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Algebraic results for the values $\vartheta_3(m\tau)$ and $\vartheta_3(n\tau)$ of the Jacobi theta-constant

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Let $\vartheta_3(\tau) = 1 + 2\sum_{\nu=1}^{\infty} e^{\pi i \nu^2 \tau}$ denote the classical Jacobi theta-constant. We prove that the two values $\vartheta_3(m\tau)$ and $\vartheta_3(n\tau)$ are algebraically independent over $\mathbb{Q}$ for any τ in the upper half-plane such that $q = e^{\pi i \tau}$ is an algebraic number, where $m, n \geq 2$ are distinct integers.

1. Introduction and statement of the results

Throughout this paper, let τ be a complex variable in the upper half-plane $\mathbb{H} := \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$. The three classical theta functions

$$\vartheta_2(\tau) = 2\sum_{\nu=0}^{\infty} q^{(\nu+1/2)^2}, \quad \vartheta_3(\tau) = 1 + 2\sum_{\nu=1}^{\infty} q^{\nu^2}, \quad \vartheta_4(\tau) = 1 + 2\sum_{\nu=1}^{\infty} (-1)^{\nu} q^{\nu^2}$$

are known as theta-constants or Thetanullwerte, where $q := e^{\pi i \tau}$. These theta-constants are holomorphic in $\mathbb{H}$ and never vanish for any $\tau \in \mathbb{H}$. In particular, the function $\vartheta_3(\tau)$ is called a Jacobi theta-constant or Thetanullwert of the Jacobi theta function $\vartheta(z \mid \tau) = \sum_{\nu=-\infty}^{\infty} e^{\pi i \nu^2 \tau + 2\pi i \nu z}$. For an extensive discussion of the Jacobi theta function and theta-constants we refer the reader to [Stein and Shakarchi 2003, Chapter 10]. Y. V. Nesterenko [2006] has improved upon a result from [Grinspan 2001] and obtained some identities for the theta-constants.

Theorem A [Nesterenko 2006, Theorem 1]. *For any odd integer $n \geq 3$ there exists an integer polynomial $P_n(X, Y)$ with $\deg_X P_n(X, Y) = \psi(n)$ such that*

$$P_n\left(n^2 \frac{\vartheta_3^4(n\tau)}{\vartheta_3^4(\tau)}, 16\frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)}\right) = 0$$

holds for any $\tau \in \mathbb{H}$, where

$$\psi(n) := n \prod_{p \mid n}\left(1 + \frac{1}{p}\right).$$

For example, the first polynomials P_3 and P_5 are given in [Nesterenko 2006] by

$$P_3 = 9 - (28 - 16Y + Y^2)X + 30X^2 - 12X^3 + X^4,$$

$$P_5 = 25 - (126 - 832Y + 308Y^2 - 32Y^3 + Y^4)X + (255 + 1920Y - 120Y^2)X^2 + (-260 + 320Y - 20Y^2)X^3 + 135X^4 - 30X^5 + X^6$$

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and the polynomials P_7, P_9, and P_{11} are listed in the appendix of [Elsner 2015]. Recently one of us (Elsner) constructed similar integer polynomials in two variables X and Y, which vanish identically at certain rational functions of theta-constants including the function $\vartheta_3(n\tau)$ for $n = 2^m$. He applied this result and Theorem A to settle the algebraic independence problem of the two values $\vartheta_3(\tau)$ and $\vartheta_3(n\tau)$ for integers $n \geq 2$, and obtained the following Theorem B.

Theorem B [Elsner 2015, Theorem 1.1]. *Let $\tau \in \mathbb{H}$ such that $e^{\pi i\tau}$ is an algebraic number. Then the two values $\vartheta_3(\tau)$ and $\vartheta_3(2^m\tau)$ are algebraically independent over $\mathbb{Q}$ for each integer $m \geq 1$. Furthermore, the same holds for the two values $\vartheta_3(\tau)$ and $\vartheta_3(n\tau)$ if $n = 3, 5, 6, 7, 9, 10, 11, 12$.*

The proof of Theorem B is based on an algebraic independence criterion, see [Elsner et al. 2011, Lemma 3.1], which requires a nonvanishing of a Jacobian determinant. In particular, to prove the latter assertion in Theorem B, he needed the explicit forms of the polynomials P_3, P_5, P_7, P_9 and P_{11} stated above. In [Elsner and Tachiya 2017], two of us obtained the following Theorem C by studying the specific properties of the polynomials P_n.

Theorem C [Elsner and Tachiya 2017, Theorem 1.2]. *Let $n \geq 2$ be an integer and $j \in \{2, 3, 4\}$. Then for any $\tau \in \mathbb{H}$ at least three of the numbers $e^{\pi i\tau}$, $\vartheta_3(\tau)$, $\vartheta_3(n\tau)$, and $D\vartheta_j(\tau)$ are algebraically independent over $\mathbb{Q}$, where $D := (\pi i)^{-1}d/d\tau$ denotes a differential operator.*

An application of Theorem C gives an improvement of Theorem B as follows:

Theorem D. *Let $\tau \in \mathbb{H}$ be such that $e^{\pi i\tau}$ is an algebraic number. Then the two numbers $\vartheta_3(\tau)$ and $\vartheta_3(n\tau)$ are algebraically independent over $\mathbb{Q}$ for each integer $n \geq 2$.*

On the other hand, the algebraic dependence result is also obtained in [Elsner and Tachiya 2017] through the properties of the polynomials P_n.

Theorem E [Elsner and Tachiya 2017, Theorem 1.4]. *Let $\ell, m, n \geq 1$ be integers and $\tau \in \mathbb{H}$ be any complex number. Then the three values $\vartheta_3(\ell\tau)$, $\vartheta_3(m\tau)$, and $\vartheta_3(n\tau)$ are algebraically dependent over $\mathbb{Q}$.*

In this paper, we fill the gap between Theorems D and E. Our main result is the following.

Theorem 1. *Let $m, n \geq 1$ be distinct integers and $\tau \in \mathbb{H}$. Then at least two of the numbers $e^{\pi i\tau}$, $\vartheta_3(m\tau)$, and $\vartheta_3(n\tau)$ are algebraically independent over $\mathbb{Q}$. In particular, the two numbers $\vartheta_3(m\tau)$ and $\vartheta_3(n\tau)$ are algebraically independent over $\mathbb{Q}$ for any $\tau \in \mathbb{H}$ such that $e^{\pi i\tau}$ is an algebraic number.*

Of course the two numbers $\vartheta_3(m\tau)$ and $\vartheta_3(n\tau)$ can be algebraically dependent over $\mathbb{Q}$ without an algebraic condition on $e^{\pi i\tau}$. Indeed, for $\tau = i \in \mathbb{H}$ the two numbers $\vartheta_3(i)$ and $\vartheta_3(2i)$ are algebraically dependent over $\mathbb{Q}$, since the nontrivial relation

$$4\vartheta_3^2(2i) - (\sqrt{2}+2)\vartheta_3^2(i) = 0 \tag{1}$$

exists; see [Berndt 1998, p. 325]. Note that the number $e^{\pi} = i^{-2i}$ was shown to be transcendental for the first time by A. O. Gelfond (1929) and, a few years later, this property of e^{π} was corroborated by the Gelfond–Schneider theorem (1934). Conversely, the above identity (1) and Theorem 1 imply the transcendence of e^{π} as well.

2. Some properties of $P_n(X, Y)$

We now discuss some properties of $P_n(X, Y)$ given in Theorem A. We start with a short description of the construction of $P_n(X, Y)$; for details, see [Nesterenko 2006]. Let $\Gamma(2)$ be the principal congruence subgroup of level 2 in $\mathrm{SL}(2, \mathbb{Z})$; that is,

$$\Gamma(2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2 \right\}.$$

Then for each odd integer $n \geq 3$ the set of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2, \quad (a, b, c, d) = 1, \; ad - bc = n,$$

is a union of $\psi(n)$ equivalence classes with respect to the left–multiplication on the elements of $\Gamma(2)$, and the class representatives are given by

$$\alpha_\nu := \begin{pmatrix} u & 2v \\ 0 & w \end{pmatrix}, \quad (u, v, w) = 1, \; uw = n, \; 0 \leq v < w. \tag{2}$$

For these $\psi(n)$ matrices $\alpha_1, \ldots, \alpha_{\psi(n)}$ in (2), we define the polynomial

$$\prod_{\nu=1}^{\psi(n)} (X - x_\nu(\tau)) =: X^{\psi(n)} + a_1(\tau) X^{\psi(n)-1} + \cdots + a_{\psi(n)-1}(\tau) X + a_{\psi(n)}(\tau),$$

where

$$x_\nu(\tau) := u^2 \frac{\vartheta_3^4((u\tau + 2v)/w)}{\vartheta_3^4(\tau)} \quad \text{with} \begin{pmatrix} u & 2v \\ 0 & w \end{pmatrix} = \alpha_\nu, \; \nu = 1, \ldots, \psi(n). \tag{3}$$

Then, using the modular method as well as Galois considerations, one finds that there exist polynomials $R_j(Y) \in \mathbb{Z}[Y]$, $j = 1, \ldots, \psi(n)$, such that

$$a_j(\tau) = R_j(16\lambda(\tau)), \quad \lambda(\tau) := \frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)}. \tag{4}$$

Thus, the integer polynomial

$$P_n(X, Y) := X^{\psi(n)} + R_1(Y) X^{\psi(n)-1} + \cdots + R_{\psi(n)-1}(Y) X + R_{\psi(n)}(Y) \tag{5}$$

vanishes identically at $X = n^2 \vartheta_3^4(n\tau)/\vartheta_3^4(\tau)$ and $Y = 16\lambda(\tau)$.

Lemma 2. *For each odd integer $n \geq 3$, the polynomial $P_n(X, 16\lambda(\tau))$ is irreducible over the field $\mathbb{C}(\lambda(\tau))$.*

Proof. The group $\Gamma(2)$ fixes the function $\lambda(\tau) = \vartheta_2^4(\tau)/\vartheta_3^4(\tau)$, since the functions $\vartheta_3^4(\tau)$ and $\vartheta_4^4(\tau)$ are modular forms of weight 2 with respect of the subgroup $\Gamma(2)$. Moreover, we have the transformation formula

$$x_\nu \left(\frac{a\tau + b}{c\tau + d} \right) = x_\mu(\tau) \tag{6}$$

for a proper matrix $\beta := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(2)$ and subscripts ν, μ such that a proper matrix $\gamma \in \Gamma(2)$ satisfies $\alpha_\nu \beta = \gamma \alpha_\mu$; see formulae (6) and (7) in [Nesterenko 2006]. This may be regarded as an equivalence relation over the matrices $\alpha_1, \alpha_2, \dots, \alpha_{\psi(n)}$ from (2). One can show that any two matrices α_ν and α_μ, $1 \le \nu, \mu \le \psi(n)$, are equivalent. Together with (6) it turns out that the group $\Gamma(2)$ permutes the $\psi(n)$ distinct functions $x_1(\tau), \dots, x_{\psi(n)}(\tau)$ transitively. This implies that $P_n(X, 16\lambda(\tau))$ is a minimal polynomial of $x_1(\tau)$ over the field $\mathbb{C}(\lambda(\tau))$. □

Remark 3. There is no complex number α such that $P_n(\alpha, Y)$ is identically zero. If such an α existed, the polynomial $P_n(X, Y)$ would be divisible by $(X-\alpha)$, which is impossible by Lemma 2. This fact can also be checked directly from the definition of $x_\nu(\tau)$; see [Elsner and Tachiya 2017, Lemma 2.1]. In particular, $P_n(X, Y)$ has positive degree in Y.

Lemma 4. *We have*

$$P_n(X, 0) = \prod_{u \mid n,\ u \ge 1} (X - u^2)^{w(u, n/u)},$$

where

$$w(a, b) := \sum_{\substack{(a,b,k)=1 \\ 0 \le k < b}} 1.$$

Proof. This follows immediately from the relation

$$P_n(X, 16\lambda(\tau)) = \prod_{\nu=1}^{\psi(n)} (X - x_\nu(\tau))$$

as $\tau \to i\infty$, since we have $\lambda(\tau) \to 0$ and $x_\nu(\tau) \to u^2$ for each $\nu = 1, \dots, \psi(n)$ in (3), respectively. □

Example 5. For the polynomial P_3 given in Section 1, we have

$$P_3(X, 0) = 9 - 28X + 30X^2 - 12X^3 + X^4 = (X-1)^3(X-3^2).$$

Here, $\psi(3) = 4$ and the four triples (u, v, w) in (2) are given by

$$(3, 0, 1), \quad (1, 0, 3), \quad (1, 1, 3), \quad (1, 2, 3).$$

More generally, $P_p(X, 0) = (X-1)^p(X-p^2)$ for any odd prime $p \ge 3$.

3. Lemmas

Let $\tau \in \mathbb{H}$. We prove in Lemmas 7 and 8 below that the number $\vartheta_3(\tau)$ is algebraic over the field $\mathbb{Q}(\vartheta_3(u\tau), \vartheta_3(v\tau))$ for certain positive integers u and v. To see this, we need the following Lemma 6. Note that $P_n(0, Y)$ is a *nonzero integer* for the polynomial $P_n(X, Y)$ in Theorem A; see [Elsner and Tachiya 2017, Lemma 2.3].

Lemma 6 [Elsner and Tachiya 2017, Lemma 2.5]. *Let $n = 2^\alpha m$ be an integer with $\alpha \ge 1$ and odd integer $m \ge 3$. Then there exists a polynomial $Q_n(X, Y) \in \mathbb{Z}[X, Y]$ such that*

$$Q_n\left(\frac{\vartheta_3^4(n\tau)}{\vartheta_3^4(\tau)}, \frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)}\right) = 0$$

for any $\tau \in \mathbb{H}$. *Furthermore, the polynomial* $Q_n(X, Y)$ *is of the form*

$$Q_n(X, Y) = c^{2^\alpha} Y^{2^\alpha \psi(m)} + \sum_{j=0}^{2^\alpha \psi(m)-1} R_{n,j}(X) Y^j, \tag{7}$$

with

$$Q_n(0, Y) = c^{2^\alpha} Y^{2^\alpha \psi(m)},$$

where c *is equal to the nonzero integer* $P_m(0, Y)$.

First we consider the case where u and v have different parity.

Lemma 7. *Let* $u \geq 1$ *be an odd integer and* $v \geq 2$ *be an even integer which is not a power of* 2. *Then for any* $\tau \in \mathbb{H}$ *the number* $\vartheta_3(\tau)$ *is algebraic over the field* $\mathbb{Q}(\vartheta_3(u\tau), \vartheta_3(v\tau))$.

Proof. The assertion is clear if $u = 1$. Let $u \geq 3$ be an odd integer and $P_u(X, Y)$ be as in Theorem A. Then

$$P_u\left(u^2 \frac{\vartheta_3^4(u\tau)}{\vartheta_3^4(\tau)}, 16\frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)}\right) = 0 \tag{8}$$

for any $\tau \in \mathbb{H}$. Noting that $P_u(X, Y)$ has positive degree in Y and $P_u(0, Y)$ is a nonzero integer, we have the form

$$P_u(X, Y) = \sum_{j=0}^{d_u} S_{u,j}(X) Y^j, \quad S_{u,d_u}(X) \not\equiv 0,$$

with

$$c_u := S_{u,0}(0) = P_u(0, 0) \in \mathbb{Z} \setminus \{0\} \quad \text{and} \quad S_{u,j}(0) = 0 \quad (1 \leq j \leq d_u). \tag{9}$$

On the other hand, since v is not a power of 2, Lemma 6 shows that there exists a nonzero polynomial $Q_v(X, Y) \in \mathbb{Z}[X, Y]$ such that

$$Q_v\left(\frac{\vartheta_3^4(v\tau)}{\vartheta_3^4(\tau)}, \frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)}\right) = 0 \tag{10}$$

for any $\tau \in \mathbb{H}$, where $Q_v(X, Y)$ is of the form (7) with

$$Q_v(0, Y) := c_v Y^{d_v}, \quad c_v \in \mathbb{Z} \setminus \{0\}. \tag{11}$$

Let $\tau \in \mathbb{H}$ be a fixed complex number. Then, by (8) and (10), the polynomials $P_u(u^2\vartheta_3^4(u\tau)/\vartheta_3^4(\tau), 16Y)$ and $Q_v(\vartheta_3^4(v\tau)/\vartheta_3^4(\tau), Y)$ have the same common root $Y_0 = \vartheta_2^4(\tau)/\vartheta_3^4(\tau)$. Hence, the resultant

$$R_1(X, Z) := \mathrm{Res}_Y(P_u(X, 16Y), Q_v(Z, Y))$$

is given by the determinant D_Y of the square $(d_u + d_v)$ Sylvester matrix depending on the coefficients of $P_u(X, 16Y)$ and $Q_v(Z, Y)$ with respect to Y. Then, $R_1(X, Z)$ (and thus D_Y) vanishes at $X := u^2\vartheta_3^4(u\tau)/\vartheta_3^4(\tau)$ and $Z := \vartheta_3^4(v\tau)/\vartheta_3^4(\tau)$, so that the polynomial

$$R_2(W) := R_1(u^2\vartheta_3^4(u\tau)W, \vartheta_3^4(v\tau)W)$$

has a root $W_0 = \vartheta_3^{-4}(\tau)$ over the field $K := \mathbb{Q}(\vartheta_3(u\tau), \vartheta_3(v\tau))$. Note that $R_2(W)$ is not identically zero, since by (9) and (11) the determinant D_Y takes the form

$$R_2(0) = R_1(0,0) = \det \begin{pmatrix} & & & c_u & 0 & 0 \\ & & & & \ddots & 0 \\ & & & & & c_u \\ c_v & & & & & \\ 0 & \ddots & & & & \\ 0 & 0 & c_v & & & \end{pmatrix} = \pm c_u^{d_v} c_v^{d_u} \neq 0.$$

Therefore the number $\vartheta_3(\tau)$ is algebraic over K and the proof of Lemma 7 is completed. □

Next we treat the case where both u and v are odd.

Lemma 8. *Let $u, v \geq 1$ be distinct odd integers. Then for any $\tau \in \mathbb{H}$ the number $\vartheta_3(\tau)$ is algebraic over the field $\mathbb{Q}(\vartheta_3(u\tau), \vartheta_3(v\tau))$.*

Proof. We may assume $u, v \geq 3$. Similarly to the proof of Lemma 7, we consider the resultant

$$R_1(X, Z) := \mathrm{Res}_Y(P_u(X, Y), P_v(Z, Y)), \tag{12}$$

and the polynomial

$$R_2(W) := R_1(u^2\vartheta_3^4(u\tau)W, v^2\vartheta_3^4(v\tau)W), \tag{13}$$

which has a root $W_0 = \vartheta_3^{-4}(\tau)$. Suppose to the contrary that the above polynomial $R_2(W)$ is identically zero for some $\tau_0 \in \mathbb{H}$. Then, putting $\alpha := u^2\vartheta_3^4(u\tau_0)$ and $\beta := v^2\vartheta_3^4(v\tau_0)$, we have by (12) and (13)

$$\mathrm{Res}_Y(P_u(\alpha W, Y), P_v(\beta W, Y)) = R_1(\alpha W, \beta W) = R_2(W) \equiv 0,$$

and so there exists a common factor $H(W, Y) \in \mathbb{C}[W, Y]$ with positive degree in Y of the two polynomials $P_u(\alpha W, Y)$ and $P_v(\beta W, Y)$. Let

$$P_u(\alpha W, Y) = H(W, Y)\, G(W, Y).$$

Substituting the function $\lambda(\tau)$ defined by (4) into Y in the above, we have

$$P_u(\alpha W, 16\lambda(\tau)) = H(W, 16\lambda(\tau))\, G(W, 16\lambda(\tau)). \tag{14}$$

In what follows, we denote by $\deg H(W, Y)$, $\deg G(W, Y)$, and $\deg P_u(\alpha W, Y)$ the total degrees of the polynomials $H(W, Y)$, $G(W, Y)$, and $P_u(\alpha W, Y)$ with respect to W and Y, respectively. Then

$$\deg_W H(W, 16\lambda(\tau)) \leq \deg H(W, Y), \quad \deg_W G(W, 16\lambda(\tau)) \leq \deg G(W, Y),$$

so that

$$\begin{aligned} \deg_W P_u(\alpha W, 16\lambda(\tau)) &= \deg_W H(W, 16\lambda(\tau)) + \deg_W G(W, 16\lambda(\tau)) \\ &\leq \deg H(W, Y) + \deg G(W, Y) \\ &= \deg P_u(\alpha W, Y). \end{aligned}$$

On the other hand, it is clear that

$$\deg_W P_u(\alpha W, 16\lambda(\tau)) = \deg P_u(\alpha W, Y),$$

since by [Nesterenko 2006, Corollary 4] the inequalities

$$\deg_Y R_k(Y) \le k \cdot \frac{n-1}{n}, \quad 1 \le k \le \psi(n),$$

hold in (5). Thus, we get

$$\deg_W H(W, 16\lambda(\tau)) = \deg H(W, Y) \ge \deg_Y H(W, Y) \ge 1. \tag{15}$$

Hence by Lemma 2 together with (14) and (15), we obtain

$$P_u(\alpha W, 16\lambda(\tau)) = c_1 H(W, 16\lambda(\tau))$$

for some nonzero complex numbers c_1. Similarly there exists a nonzero complex number c_2 such that

$$P_v(\beta W, 16\lambda(\tau)) = c_2 H(W, 16\lambda(\tau)),$$

and hence

$$P_u(\alpha W, 16\lambda(\tau)) = c P_v(\beta W, 16\lambda(\tau)), \quad c := c_1/c_2.$$

Taking $\tau \to i\infty$ in the above equality, we have by Lemma 4

$$\prod_{d \mid u,\ d \ge 1} (\alpha W - d^2)^{w(d,u/d)} = c \prod_{d \mid v,\ d \ge 1} (\beta W - d^2)^{w(d,v/d)}.$$

Assume, without loss of generality, that $u > v$. Then, comparing the multiplicity of the zeros of these polynomials at $1/\alpha$, we obtain

$$u = w(1, u) \le \max_d w(d, v/d) \le v,$$

which is a contradiction. Hence, the polynomial $R_2(W)$ is not identically zero for any $\tau \in \mathbb{H}$, and the proof of Lemma 8 is completed by $R_2(\vartheta_3^{-4}(\tau)) = 0$. □

4. Proof of Theorem 1

Proof of Theorem 1. Let m and n be distinct positive integers. Define $m_1 := m/d$ and $n_1 := n/d$, where $d := \gcd(m, n)$. Without loss of generality, we may assume that m_1 is odd. In what follows, we distinguish two cases based on the parity of n_1. We first suppose that n_1 is even. Let $\tau \in \mathbb{H}$. Then, by Lemma 7 with $u := 3m_1 \ge 3,\ v := 3n_1 \ne 2^\alpha\ (\alpha \ge 0)$, and $\tau_0 := d\tau/3 \in \mathbb{H}$, the number $\vartheta_3(\tau_0)$ is algebraic over the field $\mathbb{Q}(\vartheta_3(u\tau_0), \vartheta_3(v\tau_0))$. Hence, we obtain

$$\begin{aligned}
\operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(e^{\pi i\tau}, \vartheta_3(m\tau), \vartheta_3(n\tau)) &= \operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(e^{\pi i\tau_0}, \vartheta_3(u\tau_0), \vartheta_3(v\tau_0)) \\
&= \operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(e^{\pi i\tau_0}, \vartheta_3(\tau_0), \vartheta_3(u\tau_0), \vartheta_3(v\tau_0)) \\
&\ge \operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(e^{\pi i\tau_0}, \vartheta_3(\tau_0), \vartheta_3(u\tau_0)) \\
&\ge 2,
\end{aligned}$$

where for the last inequality we used the fact that $u > 2$ and that at least two of the numbers $e^{\pi i\tau_0}$, $\vartheta_3(\tau_0)$ and $\vartheta_3(u\tau_0)$ are algebraically independent over $\mathbb{Q}$; see [Elsner and Tachiya 2017, Theorem 1.2]. In the case where n_1 is odd, we can deduce the same inequality as above by applying Lemma 8 with the same quantities u, v, τ_0 as above.

Therefore, at least two of the numbers $e^{\pi i\tau}$, $\vartheta_3(m\tau)$, and $\vartheta_3(n\tau)$ are algebraically independent over $\mathbb{Q}$, and the proof of Theorem 1 is complete. □

In the case where $m > n$ with two odd integers m, n, we obtain a stronger result based on [Elsner and Tachiya 2017, Theorem 1.2] and on Lemma 8.

Theorem 9. *Let $m > n \geq 1$ be odd integers, $j \in \{2, 3, 4\}$ and $\tau \in \mathbb{H}$. Then we have*

$$\text{trans.}\deg_{\mathbb{Q}} \mathbb{Q}(e^{i\pi\tau}, \vartheta_3(m\tau), \vartheta_3(n\tau), D\vartheta_j(\tau)) \geq 3.$$

Proof. We apply Lemma 8 with $u = m$ and $v = n$. Therefore, we know that $\vartheta_3(\tau)$ is algebraic over the field $\mathbb{Q}(\vartheta_3(m\tau), \vartheta_3(n\tau))$. Hence we obtain with Theorem C,

$$\begin{aligned}\text{trans.}\deg_{\mathbb{Q}} \mathbb{Q}(e^{i\pi\tau}, \vartheta_3(m\tau), \vartheta_3(n\tau), D\vartheta_j(\tau)) &= \text{trans.}\deg_{\mathbb{Q}} \mathbb{Q}(e^{\pi i\tau}, \vartheta_3(\tau), \vartheta_3(m\tau), \vartheta_3(n\tau), D\vartheta_j(\tau))\\ &\geq \text{trans.}\deg_{\mathbb{Q}} \mathbb{Q}(e^{i\pi\tau}, \vartheta_3(\tau), \vartheta_3(m\tau), D\vartheta_j(\tau))\\ &\geq 3,\end{aligned}$$

as desired. This proves the theorem. □

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