2^{r+1}}, \sigma_{2^{r+1}-2}, \ldots, \sigma_{2n+2}, \sigma_{2n-1}, \sigma_{2n-3}, \ldots, \sigma_3, \sigma_2$. Here the total length of the cycle remains correct and again there is no symmetry σ_{2n} with $2n \cdot 2^{r-1}$ ovals, this symmetry was replaced by σ_0. As for the other symmetries, the only difference is that the symmetry σ_2 appeared once more at the end of the cycle, which gave her the lacking 2^{r-1} ovals - the ones that were taken during the modification of the last segment.

Now we shall construct surfaces X_{4n+3} for $0 \leq n < 2^{r-1} - 1$. Consider an NEC group Λ with signature (2) where $m = (u - 2^{r+1}(2^{r-1} + 1) + 5)/2 + (n + 1)$ and $s = 2^{r+1}(2^{r-1} + 1) - 1 - 2(n + 1)$. We define $\theta_{4n+3} : \Lambda \to G$ in the similar way as in the previous case. Like before, we divide our cycle into pieces such that first we have $2^r - 1$ segments and after these segments we have the tail of the cycle. On the first segments, up to the number $2^r - 2n - 2$, we define the epimorphism as in the case of θ_0, which means that a-th segment has length $2^{r+1} - 2(a - 1)$ and its reflections are sent alternatively to $\sigma_{2^{r+1}-2(a-1)}$ and $\sigma_{2^{r+1}-2a+1}$, starting with the former and finishing with the latter. We change the epimorphism on the next segment, with number $2^r - 2n - 1$, by replacing all the entries of symmetry σ_{4n+3} by symmetries σ_{2n+2} and σ_{2n+1}. As a result, the epimorphism θ_{4n+3} sends the reflections of this segment respectively to:

$$\underbrace{\sigma_{4n+4}, \sigma_{2n+2}, \ldots \sigma_{4n+4}, \sigma_{2n+2}}_{2n+2} \underbrace{\sigma_{4n+4}, \sigma_{2n+1}, \ldots \sigma_{2n+1}, \sigma_{4n+4}, \sigma_{2n+1}}_{2n+2}.$$

Observe that in fact the symmetries $\sigma_{2n+2}, \sigma_{2n+1}$ cannot appear again in the sequence of images of the canonical reflections. Moreover, the symmetry σ_{4n+4} lost 2^{r-1} ovals. Observe also that for $n = 0$ this modified segment is the last one and it consists of $\sigma_4, \sigma_2, \sigma_4, \sigma_1$, while in the other cases the last segment consists of two reflections which are mapped to σ_2, σ_1. On the segments with numbers from $2^r - 2n$ up to $2^r - n - 1$, if there are any, again we define the epimorphism as in the case of θ_0: the a-th segment has length $2^{r+1} - 2(a - 1)$ and its reflections are sent alternatively to $\sigma_{2^{r+1}-2(a-1)}$ and $\sigma_{2^{r+1}-2a+1}$, starting with the former and finishing with the latter. Now for the segment with number $2^r - n$ we should have symmetries $\sigma_{2n+2}, \sigma_{2n+1}$, but these were already used before. Therefore we skip these symmetries and take the next pair, if there is any. As a result, all the segments with numbers from $2^r - n$ up to $2^r - 1$ are, roughly speaking, 'shifted' compared to what we had with θ_0 and a-th segment has length $2^{r+1} - 2a$, its reflections being sent alternatively to $\sigma_{2^{r+1}-2a}$ and $\sigma_{2^{r+1}-2a-1}$, starting with the former and finishing with the latter. The last thing here is to define the epimorphism on the tail of the cycle and we do this by mapping the consecutive reflections respectively to

$$\sigma_{2^{r+1}-2}, \sigma_{2^{r+1}-4}, \ldots, \sigma_{4n+6}, \widehat{\sigma_{4n+4}}, \ldots, \sigma_{2n+4}, \widehat{\sigma_{2n+2}}, \sigma_{2n}, \ldots, \sigma_2, \sigma_{2^{r+1}}, \sigma_{4(n+1)}$$

for $0 \le n \le 2^{r-1} - 2$ (note that for $n = 0$ we have $\sigma_{4n+4} = \sigma_{2n+4}$ and the last appears only at the end of the cycle and σ_2 does not appear at all). Here it is easy to see that we obtained a configuration we were looking for, that is a set of 2^{r+1} symmetries, on a Riemann surface of genus g, where the symmetries have $i \cdot 2^{r-1}$ ovals for $0 \le i \le 2^{r+1}$ and $i \ne 4n + 3$ for some $0 \le n < 2^{r-1} - 1$.

Now if $n = 2^{r-1} - 1$, we use the same definitions for Λ and in the epimorphism we only change the last segment and the tail of the sequence. That is, we take the last segment, with number $2^r - 1$, to be σ_1, σ_2 and the tail of the sequence to be:

$$\sigma_{2^{r+1}}, \sigma_2, \sigma_4, \ldots, \widehat{\sigma_{2^r}}, \ldots, \sigma_{2^{r+1}-4}, \sigma_{2^{r+1}-2}.$$

With these definitions we obtain a desired configuration of symmetries.

Now we shall construct X_{4n+1} for $1 \le n \le 2^{r-1} - 1$. Consider an NEC group Λ with signature (2), where $m = (u-2^{r+1}(2^{r-1}+1)+5)/2+(n+1)$, $s = 2^{r+1}(2^{r-1}+1)-1-2(n+1)$ We define $\theta_{4n+1} : \Lambda \to G$ in the following way: we divide our cycle into pieces such that first we have $2^r - 2$ segments and after these segments we have the tail of the cycle. On the first segments, up to the number $2^r - 2n - 1$, we define the epimorphism as in the case of θ_0. We change the next segment, with number $2^r - 2n$ and length $4n + 2$, by changing its length to $4n + 4$ and the epimorphism on this segment by replacing symmetry σ_{4n+1} by symmetries σ_{2n+2} and σ_{2n+1}. As a result, the epimorphism θ_{4n+1} sends the reflections of this segment respectively to:

$$\underbrace{\sigma_{4n+2}, \sigma_{2n+2}, \ldots \sigma_{4n+2}, \sigma_{2n+2}}_{2n+2} \underbrace{\sigma_{4n+2}, \sigma_{2n+1}, \ldots \sigma_{2n+1}\ \sigma_{4n+2}, \sigma_{2n+1}}_{2n+2}.$$

Again symmetries $\sigma_{2n+2}, \sigma_{2n+1}$ already have $(2n + 2) \cdot 2^{r-1}$ and $(2n + 1) \cdot 2^{r-1}$ ovals and will not appear again in the sequence of images of the canonical reflections. Moreover, the symmetry σ_{4n+2} gained 2^{r-1} ovals and also will not appear again. Now for the segments with numbers from 2^r-2n+1 up to 2^r-n-1, if any exist, again we define the epimorphism as in the case of θ_0. Similarly, for the segment with number $2^r - n$, if it exists, we should have symmetries $\sigma_{2n+2}, \sigma_{2n+1}$, but as these were already used before, we take the next pair. As a result, all the segments (if there are any) with numbers from $2^r - n$ up to $2^r - 2$ are 'shifted' compared to what we had with θ_0 and a-th segment has length $2^{r+1} - 2a$, its reflections being sent alternatively to $\sigma_{2^{r+1}-2a}$ and $\sigma_{2^{r+1}-2a-1}$, starting with the former and finishing with the latter. The last thing here is to define the epimorphism on the last part of the cycle and we do this by mapping the consecutive reflections respectively to

$$\sigma_1, \sigma_4, \sigma_6, \ldots, \widehat{\sigma_{2n+2}}, \sigma_{2n+4}, \ldots, \widehat{\sigma_{4n+2}}, \sigma_{4n+4}, \ldots, \sigma_{2^{r+1}}, \sigma_2$$

for $1 \leq n \leq 2^{r-1} - 1$. In this sequence any symmetry appears only once and we remove σ_{2n+2} and σ_{4n+2}. Note that for $n = 1$ we have $2n + 2 = 4, 4n + 2 = 6$ and so σ_4 and σ_6 do not appear at all in the sequence above. This again leads to the configuration we were looking for.

The last thing is to construct X_1. Here again we take an NEC group with signature (2), where $m = (u - 2^{r+1}(2^{r-1} + 1) + 5)/2$ and $s = 2^{r+1}(2^{r-1} + 1) - 1$. The epimorphism $\theta_1 : \Lambda \to G$ is defined in the same way as above but we need to change the tail of the sequence slightly, as the symmetry removed is σ_1. We start with $2^r - 1$ segments, where the a-th segment has length $2^{r+1} - 2(a - 1)$ and maps the consecutive reflections alternatively to $\sigma_{2^{r+1}-2(a-1)}$ and $\sigma_{2^{r+1}-2(a-1)-1}$, starting with the former and finishing with the latter. The last segment has length 4 and its symmetries are $\sigma_4, \sigma_3, \sigma_4, \sigma_3$. Now for the tail of the sequence we take:

$$\sigma_2,\ \sigma_{2^{r+1}},\ \sigma_{2^{r+1}-2}, \ldots, \sigma_4,\ \sigma_2.$$

This construction gives a Riemann surface X_1, having 2^{r+1} nonconjugate symmetries of distinct topological types, as the symmetries have $i \cdot 2^{r-1}$ ovals for $i \neq 1$ and $0 \leq i \leq 2^{r+1}$. □

Remark 4.3. In the two main theorems of this paper we constructed surfaces having the desired configuration of symmetries with distinct numbers of ovals and hence with distinct types. However, as in our constructions we used the abelian group G as the automorphism group of the surface, by the results of [1] it is not hard to determine that all the symmetries constructed are nonseparating.

5. Concluding remarks, open problems and conjectures

We would like to finish this paper with some conjectures, which are being investigated right now and as far as we are concerned at the moment, they are most probably true.

The first conjecture concerns the structure of the group generated by the so-called *extremal configuration of symmetries.*

Conjecture 5.1. *Let G be an abstractly oriented 2-group generated by orientation reversing involutions. If G contains a dihedral group D_n as a subgroup of index 2^r, an element of order n in D_n preserves the orientation and G has 2^{r+1} conjugacy classes of orientation reversing involutions, then it is a direct product* $D_n \oplus Z_2 \oplus \overset{r}{\ldots} \oplus Z_2$.

This basically means, by theorem 2.4, that given a Riemann surface of genus $g = 2^{r-1}u + 1$, for some odd integer u and such that it has 2^{r+1} nonconjugate symmetries, the structure of the group generated by the symmetries is just a direct product of a dihedral group and respective amount of cyclic groups of order 2. This result can be seen as the generalization of one of the theorems of Gromadzki and Izquierdo from [11]. Being given the structure of the group generated by symmetries in question, we should have enough data to prove the next conjecture to be true.

Conjecture 5.2. *The sufficient conditions for the maximal geometrical and homological dimensions of* $\mathcal{N}_g$*, given in theorems* 3.1 *and* 4.1*, are also necessary.*

We claim that actually the lower bounds on integer u, and hence the lower bounds on g cannot be improved. As for the geometrical dimension, it seems that actually it is enough to consider an abelian group G, being a direct product Z_2^{r+2} and an NEC group signature (2). The important thing here is that when we get rid of the non-zero genus and multiple period cycles in the signature of Λ, we are left with consecutive reflections situated on a circle. This basically means, that we only have to solve the following combinatorial problem for $k = 2^{r+1} - 1$.

Problem 5.3. Let us consider a number of points situated on a circle, coloured by $k \geq 3$ colours in such a way that no two consecutive points have the same colour. Moreover, we put weights on our points in such a way that the weight is 2 if a point has neighbours with the same colour and the weight is 1 otherwise. Next, for every colour we define its weight as the sum of all the weights of points coloured with it. What is the smallest possible number of points $\varphi(k)$, for which there exists such a colouring and all the colours have distinct weights? For our principal goals, the most important is the case $k = 2^{r+1} - 1$ for which we have some evidence for the following conjecture to be true.

Conjecture 5.4. $\varphi(k) = 2^r(2^r + 1) - 1$.

The point is that this problem, $k = 2^{r+1} - 1$, describes exactly the situation, when we need to have $k = 2^{r+1} - 1$ symmetries with fixed points (colours), in addition one fixed point free symmetry, and the group epimorphism can be seen as the colouring of the points (consecutive canonical reflections). Looking for the smallest possible number of points is just looking for the shortest possible period cycle, hence the group Λ with smallest area and hence, by the Hurwitz-Riemann formula, the smallest possible g for which there exists a Riemann surface X of genus g, realizing the maximal geometrical dimension of the nerve $\mathcal{N}_g$. We also believe that it will be possible to employ this method to solve similar, but more difficult, problem for the homological dimension of $\mathcal{N}_g$. Finally, given k and $s \geq \varphi(k)$, a systematic procedure of producing all colourings, described in Problem 5.3, is also crucial to find higher Betti numbers of $\mathcal{N}_g$.

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