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On Pillai's problem with the Fibonacci and Pell sequences

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Abstract Let $(F_n)_{n\geq 0}$ and $(P_n)_{n\geq 0}$ be the Fibonacci and Pell sequences given by the inicial conditions $F_0 = 0,\ F_1 = 1,\ P_0 = 0,\ P_1 = 1$ and the recurrences formulas $F_{n+2} = F_{n+1} + F_n,\ P_{n+2} = 2P_{n+1} + P_n$ for all $n \geq 0$, respectively. In this note we shall study the Pillai's type problem

$$F_n - P_m = F_{n_1} - P_{m_1}$$

in non-negative integer pairs $(n,m) \neq (n_1,m_1)$. We completely solve this equation.

Keywords Fibonacci, Pell sequences · Pillai's type problem · Linear form in logarithms

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1 Introduction

Let $\mathbf{U} := (U_n)_{n\geq 0}$ and $\mathbf{V} := (V_n)_{n\geq 0}$ be two linearly recurrent sequences of integers. Recently, the following variation of a problem of Pillai has been studied. Find all non-negative integer solutions (n,m,n_1,m_1) of the equation

$$U_n - V_m = U_{n_1} - V_{m_1}, \qquad (n,m) \neq (n_1,m_1). \tag{1}$$

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In particular, find also all integers c which can be written as the difference between an element of $\mathbf{U}$ and an element of $\mathbf{V}$ in at least two different ways. Pillai [12], studied this problem when $\mathbf{U}$ and $\mathbf{V}$ are the sequences of powers of a, and powers of b, respectively, where $a,\ b$ are two given coprime integers different than $0,\ \pm 1$. It has been shown in [6] that, under some technical but natural conditions, equation (1) has only finitely many non-negative integer solutions and all of them are effectively computable. This version of Pillai's problem was initiated in [8] by Ddamulira, Luca and Rakotomalala who studied equation (1) when $\mathbf{U}$ and $\mathbf{V}$ are the sequences of Fibonacci numbers and powers of 2, respectively. Many other particular cases have been studied. See, for example [5], [3], [7]. We recall that the Fibonacci sequence $(F_n)_{n\geq 0}$ is given by $F_0 = 0$, $F_1 = 1$ and the recurrence formula

$$F_{n+2} = F_{n+1} + F_n \qquad \text{for all} \qquad n \geq 0.$$

Let $(P_n)_{n\geq 0}$ be the Pell sequence given by $P_0 = 0$, $P_1 = 1$, and the recurrence formula

$$P_{n+2} = 2P_{n+1} + P_n \qquad \text{for all} \qquad n \geq 0.$$

Their first terms are,

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ 377,\ 610,\ 987,\ 1597,\ \ldots$$

and

$$0,\ 1,\ 2,\ 5,\ 12,\ 29,\ 70,\ 169,\ 408,\ 985,\ 2378,\ 5741,\ 13860,\ 33461,\ \ldots,$$

respectively. In this note, we study another particular case of this problem, namely equation (1) with Fibonacci and Pell numbers. More precisely, we look at the equation

$$F_n - P_m = F_{n_1} - P_{m_1} \tag{2}$$

in integer pairs $(n,m) \neq (n_1,m_1)$. Since $F_1 = F_2 = 1$, we assume that $n \neq 1,\ n_1 \neq 1$. That is, whenever we think of 1 as a member of the Fibonacci sequence, we think of it as being F_2. Our result is then the following

Theorem 1 *All non-negative integer solutions (n,m,n_1,m_1) of (2) with $n \neq 1,\ n_1 \neq 1$ belong to the set*

$$\left\{\begin{array}{llll} (2,1,0,0), & (2,2,0,1), & (3,1,2,0), & (3,2,0,0), \\ (3,2,2,1), & (4,1,3,0), & (4,2,2,0), & (4,2,3,1), \\ (4,3,0,2), & (5,2,4,0), & (5,3,0,0), & (5,3,2,1), \\ (5,3,3,2), & (6,3,4,0), & (6,3,5,2), & (6,4,2,3), \\ (7,3,6,0), & (7,4,2,0), & (7,4,3,1), & (7,4,4,2), \\ (9,5,5,0), & (11,6,8,2), & (16,9,3,0), & (16,9,4,1) \end{array}\right\}.$$

The set of integers c admitting two representations as a difference between a Fibonacci and a Pell number in at least two different ways is

$$\{-4,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 5,\ 8,\ 19\}.$$

The representations of the above c are

$$\begin{aligned}
-4 &= F_6 - P_4 = F_2 - P_3;\\
-2 &= F_4 - P_3 = F_0 - P_2;\\
-1 &= F_2 - P_2 = F_0 - P_1;\\
0 &= F_5 - P_3 = F_3 - P_2 = F_2 - P_1 = F_0 - P_0;\\
1 &= F_7 - P_4 = F_4 - P_2 = F_3 - P_1 = F_2 - P_0;\\
2 &= F_{16} - P_9 = F_4 - P_1 = F_3 - P_0;\\
3 &= F_6 - P_3 = F_5 - P_2 = F_4 - P_0;\\
5 &= F_9 - P_5 = F_5 - P_0;\\
8 &= F_7 - P_3 = F_6 - P_0;\\
19 &= F_{11} - P_6 = F_8 - P_2.
\end{aligned}$$

2 Tools

The first one is a lower bound for a linear form in logarithms due to Matveev [11]. Let α be an algebraic number of degree d. Let a be the leading coefficient of its minimal polynomial over $\mathbf{Z}$ and let $\alpha_1 = \alpha, \ldots, \alpha_d$ denote the conjugates of α. The *Weil height* of α is defined as

$$h(\alpha) = \frac{1}{d}\left(\log a + \sum_{i=1}^{d} \log\max\{|\alpha_i|, 1\}\right).$$

The height has the following basic properties. For α, β algebraic numbers and $m \in \mathbf{Z}$, we have:

- $h(\alpha+\beta) \le h(\alpha) + h(\beta) + \log 2$.
- $h(\alpha\beta) \le h(\alpha) + h(\beta)$.
- $h(\alpha^m) = |m| h(\alpha)$.

Now let $\mathbf{L}$ be a real number field of degree $d_{\mathbf{L}}$, $\alpha_1, \ldots, \alpha_\ell \in \mathbf{L}$ and $b_1, \ldots, b_\ell \in \mathbf{Z}\setminus\{0\}$. Let $B \ge \max\{|b_1|, \ldots, |b_\ell|\}$ and

$$\Lambda = \alpha_1^{b_1} \cdots \alpha_\ell^{b_\ell} - 1.$$

Let $A_1, \ldots, A_\ell$ be real numbers such that

$$A_i \ge \max\{\, d_{\mathbf{L}} h(\alpha_i),\ |\log \alpha_i|,\ 0.16\} \qquad \text{for all} \qquad i = 1, \ldots, \ell.$$

The following result is due to Matveev in [11] (see also Theorem 9.4 in [4]).

Theorem 2 *Assume that* $\Lambda \ne 0$. *Then*

$$\log|\Lambda| > -1.4 \times 30^{\ell+3} \times \ell^{4.5} \times d_{\mathbf{L}}^2 (1 + \log d_{\mathbf{L}})(1 + \log B)\, A_1 \cdots A_\ell.$$

In this paper, we always use $\ell = 3$. Further, $\mathbf{L} = \mathbf{Q}[\sqrt{2}, \sqrt{5}]$ has degree $d_{\mathbf{L}} = 4$. Thus, once for all we fix the constant

$$C := 5.46696 \times 10^{12} > 1.4 \times 30^{3+3} \times 3^{4.5} \times 4^2 (1 + \log 4).$$

Matveev's bound gives us some large bounds on our parameters. In order to lower such bounds, we use a version of a reduction method of Baker-Davenport based on Lemma in [1]. We shall use the one given by Bravo, Gómez and Luca in [2]. For a real number x, we write

$$\|x\| = \min\{|x - n| \ : \ n \in \mathbf{Z}\}.$$

Lemma 1 *Let M be a positive integer. Let $\tau, \ \mu, \ A > 0, \ B > 1$ be given real numbers. Assume that p/q is a convergent of τ such that $q > 6M$ and $\varepsilon := \|q\mu\| - M\|q\tau\| > 0$. Then the inequality*

$$0 < |n\tau - m + \mu| < \frac{A}{B^w}$$

does not has a solution in positive integers $n, \ m$ and w in the ranges

$$n \leq M \qquad \text{and} \qquad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

This lemma is a slightly variation of the one given by Dujella and Pethő in [9]. The following lemma is also useful. It is Lemma 7 in [10].

Lemma 2 *If $m \geq 1$, $T > (4m^2)^m$ and $T > x/(\log x)^m$, then*

$$x < 2^m T (\log T)^m.$$

3 Proof of Theorem 1

We start with some basic properties of our sequences. Put

$$\alpha := \frac{1+\sqrt{5}}{2}, \quad \beta := \frac{1-\sqrt{5}}{2}\ ; \qquad \text{and} \qquad \gamma := 1 + \sqrt{2}, \quad \delta := 1 - \sqrt{2}.$$

We have the well–known Binet's formulas

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{and} \quad P_n = \frac{\gamma^n - \delta^n}{2\sqrt{2}} \tag{3}$$

which hold for all $n \geq 0$. Further, the inequalities

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \quad \text{and} \quad \gamma^{n-2} \leq P_n \leq \gamma^{n-1} \tag{4}$$

also hold for all $n \geq 1$.

Now, we start to study our equation (2) in non-negative integers (n, m, n_1, m_1) with $(n, m) \neq (n_1, m_1)$. As we said, we assume $n \neq 1, \ n_1 \neq 1$. It could happen that $\min\{n, n_1\} = 0$. At any rate, $\max\{n, n_1\} \geq 2$. If in (2) we have $m = m_1$, then $F_n = F_{n_1}$, implies that $n = n_1$, a contradiction. Thus, from now on we assume $m > m_1$. Rewriting (2) as

$$F_n - F_{n_1} = P_m - P_{m_1}, \tag{5}$$

we observe the right-hand side is positive. Hence, so is the left-hand side, therefore $n > n_1$. We now compare the two sides of (5) using (4). We have

$$\alpha^{n-4} \le F_n - F_{n_1} = P_m - P_{m_1} \le P_m \le \gamma^{n-1}.$$

The left–hand side inequality is clear if $n_1 = 0$. It is also clear if $n_1 \neq 0$, since in that case $n_1 \ge 2$, so $n \ge 3$, so $F_n - F_{n_1} \ge F_n - F_{n-1} = F_{n-2} \ge \alpha^{n-4}$. Thus, $\alpha^{n-4} \le \gamma^{n-1}$. In a similar way,

$$\alpha^{n-1} \ge F_n \ge F_n - F_{n_1} = P_m - P_{m_1} \ge P_{m-1} \ge \gamma^{m-3},$$

where the right–most inequality is clear (both for $m_1 = 0$ and for $m_1 > 0$). We thus have

$$n - 4 \le (m-1)\frac{\log\gamma}{\log\alpha} \qquad \text{and} \qquad n - 1 \ge (m-3)\frac{\log\gamma}{\log\alpha}. \tag{6}$$

Since $\log\gamma/\log\alpha = 1.831570923\ldots$ it follows that if $n \le 300$, then $m \le 166$. Running a *Mathematica* program in the range $0 \le n_1 < n \le 300$ and $0 \le m_1 < m \le 166$, with our convention, we obtain all the possibilities listed in Theorem 1.

From now on, $n > 300$. Further, by (6) we get $m > 162$ and also $n > m$. From Binet's formulas (3), we obtain

$$\begin{aligned} \left|\frac{\alpha^n}{\sqrt{5}} - \frac{\gamma^m}{2\sqrt{2}}\right| &= \left|\frac{\alpha^{n_1} + \beta^n - \beta^{n_1}}{\sqrt{5}} - \frac{\gamma^{m_1} - \delta^{m_1} + \delta^m}{2\sqrt{2}}\right| \le \frac{\alpha^{n_1}+2}{\sqrt{5}} + \frac{\gamma^{m_1}+2}{2\sqrt{2}} \\ &\le 2\max\{\alpha^{n_1+2}, \gamma^{m_1+1}\}. \end{aligned} \tag{7}$$

Dividing through by $\gamma^m/2\sqrt{2}$ we get

$$\left|\frac{4}{\sqrt{10}}\gamma^{-m}\alpha^n - 1\right| \le \max\{\alpha^{n_1-n+9}, \gamma^{m_1-m+3}\}, \tag{8}$$

where we have used that $\alpha^{n-4} \le \gamma^{m-1}$ as well as the fact that $4\sqrt{2} < \gamma^2 < \alpha^4$. Let Λ be the expression inside the absolute value in in the left–hand side above. Observe that Λ is not zero. Indeed, otherwise $8/5 = \gamma^{2m}/\alpha^{2n}$ is both a unit (an algebraic integer whose reciprocal is also an algebraic integer) and a rational number, which is false since the only rational units are ± 1.

Now we apply Matveev's inequality with

$$\alpha_1 = \frac{4}{\sqrt{10}}, \quad \alpha_2 = \gamma, \quad \alpha_3 = \alpha, \quad b_1 = 1, \quad b_2 = -m, \quad b_3 = n.$$

We have $B = n$. Further, we have $h(\alpha_1) = \log 8/2$, $h(\alpha_2) = \log\gamma/2$ and $h(\alpha_3) = \log\alpha/2$. Thus, we may take $A_1 := 4.2$, $A_2 := 1.8$ and $A_3 := 1$ we obtain that

$$\log|\Lambda| > -C(1+\log n) \times 4.2 \times 1.8.$$

Comparing with (8) we obtain

$$\min\{(n-n_1-9)\log\alpha, (m-m_1-3)\log\gamma\} \le 4.13302\times 10^{13}(1+\log n). \quad (9)$$

We next study each of these two possibilities.

Case 1. $\min\{(n-n_1)\log\alpha, (m-m_1)\log\gamma\} = (n-n_1)\log\alpha$.

To this case, we rewrite our equation as follows:

$$\begin{aligned}\left|\left(\frac{\alpha^{n-n_1}-1}{\sqrt{5}}\right)\alpha^{n_1} - \frac{\gamma^m}{2\sqrt{2}}\right| &= \left|\frac{\beta^n-\beta^{n_1}}{\sqrt{5}} - \frac{\gamma^{m_1}-\delta^{m_1}+\delta^m}{2\sqrt{2}}\right| \\ &\le \frac{2}{\sqrt{5}} + \frac{\gamma^{m_1}+2}{2\sqrt{2}} < \gamma^{m_1+2}.\end{aligned}$$

Thus,

$$\left|\left(\frac{4(\alpha^{n-n_1}-1)}{\sqrt{10}}\right)\alpha^{n_1}\gamma^{-m} - 1\right| < \gamma^{m_1-m+4}. \quad (10)$$

Let Λ_1 be the expression inside the absolute value which is in the left–hand side. We note that $\Lambda_1 \ne 0$, for if this is not so then we would get

$$\frac{\alpha^n-\alpha^{n_1}}{\gamma^m} = \frac{\sqrt{10}}{4},$$

which implies that the right–hand side is an algebraic integer, which it isn't (it's square is $5/8$). We apply again Matveev's inequality by taking

$$\alpha_1 = \frac{4(\alpha^{n-n_1}-1)}{\sqrt{10}}, \quad \alpha_2=\gamma, \quad \alpha_3=\alpha, \quad b_1=1, \quad b_2=-m, \quad b_3=n_1.$$

Thus, $B=n$. The heights of α_2 and α_3 have already been calculated. As for $h(\alpha_1)$, we have

$$\begin{aligned} h\left(\frac{4(\alpha^{n-n_1}-1)}{\sqrt{10}}\right) &\le h\left(\frac{4}{\sqrt{10}}\right) + h\left(\alpha^{n-n_1}-1\right) \le \frac{\log 8}{2} + h(\alpha^{n-n_1}) + \log 2 \\ &= \frac{\log 32}{2} + (n-n_1)\frac{\log\alpha}{2} \le \frac{4.13304\times 10^{13}(1+\log n)}{2},\end{aligned}$$

where we have used (9). Thus, we can take $A_1 := 8.26608\times 10^{13}(1+\log n)$, A_2 and A_3 as in the analysis of Λ, and get

$$\log|\Lambda_1| > -C\times(8.26608\times 10^{13}(1+\log n)^2)\times 1.8.$$

Combining this with (10), we get

$$(m-m_1)\log\gamma < 8.13427\times 10^{26}(1+\log n)^2.$$

Case 2. $\min\{(n-n_1)\log\alpha, (m-m_1)\log\gamma\} = (m-m_1)\log\gamma$.

Here, we rewrite our equation as

$$\begin{aligned}\left|\frac{\alpha^n}{\sqrt{5}}-\left(\frac{\gamma^{m-m_1}-1}{2\sqrt{2}}\right)\gamma^{m_1}\right| &= \left|\frac{\beta^n+\alpha^{n_1}-\beta^{n_1}}{\sqrt{5}}-\frac{\delta^m-\delta^{m_1}}{2\sqrt{2}}\right|\\ &\le \frac{\alpha^{n_1}+2}{\sqrt{5}}+\frac{1}{\sqrt{2}}<\alpha^{n_1+5}.\end{aligned}$$

Thus,

$$\left|1-\left(\frac{\sqrt{10}(\gamma^{m-m_1}-1)}{4}\right)\gamma^{m_1}\alpha^{-n}\right|<\alpha^{n_1-n+7}. \tag{11}$$

We let Λ_2 be the expression inside the absolute value in the left-hand side. As before, $\Lambda_2\neq 0$, for otherwise we get that $8/5$ is an algebraic integer, which is false. We apply again Matveev's inequality by taking

$$\alpha_1=\frac{\sqrt{10}(\gamma^{m-m_1}-1)}{4},\quad \alpha_2=\gamma,\quad \alpha_3=\alpha,\quad b_1=1,\quad b_2=m_1,\quad b_3=-n.$$

Thus, $B=n$. The heights of α_2 and α_3 have already been calculated. As for $h(\alpha_1)$, we have

$$\begin{aligned}h\left(\frac{\sqrt{10}(\gamma^{m-m_1}-1)}{4}\right) &\le h\left(\frac{\sqrt{10}}{4}\right)+h\left(\gamma^{m-m_1}-1\right)\\ &\le \frac{4.13304\times 10^{13}(1+\log n)}{2},\end{aligned}$$

Thus, we can take the same A_1 as in Case 1, and so we get the same lower bound for $\log|\Lambda_2|$. Therefore,

$$(n-n_1)\log\alpha<8.13427\times 10^{26}(1+\log n)^2.$$

So, we have proved that

$$\max\{(n-n_1)\log\alpha,(m-m_1)\log\gamma\}\le 8.13427\times 10^{26}(1+\log n)^2. \tag{12}$$

We now get a bound on n. Using Binet's formulas (3), we write our equation as follows:

$$\left|\frac{\alpha^{n-n_1}-1}{\sqrt{5}}\alpha^{n_1}-\frac{\gamma^{m-m_1}-1}{2\sqrt{2}}\gamma^{m_1}\right|=\left|\frac{\beta^n-\beta^{n_1}}{\sqrt{5}}-\frac{\delta^m-\delta^{m_1}}{2\sqrt{2}}\right|<\frac{2}{\sqrt{5}}+\frac{1}{\sqrt{2}}<2.$$

Dividing across by $(\gamma^m-\gamma^{m-1})/2\sqrt{2}$, we obtain

$$\left|\left(\frac{4}{\sqrt{10}}\left(\frac{\alpha^{n-n_1}-1}{\gamma^{m-m_1}-1}\right)\right)\gamma^{-m_1}\alpha^{n_1}-1\right|<\frac{4\sqrt{2}}{\gamma^m-\gamma^{m_1}}<\frac{8\sqrt{2}}{\gamma^m}<\frac{1}{\alpha^{n-8}}, \tag{13}$$

where we used $\alpha^{n-4}<\gamma^{m-1}$, as well as the fact that $8\sqrt{2}<\alpha^4\gamma$. We let Λ_3 be the expression inside the absolute value in (13). We apply Matveev's inequality with

$$\alpha_1=\frac{4}{\sqrt{10}}\left(\frac{\alpha^{n-n_1}-1}{\gamma^{m-m_1}-1}\right),\ \alpha_2=\gamma,\ \alpha_3=\alpha,\ b_1=1,\ b_2=-m_1,\ b_3=-n_1.$$

Thus, we take $B = n$. We need to show that $\Lambda_3 \neq 0$. To do this we take the **Q**-automorphism σ of **L** given by $\sigma(\sqrt{5}) = -\sqrt{5}$ and $\sigma(\sqrt{2}) = \sqrt{2}$. Under this automorphism, we have $\sigma(\alpha) = \beta$, $\sigma(\gamma) = \gamma$ and $\sigma(\sqrt{10}) = -\sqrt{10}$. Thus, if $\Lambda_3 = 0$, then $\sigma(\Lambda_3) = 0$, which implies, in particular, that

$$\frac{\sqrt{10}}{4} = \left|\frac{\beta^n - \beta^{n_1}}{\gamma^m - \gamma^{m_1}}\right| < \frac{2}{\gamma^{m-1}(\gamma - 1)} < \frac{1}{2},$$

since $m > 162$, which is a contradiction. As before, the heights of α_2 and α_3 have already been calculated. For $h(\alpha_1)$, we have

$$\begin{aligned} h\left(\frac{4}{\sqrt{10}}\left(\frac{\alpha^{n-n_1}-1}{\gamma^{m-m_1}-1}\right)\right) &\leq h\left(\frac{4}{\sqrt{10}}\right) + h\left(\alpha^{n-n_1}+1\right) + h\left(\gamma^{m-m_1}+1\right) \\ &\leq \frac{\log 128}{2} + (n-n_1)\frac{\log\alpha}{2} + (m-m_1)\frac{\log\gamma}{2} \\ &\leq 8.13428 \times 10^{26}(1+\log n)^2. \end{aligned}$$

Thus, we can take $A_1 := 3.25371 \times 10^{27}(1+\log n)^2$, and A_2, A_3 as before. Therefore, we get

$$\begin{aligned} \log|\Lambda_3| &> -C(1+\log n) \times (3.25371 \times 10^{27}(1+\log n)^2) \times 1.8 \\ &> -3.20183 \times 10^{40}(1+\log n)^3, \end{aligned}$$

which, upon comparing it to (13) and applying Lemma 2, we obtain

$$n < 3.77671 \times 10^{48}. \tag{14}$$

Now, we will reduce the upper bound of n. To do this, let Γ be defined as

$$\Gamma = n\log\alpha - m\log\gamma + \log\left(\frac{4}{\sqrt{10}}\right).$$

Assume first that $\min\{n-n_1, m-m_1\} \geq 20$. We go to (8). Note that $e^\Gamma - 1 = \Lambda \neq 0$, so $\Gamma \neq 0$. If $\Gamma > 0$ then

$$0 < \Gamma < e^\Gamma - 1 = \Lambda = |\Lambda| < \max\{\alpha^{n_1-n+9}, \gamma^{m_1-m+3}\}.$$

On the other hand, if $\Gamma < 0$, we then have $1 - e^\Gamma = |e^\Gamma - 1| < 1/2$ which implies $e^{|\Gamma|} < 2$. Thus,

$$0 < |\Gamma| < e^{|\Gamma|} - 1 = e^{|\Gamma|}|\Lambda| < 2\max\{\alpha^{n_1-n+9}, \gamma^{m_1-m+3}\}.$$

So, in both cases we have

$$0 < |\Gamma| < 2\max\{\alpha^{n_1-n+9}, \gamma^{m_1-m+3}\}. \tag{15}$$

Dividing through by $\log\gamma$ in the above inequality, we get

$$0 < |n\tau - m + \mu| < \max\left\{\frac{173}{\alpha^{n-n_1}}, \frac{32}{\gamma^{m-m_1}}\right\},$$

where
$$\tau := \frac{\log \alpha}{\log \gamma}, \qquad \mu := \frac{\log\left(4/\sqrt{10}\right)}{\log \gamma}.$$

Now we apply Lemma 1. To do this, we take $M := 3.77671 \times 10^{48}$ (a bound on m and n by (14)), our τ and, with a *Mathematica* program, we find that the denominator of the convergent

$$\frac{p_{112}}{q_{112}} = \frac{111842821415068814601069451383096958405345992106163812}{204848059751598401563305907296432335323118859258712413}$$

of τ satisfies $q_{112} > 6M$ and that $\varepsilon = \|q\mu\| - M\|q\tau\| = 0.105822 > 0$. This implies, with $(A,B) = (173, \alpha)$ or $(32, \gamma)$, that either

$$n - n_1 \le 270, \qquad \text{or} \qquad m - m_1 \le 145.$$

We now look at each one of these two cases. First, we assume that $n - n_1 \le 270$ and $m - m_1 \ge 20$. In this case, we consider

$$\Gamma_1 = n_1 \log \alpha - m \log \gamma + \log\left(\frac{4(\alpha^{n-n_1} - 1)}{\sqrt{10}}\right).$$

As before, $e^{\Gamma_1} - 1 = \Lambda_1 \neq 0$, so $\Gamma_1 \neq 0$. We go to (10). With an argument similar to a previous one, we have that
$$0 < |\Gamma_1| < \frac{2\gamma^4}{\gamma^{m-m_1}}.$$

Dividing through by $\log \gamma$ we obtain

$$0 < |n_1 \tau - m + \mu| < \frac{78}{\gamma^{m-m_1}},$$

where τ is the same one as above and

$$\mu := \frac{\log\left(4(\alpha^{n-n_1} - 1)/\sqrt{10}\,\right)}{\log \gamma}.$$

We apply again Lemma 1 noting that $n_1 > 0$, for otherwise we would have that $n \le 270$ which contradicts our hypothesis that $n > 300$. Consider

$$\mu_k := \frac{\log\left(4(\alpha^k - 1)/\sqrt{10}\,\right)}{\log \gamma}, \qquad \text{for} \qquad k = 1, \ldots, 270.$$

We ran a *Mathematica* program and found that the same convergent p_{112}/q_{112} satisfies $q_{112} > 6M$. Further, $\varepsilon_k \ge 0.00119532$ for all $1 \le k \le 270$. For each of the values of ε_k and with $(A,B) = (78, \gamma)$, we calculate $\log(78 q_{112}/\varepsilon_k)/\log \gamma$ and found that each of them is at most 151. Thus, $m - m_1 \le 151$.

Now let us look at the other case. Assume that $m - m_1 \le 145$ and $n - n_1 \ge 20$. We consider

$$\Gamma_2 = n \log \alpha - m_1 \log \gamma + \log\left(\frac{4}{\sqrt{10}(\gamma^{m-m_1} - 1)}\right).$$

We note that $1-e^{-\Gamma_2}=\Lambda_2\neq 0$, so $\Gamma_2\neq 0$. We go to (11). With an argument similar to one above, we obtain

$$0<|\Gamma_2|<\frac{2\alpha^7}{\alpha^{n-n_1}}.$$

Dividing through by $\log\lambda$, we get

$$0<|n\tau-m_1+\mu|<\frac{66}{\alpha^{n-n_1}},$$

where τ is the same one as above and

$$\mu:=\frac{\log\left(4/(\sqrt{10}(\gamma^{n-m_1}-1))\right)}{\log\gamma}.$$

Now we use again Lemma 1 noting that $m_1>0$, which is the case, since otherwise we have $m\leq 145$, which contradicts our hypothesis $m>162$. As above, by considering now

$$\mu_\ell:=\frac{\log\left(4/(\sqrt{10}(\gamma^{\ell}-1))\right)}{\log\gamma},\qquad \text{for all}\qquad \ell=1,\ldots,145$$

and running a *Mathematica* program, we find that $q_{112}>6M$, and that for this convergent $\varepsilon_\ell\geq 0.0000620746$ for all $1\leq\ell\leq 145$. For each of these ε_ℓ and with $(A,B):=(66,\alpha)$, we calculated $\log(66q_{112}/\varepsilon_\ell)/\log\alpha$ and found that all these numbers are at most 283. Thus $n-n_1\leq 283$.

So, we got that either $n-n_1\leq 270$ or $m-m_1\leq 145$. Assuming the first one we deduced $m-m_1\leq 151$, and assuming the second one, we deduced $n-n_1\leq 283$. Altogether, we have $n-n_1\leq 283$, $m-m_1\leq 151$. So, it remains to study this case. We consider

$$\Gamma_3=n_1\log\alpha-m_1\log\gamma+\log\left(\frac{4}{\sqrt{10}}\left(\frac{\alpha^{n-n_1}-1}{\gamma^{m-m_1}-1}\right)\right).$$

We note that $e^{\Gamma_3}-1=\Lambda_3$. Again, since $\Lambda_3\neq 0$, we have that $\Gamma_3\neq 0$. Since $n>300$, we get

$$0<|\Gamma_3|<\frac{2\alpha^8}{\alpha^n}.$$

Dividing through by $\log\gamma$, we get

$$0<|n_1\tau-m_1+\mu|<\frac{107}{\alpha^n},$$

where τ is as above and

$$\mu:=\frac{\log\left(4(\alpha^{n-n_1}-1)/\sqrt{10}(\gamma^{m-m_1}-1)\right)}{\log\gamma}.$$

We apply for the last time Lemma 1. As above, we have that $n_1,\ m_1>0$. Thus, we consider

$$\mu_{k,\ell}:=\frac{\log\left(4(\alpha^{k}-1)/\sqrt{10}(\gamma^{\ell}-1)\right)}{\log\gamma},\quad k=1,\ldots,283,\quad \ell=1,\ldots,151.$$

Running a *Mathematica* program, we find again that the same convergent works namely $q_{112} > 6M$ and $\varepsilon_{k,\ell} \geq 0.0000307767$ for all $1 \leq k \leq 283$ and $1 \leq \ell \leq 151$. With $(A,B) := (107,\alpha)$ we calculated $\log\left(107q_{112}/\varepsilon_{k,\ell}\right)/\log\alpha$ for each of these values $\varepsilon_{k,\ell}$, and found that the maximum value of them is ≤ 286. Thus, $n \leq 286$, which contradicts our assumption on n.

This finishes the proof of our theorem.

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