On Pillai's problem with the Fibonacci and Pell sequences

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Abstract Let $(F_n)_{n \geq 0}$ and $(P_n)_{n \geq 0}$ be the Fibonacci and Pell sequences given by the initial conditions $F_0 = 0$, $F_1 = 1$, $P_0 = 0$, $P_1 = 1$ and the recurrences formulas $F_{n+2} = F_{n+1} + F_n$, $P_{n+2} = 2P_{n+1} + P_n$ for all $n \geq 0$, respectively. In this note we shall study the Pillai's type problem

$$F_n - P_m = F_{n_1} - P_{m_1}$$

in non-negative integer pairs $(n, m) \neq (n_1, m_1)$. We completely solve this equation.

Keywords Fibonacci, Pell sequences · Pillai's type problem · Linear form in logarithms

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1 Introduction

Let $\mathbf{U} := (U_n)_{n \geq 0}$ and $\mathbf{V} := (V_n)_{n \geq 0}$ be two linearly recurrent sequences of integers. Recently, the following variation of a problem of Pillai has been studied. Find all non-negative integer solutions (n, m, n_1, m_1) of the equation

$$U_n - V_m = U_{n_1} - V_{m_1}, \quad (n, m) \neq (n_1, m_1). \quad (1)$$

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In particular, find also all integers c which can be written as the difference between an element of $\mathbf{U}$ and an element of $\mathbf{V}$ in at least two different ways. Pillai [12], studied this problem when $\mathbf{U}$ and $\mathbf{V}$ are the sequences of powers of a , and powers of b , respectively, where a, b are two given coprime integers different than 0, ± 1 . It has been shown in [6] that, under some technical but natural conditions, equation (1) has only finitely many non-negative integer solutions and all of them are effectively computable. This version of Pillai's problem was initiated in [8] by Ddamulira, Luca and Rakotomalala who studied equation (1) when $\mathbf{U}$ and $\mathbf{V}$ are the sequences of Fibonacci numbers and powers of 2, respectively. Many other particular cases have been studied. See, for example [5], [3], [7]. We recall that the Fibonacci sequence $(F_n)_{n \geq 0}$ is given by $F_0 = 0, F_1 = 1$ and the recurrence formula

$$F_{n+2} = F_{n+1} + F_n \quad \text{for all } n \geq 0.$$

Let $(P_n)_{n \geq 0}$ be the Pell sequence given by $P_0 = 0, P_1 = 1$, and the recurrence formula

$$P_{n+2} = 2P_{n+1} + P_n \quad \text{for all } n \geq 0.$$

Their first terms are,

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \dots$$

and

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, \dots,$$

respectively. In this note, we study another particular case of this problem, namely equation (1) with Fibonacci and Pell numbers. More precisely, we look at the equation

$$F_n - P_m = F_{n_1} - P_{m_1} \quad (2)$$

in integer pairs $(n, m) \neq (n_1, m_1)$. Since $F_1 = F_2 = 1$, we assume that $n \neq 1, n_1 \neq 1$. That is, whenever we think of 1 as a member of the Fibonacci sequence, we think of it as being F_2 . Our result is then the following

Theorem 1 *All non-negative integer solutions (n, m, n_1, m_1) of (2) with $n \neq 1, n_1 \neq 1$ belong to the set*

$$\left\{ \begin{array}{l} (2, 1, 0, 0), (2, 2, 0, 1), (3, 1, 2, 0), (3, 2, 0, 0), \\ (3, 2, 2, 1), (4, 1, 3, 0), (4, 2, 2, 0), (4, 2, 3, 1), \\ (4, 3, 0, 2), (5, 2, 4, 0), (5, 3, 0, 0), (5, 3, 2, 1), \\ (5, 3, 3, 2), (6, 3, 4, 0), (6, 3, 5, 2), (6, 4, 2, 3), \\ (7, 3, 6, 0), (7, 4, 2, 0), (7, 4, 3, 1), (7, 4, 4, 2), \\ (9, 5, 5, 0), (11, 6, 8, 2), (16, 9, 3, 0), (16, 9, 4, 1) \end{array} \right\}.$$

The set of integers c admitting two representations as a difference between a Fibonacci and a Pell number in at least two different ways is

$$\{-4, -2, -1, 0, 1, 2, 3, 5, 8, 19\}.$$

The representations of the above c are

$$\begin{aligned}
-4 &= F_6 - P_4 = F_2 - P_3; \\
-2 &= F_4 - P_3 = F_0 - P_2; \\
-1 &= F_2 - P_2 = F_0 - P_1; \\
0 &= F_5 - P_3 = F_3 - P_2 = F_2 - P_1 = F_0 - P_0; \\
1 &= F_7 - P_4 = F_4 - P_2 = F_3 - P_1 = F_2 - P_0; \\
2 &= F_{16} - P_9 = F_4 - P_1 = F_3 - P_0; \\
3 &= F_6 - P_3 = F_5 - P_2 = F_4 - P_0; \\
5 &= F_9 - P_5 = F_5 - P_0; \\
8 &= F_7 - P_3 = F_6 - P_0; \\
19 &= F_{11} - P_6 = F_8 - P_2.
\end{aligned}$$

2 Tools

The first one is a lower bound for a linear form in logarithms due to Matveev [11]. Let α be an algebraic number of degree d . Let a be the leading coefficient of its minimal polynomial over $\mathbf{Z}$ and let $\alpha_1 = \alpha, \dots, \alpha_d$ denote the conjugates of α . The *Weil height* of α is defined as

$$h(\alpha) = \frac{1}{d} \left(\log a + \sum_{i=1}^d \log \max\{|\alpha_i|, 1\} \right).$$

The height has the following basic properties. For α, β algebraic numbers and $m \in \mathbf{Z}$, we have:

- $h(\alpha + \beta) \leq h(\alpha) + h(\beta) + \log 2$.
- $h(\alpha\beta) \leq h(\alpha) + h(\beta)$.
- $h(\alpha^m) = |m|h(\alpha)$.

Now let $\mathbf{L}$ be a real number field of degree $d_{\mathbf{L}}$, $\alpha_1, \dots, \alpha_{\ell} \in \mathbf{L}$ and $b_1, \dots, b_{\ell} \in \mathbf{Z} \setminus \{0\}$. Let $B \geq \max\{|b_1|, \dots, |b_{\ell}|\}$ and

$$\Lambda = \alpha_1^{b_1} \cdots \alpha_{\ell}^{b_{\ell}} - 1.$$

Let $A_1, \dots, A_{\ell}$ be real numbers such that

$$A_i \geq \max\{d_{\mathbf{L}} h(\alpha_i), |\log \alpha_i|, 0.16\} \quad \text{for all } i = 1, \dots, \ell.$$

The following result is due to Matveev in [11] (see also Theorem 9.4 in [4]).

Theorem 2 Assume that $\Lambda \neq 0$. Then

$$\log |\Lambda| > -1.4 \times 30^{\ell+3} \times \ell^{4.5} \times d_{\mathbf{L}}^2 (1 + \log d_{\mathbf{L}}) (1 + \log B) A_1 \cdots A_{\ell}.$$

In this paper, we always use $\ell = 3$. Further, $\mathbf{L} = \mathbf{Q}[\sqrt{2}, \sqrt{5}]$ has degree $d_{\mathbf{L}} = 4$. Thus, once for all we fix the constant

$$C := 5.46696 \times 10^{12} > 1.4 \times 30^{3+3} \times 3^{4.5} \times 4^2(1 + \log 4).$$

Matveev's bound gives us some large bounds on our parameters. In order to lower such bounds, we use a version of a reduction method of Baker-Davenport based on Lemma in [1]. We shall use the one given by Bravo, Gómez and Luca in [2]. For a real number x , we write

$$\|x\| = \min\{|x - n| : n \in \mathbf{Z}\}.$$

Lemma 1 *Let M be a positive integer. Let $\tau, \mu, A > 0, B > 1$ be given real numbers. Assume that p/q is a convergent of τ such that $q > 6M$ and $\varepsilon := \|q\mu\| - M\|q\tau\| > 0$. Then the inequality*

$$0 < |n\tau - m + \mu| < \frac{A}{B^w}$$

does not has a solution in positive integers n, m and w in the ranges

$$n \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

This lemma is a slightly variation of the one given by Dujella and Pethő in [9]. The following lemma is also useful. It is Lemma 7 in [10].

Lemma 2 *If $m \geq 1, T > (4m^2)^m$ and $T > x/(\log x)^m$, then*

$$x < 2^m T (\log T)^m.$$

3 Proof of Theorem 1

We start with some basic properties of our sequences. Put

$$\alpha := \frac{1 + \sqrt{5}}{2}, \quad \beta := \frac{1 - \sqrt{5}}{2}; \quad \text{and} \quad \gamma := 1 + \sqrt{2}, \quad \delta := 1 - \sqrt{2}.$$

We have the well-known Binet's formulas

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{and} \quad P_n = \frac{\gamma^n - \delta^n}{2\sqrt{2}} \quad (3)$$

which hold for all $n \geq 0$. Further, the inequalities

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \quad \text{and} \quad \gamma^{n-2} \leq P_n \leq \gamma^{n-1} \quad (4)$$

also hold for all $n \geq 1$.

Now, we start to study our equation (2) in non-negative integers (n, m, n_1, m_1) with $(n, m) \neq (n_1, m_1)$. As we said, we assume $n \neq 1, n_1 \neq 1$. It could happen that $\min\{n, n_1\} = 0$. At any rate, $\max\{n, n_1\} \geq 2$. If in (2) we have $m = m_1$, then $F_n = F_{n_1}$,

implies that $n = n_1$, a contradiction. Thus, from now on we assume $m > m_1$. Rewriting (2) as

$$F_n - F_{n_1} = P_m - P_{m_1}, \quad (5)$$

we observe the right-hand side is positive. Hence, so is the left-hand side, therefore $n > n_1$. We now compare the two sides of (5) using (4). We have

$$\alpha^{n-4} \leq F_n - F_{n_1} = P_m - P_{m_1} \leq P_m \leq \gamma^{m-1}.$$

The left-hand side inequality is clear if $n_1 = 0$. It is also clear if $n_1 \neq 0$, since in that case $n_1 \geq 2$, so $n \geq 3$, so $F_n - F_{n_1} \geq F_n - F_{n-1} = F_{n-2} \geq \alpha^{n-4}$. Thus, $\alpha^{n-4} \leq \gamma^{m-1}$. In a similar way,

$$\alpha^{n-1} \geq F_n \geq F_n - F_{n_1} = P_m - P_{m_1} \geq P_{m-1} \geq \gamma^{m-3},$$

where the right-most inequality is clear (both for $m_1 = 0$ and for $m_1 > 0$). We thus have

$$n - 4 \leq (m - 1) \frac{\log \gamma}{\log \alpha} \quad \text{and} \quad n - 1 \geq (m - 3) \frac{\log \gamma}{\log \alpha}. \quad (6)$$

Since $\log \gamma / \log \alpha = 1.831570923 \dots$ it follows that if $n \leq 300$, then $m \leq 166$. Running a *Mathematica* program in the range $0 \leq n_1 < n \leq 300$ and $0 \leq m_1 < m \leq 166$, with our convention, we obtain all the possibilities listed in Theorem 1.

From now on, $n > 300$. Further, by (6) we get $m > 162$ and also $n > m$. From Binet's formulas (3), we obtain

$$\left| \frac{\alpha^n}{\sqrt{5}} - \frac{\gamma^m}{2\sqrt{2}} \right| = \left| \frac{\alpha^{n_1} + \beta^n - \beta^{n_1}}{\sqrt{5}} - \frac{\gamma^{m_1} - \delta^{m_1} + \delta^m}{2\sqrt{2}} \right| \leq \frac{\alpha^{n_1} + 2}{\sqrt{5}} + \frac{\gamma^{m_1} + 2}{2\sqrt{2}} \leq 2 \max\{\alpha^{n_1+2}, \gamma^{m_1+1}\}. \quad (7)$$

Dividing through by $\gamma^m / 2\sqrt{2}$ we get

$$\left| \frac{4}{\sqrt{10}} \gamma^{-m} \alpha^n - 1 \right| \leq \max\{\alpha^{n_1-n+9}, \gamma^{m_1-m+3}\}, \quad (8)$$

where we have used that $\alpha^{n-4} \leq \gamma^{m-1}$ as well as the fact that $4\sqrt{2} < \gamma^2 < \alpha^4$. Let Λ be the expression inside the absolute value in the left-hand side above. Observe that Λ is not zero. Indeed, otherwise $8/5 = \gamma^{2m}/\alpha^{2n}$ is both a unit (an algebraic integer whose reciprocal is also an algebraic integer) and a rational number, which is false since the only rational units are ± 1 .

Now we apply Matveev's inequality with

$$\alpha_1 = \frac{4}{\sqrt{10}}, \quad \alpha_2 = \gamma, \quad \alpha_3 = \alpha, \quad b_1 = 1, \quad b_2 = -m, \quad b_3 = n.$$

We have $B = n$. Further, we have $h(\alpha_1) = \log 8/2$, $h(\alpha_2) = \log \gamma/2$ and $h(\alpha_3) = \log \alpha/2$. Thus, we may take $A_1 := 4.2$, $A_2 := 1.8$ and $A_3 := 1$ we obtain that

$$\log |\Lambda| > -C(1 + \log n) \times 4.2 \times 1.8.$$

Comparing with (8) we obtain

$$\min\{(n - n_1 - 9) \log \alpha, (m - m_1 - 3) \log \gamma\} \leq 4.13302 \times 10^{13}(1 + \log n). \quad (9)$$

We next study each of these two possibilities.

Case 1. $\min\{(n - n_1) \log \alpha, (m - m_1) \log \gamma\} = (n - n_1) \log \alpha$.

To this case, we rewrite our equation as follows:

$$\begin{aligned} \left| \left(\frac{\alpha^{n-n_1} - 1}{\sqrt{5}} \right) \alpha^{n_1} - \frac{\gamma^m}{2\sqrt{2}} \right| &= \left| \frac{\beta^n - \beta^{n_1}}{\sqrt{5}} - \frac{\gamma^{m_1} - \delta^{m_1} + \delta^m}{2\sqrt{2}} \right| \\ &\leq \frac{2}{\sqrt{5}} + \frac{\gamma^{m_1} + 2}{2\sqrt{2}} < \gamma^{m_1+2}. \end{aligned}$$

Thus,

$$\left| \left(\frac{4(\alpha^{n-n_1} - 1)}{\sqrt{10}} \right) \alpha^{n_1} \gamma^{-m} - 1 \right| < \gamma^{m_1-m+4}. \quad (10)$$

Let Λ_1 be the expression inside the absolute value which is in the left-hand side. We note that $\Lambda_1 \neq 0$, for if this is not so then we would get

$$\frac{\alpha^n - \alpha^{n_1}}{\gamma^m} = \frac{\sqrt{10}}{4},$$

which implies that the right-hand side is an algebraic integer, which it isn't (it's square is $5/8$). We apply again Matveev's inequality by taking

$$\alpha_1 = \frac{4(\alpha^{n-n_1} - 1)}{\sqrt{10}}, \quad \alpha_2 = \gamma, \quad \alpha_3 = \alpha, \quad b_1 = 1, \quad b_2 = -m, \quad b_3 = n_1.$$

Thus, $B = n$. The heights of α_2 and α_3 have already been calculated. As for $h(\alpha_1)$, we have

$$\begin{aligned} h\left(\frac{4(\alpha^{n-n_1} - 1)}{\sqrt{10}}\right) &\leq h\left(\frac{4}{\sqrt{10}}\right) + h(\alpha^{n-n_1} - 1) \leq \frac{\log 8}{2} + h(\alpha^{n-n_1}) + \log 2 \\ &= \frac{\log 32}{2} + (n - n_1) \frac{\log \alpha}{2} \leq \frac{4.13304 \times 10^{13}(1 + \log n)}{2}, \end{aligned}$$

where we have used (9). Thus, we can take $A_1 := 8.26608 \times 10^{13}(1 + \log n)$, A_2 and A_3 as in the analysis of Λ , and get

$$\log |\Lambda_1| > -C \times (8.26608 \times 10^{13}(1 + \log n)^2) \times 1.8.$$

Combining this with (10), we get

$$(m - m_1) \log \gamma < 8.13427 \times 10^{26}(1 + \log n)^2.$$

Case 2. $\min\{(n - n_1) \log \alpha, (m - m_1) \log \gamma\} = (m - m_1) \log \gamma$.

Here, we rewrite our equation as

$$\left| \frac{\alpha^n}{\sqrt{5}} - \left(\frac{\gamma^{n-m_1} - 1}{2\sqrt{2}} \right) \gamma^{m_1} \right| = \left| \frac{\beta^n + \alpha^{n_1} - \beta^{n_1}}{\sqrt{5}} - \frac{\delta^m - \delta^{m_1}}{2\sqrt{2}} \right| \leq \frac{\alpha^{n_1} + 2}{\sqrt{5}} + \frac{1}{\sqrt{2}} < \alpha^{n_1+5}.$$

Thus,

$$\left| 1 - \left(\frac{\sqrt{10}(\gamma^{n-m_1} - 1)}{4} \right) \gamma^{m_1} \alpha^{-n} \right| < \alpha^{n_1-n+7}. \quad (11)$$

We let Λ_2 be the expression inside the absolute value in the left-hand side. As before, $\Lambda_2 \neq 0$, for otherwise we get that $8/5$ is an algebraic integer, which is false. We apply again Matveev's inequality by taking

$$\alpha_1 = \frac{\sqrt{10}(\gamma^{n-m_1} - 1)}{4}, \quad \alpha_2 = \gamma, \quad \alpha_3 = \alpha, \quad b_1 = 1, \quad b_2 = m_1, \quad b_3 = -n.$$

Thus, $B = n$. The heights of α_2 and α_3 have already been calculated. As for $h(\alpha_1)$, we have

$$\begin{aligned} h\left(\frac{\sqrt{10}(\gamma^{n-m_1} - 1)}{4}\right) &\leq h\left(\frac{\sqrt{10}}{4}\right) + h(\gamma^{n-m_1} - 1) \\ &\leq \frac{4.13304 \times 10^{13}(1 + \log n)}{2}, \end{aligned}$$

Thus, we can take the same A_1 as in Case 1, and so we get the same lower bound for $\log |\Lambda_2|$. Therefore,

$$(n - n_1) \log \alpha < 8.13427 \times 10^{26}(1 + \log n)^2.$$

So, we have proved that

$$\max\{(n - n_1) \log \alpha, (m - m_1) \log \gamma\} \leq 8.13427 \times 10^{26}(1 + \log n)^2. \quad (12)$$

We now get a bound on n . Using Binet's formulas (3), we write our equation as follows:

$$\left| \frac{\alpha^{n-n_1} - 1}{\sqrt{5}} \alpha^{n_1} - \frac{\gamma^{n-m_1} - 1}{2\sqrt{2}} \gamma^{m_1} \right| = \left| \frac{\beta^n - \beta^{n_1}}{\sqrt{5}} - \frac{\delta^m - \delta^{m_1}}{2\sqrt{2}} \right| < \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{2}} < 2.$$

Dividing across by $(\gamma^n - \gamma^{n-1})/2\sqrt{2}$, we obtain

$$\left| \left(\frac{4}{\sqrt{10}} \left(\frac{\alpha^{n-n_1} - 1}{\gamma^{n-m_1} - 1} \right) \right) \gamma^{-m_1} \alpha^{n_1} - 1 \right| < \frac{4\sqrt{2}}{\gamma^n - \gamma^{m_1}} < \frac{8\sqrt{2}}{\gamma^n} < \frac{1}{\alpha^{n-8}}, \quad (13)$$

where we used $\alpha^{n-4} < \gamma^{n-1}$, as well as the fact that $8\sqrt{2} < \alpha^4 \gamma$. We let Λ_3 be the expression inside the absolute value in (13). We apply Matveev's inequality with

$$\alpha_1 = \frac{4}{\sqrt{10}} \left(\frac{\alpha^{n-n_1} - 1}{\gamma^{n-m_1} - 1} \right), \quad \alpha_2 = \gamma, \quad \alpha_3 = \alpha, \quad b_1 = 1, \quad b_2 = -m_1, \quad b_3 = -n_1.$$

Thus, we take $B = n$. We need to show that $\Lambda_3 \neq 0$. To do this we take the $\mathbf{Q}$ -automorphism σ of $\mathbf{L}$ given by $\sigma(\sqrt{5}) = -\sqrt{5}$ and $\sigma(\sqrt{2}) = \sqrt{2}$. Under this automorphism, we have $\sigma(\alpha) = \beta$, $\sigma(\gamma) = \gamma$ and $\sigma(\sqrt{10}) = -\sqrt{10}$. Thus, if $\Lambda_3 = 0$, then $\sigma(\Lambda_3) = 0$, which implies, in particular, that

$$\frac{\sqrt{10}}{4} = \left| \frac{\beta^n - \beta^{n_1}}{\gamma^m - \gamma^{m_1}} \right| < \frac{2}{\gamma^{m-1}(\gamma-1)} < \frac{1}{2},$$

since $m > 162$, which is a contradiction. As before, the heights of α_2 and α_3 have already been calculated. For $h(\alpha_1)$, we have

$$\begin{aligned} h\left(\frac{4}{\sqrt{10}} \left(\frac{\alpha^{n-n_1}-1}{\gamma^{m-m_1}-1}\right)\right) &\leq h\left(\frac{4}{\sqrt{10}}\right) + h(\alpha^{n-n_1}+1) + h(\gamma^{m-m_1}+1) \\ &\leq \frac{\log 128}{2} + (n-n_1)\frac{\log \alpha}{2} + (m-m_1)\frac{\log \gamma}{2} \\ &\leq 8.13428 \times 10^{26}(1+\log n)^2. \end{aligned}$$

Thus, we can take $A_1 := 3.25371 \times 10^{27}(1+\log n)^2$, and A_2, A_3 as before. Therefore, we get

$$\begin{aligned} \log |\Lambda_3| &> -C(1+\log n) \times (3.25371 \times 10^{27}(1+\log n)^2) \times 1.8 \\ &> -3.20183 \times 10^{40}(1+\log n)^3, \end{aligned}$$

which, upon comparing it to (13) and applying Lemma 2, we obtain

$$n < 3.77671 \times 10^{48}. \quad (14)$$

Now, we will reduce the upper bound of n . To do this, let Γ be defined as

$$\Gamma = n \log \alpha - m \log \gamma + \log \left(\frac{4}{\sqrt{10}} \right).$$

Assume first that $\min\{n-n_1, m-m_1\} \geq 20$. We go to (8). Note that $e^\Gamma - 1 = \Lambda \neq 0$, so $\Gamma \neq 0$. If $\Gamma > 0$ then

$$0 < \Gamma < e^\Gamma - 1 = \Lambda = |\Lambda| < \max\{\alpha^{n_1-n+9}, \gamma^{m_1-m+3}\}.$$

On the other hand, if $\Gamma < 0$, we then have $1 - e^\Gamma = |e^\Gamma - 1| < 1/2$ which implies $e^{|\Gamma|} < 2$. Thus,

$$0 < |\Gamma| < e^{|\Gamma|} - 1 = e^{|\Gamma|} |\Lambda| < 2 \max\{\alpha^{n_1-n+9}, \gamma^{m_1-m+3}\}.$$

So, in both cases we have

$$0 < |\Gamma| < 2 \max\{\alpha^{n_1-n+9}, \gamma^{m_1-m+3}\}. \quad (15)$$

Dividing through by $\log \gamma$ in the above inequality, we get

$$0 < |n\tau - m + \mu| < \max \left\{ \frac{173}{\alpha^{n-n_1}}, \frac{32}{\gamma^{m-m_1}} \right\},$$

where

$$\tau := \frac{\log \alpha}{\log \gamma}, \quad \mu := \frac{\log(4/\sqrt{10})}{\log \gamma}.$$

Now we apply Lemma 1. To do this, we take $M := 3.77671 \times 10^{48}$ (a bound on m and n by (14)), our τ and, with a *Mathematica* program, we find that the denominator of the convergent

$$\frac{p_{112}}{q_{112}} = \frac{111842821415068814601069451383096958405345992106163812}{204848059751598401563305907296432335323118859258712413}$$

of τ satisfies $q_{112} > 6M$ and that $\varepsilon = \|q\mu\| - M\|q\tau\| = 0.105822 > 0$. This implies, with $(A, B) = (173, \alpha)$ or $(32, \gamma)$, that either

$$n - n_1 \leq 270, \quad \text{or} \quad m - m_1 \leq 145.$$

We now look at each one of these two cases. First, we assume that $n - n_1 \leq 270$ and $m - m_1 \geq 20$. In this case, we consider

$$\Gamma_1 = n_1 \log \alpha - m \log \gamma + \log \left(\frac{4(\alpha^{n-n_1} - 1)}{\sqrt{10}} \right).$$

As before, $e^{\Gamma_1} - 1 = \Lambda_1 \neq 0$, so $\Gamma_1 \neq 0$. We go to (10). With an argument similar to a previous one, we have that

$$0 < |\Gamma_1| < \frac{2\gamma^4}{\gamma^{m-m_1}}.$$

Dividing through by $\log \gamma$ we obtain

$$0 < |n_1 \tau - m + \mu| < \frac{78}{\gamma^{m-m_1}},$$

where τ is the same one as above and

$$\mu := \frac{\log(4(\alpha^{n-n_1} - 1)/\sqrt{10})}{\log \gamma}.$$

We apply again Lemma 1 noting that $n_1 > 0$, for otherwise we would have that $n \leq 270$ which contradicts our hypothesis that $n > 300$. Consider

$$\mu_k := \frac{\log(4(\alpha^k - 1)/\sqrt{10})}{\log \gamma}, \quad \text{for } k = 1, \dots, 270.$$

We ran a *Mathematica* program and found that the same convergent p_{112}/q_{112} satisfies $q_{112} > 6M$. Further, $\varepsilon_k \geq 0.00119532$ for all $1 \leq k \leq 270$. For each of the values of ε_k and with $(A, B) = (78, \gamma)$, we calculate $\log(78q_{112}/\varepsilon_k)/\log \gamma$ and found that each of them is at most 151. Thus, $m - m_1 \leq 151$.

Now let us look at the other case. Assume that $m - m_1 \leq 145$ and $n - n_1 \geq 20$. We consider

$$\Gamma_2 = n \log \alpha - m_1 \log \gamma + \log \left(\frac{4}{\sqrt{10}(\gamma^{m-m_1} - 1)} \right).$$

We note that $1 - e^{-I_2} = \Lambda_2 \neq 0$, so $I_2 \neq 0$. We go to (11). With an argument similar to one above, we obtain

$$0 < |I_2| < \frac{2\alpha^7}{\alpha^{n-n_1}}.$$

Dividing through by $\log \lambda$, we get

$$0 < |n\tau - m_1 + \mu| < \frac{66}{\alpha^{n-n_1}},$$

where τ is the same one as above and

$$\mu := \frac{\log(4/(\sqrt{10}(\gamma^{n-m_1} - 1)))}{\log \gamma}.$$

Now we use again Lemma 1 noting that $m_1 > 0$, which is the case, since otherwise we have $m \leq 145$, which contradicts our hypothesis $m > 162$. As above, by considering now

$$\mu_\ell := \frac{\log(4/(\sqrt{10}(\gamma^\ell - 1)))}{\log \gamma}, \quad \text{for all } \ell = 1, \dots, 145$$

and running a *Mathematica* program, we find that $q_{112} > 6M$, and that for this convergent $\varepsilon_\ell \geq 0.0000620746$ for all $1 \leq \ell \leq 145$. For each of these ε_ℓ and with $(A, B) := (66, \alpha)$, we calculated $\log(66q_{112}/\varepsilon_\ell)/\log \alpha$ and found that all these numbers are at most 283. Thus $n - n_1 \leq 283$.

So, we got that either $n - n_1 \leq 270$ or $m - m_1 \leq 145$. Assuming the first one we deduced $m - m_1 \leq 151$, and assuming the second one, we deduced $n - n_1 \leq 283$. Altogether, we have $n - n_1 \leq 283$, $m - m_1 \leq 151$. So, it remains to study this case. We consider

$$I_3 = n_1 \log \alpha - m_1 \log \gamma + \log \left(\frac{4}{\sqrt{10}} \left(\frac{\alpha^{n-n_1} - 1}{\gamma^{m-m_1} - 1} \right) \right).$$

We note that $e^{I_3} - 1 = \Lambda_3$. Again, since $\Lambda_3 \neq 0$, we have that $I_3 \neq 0$. Since $n > 300$, we get

$$0 < |I_3| < \frac{2\alpha^8}{\alpha^n}.$$

Dividing through by $\log \gamma$, we get

$$0 < |n_1\tau - m_1 + \mu| < \frac{107}{\alpha^n},$$

where τ is as above and

$$\mu := \frac{\log(4(\alpha^{n-n_1} - 1)/\sqrt{10}(\gamma^{m-m_1} - 1))}{\log \gamma}.$$

We apply for the last time Lemma 1. As above, we have that $n_1, m_1 > 0$. Thus, we consider

$$\mu_{k,\ell} := \frac{\log(4(\alpha^k - 1)/\sqrt{10}(\gamma^\ell - 1))}{\log \gamma}, \quad k = 1, \dots, 283, \quad \ell = 1, \dots, 151.$$

Running a *Mathematica* program, we find again that the same convergent works namely $q_{112} > 6M$ and $\varepsilon_{k,\ell} \geq 0.0000307767$ for all $1 \leq k \leq 283$ and $1 \leq \ell \leq 151$. With $(A, B) := (107, \alpha)$ we calculated $\log(107q_{112}/\varepsilon_{k,\ell})/\log \alpha$ for each of these values $\varepsilon_{k,\ell}$, and found that the maximum value of them is ≤ 286 . Thus, $n \leq 286$, which contradicts our assumption on n .

This finishes the proof of our theorem.

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