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Research Article

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A construction of residues of Eisenstein series and related square-integrable classes in the cohomology of arithmetic groups of low *k*-rank

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Abstract: The cohomology of an arithmetic congruence subgroup of a connected reductive algebraic group defined over a number field is captured in the automorphic cohomology of that group. The residual Eisenstein cohomology is by definition the part of the automorphic cohomology represented by square-integrable residues of Eisenstein series. The existence of residual Eisenstein cohomology classes depends on a subtle combination of geometric conditions (coming from cohomological reasons) and arithmetic conditions in terms of analytic properties of automorphic L-functions (coming from the study of poles of Eisenstein series). Hence, there are almost no unconditional results in the literature regarding the very existence of non-trivial residual Eisenstein cohomology classes. In this paper, we show the existence of certain non-trivial residual cohomology classes in the case of the split symplectic, and odd and even special orthogonal groups of rank two, as well as the exceptional group of type G_2, defined over a totally real number field. The construction of cuspidal automorphic representations of GL_2 with prescribed local and global properties is decisive in this context.

Keywords: Eisenstein cohomology, square-integrable cohomology classes, construction of non-trivial classes, automorphic forms, Eisenstein series, automorphic L-functions, split groups of rank two

MSC 2010: Primary 11F75; secondary 11F67, 11F70, 22E40, 22E55

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1 Introduction

1.1 Prelude – The residual spectrum through a cohomological lens

Let G be a connected algebraic group defined over an algebraic number field k. For simplicity of exposition, we assume in this subsection that G is semisimple. Let G_∞ be the group of real points of the algebraic $\mathbb{Q}$-group $\mathrm{Res}_{k/\mathbb{Q}}\, G$ obtained from G by restriction of scalars. Let $\Gamma \subset G(k)$ be a torsion-free arithmetic subgroup of G, viewed as a discrete subgroup of the real Lie group G_∞ via the diagonal embedding.[1]

1 The group G_∞ is isomorphic to the product of the Lie groups $G_v = G^{\iota_v}(k_v)$, where, given an archimedean place $v \in V_\infty$ of k, $\iota_v : k \to k_v$ denotes the corresponding embedding of k into the completion with respect to v.

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Let $L^2(\Gamma\backslash G_\infty)$ be the space of square-integrable functions (modulo the center) on $\Gamma\backslash G_\infty$, viewed as usual as a unitary G_∞-module via right-translations. The theory of Eisenstein series plays a fundamental role in the description of the spectral decomposition of $L^2(\Gamma\backslash G_\infty)$. This space is the direct sum of the discrete spectrum $L^2_{\mathrm{dis}}(\Gamma\backslash G_\infty)$, i.e., the span of the irreducible closed G_∞-submodules of $L^2(\Gamma\backslash G_\infty)$, and the continuous spectrum $L^2_{\mathrm{ct}}(\Gamma\backslash G_\infty)$. The former space contains as a G_∞-invariant subspace the space $L^2_{\mathrm{cusp}}(\Gamma\backslash G_\infty)$ of cuspidal automorphic forms, the so-called cuspidal spectrum. The orthogonal complement in $L^2_{\mathrm{dis}}(\Gamma\backslash G_\infty)$ is the residual spectrum, to be denoted $L^2_{\mathrm{res}}(\Gamma\backslash G_\infty)$, thus, there is a direct sum decomposition

$$L^2_{\mathrm{dis}}(\Gamma\backslash G_\infty) = L^2_{\mathrm{cusp}}(\Gamma\backslash G_\infty) \oplus L^2_{\mathrm{res}}(\Gamma\backslash G_\infty).$$

The discrete spectrum is a countable Hilbert direct sum of irreducible G_∞-modules with finite multiplicities. By the work of Langlands each of the constituents of the residual spectrum $L^2_{\mathrm{res}}(\Gamma\backslash G_\infty)$ can be structurally described in terms of residues of Eisenstein series attached to irreducible representations occurring in the discrete spectra of the Levi components of proper parabolic k-subgroups of G.

Given a rational finite-dimensional representation (η, E) of G_∞, our object of concern is the cohomology of Γ with values in E, to be given in terms of relative Lie algebra cohomology as

$$H^*(\Gamma, E) = H^*(\mathfrak{g}_\infty, K_\infty; C^\infty(\Gamma\backslash G_\infty) \otimes_{\mathbb{C}} E),$$

where $C^\infty(\Gamma\backslash G_\infty)$ denotes the space of C^∞-functions on $\Gamma\backslash G_\infty$.[2] This cohomology space contains as a natural subspace the so-called square integrable cohomology $H^*_{(\mathrm{sq})}(\Gamma, E)$ to be defined as the image of the homomorphism

$$j_{\mathrm{dis}} : H^*(\mathfrak{g}_\infty, K_\infty; L^{2,\infty}_{\mathrm{dis}}(\Gamma\backslash G_\infty) \otimes E) \to H^*(\mathfrak{g}_\infty, K_\infty; C^\infty(\Gamma\backslash G_\infty) \otimes E)$$

induced in cohomology by the natural inclusion of the space of C^∞-vectors in the discrete spectrum of $\Gamma\backslash G_\infty$ into $C^\infty(\Gamma\backslash G_\infty)$. In general, the homomorphism j_{dis} is not injective whereas the homomorphism induced by the inclusion of the space of C^∞-vectors in the cuspidal spectrum into $C^\infty(\Gamma\backslash G_\infty)$ is injective [see Section 3]; its image is called the cuspidal cohomology of Γ.

We are interested in the contribution of the residual spectrum of $\Gamma\backslash G_\infty$ to the cohomology groups $H^*_{(\mathrm{sq})}(\Gamma, E) \subset H^*(\Gamma, E)$, that is, we aim at

- constructing non-trivial elements in $L^2_{\mathrm{res}}(\Gamma\backslash G_\infty)$ via residues of Eisenstein series and, by using these,
- constructing non-trivial cohomology classes in $H^*(\mathfrak{g}_\infty, K_\infty, L^{2,\infty}_{\mathrm{res}}(\Gamma\backslash G_\infty) \otimes E)$, and finally
- showing that the classes so constructed are carried over to non-trivial classes under the map

$$j_{\mathrm{res}} : H^*(\mathfrak{g}_\infty, K_\infty; L^{2,\infty}_{\mathrm{res}}(\Gamma\backslash G_\infty) \otimes E) \to H^*(\mathfrak{g}_\infty, K_\infty; C^\infty(\Gamma\backslash G_\infty) \otimes E)$$

whose image is contained in the square integrable cohomology $H^*_{(\mathrm{sq})}(\Gamma, E)$.

The general specification of the residual spectrum via residues of Eisenstein series, which in turn depends in a recursive way on the description of the cuspidal spectra of groups of lower rank, is dealt with in [43] and in the context of adele groups in [48]. Besides the work [47] on the detailed description of the residual spectrum for GL_n, there are only complete results for the groups G_2 by Žampera [71] resp. Kim [34], the symplectic group Sp_2 by Kim [33] resp. Konno [37] and the special orthogonal group SO_5 by Kim [35]. However, these latter results account which residues of Eisenstein series can possibly occur in the residual spectrum, depending essentially on the analytic properties of certain Euler products attached to the cuspidal automorphic forms which are used to exhibit the Eisenstein series in question. Thus, our main focus is on the explicit construction of residues of suitable Eisenstein series subject to the condition that these residues give rise to non-trivial classes in $H^*(\mathfrak{g}_\infty, K_\infty; L^{2,\infty}_{\mathrm{res}}(\Gamma\backslash G_\infty) \otimes E)$. The quest for residues of Eisenstein series which are cohomologically relevant puts additional constraints on the cuspidal data used for the Eisenstein series involved. Our main results concern constructions of such non-trivial classes in the cases of the split k-groups Sp_2/k, SO_5/k,

2 For a differentiable G_∞-module F we usually put $H^*(\mathfrak{g}_\infty, K_\infty, F) = H^*(\mathfrak{g}_\infty, K_\infty, F_{K_\infty})$, where F_{K_∞} denotes the space of all K_∞-finite vectors in F, K_∞ a maximal compact subgroup in G_∞.

and SO_4/k of k-rank two and the exceptional group G_2/k, k a totally real number field. These results rely essentially on various explicit constructions of cuspidal automorphic representations of GL_2/k with prescribed local and global properties, the latter ones expressed in terms of a specific automorphic L-function.

This work has to be carried through in the framework of adele groups. Thus, in the next subsection, we set up the framework and describe our results in a more precise way.

1.2 Adele groups and automorphic cohomology

The cohomology of an arithmetic subgroup Γ of a connected reductive algebraic group G defined over an algebraic number field k can be interpreted in terms of the automorphic spectrum of Γ. There is a sum decomposition of the cohomology into the cuspidal cohomology (i.e. classes represented by cuspidal automorphic forms) and the so-called Eisenstein cohomology constructed as the span of appropriate residues or derivatives of Eisenstein series. These are attached to cuspidal automorphic forms π on the Levi components of proper parabolic k-subgroups of G. Taking into account the cuspidal support of each of these Eisenstein series results in an even finer decomposition of the Eisenstein cohomology. More precisely [16, Theorem 1.4 resp. 2.3], the automorphic cohomology $H^*(G,E)$ with a coefficient system originating in an irreducible finite-dimensional algebraic representation of G has a direct sum decomposition[3]

$$H^*(G,E) = \bigoplus_{\{P\}\in\mathcal{C}} \bigoplus_{\phi\in\Phi_{E,\{P\}}} H^*(\mathfrak{m}_G, K_\infty; \mathcal{A}_{E,\{P\},\phi} \otimes_{\mathbb{C}} E),$$

where $\mathcal{C}$ denotes the set of classes of associate parabolic k-subgroups of G, and the second sum ranges over the set $\Phi_{E,\{P\}}$ of classes of associate irreducible cuspidal automorphic representations of the Levi components of elements of $\{P\}$. The summand that is indexed by the full group $\{G\}$ accounts for the *cuspidal cohomology of G with coefficients in E*, to be denoted $H^*_{\mathrm{cusp}}(G,E)$. The *Eisenstein cohomology* $H^*_{\mathrm{Eis}}(G,E)$ ranging over the summands indexed by $\{P\}\in\mathcal{C}$, $\{P\}\neq\{G\}$ exhibits a natural complement to the cuspidal cohomology.

The square integrable cohomology $H^*_{(\mathrm{sq})}(G,E)$ is a natural subspace of $H^*(G,E)$. Since cuspidal automorphic forms are all square-integrable, we have for the summand indexed by $\{P\}=\{G\}$ that

$$H^*_{\mathrm{cusp}}(G,E) = H^*_{(\mathrm{sq})}(\mathfrak{m}_G, K_\infty; \mathcal{A}_{E,\{G\}} \otimes E).$$

The remaining part of the square-integrable cohomology is inside the Eisenstein cohomology, and we may write

$$H^*_{\mathrm{Eis},(\mathrm{sq})}(G,E) = \bigoplus_{\substack{\{P\}\in\mathcal{C}\\ \{P\}\neq\{G\}}} \bigoplus_{\phi\in\Phi_{E,\{P\}}} H^*_{(\mathrm{sq})}(\mathfrak{m}_G, K_\infty; \mathcal{A}_{E,\{P\},\phi} \otimes E)$$

with $\mathcal{L}_{E,\{P\},\phi}$ just being the (possibly trivial) subspace of square-integrable forms in $\mathcal{A}_{E,\{P\},\phi}$, the summands on the right-hand side are the images of the map induced in cohomology by the inclusions $\mathcal{L}_{E,\{P\},\phi} \hookrightarrow \mathcal{A}_{E,\{P\},\phi}$.

1.3 Residual Eisenstein cohomology classes

Our goal in this paper is, in the case of a given group G of low k-rank, to carry through a construction of non-trivial cohomology classes in $H^*_{(\mathrm{sq})}(G,E)$ which are represented by residues of Eisenstein series whose cuspidal support is a class of maximal proper parabolic k-subgroups $\{P\}$ of G. Thereby, we exhibit explicit examples of non-trivial classes in some of the summands $H^*_{(\mathrm{sq})}(\mathfrak{m}_G, K_\infty; \mathcal{A}_{E,\{P\},\phi} \otimes E)$ in $H^*_{(\mathrm{sq})}(G,E)$. Given a totally real algebraic number field k, the results obtained concern the split classical k-groups Sp_2/k, SO_5/k, and SO_4/k of k-rank two and the exceptional group G_2/k.

As a result of previous work [24, 25, 56], given a class $\{P\}$ of associate maximal parabolic k-subgroups of G, one can describe in detail which types (in the sense of [53]) of Eisenstein cohomology classes occur

3 We refer to Section 3 for details and unexplained notation.

in $H^*(\mathfrak{m}_G, K_\infty; \mathcal{A}_{E,\{P\},\phi} \otimes_{\mathbb{C}} E)$ and how their actual construction is related to the analytic properties of certain Euler products (or automorphic L-functions) attached to the cuspidal automorphic representations π one starts with. Furthermore, one can determine in which way residues of the Eisenstein series in question may possibly give rise to non-trivial classes in the cohomology of Γ. The very existence of these residual Eisenstein cohomology classes is subject to a quite restrictive set of conditions, a subtle combination of arithmetic and geometric conditions. The former assure that the Eisenstein series in question has a pole, and the latter are the necessary conditions for the cohomology class so obtained to be non-vanishing. We refer, for example, to the results concerning the symplectic group of k-rank n in [24]. In particular, a non-vanishing condition on the central value of a certain Euler product attached to π plays an important role in this discussion. These L-functions naturally appear in the constant terms of the Eisenstein series under consideration.

In view of this general situation there are only very few scattered results concerning the actual existence of residual Eisenstein cohomology classes, e.g., [15, 29, 49], the other ones are all conditional, subject to conditions on the (non)-vanishing of an automorphic L-function or the existence of a residue of an Eisenstein series, see [22, 24, 27, 51, 54]. The only case in which the existence of non-trivial residues of Eisenstein series is unconditional, is the case of the general linear group [47] and its inner forms [2, 3]. In the case of GL_n, the actual existence of non-trivial residual cohomology classes supported in a maximal parabolic subgroup was treated in [16], see also [28]. In the case of the inner form $\mathrm{GL}_2(D)$, D a quaternion division algebra, of GL_4/k, the existence of residual cohomology classes was studied in [23], and for $k = \mathbb{Q}$ in [26].

Our construction of non-trivial residual Eisenstein cohomology classes for the groups G/k, k a totally real number field, we deal with, relies on three different results regarding the actual existence of cuspidal automorphic representations $\pi = \bigotimes'_{v\in V} \pi_v$ of $\mathrm{GL}_2(\mathbb{A}_k)$ with, on one hand, very specific local components and, on the other hand, a prescribed analytic behavior of a specific automorphic L-functions attached to π. We refer to Section 6 for details.

Firstly, using a result of D. Trotabas [65] regarding the non-vanishing of L-functions attached to Hilbert modular forms at the central value we derive the following:

Proposition 1.1. *Given an irreducible finite-dimensional algebraic representation (η, E) of the real Lie group $G_\infty = \prod_{v\in V_\infty} G_v$ with $G_v \cong \mathrm{GL}_2(\mathbb{R})$ of even highest weight μ, there exists an irreducible cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_k)$ whose central character ω_π is trivial, whose archimedean components π_v in $\pi_\infty = \bigotimes_{v\in V_\infty} \pi_v$ are discrete series representations of $\mathrm{GL}_2(\mathbb{R})$ compatible with μ, and whose corresponding L-function $L(s, \pi, \rho_2)$ does not vanish at $s = \frac{1}{2}$. Such a representation π of $\mathrm{GL}_2(\mathbb{A}_k)$ contributes non-trivially to the cuspidal cohomology $H^*_{\mathrm{cusp}}(\mathrm{GL}_2, E)$ in degree $[k : \mathbb{Q}]$.*

The other two results regard the existence of specific monomial cuspidal representations of $\mathrm{GL}_2(\mathbb{A}_k)$ tailored by the needs of the actual construction of residual Eisenstein cohomology classes. Here is one of them (see Section 6):

Proposition 1.2. *Suppose that the highest weight μ of (η, E) is odd, then there exists an irreducible monomial cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_k)$ which is selfdual with a non-trivial central character ω_π and whose archimedean components π_v in $\pi_\infty = \bigotimes_{v\in V_\infty} \pi_v$ are discrete series representations of $\mathrm{GL}_2(\mathbb{R})$ compatible with μ. Such a representation π of $\mathrm{GL}_2(\mathbb{A}_k)$ contributes non-trivially to the cuspidal cohomology $H^*_{\mathrm{cusp}}(\mathrm{GL}_2, E)$ in degree $[k : \mathbb{Q}]$.*

As an example we now describe in the case $G = \mathrm{G}_2$ one result pertaining to the actual construction of non-trivial cohomology classes in $H^*_{(\mathrm{sq})}(G, E)$ which are represented by residues of Eisenstein series.

1.4 The case G_2

Given a totally real algebraic number field k of degree d, let G be the k-split algebraic k-group of type G_2; the k-rank of G is two. We fix a minimal parabolic k-subgroup P_0 with Levi decomposition $P_0 = L_0N_0$. Let Φ, Φ^+, Δ denote the corresponding sets of roots, positive roots, simple roots, respectively. We write $\Delta = \{\alpha_1, \alpha_2\}$, where α_1 resp. α_2 denotes the short resp. long root; the half-sum of the positive roots is $\rho_{P_0} = 5\alpha_1 + 3\alpha_2$.

For $r = 1, 2$, the maximal proper standard parabolic k-subgroup $P_{\Delta\setminus\{\alpha_r\}}$ corresponding to the subset $\Delta \setminus \{\alpha_r\}$ of Δ is denoted by P_r, and its Levi decomposition by $P_r = L_r N_r$, where L_r is the Levi subgroup containing L_0, and N_r the unipotent radical. In both cases we have $L_r \cong \mathrm{GL}_2$. Observe that the parabolic subgroups P_r are self-associate.

Given the irreducible finite-dimensional representation (η, E) of the group $G_\infty = \mathrm{Res}_{k/\mathbb{Q}}(\mathrm{G}_2)(\mathbb{R})$ in a complex vector space, its highest weight can be written as $\Lambda = (\Lambda)_{\iota_v}$, $v \in V_\infty$, where ι_v denotes the embedding $k \to \mathbb{R}$ which corresponds to an archimedean place $v \in V_\infty$ of k. For the sake of simplicity we assume that $\Lambda_{\iota_v} = \Lambda_{\iota_{v'}}$ for all archimedean places $v, v' \in V_\infty$. Recall that this representation originates from an algebraic representation of the algebraic k-group G. We write

$$\Lambda = c_1\Lambda_1 + c_2\Lambda_2,$$

with c_1, c_2 non-negative integers, where Λ_i, $i = 1, 2$, denote the fundamental dominant weights.

The following result concerns the square integrable cohomology $H^*_{(\mathrm{sq})}(\mathfrak{m}_{\mathrm{G}_2}, K_\infty; \mathcal{A}_{E,\{P_2\},\phi} \otimes E)$.

Theorem 1.3. *Suppose that the highest weight Λ of the representation (η, E) of G is of the form $\Lambda = c_1\Lambda_1$, that is, $c_2 = 0$. Then there exists a selfdual unitary cuspidal automorphic representation π of $L_2(\mathbb{A})$ such that the Eisenstein series $E(f, s)$ attached to π has a pole at $s = \frac{1}{2}\tilde{\rho}_{P_2}$ and the corresponding residue $\mathrm{Res}_{s=\frac{1}{2}} E(f, s)$ gives rise to a non-trivial class in $H^*(\mathfrak{m}_{\mathrm{G}_2}, K_\infty; \mathcal{L}_{E,\{P_2\},\phi} \otimes E)$, where ϕ is the associate class represented by $\pi \otimes e^{\langle \frac{1}{2}\tilde{\rho}_{P_2}, H_{P_2}(\cdot)\rangle}$.*

In degree $q = 3d$, the map in cohomology induced by the inclusion $\mathcal{L}_{E,\{P\},\phi} \hookrightarrow \mathcal{A}_{E,\{P\},\phi}$ is injective so that the residual Eisenstein cohomology space $H^*_{(\mathrm{sq})}(\mathfrak{m}_{\mathrm{G}_2}, K_\infty; \mathcal{A}_{E,\{P_2\},\phi} \otimes E)$ does not vanish.

Remark 1.4. By means of the global theta lifting related to the dual reductive pair (H_Q, SL_2), where H_Q denotes a suitable orthogonal group containing G_2 as a subgroup, one finds in [45] a construction of cuspidal automorphic representations which give rise to non-vanishing cohomology classes in $H^*_{\mathrm{cusp}}(G, E)$. The archimedean components of these representations are non-tempered and correspond to the irreducible unitary representations $A_{\mathfrak{q}_1}(\chi_1)$ for a suitable character χ_1. The classes so obtained are shadows of the residual cohomology classes constructed above.

Notation and conventions

Let k be an algebraic number field, i.e., an arbitrary finite extension $k/\mathbb{Q}$ of the field $\mathbb{Q}$ of rational numbers, and let $\mathcal{O}_k$ denote its ring of integers. The set of places of k will be denoted by V_k, and $V_{\infty,k}$ (resp. $V_{f,k}$) refers to the subsets of archimedean (resp. non-archimedean) places of k. Given a place $v \in V_k$, the completion of k with respect to v is denoted k_v. For a finite place $v \in V_{f,k}$ we write $\mathcal{O}_{k,v}$ for the valuation ring in k_v. If the field k is fixed, we write $V = V_k$, etc.

We denote by $\mathbb{A} = \mathbb{A}_k$ (resp. $\mathbb{I} = \mathbb{I}_k$) the ring of adeles (resp. the group of ideles) of k. There is the usual decomposition of $\mathbb{A}$ (resp. $\mathbb{I}$) into the archimedean and the finite part $\mathbb{A} = \mathbb{A}_\infty \times \mathbb{A}_f$ (resp. $\mathbb{I} = \mathbb{I}_\infty \times \mathbb{I}_f$).

2 Preliminaries

2.1 The group G

Let G be a connected reductive linear algebraic group over an algebraic number field k. Fix a minimal parabolic subgroup P_0 of G defined over k and a Levi subgroup L_0 of P_0 defined over k. One has the Levi decomposition $P_0 = L_0N_0$ with unipotent radical N_0. By definition, a standard parabolic k-subgroup P of G is a parabolic subgroup P of G defined over k that contains P_0. Analogously, a standard Levi subgroup L of G is a Levi subgroup of any standard parabolic k-subgroup P of G such that L contains L_0. A given standard parabolic k-subgroup P of G has a unique standard Levi subgroup L. We denote by $P = LN$ the corresponding Levi decomposition of P over k.

By definition, the adele group $G(\mathbb{A})$ of the group G is the restricted product $G(\mathbb{A}) = \prod'_{v\in V} G(k_v)$ with respect to the maximal compact subgroups $G(\mathcal{O}_{k,v}) \subset G(k_v)$, for almost all $v \in V_f$. Let G_∞ denote the group $R_{k/\mathbb{Q}}(G)(\mathbb{R})$ of real points of the algebraic $\mathbb{Q}$-group $R_{k/\mathbb{Q}}(G)$ obtained from the k-group G by the restriction of scalars from k to $\mathbb{Q}$. Then the locally compact group $G(\mathbb{A})$ is the direct product of the group G_∞ and the restricted product $\prod'_{v\in V_f} G(k_v) =: G(\mathbb{A}_f)$. We fix a maximal compact subgroup K of $G(\mathbb{A})$ subject to the following condition. Since it is of the form $K = \prod_{v\in V} K_v$, where K_v is a maximal compact subgroup of $G(k_v)$, $v \in V$, we suppose (as we may) that $K_v = G(\mathcal{O}_{k,v})$ for almost all finite places $v \in V_f$. We write $K_\infty = \prod_{v\in V_\infty} K_v$ and we write $K_f = \prod_{v\in V_f} K_v$.

We may assume that the group K is in good position relative to P_0, that is, K satisfies the following requirements:

- $G(\mathbb{A}) = P_0(\mathbb{A})K$,
- given a standard parabolic k-subgroup $P = LN$ of G one has the decomposition

$$P(\mathbb{A}) \cap K = (L(\mathbb{A}) \cap K)(N(\mathbb{A}) \cap K),$$

  and $L(\mathbb{A}) \cap K$ is a maximal compact subgroup of $L(\mathbb{A})$.

2.2 Parabolics, Levi subgroups and characters

Let P be a standard parabolic k-subgroup of G. Fix the Levi decomposition $P = LN$, where L is the unique standard Levi subgroup of P. We denote by $X^*(L)$ the group of k-rational characters of L. Since L is a connected group, $X^*(L)$ is a free $\mathbb{Z}$-module of finite rank r. We put $\check{\mathfrak{a}}_{P,\mathbb{C}} = X^*(L) \otimes_{\mathbb{Z}} \mathbb{C}$. Analogously we put $\mathfrak{a}_{P,\mathbb{C}} = X_*(A_L) \otimes_{\mathbb{Z}} \mathbb{C}$, where $X_*(A_L)$ denotes the group of k-rational cocharacters of the maximal k-split torus A_L in the center of L. The complex vector spaces $\mathfrak{a}_{P,\mathbb{C}}$ and $\check{\mathfrak{a}}_{P,\mathbb{C}}$ are in a natural way in duality with one another. These spaces come equipped with a $\mathbb{Q}$-structure, given by $\check{\mathfrak{a}}_{P,\mathbb{Q}} = X^*(L) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathfrak{a}_{P,\mathbb{Q}} = X_*(A_L) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then one has $\check{\mathfrak{a}}_{P,\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} = \check{\mathfrak{a}}_{P,\mathbb{C}}$ resp. $\mathfrak{a}_{P,\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} = \mathfrak{a}_{P,\mathbb{C}}$. We also have to consider the real spaces $\check{\mathfrak{a}}_{P,\mathbb{R}} = X^*(L) \otimes \mathbb{R}$ resp. $\mathfrak{a}_{P,\mathbb{R}} = X_*(A_L) \otimes \mathbb{R}$.

Given a place $v \in V$, a k-rational character $\chi \in X^*(L)$ defines an algebraic character $\chi_v : L(k_v) \to k_v^*$. Then the assignment $y = (y_v) \mapsto \prod_v |\chi_v(y_v)|_v$ defines a continuous homomorphism $L(\mathbb{A}) \to \mathbb{C}^*$, to be denoted $|\chi|$. The group

$$L^1 = \bigcap_{\chi\in X^*(L)} \ker |\chi|$$

is a normal subgroup of $L(\mathbb{A})$. We denote the group of continuous homomorphisms of $L(\mathbb{A})$ into $\mathbb{C}^*$ which are trivial on L^1 by X_P. Let X_P^G be the subgroup of X_P which consists of those continuous homomorphisms of $L(\mathbb{A})/L^1$ into $\mathbb{C}^*$ which are trivial on the center Z_G of G. This group plays a decisive role [as parameter space] in the final construction of Eisenstein series.

The group X_P can also be described in the following way: Given an element $\lambda \in X_P$, there exist characters $\chi_1, \dots, \chi_r \in X^*(L)$ and complex numbers $s_1, \dots, s_r \in \mathbb{C}$ such that for all $l \in L(\mathbb{A})$, $\lambda(l) = |\chi_1|(l)^{s_1} \cdots |\chi_r|(l)^{s_r}$. This result gives rise to an isomorphism of groups

$$\kappa : \check{\mathfrak{a}}_{P,\mathbb{C}} \to X_P.$$

As in [48, p. 7] we put $\mathrm{Re} X_P := \kappa(\check{\mathfrak{a}}_{P,\mathbb{R}})$. This can be seen to be the group of continuous homomorphisms of $L(\mathbb{A})/L^1$ into $\mathbb{C}^*$ with values in $(\mathbb{R}^*)^+$.

Given the minimal parabolic k-subgroup P_0 of G with Levi subgroup L_0 defined over k, let T_0 be the maximal split torus in the center of L_0. We denote by $\Phi(G, T_0)$ the set of k-roots of G with respect to T_0. Given a root $\alpha \in \Phi(G, T_0)$, there is a corresponding coroot, denoted $\check{\alpha}$, which is a one-parameter subgroup of T_0. Note that the choice of the minimal parabolic subgroup P_0 determines in $\Phi(G, T_0)$ a set of positive roots, to be denoted $\Phi^+(G, T_0)$. We denote by Δ_0 the set of simple roots in $\Phi(G, T_0)$.

Let $\mathrm{res}_{L/T} : X^*(L_0) \to X^*(T_0)$ be the natural restriction map from L_0 to T_0. Then, using the natural duality $\langle \cdot , \cdot \rangle$ with values in $\mathbb{Z}$ between rational characters of a split torus and one-parameter subgroups, we can define for every $\chi \in X^*(L_0)$ and every coroot $\check{\alpha}$ the pairing $\langle \chi, \check{\alpha} \rangle := \langle \mathrm{res}_{L/T}\, \chi, \check{\alpha} \rangle$. By $\mathbb{R}$-linear extension this pairing is also defined for all $\lambda \in \operatorname{Re} X_{P_0} \cong \check{\mathfrak{a}}_{P_0,\mathbb{R}}$. In the same way, by $\mathbb{C}$-linear extension, $\langle \lambda, \check{\alpha} \rangle$ is defined for all $\lambda \in \check{\mathfrak{a}}_{P_0,\mathbb{C}} \cong X_{L_0}$ and all coroots $\check{\alpha}$.

Using the isomorphisms $\check{\mathfrak{a}}_{P_0,\mathbb{R}} \cong X^*(L_0) \otimes_{\mathbb{Z}} \mathbb{R} \cong X^*(T_0) \otimes_{\mathbb{Z}} \mathbb{R}$, we may (and will) interpret roots as elements of $\check{\mathfrak{a}}_{P_0,\mathbb{R}} \cong \operatorname{Re} X_{P_0}$. In fact, roots are already contained in the underlying $\mathbb{Q}$-structure $X^*(L_0) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Now we compare in this context the standard parabolic k-subgroup $P = LN$ as above with $P_0 = L_0 N_0$. The intersection $P_0 \cap L$ is a minimal parabolic subgroup of L. We denote by $\Phi(L, T_0)$ the set of roots of L with respect to T_0, and we define $\Delta_0^L := \Delta_0 \cap \Phi(L, T_0)$.

The maximal split torus T_L in the center of L is contained in T_0. The set $\Phi(G, T_L)$ of “roots” of G with respect to T_L is in general not a root system. However, we can identify this set with a subset of $\check{\mathfrak{a}}_{P,\mathbb{R}} \cong \operatorname{Re} X_P$. Moreover, this set generates this latter space. Now consider the restriction map

$$\Phi(G, T_0) \to \Phi(G, T_L) \cup \{0\};$$

it is trivial on $\Phi(L, T_0)$. The set of non-trivial restrictions of elements in Δ_0 under this map is denoted by Δ_L. Observe that Δ_L generates $\check{\mathfrak{a}}_{P,\mathbb{R}}$ as well.

Via the restriction from L to L_0, we identify $\check{\mathfrak{a}}_{P,\mathbb{R}}$ with a subspace of the vectorspace $\check{\mathfrak{a}}_{P_0,\mathbb{R}}$. If $\check{\mathfrak{a}}^P_{P_0,\mathbb{R}}$ denotes the subspace of $\check{\mathfrak{a}}_{P_0,\mathbb{R}}$ which is generated by $\Phi(L, T_0)$, then we have a direct sum decomposition

$$\check{\mathfrak{a}}_{P_0,\mathbb{R}} = \check{\mathfrak{a}}_{P,\mathbb{R}} \oplus \check{\mathfrak{a}}^P_{P_0,\mathbb{R}}. \tag{2.1}$$

In view of the identification $\check{\mathfrak{a}}_{P_0,\mathbb{R}} \cong \operatorname{Re} X_{P_0}$, we identify the elements of $\check{\mathfrak{a}}^P_{P_0,\mathbb{R}}$ with the set of those elements in $\operatorname{Re} X_{P_0}$ which are trivial on the center of $L(\mathbb{A})$. This latter set is denoted by $\operatorname{Re} X^P_{P_0}$.

Given a pair $P \subset P'$ of standard parabolic k-subgroups of G, there is a generalization of the decomposition (2.1). Fix the Levi decompositions $P = LN$ resp. $P' = L'N'$, where L resp. L' is the unique standard Levi subgroup of P resp. P'. We define $\Phi(L', T_L)$ as the set of “roots” of L' with respect to T_L; this is a subset of $\Phi(G, T_L)$. Then we have as above the direct sum decomposition

$$\check{\mathfrak{a}}_{P,\mathbb{R}} = \check{\mathfrak{a}}_{P',\mathbb{R}} \oplus \check{\mathfrak{a}}^{P'}_{P,\mathbb{R}},$$

where $\check{\mathfrak{a}}^{P'}_{P,\mathbb{R}}$ is the real subspace generated by $\Phi(L', T_L)$. We may identify the elements in the space $\check{\mathfrak{a}}^{P'}_{P,\mathbb{R}}$ with the elements in $\operatorname{Re} X_P$ which are trivial on the center of $L'(\mathbb{A})$. The set of these elements is denoted by $\operatorname{Re} X^{P'}_P$.

For a given standard parabolic subgroup $P = LN$ of G one has the Iwasawa decomposition

$$G(\mathbb{A}) = L(\mathbb{A})N(\mathbb{A})K.$$

Then we can define the standard height function $H_P : G(\mathbb{A}) \to \mathfrak{a}_{P,\mathbb{R}}$ on $G(\mathbb{A})$ by $|\chi|(l) = e^{\langle \chi, H_P(lnk) \rangle}$ for any character $\chi \in X^*(L) \subset \check{\mathfrak{a}}_{P,\mathbb{R}}$, where $g = lnk$, $l \in L(\mathbb{A})$, $n \in N(\mathbb{A})$, $k \in K$, is the Iwasawa decomposition of $g \in G(\mathbb{A})$. The definition does not depend on the choice of the Iwasawa decomposition.

2.3 Weyl group

Given the minimal parabolic k-subgroup P_0 of the connected reductive k-group G with Levi subgroup L_0 defined over k, let T_0 be the maximal split torus in the center of L_0. The Weyl group of G is defined to be

$$W := N_{G(k)}(T_0(k))/Z_{G(k)}(T_0(k)).$$

The simple reflection in W which corresponds to a simple root α is denoted by w_α. Given $w \in W$, the length $\ell(w)$ of w is defined to be the smallest number s such that w can be written as a product of s simple refections.

Let P be a standard parabolic k-subgroup of G. Fix the Levi decomposition $P = LN$, where L is the unique standard Levi subgroup of P. Let W_P be the Weyl group of L. There exists in any right coset of W_P in W a unique element w of smallest length in the coset $W_P w$, cf. [6, 3.9]. Thus, the projection $W \to W_P \setminus W$ has a canonical splitting. Let W^P be its image, to be called the set of minimal coset representatives.

2.4 Notation

We denote by M_G the connected component of the intersection of the kernels of all k-rational characters of G, and by $\mathfrak{m}_G$ the Lie algebra of the Lie group $R_{k/\mathbb{Q}}(M_G)(\mathbb{R})$. Note that the maximal k-split torus A_G in the center of G reduces to the identity if G is a semisimple group. In such a case, $\mathfrak{m}_G = \mathrm{Lie}(G_\infty)$. Given a k-parabolic subgroup P, we denote by A_P the maximal k-split torus in the center of the Levi subgroup L_P. We write $A_{P,\infty}$ for the group of real points $R_{k/\mathbb{Q}}(A_P)(\mathbb{R})$ and $A^0_{P,\infty}$ for its connected component of the identity.

3 Automorphic cohomology

3.1 The cohomology group

Let A_G denote the maximal k-split torus in the center Z_G of G. We write $A_{G,\infty}$ for the group of real points $R_{k/\mathbb{Q}}(A_G)(\mathbb{R})$. Let (η, E) be an irreducible finite-dimensional algebraic representation of G_∞ in a complex vector space E. We assume that this representation originates from an algebraic representation of the algebraic k-group G. We suppose that $A^0_{G,\infty}$ acts by a character on E, to be denoted by χ^{-1}. Let $J_E \subset Z(\mathfrak{g}_{\infty,\mathbb{C}})$ be the annihilator of the dual representation of E in the center of the universal enveloping algebra $U(\mathfrak{g}_{\infty,\mathbb{C}})$ of the complexified Lie algebra of G_∞.

We denote by $V_G = C^\infty_{\mathrm{umg}}(G(k)\backslash G(\mathbb{A}))$ the space of smooth complex-valued functions f of uniform moderate growth on $G(k)\backslash G(\mathbb{A})$, that is, $f \in V_G$ is K-finite, and f resp. its derivatives have uniformly moderate growth (cf. [48, I.2.3]). Let $\mathcal{A}_E \subset V_G$ be the subspace of functions $f \in V_G$ which are annihilated by a power of J_E. The space $\mathcal{A}_E \otimes_\mathbb{C} E$ is naturally equipped with a $(\mathfrak{m}_G, K_\infty; G(\mathbb{A}_f))$-module structure. We define the automorphic cohomology of G with coefficients in E by

$$H^*(G, E) := H^*(\mathfrak{m}_G, K_\infty; \mathcal{A}_E \otimes_\mathbb{C} E).$$

We keep in mind that these cohomology groups have an interpretation as the inductive limit of the de Rham cohomology groups $H^*(X_C, E)$ of the orbit space $X_C := G(k) \setminus G(\mathbb{A})/K_\infty C$ with coefficients in the local system given by the representation (η, E), where C ranges over the open compact subgroups of $G(\mathbb{A}_f)$.

Two parabolic k-subgroups P and P' of G are said to be associate if their Levi subgroups are conjugate by an element in $G(k)$. This notion induces an equivalence relation on the set $\mathcal{P}(G)$ of parabolic k-subgroups of G. Given $P \in \mathcal{P}(G)$, we denote its equivalence class by $\{P\}$, to be called the associate class of P. Let $\mathcal{C}$ be the set of classes of associate parabolic k-subgroups of G.

Given a class $\{P\} \in \mathcal{C}$, we denote by $\mathcal{A}_{E,\{P\}}$ the subspace of elements in $\mathcal{A}_E$ which are negligible along Q for every parabolic k-subgroup Q in G, $Q \notin \{P\}$. The spaces $\mathcal{A}_{E,\{P\}}$, $\{P\} \in \mathcal{C}$, form a direct sum, and one has a decomposition $\mathcal{A}_E = \bigoplus_{\{P\}\in\mathcal{C}} \mathcal{A}_{E,\{P\}}$ as a direct sum of $(\mathfrak{m}_G, K_\infty; G(\mathbb{A}_f))$-modules. This was first proved in [42], see [5, Theorem 2.4], for a variant of the original proof. This direct sum decomposition induces a direct sum decomposition

$$H^*(G, E) = \bigoplus_{\{P\}\in\mathcal{C}} H^*(\mathfrak{m}_G, K_\infty; \mathcal{A}_{E,\{P\}} \otimes E)$$

in cohomology. The summand in this decomposition of the cohomology $H^*(G, E)$ that is indexed by the full group $\{G\}$ will be called the *cuspidal cohomology of G with coefficients in E*, to be denoted $H^*_{\mathrm{cusp}}(G, E)$. We call the direct sum over the classes $\{P\} \in \mathcal{C}$, $\{P\} \neq \{G\}$, the *Eisenstein cohomology of G with coefficients in E*, denoted $H^*_{\mathrm{Eis}}(G, E)$.

3.2 Decomposition along the cuspidal support

Given a class $\{P\} \in \mathcal{C}$, let $\phi = \{\phi_Q\}_{Q\in\{P\}}$ be a class of associate irreducible cuspidal automorphic representations of the Levi subgroups of elements of $\{P\}$ as defined in [16, Section 1.2]. Observe that the elements of the associate class are not necessarily unitary. The set of all such collections $\phi = \{\phi_Q\}_{Q\in\{P\}}$ compatible with E is denoted by $\Phi_{E,\{P\}}$. Given any $\phi \in \Phi_{E,\{P\}}$, we let

$$\mathcal{A}_{E,\{P\},\phi} = \Big\{ f \in \mathcal{A}_{E,\{P\}} : f_Q \in \bigoplus_{\pi\in\phi_Q} L^2_{\mathrm{cusp},\pi}(L_Q(k) \setminus L_Q(\mathbb{A}))_{\chi_\pi} \otimes S(\check{\mathfrak{a}}_Q^G) \Big\}$$

be the space of functions of uniform moderate growth whose constant term along each $Q \in \{P\}$ belongs to the isotypic components attached to the elements $\pi \in \phi_Q$. Finally, we have the following:

Theorem 3.1. *The automorphic cohomology $H^*(G,E)$ has a direct sum decomposition*

$$H^*(G,E) = \bigoplus_{\{P\}\in\mathcal{C}} \bigoplus_{\phi\in\Phi_{E,\{P\}}} H^*(\mathfrak{m}_G, K_\infty; \mathcal{A}_{E,\{P\},\phi} \otimes_{\mathbb{C}} E),$$

where, given $\{P\} \in \mathcal{C}$, the second sum ranges over the set $\Phi_{E,\{P\}}$ of classes of associate irreducible cuspidal automorphic representations of the Levi components of elements of $\{P\}$.

For a proof of this result we refer to [16, Theorem 1.4 resp. 2.3], or [48, Theorem in III, 2.6], where a different approach to the decomposition of the space of automorphic forms along the cuspidal support is given.

3.3 Square-integrable cohomology

Let $\mathcal{L}_E$ be the subspace consisting of all square-integrable automorphic forms in $\mathcal{A}_E$. Note that it is an $(\mathfrak{m}_G, K_\infty; G(\mathbb{A}_f))$-submodule and the inclusion $\mathcal{L}_E \hookrightarrow \mathcal{A}_E$ gives rise to a map

$$H^*(\mathfrak{m}_G, K_\infty; \mathcal{L}_E \otimes E) \to H^*(\mathfrak{m}_G, K_\infty; \mathcal{A}_E \otimes E)$$

in cohomology. It is the image of this map that we call the *square-integrable (automorphic) cohomology* and denote by $H^*_{(\mathrm{sq})}(G,E)$. According to the decomposition of $\mathcal{A}_E$ over associate classes of parabolic k-subgroups and along the cuspidal support, we obtain a decomposition

$$\begin{aligned}\mathcal{L}_E &= \bigoplus_{\{P\}\in\mathcal{C}} \mathcal{L}_{E,\{P\}} \\ &= \bigoplus_{\{P\}\in\mathcal{C}} \bigoplus_{\phi\in\Phi_{E,\{P\}}} \mathcal{L}_{E,\{P\},\phi},\end{aligned}$$

where $\mathcal{L}_{E,\{P\}}$, resp. $\mathcal{L}_{E,\{P\},\phi}$ is just the (possibly trivial) subspace of square-integrable forms in $\mathcal{A}_{E,\{P\}}$, resp. $\mathcal{A}_{E,\{P\},\phi}$. Then the square-integrable cohomology decomposes accordingly into

$$\begin{aligned}H^*_{(\mathrm{sq})}(G,E) &= \bigoplus_{\{P\}\in\mathcal{C}} H^*_{(\mathrm{sq})}(\mathfrak{m}_G, K_\infty; \mathcal{A}_{E,\{P\}} \otimes E) \\ &= \bigoplus_{\{P\}\in\mathcal{C}} \bigoplus_{\phi\in\Phi_{E,\{P\}}} H^*_{(\mathrm{sq})}(\mathfrak{m}_G, K_\infty; \mathcal{A}_{E,\{P\},\phi} \otimes E),\end{aligned}$$

where the summands on the right-hand side are the images of the map induced in cohomology by the inclusions $\mathcal{L}_{E,\{P\}} \hookrightarrow \mathcal{A}_{E,\{P\}}$ and $\mathcal{L}_{E,\{P\},\phi} \hookrightarrow \mathcal{A}_{E,\{P\},\phi}$.

Since cuspidal automorphic forms are all square-integrable, we have for the summand indexed by $\{P\} = \{G\}$ that $\mathcal{L}_{E,\{G\}} = \mathcal{A}_{E,\{G\}}$ and hence $H^*_{\mathrm{cusp}}(G,E) = H^*_{(\mathrm{sq})}(\mathfrak{m}_G, K_\infty; \mathcal{A}_{E,\{G\}} \otimes E)$. The remaining part of the square-integrable cohomology is inside the Eisenstein cohomology and we write

$$H^*_{\mathrm{Eis},(\mathrm{sq})}(G,E) = \bigoplus_{\substack{\{P\}\in\mathcal{C} \\ \{P\}\neq\{G\}}} \bigoplus_{\phi\in\Phi_{E,\{P\}}} H^*_{(\mathrm{sq})}(\mathfrak{m}_G, K_\infty; \mathcal{A}_{E,\{P\},\phi} \otimes E).$$

This Eisenstein part of the square-integrable cohomology is often called the residual Eisenstein cohomology, even though there could be residues of Eisenstein series that are not square-integrable automorphic forms.

Our goal in this paper is to construct explicitly examples of non-trivial cohomology classes in some of the summands $H^*_{(\mathrm{sq})}(\mathfrak{m}_G, K_\infty; \mathcal{A}_{E,\{P\},\phi} \otimes E)$ in the residual Eisenstein cohomology for low-rank groups G and a class of maximal proper parabolic k-subgroups $\{P\}$.

4 Eisenstein series of relative rank one

Given an associate class $\{P\}$ of maximal proper parabolic k-subgroups of G and a class $\phi = \{\phi_Q\}_{Q\in\{P\}} \in \Phi_{E,\{P\}}$ of associate irreducible cuspidal automorphic representations of the Levi subgroups of elements of $\{P\}$ as defined in [16, Section 1.2], we would like to study the space $\mathcal{L}_{E,\{P\},\phi}$ in some detail, i.e., to determine its possible constituents. Let ϕ be represented by a cuspidal automorphic representation $\pi \otimes e^{\langle\lambda, H_P(\cdot)\rangle}$ of $L(\mathbb{A})$, where π is a unitary cuspidal automorphic representation of $L(\mathbb{A})$ and $\lambda \in \operatorname{Re} X_P^G$ whose real part belongs to a suitable positive cone. The space $\mathcal{L}_{E,\{P\},\phi}$ is spanned by the residues of Eisenstein series attached to π at the value of its complex parameter $\nu = \lambda$, as these residues are always square-integrable.

4.1 Eisenstein series

Let P be a standard maximal parabolic k-subgroup of G with Levi decomposition $P = LN$, where L is the unique standard Levi subgroup of P. Then X_P^G is a one-dimensional complex vector space. The subset Δ_L of $\Phi(G, T_L)$ consists of a unique reduced root α; it is obtained as a non-trivial restriction of an element in Δ_0.

Let ρ_P be the half-sum of k-roots which generate the unipotent radical N. As a suitable basis for $\check{\mathfrak{a}}_{P,\mathbb{C}} \cong \mathbb{C}$ we choose

$$\widetilde{\rho}_P = \langle \rho_P, \check{\alpha}\rangle^{-1}\rho_P,$$

as in the work of Shahidi [59]. We always identify accordingly $s \in \mathbb{C}$ with $\nu_s = \widetilde{\rho}_P \otimes s \in \check{\mathfrak{a}}_{P,\mathbb{C}}$.

Let π be a unitary cuspidal automorphic representation of $L(\mathbb{A})$.[4] We denote by V_π the space of smooth K-finite functions in the π-isotypic component of the space of cuspidal automorphic forms on $L(k)\backslash L(\mathbb{A})$.[5]

We consider the space W_π of right K-finite smooth functions $f : N(\mathbb{A})L(k)\backslash G(\mathbb{A}) \to \mathbb{C}$ such that for every $g \in G(\mathbb{A})$ the function $f_g(l) = f(lg)$ on $L(k)\backslash L(\mathbb{A})$ belongs to the subspace V_π of the space of cuspidal automorphic forms on $L(\mathbb{A})$, see, e.g., [16, Section 1.3]. Then, for $f \in W_\pi$, and $\nu \in X_P^G$, and for each $g \in G(\mathbb{A})$, one defines (at least formally) the Eisenstein series as

$$E_P^G(f,\nu)(g) = \sum_{\gamma\in P(k)\backslash G(k)} e^{\langle \nu+\rho_P, H_P(\gamma g)\rangle} f(\gamma g) = \sum_{\gamma\in P(k)\backslash G(k)} f_\nu(\gamma g),$$

where $f_\nu(g) = f(g)e^{\langle\nu+\rho_P, H_P(g)\rangle}$. This Eisenstein series converges absolutely and locally uniformly in g for all $\nu \in X_P^G$ whose real part belongs to the positive cone

$$\{\nu \in X_P^G : \langle \operatorname{Re}\nu, \check{\alpha}\rangle > \langle\rho_P, \check{\alpha}\rangle \text{ for all } \alpha \in \Phi^+(G, T_L)\}.$$

The assignment $s \mapsto E_P^G(f, \nu_s)(g)$ defines a map that is holomorphic in the region of absolute convergence of the defining series and has a meromorphic continuation to all of $\check{\mathfrak{a}}_{P,\mathbb{C}}$. If $\nu \in X_P^G$ is purely imaginary, then the Eisenstein series is holomorphic. Because of our normalization its singularities all lie on the real axis; more precisely, it has a finite number of simple poles in the real interval

$$\{\nu \in \operatorname{Re} X_P^G : 0 < \langle \nu, \check{\alpha}\rangle \le \langle\rho_P, \check{\alpha}\rangle\}.$$

All the remaining poles lie in the region $\{\nu \in X_P^G : \langle \operatorname{Re}\nu, \check{\alpha}\rangle < 0\}$. Given a specific reductive k-group G and a maximal parabolic k-subgroup $P \subset G$, these intervals can be made explicit in terms of the complex parameter $s \in \mathbb{C}$ with reference to the coordinate $\widetilde{\rho}_P$.[6]

4 Throughout the paper we mean by a cuspidal automorphic representation of $H(\mathbb{A})$, where H is a reductive linear group defined over k, an irreducible $(\mathfrak{h}, K_\infty; H(\mathbb{A}_f))$-module realized on a subspace of the space of cuspidal automorphic forms on $H(k)\backslash H(\mathbb{A})$ (see [48, Section I.2.17]).

5 When computing the Eisenstein cohomology, one considers only the real poles of the Eisenstein series. Hence, we make the following convention. We assume that π is normalized in such a way that the differential of the restriction of the central character of π to $A_{P,\infty}^0$ is trivial. This assumption is just a convenient choice of coordinates, which makes the poles of the Eisenstein series attached to π real. As explained in [16, Section 1.3], it can be achieved by replacing π by an appropriate twist. The twist just moves the poles of the Eisenstein series along the imaginary axis.

6 The reference for these facts concerning Eisenstein series is [48, Section IV.1].

The space $\mathcal{A}_{E,\{P\}}$ introduced in Section 2 has a two-step filtration defined in [14, Section 6]. However, we use a slight modification as in [16, Section 5.2] and [21]. According to the decomposition of $\mathcal{A}_{E,\{P\}}$ along the cuspidal support as in Section 2, it suffices to give the filtration of the spaces $\mathcal{A}_{E,\{P\},\phi}$, where ϕ is the associate class of $\pi \otimes e^{\langle \lambda, H_P(\cdot)\rangle}$. Then, the filtration is given by $\mathcal{L}_{E,\{P\},\phi} \subset \mathcal{A}_{E,\{P\},\phi}$, where $\mathcal{L}_{E,\{P\},\phi}$ is the subspace of $\mathcal{A}_{E,\{P\},\phi}$ consisting of square integrable automorphic forms. The space $\mathcal{L}_{E,\{P\},\phi}$ is spanned by the residues at $\nu_s = \lambda$ of the Eisenstein series attached to a function f such that for every $g \in G(\mathbb{A})$ the functions f_g on $L(k)\backslash L(\mathbb{A})$ defined above belong to the π-isotypic subspace of the space of cuspidal automorphic forms on $L(\mathbb{A})$. Those residues are square-integrable automorphic forms by the Langlands criterion [48, Section I.4.11]. The quotient $\mathcal{A}_{E,\{P\},\phi}/\mathcal{L}_{E,\{P\},\phi}$ is spanned by the principal value of the derivatives of such Eisenstein series at $\nu_s = \lambda$.

We also consider a subspace of $\mathcal{L}_{E,\{P\},\phi}$, to be denoted $\mathcal{L}_{E,\{P\},\phi,V_\pi}$, spanned by the residues at poles $\nu_s = \lambda$ of the Eisenstein series $E_P^G(f, \nu_s)(g)$ attached as above to a fixed (irreducible) realization V_π of a unitary cuspidal automorphic representation π of $L(\mathbb{A})$.

4.2 Intertwining operators

If the parabolic k-subgroup P in G is self-associate, the poles of the Eisenstein series coincide with the poles of its constant term $E_P^G(f, \nu_s)_P$ along P (see [48, Section II.1.7]). The constant term along P is given by, using the notation $f_s := f_{\nu_s}$,

$$E_P^G(f, \nu_s)_P(g) = f_s(g) + M(\nu_s, \pi, w_0)f_s(g),$$

where $w_0 \in W$ is the unique non-trivial Weyl group element such that $w_0(\Delta_0 \setminus \{\alpha\}) \subset \Delta_0$, while $w_0(\alpha)$ is a negative root, and $M(\nu_s, \pi, w_0)$ is the standard intertwining operator defined as the analytic continuation from the domain of convergence of the integral

$$M(\nu_s, \pi, w_0)f_s(g) = \int\limits_{N(\mathbb{A})} f_s(\tilde{w}_0^{-1} n g)\, dn, \tag{4.1}$$

where $\tilde{w}_0$ is the representative for w_0 in $G(k) \cap K$ chosen as in [58]. Away from the poles it intertwines the induced representation

$$\begin{aligned} I(\nu_s, \pi) &= \operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi \otimes e^{\langle \nu_s, H_P(\cdot)\rangle}) \\ &\cong \{f_s = f \cdot e^{\langle \nu_s + \rho_P, H_P(\cdot)\rangle} : f \in W_\pi\} \end{aligned}$$

and $I(\nu_{-s}, w_0(\pi))$, where the action of w_0 on π is given by $w_0(\pi)(l) = \pi(\tilde{w}_0^{-1} l \tilde{w}_0)$ for $l \in L(\mathbb{A})$. Observe that in our notation $\operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}$ includes the normalization by ρ_P, and thus ρ_P does not appear in the first line but appears in the second line of the above equation.

The poles of the constant term $E_P^G(f, \nu_s)_P(g)$ of the Eisenstein series coincide with the poles of the integral $M(\nu_s, \pi, w_0)f_s(g)$.

Let $\pi \cong \bigotimes_v' \pi_v$ be the decomposition into a restricted tensor product, where π_v is a unitary irreducible representation of $L(k_v)$, $v \in V$. At almost all non-archimedean places $v \in V_f$, π_v is unramified, and we denote by $f_{s,v}^\circ$ the unique K_v-invariant vector in $I(\nu_s, \pi_v)$ normalized by the condition $f_{s,v}^\circ(e) = 1$, where e is the identity in $G(k_v)$. Let S be the finite set of places $v \in V$ of k which contains all archimedean places and such that for $v \notin S$ we have $G(k_v)$ is unramified and π_v is unramified. For $v \notin S$, by [41, Section 5], the standard local intertwining operator $M(\nu_s, \pi_v, w_0)$, defined as the analytic continuation of the local analogue of the integral in (4.1), acts on $f_{s,v}^\circ$ as

$$M(\nu_s, \pi_v, w_0)f_{s,v}^\circ = r(\nu_s, \pi_v, w_0)\widetilde{f}_{-s,v}^\circ,$$

where $r(\nu_s, \pi_v, w_0)$ is the local normalizing factor given as a certain ratio of local L-functions, and $\widetilde{f}_{-s,v}^\circ$ is the normalized K_v-invariant vector in $I(\nu_{-s}, w_0(\pi_v))$. Given a place $v \notin S$, we write

$$M(\nu_s, \pi, w_0) = r(\nu_s, \pi, w_0)N(\nu_s, \pi, w_0),$$

where $N(\nu_s, \pi, w_0)$ is called the normalized intertwining operator.

If $f_s = \bigotimes_v f_{s,v}$ is decomposable, let $T(f)$ be the finite set of places which contains all archimedean places V_∞ and such that $f_{s,v} = f^\circ_{s,v}$ for all $v \in V_f \setminus T(f)$. Then the global standard intertwining operator acts on f_s as

$$M(\nu_s, \pi, w_0)f_s = r^S(\nu_s, \pi, w_0)\Big[\Big(\bigotimes_{v\in S} M(\nu_s, \pi_v, w_0)f_{s,v}\Big) \otimes \Big(\bigotimes_{v\in T(f)\setminus S} N(\nu_s, \pi_v, w_0)f_{s,v}\Big) \otimes \Big(\bigotimes_{v\notin T(f)} \widetilde{f}^\circ_{-s,v}\Big)\Big],$$

where

$$r^S(\nu_s, \pi, w_0) = \prod_{v\notin S} r(\nu_s, \pi_v, w_0) \tag{4.2}$$

is a certain ratio of partial L-functions attached to π.

4.3 The case of a quasi-split group

We now suppose that the k-group G is quasi-split. In [61], the local normalizing factors $r(\nu_s, \pi_v, w_0)$ are defined at all places for a globally ψ-generic representation π. Let $N(\nu_s, \pi_v, w_0)$ be the local normalized intertwining operator defined by

$$M(\nu_s, \pi_v, w_0) = r(\nu_s, \pi_v, w_0)N(\nu_s, \pi_v, w_0).$$

It intertwines the induced representations $I(\nu_s, \pi_v)$ and $I(\nu_{-s}, w_0(\pi_v))$. Note that at a place $v \in V_f$ where π_v is unramified $N(\nu_s, \pi_v, w_0)$ maps $f^\circ_{s,v}$ to $\widetilde{f}^\circ_{-s,v}$. Hence,

$$M(\nu_s, \pi, w_0)f_s = r(\nu_s, \pi, w_0)\Big[\bigotimes_{v\in T(f)} N(\nu_s, \pi_v, w_0)f_{s,v}\Big] \otimes \Big[\bigotimes_{v\notin T(f)} \widetilde{f}^\circ_{-s,v}\Big],$$

where

$$r(\nu_s, \pi, w_0) = \prod_{v\in V} r(\nu_s, \pi_v, w_0)$$

is the global normalizing factor given as a certain ratio of automorphic L-functions attached to π.

Given a fixed connected reductive algebraic k-group, a maximal parabolic k-subgroup $P \subset G$ with Levi decomposition $P = LN$, and a unitary cuspidal automorphic representation of $L(\mathbb{A})$, this ratio can be made explicit under the assumption that π is global generic with respect to some ψ; see the examples below.

4.4 *L*-functions

Given a connected algebraic group H defined over k, we denote its L-group by ${}^LH$. It is the semidirect product of a complex group ${}^LH^o$ and the absolute Weil group $W(\overline{k}/k)$, where $\overline{k}$ denotes the algebraic closure of k. For every place $v \in V$ of k let ${}^LH_v$ denote the L-group of H viewed as a group over k_v. There is a natural homomorphism $\iota_v : {}^LH_v \to {}^LH$. Let r be a finite-dimensional complex representation of ${}^LH$. Given $v \in V$, there is the representation $r_v := r \circ \iota_v$ of ${}^LH_v$.

Let $\pi \cong \bigotimes'_v \pi_v$ be an irreducible unitary representation of $G(\mathbb{A})$, G a connected reductive group defined over k. Given a place $v \in V$ so that $G(k_v)$ and π_v are both unramified at v, there is the local Langlands L-function $L(s, \pi_v, r_v)$ attached to π_v and r_v with complex parameter s, see [41]. Let S be a finite set of places containing V_∞ so that for every $v \notin S$ the group $G(k_v)$ and π_v are both unramified at v. Then one can define the global partial L-function by the infinite product

$$L^S(s, \pi, r) := \prod_{v\notin S} L(s, \pi_v, r_v);$$

it is absolutely convergent for $\mathrm{Re}(s)$ sufficiently large and can be analytically continued.

Given a parabolic k-subgroup $P = L_P N_P$ of G, we denote the Lie algebra of the unipotent radical ${}^LN_P$ of the L-group ${}^LP$ by ${}^L\mathfrak{n}_P$. The L-group of L_P acts on ${}^L\mathfrak{n}_P$ by the adjoint action. If $\check{\beta}$ ranges through the set of dual roots for which $X_{\check\beta} \in {}^L\mathfrak{n}_P$ holds, then the numbers $\langle \tilde\rho_P, \check\beta\rangle$ take a string of integers from 1 to a positive integer m. For a given j, $1 \le j \le m$, we define

$$V_j := \{X_{\check\beta} \in {}^L\mathfrak{n}_P : \langle \tilde\rho_P, \check\beta\rangle = j\}.$$

For each j the adjoint action of the L-group of L_P on ${}^L\mathfrak{n}_P$ leaves the space V_j stable; the corresponding representation on V_j obtained by restriction is denoted by r_j. This representation is irreducible.

Then the ratio of partial L-functions occurring in the formula (4.2) has the following form:

$$r^S(v_s, \pi, w_0) = \prod_{j=1}^{m} \frac{L^S(js, \pi, r_j)}{L^S(js+1, \pi, r_j)}.$$

Given G and a maximal parabolic subgroup P of G, the representations r_j, $j = 1, \dots, m$, and the types of the corresponding partial L-functions $L^S(s, \pi, r_j)$ are determined by Langlands in [41], see also [59]. Later on we will give explicit examples.

5 Construction of Eisenstein cohomology classes

Given an associate class $\{P\} \in \mathcal{C}$, represented by $P = LN$, of maximal parabolic k-subgroups in G and a class $\phi = \{\phi_Q\}_{Q \in \{P\}}$ of associate irreducible cuspidal automorphic representations of the Levi subgroups of elements of $\{P\}$, we now analyze the actual construction of cohomology classes in the corresponding summand

$$H^*(\mathfrak{m}_G, K_\infty; \mathcal{A}_{E,\{P\},\phi} \otimes_{\mathbb{C}} E)$$

in the direct sum decomposition of the automorphic cohomology of G. Since all our examples in the rest of the paper are split groups, from this point on we always assume that G is split over k. Suppose $\phi \in \Phi_{E,\{P\}}$ is represented by $\pi \otimes e^{\langle \lambda, H_P(\cdot)\rangle} \in \phi_P$, where π is an irreducible unitary cuspidal automorphic representation of the Levi subgroup $L(\mathbb{A})$ and $\lambda \in \operatorname{Re} X_P^G$. Let π be realized on the subspace V_π of the space of cuspidal forms on $L(\mathbb{A})$. By carrying through the construction of residues or derivatives of Eisenstein series attached to (π, V_π) (as in [46], Section 3), the corresponding contribution to $H^*(\mathfrak{m}_G, K_\infty, \mathcal{A}_{E,\{P\},\phi} \otimes_{\mathbb{C}} E)$ is embodied in the cohomology

$$H^*\big(\mathfrak{m}_G, K_\infty; \operatorname{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \operatorname{Ind}_{(\mathfrak{g}\cap\mathfrak{p}, K_\infty \cap L_\infty)}^{(\mathfrak{g}, K_\infty)} (V_\pi \otimes E \otimes S(\check{\mathfrak{a}}_{P,\mathbb{C}}^G))\big),$$

where $S(\check{\mathfrak{a}}_{P,\mathbb{C}}^G)$ is the symmetric algebra of $\check{\mathfrak{a}}_{P,\mathbb{C}}^G$ with the $(\mathfrak{m}_G, K_\infty)$-module structure as defined in [14, p. 218] (see also [46, Section 3.1]).

Using Frobenius reciprocity, the study of this space is reduced to an analysis of the $G(\mathbb{A}_f)$-module

$$\operatorname{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} H^*\big(\mathfrak{l}, K_\infty \cap L_\infty; V_\pi \otimes H^*(\mathfrak{n}, E) \otimes S(\check{\mathfrak{a}}_{P,\mathbb{C}}^G)\big). \tag{5.1}$$

Following Kostant [38, Theorem 5.13], the Lie algebra cohomology $H^*(\mathfrak{n}, E)$ of $\mathfrak{n}$ with coefficients in the irreducible representation (η, E) of G_∞ is given as an $(\mathfrak{l}, K_\infty \cap L_\infty)$-module as the sum

$$H^*(\mathfrak{n}, E) = \bigoplus_{w \in W^P} F_{\mu_w}$$

where the sum ranges over w in the set W^P of the minimal coset representatives for the right cosets of W modulo the Weyl group W_P of the Levi factor L of P, and F_{μ_w} denotes the irreducible finite-dimensional $(\mathfrak{l}, K_\infty \cap L_\infty)$-module of highest weight

$$\mu_w = w(\Lambda + \rho_{P_0}) - \rho_{P_0},$$

where $\Lambda \in \check{\mathfrak{a}}_{P_0,\mathbb{C}}$ is the highest weight of (η, E). The weights μ_w are all dominant and distinct and, given a fixed degree q, only the weights μ_w with length $\ell(w) = q$ occur in the decomposition of $H^q(\mathfrak{n}, E)$ into irreducibles. As in [53, Section 3.2], see also [55], we call a cohomology class in (5.1) which gives rise to a non-trivial class in

$$H^*(\mathfrak{l}, K_\infty \cap L_\infty; V_{\pi_\infty} \otimes F_{\mu_w})$$

a class of type (π, w), $w \in W^P$. If the infinitesimal character χ_{π_∞} of the archimedean component π_∞ of π does not coincide with the infinitesimal character of the representation contragredient to F_{μ_w}, the cohomology space $H^*(\mathfrak{l}, K_\infty \cap L_\infty; V_{\pi_\infty} \otimes F_{\mu_w})$ vanishes, that is, there are no classes of type (π, w).

Moreover, if the module F_{μ_w} is not isomorphic to its complex conjugate contragredient $\overline{F}^*_{\mu_w}$, then $H^*(\mathfrak{l}, K_\infty \cap L_\infty; V_{\pi_\infty} \otimes F_{\mu_w}) = (0)$, since this condition implies that the complex contragredient of F_{μ_w} and V_π have distinct infinitesimal character. Following [4, Section 1], $F_{\mu_w} \not\cong \overline{F}^*_{\mu_w}$ is equivalent to the condition that $-w_{l,L}(\mu_w|_{\check{\mathfrak{a}}^P_{P_0,\mathbb{R}}})$ is distinct from $\mu_w|_{\check{\mathfrak{a}}^P_{P_0,\mathbb{R}}}$, where $w_{l,L}$ is the longest element in the Weyl group W_P of the Levi component L. We recall that the transformation $-w_{l,L}$ maps the highest weight of an irreducible $\mathfrak{l}_\mathbb{C}$-module into that of the contragredient one.

Suppose there is a non-trivial cohomology class of type (π, w), $w \in W^P$. In order to understand the cohomological contribution of the corresponding Eisenstein series $E^G_P(f, \nu_s)$ or a residue of such in the cohomology space $H^*(\mathfrak{g}, K_\infty; \mathcal{A}_{E,\{P\},\phi} \otimes_\mathbb{C} E)$, following [53, Corollary 3.5], we have to analyze the analytic behavior of $E^G_P(f, \nu_s)$ at the point

$$\lambda_{[w]} := -w(\Lambda + \rho_{P_0})|_{\check{\mathfrak{a}}_{P,\mathbb{R}}}.$$

This evaluation point is real and uniquely determined by the datum (π, w). It only depends on w and the highest weight $\Lambda \in \check{\mathfrak{a}}_{P_0,\mathbb{C}}$. As a consequence of the description of the space $\mathcal{A}_{E,\{P\},\phi}$ of automorphic forms in [16, Section 1.3], only the points $\lambda_{[w]}$ with $\langle \lambda_{[w]}, \check{\alpha} \rangle \geq 0$ matter in our analysis. In other words, it suffices to consider only the evaluation points $\lambda_{[w]}$ such that in the basis $\tilde{\rho}_P$ of $\check{\mathfrak{a}}_{P,\mathbb{C}}$ we have $\lambda_{[w]} = \lambda_{s_w} = \tilde{\rho}_P \otimes s_w$ with $s_w \geq 0$.

In the following, under the assumption that $H^*(\mathfrak{l}, K_\infty \cap L_\infty; V_{\pi_\infty} \otimes F_{\mu_w})$ is non-trivial for a given $\{P\} \in \mathcal{C}$, and a pair (π, w), we make explicit the two necessary conditions this assumption implies by the discussion above, namely

$$-w_{l,L}(\mu_w|_{\check{\mathfrak{a}}^P_{P_0,\mathbb{R}}}) = \mu_w|_{\check{\mathfrak{a}}^P_{P_0,\mathbb{R}}},$$

and the infinitesimal character χ_{π_∞} of π_∞ is of the form

$$\chi_{\pi_\infty} = -w(\Lambda + \rho_{P_0})|_{\check{\mathfrak{a}}^P_{P_0,\mathbb{R}}}.$$

6 Construction of specific cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_k)$

6.1 Existence

One can use the Langlands functoriality principle to construct specific cuspidal automorphic representations for the general linear group GL_2 (or variants thereof) over a given totally real algebraic number field k such that these representations give rise to non-vanishing classes in the cuspidal cohomology

$$H^*_{\mathrm{cusp}}(\mathrm{GL}_2, E) = H^*_{(\mathrm{sq})}(\mathfrak{m}_{\mathrm{GL}_2}, K_\infty; \mathcal{A}_{E,\{\mathrm{GL}_2\}} \otimes E)$$

with a suitable coefficient system E. This result can be extended to cases of the group GL_2 defined over an extension k' of k, where the base change lift is well understood, for example, an extension k'/k of cyclic prime degree. This non-vanishing result relies on the fact that there always exist cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_k)$ whose archimedean components are discrete series representations of $\mathrm{GL}_2(\mathbb{R})$ and which is special at a given finite number of places $v \in V_f$, i.e., its local component at $v \in V_f$ is the Steinberg representation. Then the base change lift is compatible with cohomology. This result is obtained by inserting so-called pseudo-coefficients in the Selberg trace formula, see [40, 2.5].

Another approach dates back to work of Chevalley [9] and Weil [70]; it is used in combination with automorphic induction by Clozel in [10].

These general results are not sufficient for our purpose. It is decisive to exhibit cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_k)$ which are cohomological, have specific prescribed local and global behavior, the latter encoded in its automorphic L-function. In view of this task it is necessary to recall some facts which are part of the classification of irreducible admissible $(\mathfrak{gl}_2(\mathbb{R}), \mathrm{O}(2))$-modules (cf. [30] or [7, Chapter 2]).

6.2 Discrete series representations

Given an integer $m \geq 2$, we denote by D_m the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of lowest $\mathrm{O}(2)$-type m, i.e., the square-integrable representation D_m is characterized by the fact that the restriction to the maximal compact subgroup $\mathrm{O}(2)$ of $\mathrm{GL}_2(\mathbb{R})$ decomposes as an algebraic sum of the form

$$D_m|_{\mathrm{O}(2)} \cong \bigoplus_{r \in \Sigma(m)} V(r), \qquad \Sigma(m) = \{l \in \mathbb{Z} : l \equiv m \bmod 2,\ l \geq m\},$$

where $V(r)$, $r \geq 2$, is the irreducible two-dimensional representation of $\mathrm{O}(2)$ fully induced by the character $k_\theta \mapsto e^{ir\theta}$ of the subgroup $\mathrm{SO}(2)$ of rotations k_θ, $\theta \in [0, 2\pi]$ in $\mathrm{O}(2)$ of index two.

The discrete series representation D_m naturally appears as the unique irreducible subrepresentation of the representation $I(m)$ induced from the character $|\cdot|^{\frac{m-1}{2}} \mathrm{sign}^m \otimes |\cdot|^{-\frac{m-1}{2}}$ of the maximal split torus in $\mathrm{GL}_2(\mathbb{R})$. The quotient is irreducible and finite-dimensional. More precisely, in terms of the underlying $(\mathfrak{gl}_2(\mathbb{R}), \mathrm{O}(2))$-modules, one has a short exact sequence

$$0 \to D_m \to I(m) \to F_{m-2} \to 0,$$

where (σ_k, F_k), $k \geq 0$, denotes the irreducible finite-dimensional representation of $\mathrm{GL}_2(\mathbb{R})$ of highest weight $\mu_k = k \cdot \omega$ (ω denotes the fundamental dominant weight of $\mathrm{GL}_2(\mathbb{R})$), thus, $\dim F_k = k + 1$. Consequently, the infinitesimal characters of the representations D_m and F_{m-2} match. With regard to relative Lie algebra cohomology one has $H^q(\mathfrak{m}_{\mathrm{GL}_2(\mathbb{R})}, \mathrm{O}(2); D_m \otimes F_{m-2}) = \mathbb{C}$ if $q = 1$, otherwise it vanishes. In general, given a finite-dimensional representation (σ_k, F_k), $k \geq 0$, of $\mathrm{GL}_2(\mathbb{R})$ of highest weight μ_k, the relative Lie algebra cohomology

$$H^*(\mathfrak{m}_{\mathrm{GL}_2(\mathbb{R})}, \mathrm{O}(2); D_m \otimes F_k)$$

vanishes if $k \neq m - 2$ since the infinitesimal character χ_{D_m} differs from the one of the contragredient representation of (σ_k, F_k).

Finally, one observes that in this labelling of the discrete series representations of $\mathrm{GL}_2(\mathbb{R})$ the Harish-Chandra parameter of D_m, $m \geq 2$, is $m - 1$.

6.3 A result of Trotabas

Let k be a totally real algebraic number field of degree $[k : \mathbb{Q}] = d$, and let $\mathfrak{q} \subset \mathcal{O}_k$ be a prime ideal in its ring of integers. Let $\pi = \pi_\infty \otimes \pi_f$ be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_k)$ with trivial central character ω_π. Given a d-tuple $\kappa = (k_1, \ldots, k_d)$ of even non-zero integers $k_1, \ldots, k_d$, we denote by $\mathcal{D}(\kappa, \mathfrak{q})$ the set of irreducible cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_k)$ (up to infinitesimal equivalence) so that

$$\pi_\infty = \bigotimes_{i=1}^{d} D_{k_i},$$

where D_{k_i} denotes as above the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of lowest $\mathrm{O}(2)$-type k_i and so that π corresponds to a cuspidal Hilbert modular form of conductor $\mathfrak{q}$. Given κ and $\mathfrak{q}$, the set $\mathcal{D}(\kappa, \mathfrak{q})$ has finite cardinality.[7] We observe that for any κ, there exists $\mathfrak{q}$ such that $\mathcal{D}(\kappa, \mathfrak{q})$ is non-empty.

In the classical setting these are cuspidal automorphic representations associated to cuspidal Hilbert modular forms of weight κ, trivial Nebentypus and level $\mathfrak{q}$.

The following result in due to Trotabas [65].

7 We note that the labelling of the discrete series representation in [65] is via the corresponding Harish-Chandra parameter, thus it differs from the one used in this paper.

Theorem 6.1. *Given a d-tuple κ of even non-zero integers, the following estimate for $\mathfrak{q}$ ranging over the prime ideals in $\mathcal{O}_k$, k a totally real algebraic number field, is true:*

$$\liminf_{N(\mathfrak{q})\to\infty} \frac{|\{\pi \in \mathcal{D}(\kappa, \mathfrak{q}) : L(\frac{1}{2}, \pi, \rho_2) \neq 0\}|}{|\mathcal{D}(\kappa, \mathfrak{q})|} \geq \frac{1}{4},$$

where $L(s, \pi, \rho_2) = L(s, \pi)$ is the principal L-function.

We can derive the following result regarding the existence of cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_k)$ which provide non-trivial cohomology classes in the cuspidal cohomology

$$H^*_{\mathrm{cusp}}(\mathrm{GL}_2, E) := H^*(\mathfrak{m}_{\mathrm{GL}_2}, K_\infty; \mathcal{A}_{E,\{\mathrm{GL}_2\}} \otimes E) = H^*_{(\mathrm{sq})}(\mathfrak{m}_{\mathrm{GL}_2}, K_\infty; \mathcal{A}_{E,\{\mathrm{GL}_2\}} \otimes E).$$

Recall that (η, E) is an irreducible finite-dimensional algebraic representation of $\mathrm{GL}_{2,\infty}$ in a complex vector space E. We assume that this representation originates from an algebraic representation of the algebraic k-group GL_2. Its highest weight can be written as $\mu = (\mu)_{\iota_v}$, $v \in V_\infty$, where ι_v denotes the embedding $k \to \mathbb{R}$ which corresponds to $v \in V_\infty$. Each of the weights $(\mu)_{\iota_v}$ is of the form $k_v\omega_v$, $k_v \in \mathbb{Z}$, $k_v \geq 0$, where ω_v denotes the fundamental dominant weight of the group $G_v \cong \mathrm{GL}_2(\mathbb{R})$, $v \in V_\infty$. A weight μ is called even if all integers k_v, $v \in V_\infty$ are even.

Given a highest weight $\mu = (\mu)_{\iota_v}$, $v \in V_\infty$, we say that a family $\{D_{k_v}\}$, $k_v \in \mathbb{Z}$, $k_v \geq 2$, of discrete series representations of $\mathrm{GL}_2(\mathbb{R})$, parametrized by $v \in V_\infty$, is compatible with μ if $(\mu)_{\iota_v} = (k_v - 2)\omega_v$ for all $v \in V_\infty$.

Proposition 6.2. *Given an irreducible finite-dimensional algebraic representation (η, E) of the real Lie group $G_\infty = \prod_{v\in V_\infty} G_v$ with $G_v \cong \mathrm{GL}_2(\mathbb{R})$ of even highest weight μ, there exists an irreducible cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_k)$ whose central character ω_π is trivial, whose archimedean components π_v in $\pi_\infty = \bigotimes_{v\in V_\infty} \pi_v$, are discrete series representations of $\mathrm{GL}_2(\mathbb{R})$ compatible with μ, and whose corresponding L-function $L(s, \pi, \rho_2)$ does not vanish at $s = \frac{1}{2}$. Such a representation π of $\mathrm{GL}_2(\mathbb{A}_k)$ contributes non-trivially to the cuspidal cohomology $H^*_{\mathrm{cusp}}(\mathrm{GL}_2, E)$ in degree $[k : \mathbb{Q}]$.*

6.4 Monomial representations

We study now the existence of monomial cuspidal automorphic representations with a given cohomological archimedean components. By definition, a unitary cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_k)$ is monomial if there exists a non-trivial Hecke character δ of the group of ideles $\mathbb{I}_k$ such that

$$\pi \otimes \delta \cong \pi.$$

Comparing the central characters, it follows that δ is quadratic.

Monomial representations arise by automorphic induction from a Hecke character of a quadratic extension of k. Let K/k be a quadratic extension of number fields. For a unitary Hecke character θ of the group of ideles $\mathbb{I}_K$ of K, let $\pi(\theta)$ be the automorphic induction of θ to $\mathrm{GL}_2(\mathbb{A}_k)$. It is defined by $\pi(\theta) = \otimes'_v \pi(\theta)_v$, where

- if v splits in K, then $\pi(\theta)_v$ is the principal series representation of $\mathrm{GL}_2(k_v)$ induced from the character $\theta_{w_1} \otimes \theta_{w_2}$ of the torus, where w_1 and w_2 are the two places of K above v,
- if v does not split in K, then $\pi(\theta)_v$ is the local automorphic induction of θ_w to a representation of $\mathrm{GL}_2(k_v)$, where w is the unique place of K lying above v.

The following theorem is contained in Arthur–Clozel [1, Section 3.6] (see also [30] and [39]).

Theorem 6.3. *Let K/k be a quadratic extension of a number field k, c the unique non-trivial element of the Galois group $\mathrm{Gal}(K/k)$, and $\delta = \delta_{K/k}$ the quadratic character of $\mathbb{I}_k$ attached to the extension K/k by class field theory. Let θ be a unitary Hecke character of $\mathbb{I}_K$. Then the automorphic induction $\pi(\theta)$ of θ is an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_k)$ and*

$$\pi(\theta) \otimes \delta \cong \pi(\theta),$$

in particular, $\pi(\theta)$ is monomial. Moreover, $\pi(\theta)$ is cuspidal if and only if $\theta \neq \theta^c$, i.e., θ does not factor through the norm map $N_{K/k}$ from K to k. Conversely, for any monomial cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_k)$ such that $\pi \otimes \delta_{K/k} \cong \pi$, there exists a unitary Hecke character θ of $\mathbb{I}_K$ such that $\pi \cong \pi(\theta)$.

We are now in a position to show the existence of monomial representations with a given discrete series representations at all archimedean places. These are required to show the existence of non-trivial cohomology classes in the square-integrable cohomology in some of the cases. We retain the notation of the previous subsection. In particular, k is a totally real number field of degree $[k : \mathbb{Q}] = d$.

Proposition 6.4. *Given an irreducible finite-dimensional algebraic representation (η, E) of the real Lie group $G_\infty = \prod_{v\in V_\infty} G_v$ with $G_v \cong \mathrm{GL}_2(\mathbb{R})$ of highest weight $\mu = (k\omega_v)_{\sigma_v}$, where $k \in \mathbb{Z}$, $k \geq 0$, there exists an irreducible cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_k)$ which is monomial and whose archimedean components π_v in $\pi_\infty = \bigotimes_{v\in V_\infty} \pi_v$ are discrete series representations of $\mathrm{GL}_2(\mathbb{R})$ compatible with μ. Such a representation π of $\mathrm{GL}_2(\mathbb{A}_k)$ contributes non-trivially to the cuspidal cohomology $H^*_{\mathrm{cusp}}(\mathrm{GL}_2, E)$ in degree $[k : \mathbb{Q}]$.*

Proof. According to Theorem 6.3, to construct a monomial cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_k)$, it is sufficient to find a unitary Hecke character θ of the group of ideles $\mathbb{I}_K$ of a quadratic extension K/k such that $\pi(\theta)$ has the required properties. More precisely, θ should not factor though the norm map $N_{K/k}$, and all the archimedean components of θ should give by local automorphic induction $\pi(\theta)_v$ compatible with $k\omega_v$, that is, $\pi(\theta)_v \cong D_{k+2}$. The latter requirement implies that K/k is necessarily an imaginary quadratic extension, since the discrete series D_{k+2} can be obtained only if all archimedean places of k do not split in K.

The discrete series D_{k+2} of $\mathrm{GL}_2(\mathbb{R})$ corresponds, via the local Langlands correspondence, to the two-dimensional irreducible representation of the Weil group $W_\mathbb{R}$ obtained by induction from the character of $W_\mathbb{C} = \mathbb{C}^*$ given by the assignment

$$z \mapsto \left(\frac{z}{|z|}\right)^{k+1}, \quad z \in \mathbb{C}^*,$$

where $|z| = \sqrt{z \cdot \bar{z}}$. Hence, D_{k+2} is the local automorphic induction of that character, and we must construct a unitary Hecke character θ of $\mathbb{I}_K$ with that character as the archimedean component at all archimedean places. The condition that θ does not factor through the norm map $N_{K/k}$ immediately follows, because it is equivalent to the condition $\theta \neq \theta^c$, where c is the unique non-trivial element of the Galois group $\mathrm{Gal}(K/k)$, and this is obvious for the archimedean components.

The existence of a unitary Hecke character θ_0 of $\mathbb{I}_K$ with all the archimedean components given by the assignment

$$z \mapsto \frac{z}{|z|}, \quad z \in \mathbb{C}^*,$$

is well known. See [10, p. 479], and also [70]. Hence, if we let $\theta = \theta_0^{k+1}$, it is a unitary Hecke character of $\mathbb{I}_K$ with all the required properties. □

However, in some cases the previous result is not sufficient. We need to show the existence of a unitary cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_k)$ such that the symmetric square L-function $L(s, \pi, \mathrm{Sym}^2 \rho_2)$ has a simple pole at $s = 1$. From the formula

$$L(s, \pi \otimes \pi, \rho_2 \otimes \rho_2) = L(s, \pi, \wedge^2\rho_2)L(s, \pi, \mathrm{Sym}^2 \rho_2) = L(s, \omega_\pi)L(s, \pi, \mathrm{Sym}^2 \rho_2),$$

where ω_π is the central character of π, it follows that this happens if and only if π is selfdual with ω_π non-trivial. Since the contragredient $\tilde{\pi} \cong \pi \otimes \omega_\pi$, such π is necessarily monomial with $\pi \otimes \omega_\pi \cong \pi$. We now show the existence of such monomial representations π with a prescribed cohomological archimedean components.

Proposition 6.5. *Given an irreducible finite-dimensional algebraic representation (η, E) of the real Lie group $G_\infty = \prod_{v\in V_\infty} G_v$ with $G_v \cong \mathrm{GL}_2(\mathbb{R})$ of highest weight $\mu = (k\omega_v)_{\sigma_v}$, where $k \in \mathbb{Z}_{\geq 0}$, k odd, there exists an irreducible cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_k)$ which is selfdual with a non-trivial central character ω_π and whose archimedean components π_v in $\pi_\infty = \bigotimes_{v\in V_\infty} \pi_v$ are discrete series representations of $\mathrm{GL}_2(\mathbb{R})$ compatible with μ. Such a representation π of $\mathrm{GL}_2(\mathbb{A}_k)$ contributes non-trivially to the cuspidal cohomology $H^*_{\mathrm{cusp}}(\mathrm{GL}_2, E)$ in degree $[k : \mathbb{Q}]$.*

Proof. As already mentioned before the statement of the theorem, π with the required properties is necessarily monomial. Hence, by Theorem 6.3, it is an automorphic induction $\pi(\theta)$ from a character θ of a quadratic extension K/k which does not factor through the norm. Since the goal is to construct a selfdual monomial representation π with non-trivial central character, so that $\pi \otimes \omega_\pi \cong \pi$, we actually need a character θ such that the central character of $\pi(\theta)$ is the same as the character $\delta_{K/k}$ associated to the extension K/k by class field theory. Moreover, at archimedean places θ should be such that $\pi(\theta)_v \cong D_{k+2}$, in particular, K/k should be an imaginary extension.

Given an imaginary quadratic extension K/k, we first construct a character θ_0 of $\mathbb{I}_K$ such that its archimedean components are given by the assignment

$$z \mapsto \frac{z}{|z|}, \quad z \in \mathbb{C}^*,$$

and the restriction of θ_0 to $\mathbb{I}_k$ equals the character $\delta_{K/k}$ associated to K/k by class field theory. This is more subtle then in the previous proposition, but can be arranged by [13, Lemma 3.5].

Let $\theta = \theta_0^{k+1}$. Then the archimedean components of θ are given by the assignment

$$z \mapsto \left(\frac{z}{|z|}\right)^{k+1}, \quad z \in \mathbb{C}^*,$$

so that $\pi(\theta)_v \cong D_{k+2}$ at all archimedean places v, as required. In particular, looking at the archimedean places, clearly θ does not factor through the norm. Moreover, by [8, Section 29.2], the central character of the automorphic induction is given by the formula

$$\omega_{\pi(\theta)} = \delta_{K/k} \cdot \theta|_{\mathbb{I}_k}.$$

Since

$$\theta|_{\mathbb{I}_k} = (\theta_0|_{\mathbb{I}_k})^{k+1} = \delta_{K/k}^{k+1},$$

and k is odd, we obtain that $\omega_{\pi(\theta)} = \delta_{K/k}$. Hence, by Theorem 6.3, $\pi(\theta) \otimes \omega_{\pi(\theta)} \cong \pi(\theta)$, that is, $\pi(\theta)$ is selfdual with non-trivial central character. □

Remark 6.6. Note that for k even in the previous proposition, the representation π with the required properties does not exist. This is due to the fact that such π is necessarily a monomial representation, that is, obtained by automorphic induction from a character θ of $\mathbb{I}_K$, where K/k is the quadratic extension associated to ω_π by class field theory. However, the archimedean components of such π are supposed to be discrete series representations D_{k+2}, so that K/k is imaginary extension and thus archimedean components of ω_π are non-trivial. On the other hand, if k is even, then D_{k+2} has trivial central character, which is a contradiction.

7 The group G_2

7.1 Roots, weights and parabolic subgroups of G_2

Given an algebraic number field k, there is a uniquely determined Cayley algebra defined over k with divisors of zero; it is called the split Cayley algebra C over k ([52, Lemma 3.16]). The norm form of C is non-degenerate. Let G be the group of automorphisms of C. The Lie algebra of G, by definition, the derivation algebra of C is a central simple Lie algebra of dimension 14; its type is G_2. The group G is a simple algebraic group defined over k; it is split over k. The k-rank of G is two.

We fix a minimal parabolic k-subgroup P_0 with Levi decomposition $P_0 = L_0N_0$. Let Φ, Φ^+, Δ denote the corresponding sets of roots, positive roots, simple roots, respectively. The set Φ^+ can be described as $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$, where α_1 resp. α_2 denotes the short resp. long root; one has $\Delta = \{\alpha_1, \alpha_2\}$. The half-sum of the positive roots is $\rho_{P_0} = 5\alpha_1 + 3\alpha_2$.

The fundamental dominant weights are $\Lambda_1 = 2\alpha_1 + \alpha_2$ and $\Lambda_2 = 3\alpha_1 + 2\alpha_2$. One observes that ρ_{P_0} can also be written as $\Lambda_1 + \Lambda_2$.

For $r = 1, 2$, the maximal proper standard parabolic k-subgroup $P_{\Delta\setminus\{\alpha_r\}}$ corresponding to the subset $\Delta \setminus \{\alpha_r\}$ of Δ is denoted in short by P_r, and its Levi decomposition by $P_r = L_rN_r$, where L_r is the Levi factor containing L_0, and N_r the unipotent radical. The characters of L_0 in N_r are exactly those positive roots which contain at least one simple root not in $\Delta \setminus \{\alpha_r\}$. In both cases we have $L_r \cong \mathrm{GL}_2$. Observe that the parabolic subgroups P_r are self-associate, i.e. P_r itself is the only standard parabolic subgroup which is associate to P_r (see Section 3.1). However, P_r is conjugate to its opposite parabolic subgroup P_r^{opp} by a representative of the unique non-trivial Weyl group element $w_0 \in W$ with the property that $w_0(\Delta \setminus \{\alpha_r\}) \subset \Delta$.

If $P_r = L_r N_r$, $r = 1, 2$, is one of the two maximal proper standard parabolic k-subgroups of G, we identify the roots of A_{P_r} in N_r with a subset of Φ^+. Then the unique reduced root of A_{P_r} in N_r can be identified with the element $\alpha_r \in \Delta$ in the set of simple roots. Let ρ_{P_r} be the half-sum of k-roots which generate N_r (or, equivalently, of positive roots which are not the positive roots of L_r). Following the work of Shahidi [59], we choose as a suitable basis for $\check{\mathfrak{a}}_{P_r,\mathbb{C}} \cong \mathbb{C}$ the element

$$\widetilde{\rho}_{P_r} = \langle \rho_{P_r}, \check{\alpha}_r \rangle^{-1} \rho_{P_r}.$$

We obtain, as already observed in [45],

$$\widetilde{\rho}_{P_1} = \frac{2}{5}\rho_{P_1} = 2\alpha_1 + \alpha_2, \quad \widetilde{\rho}_{P_2} = \frac{2}{3}\rho_{P_2} = 3\alpha_1 + 2\alpha_2.$$

We always identify accordingly $s \in \mathbb{C}$ with $\nu_s = \widetilde{\rho}_{P_r} \otimes s \in \check{\mathfrak{a}}_{P_r,\mathbb{C}}$. Note that $\widetilde{\rho}_{P_1}$ coincides with the first fundamental weight Λ_1, and $\widetilde{\rho}_{P_2}$ coincides with the second fundamental weight Λ_2.

7.2 Classes of type (π, w), $w \in W^P$

Given a maximal parabolic k-subgroup P_r, $r = 1, 2$, of G_2, we are going to analyze which types (π, w), $w \in W^{P_r}$ occur. First, this amounts to determine the Lie algebra cohomology $H^*(\mathfrak{n}_r, E)$ of $\mathfrak{n}_r := \mathfrak{n}_{P_r}$ with coefficients in the irreducible representation (η, E) of G_∞. As explained in Section 5 it is given as an $(\mathfrak{l}_r, K_\infty \cap L_{r,\infty})$-module as the sum

$$H^*(\mathfrak{n}_r, E) = \bigoplus_{w \in W^{P_r}} F_{\mu_w}$$

where the sum ranges over w in the set W^{P_r} of the minimal coset representatives for the left cosets of W modulo the Weyl group W_{P_r} of the Levi factor L_r of P_r, and F_{μ_w} denotes the irreducible finite-dimensional $(\mathfrak{l}_r, K_\infty \cap L_{r,\infty})$-module of highest weight $\mu_w = w(\Lambda + \rho_{P_0}) - \rho_{P_0}$, where $\Lambda \in \check{\mathfrak{a}}_{P_0,\mathbb{C}}$ is the highest weight of (η, E).

As already determined in [45, 6.2], the set W^{P_r}, $r = 1, 2$, of representatives for the right cosets $W_{P_r} \cdot w$ in W characterized by the condition that the minimum of the length function ℓ on $W_{P_r} \cdot w$ is attained on $W^{P_r} \cap W_{P_r} \cdot w$, and only on that element. Let w_r denote the simple reflection corresponding to α_r. One has

$$W^{P_1} = \{1, w_1, w_1w_2, w_1w_2w_1, w_1w_2w_1w_2, w_{P_1}\},$$
$$W^{P_2} = \{1, w_2, w_2w_1, w_2w_1w_2, w_2w_1w_2w_1, w_{P_2}\},$$

where w_{P_r} denotes the uniquely determined longest element (of length 5) in W^{P_r}.

It is useful to parametrize the maximal k-split torus L_0 in two convenient ways, the first one will be adjusted to the short root α_1, the second one to the long root α_2. We define (as in [71]) $t : k^* \times k^* \to L_0$ by the assignment $(a, b) \mapsto t(a, b)$ such that $\alpha_1(t(a, b)) = ab^{-1}$, $\alpha_2(t(a, b)) = a^{-1}b^2$. The other positive roots take the following values on $t(a, b)$ in this parametrization:

$$(\alpha_1 + \alpha_2)(t(a, b)) = b, \qquad (2\alpha_1 + \alpha_2)(t(a, b)) = a,$$
$$(3\alpha_1 + \alpha_2)(t(a, b)) = a^2b^{-1}, \quad (3\alpha_1 + 2\alpha_2)(t(a, b)) = ab.$$

The second parametrization, denoted by $t' : k^* \times k^* \to L_0$, is given by the assignment $(a, b) \mapsto t'(a, b)$ such that $\alpha_1(t'(a, b)) = b$, $\alpha_2(t'(a, b)) = ab^{-1}$. In this parametrization the other positive roots take the following values on $t'(a, b)$:

$$(\alpha_1 + \alpha_2)(t'(a, b)) = a, \qquad (2\alpha_1 + \alpha_2)(t'(a, b)) = ab,$$
$$(3\alpha_1 + \alpha_2)(t'(a, b)) = ab^2, \quad (3\alpha_1 + 2\alpha_2)(t'(a, b)) = a^2b.$$

As already indicated in [45, 7.4, 7.5], the elements of length 3 in W^{P_r} are the ones which matter for a possible existence of residues of Eisenstein series (see also below).

Proposition 7.1. *Let $\Lambda = c_1\Lambda_1 + c_2\Lambda_2$ be the highest weight of the algebraic representation (η, E), where $c_1, c_2 \in \mathbb{Z}$, $c_1, c_2 \geq 0$. For the element $w_1w_2w_1 \in W^{P_1}$ of length 3 the highest weight*

$$\mu_{w_1w_2w_1} = w_1w_2w_1(\Lambda + \rho_{P_0}) - \rho_{P_0}$$

of the $(\mathfrak{l}_1, K_\infty \cap L_{1,\infty})$-module $F_{\mu_{w_1w_2w_1}}$ is given by

$$(2c_2 + c_1 + 2)\omega_1,$$

where ω_1 denotes the fundamental dominant weight for L_1.

Proof. For the sake of notational brevity we write in this computation $v_1 := w_1w_2w_1$. First, for the action of v_1, we have the general formula (see also [48, Appendix III, p. 299]), $v_1(x\Lambda_2 + y\Lambda_1) = (2x + y)\Lambda_2 + (-3x - 2y)\Lambda_1$, $x, y \in \mathbb{Z}$. Thus, using the identity $\rho_{P_0} = \Lambda_1 + \Lambda_2$, we obtain

$$\mu_{v_1} = v_1(\Lambda + \rho_{P_0}) - \rho_{P_0} = (2c_2 + c_1 + 2)\Lambda_2 + (-3c_2 - 2c_1 - 6)\Lambda_1.$$

Now we use the second parametrization $t' : k^* \times k^* \to L_0$ which is adjusted to the long root α_2, the unique simple root of L_1. The fundamental dominant weight for L_1 in this parametrization is given by

$$\omega_1(t'(a, b)) = \frac{1}{2}\alpha_2(t'(a, b)) = a^{\frac{1}{2}}b^{-\frac{1}{2}}.$$

As alluded to above, $\Lambda_1 = 2\alpha_1 + \alpha_2$ and $\Lambda_2 = 3\alpha_1 + 2\alpha_2$. Hence it follows that

$$\mu_{v_1}(t'(a, b)) = (a^{\frac{1}{2}}b^{-\frac{1}{2}})^{2c_2+c_1+2} \cdot (ab)^{\frac{-c_1-6}{2}}.$$

Thus, the highest weight of the L_1-module in question is $(2c_2 + c_1 + 2)\omega_1$. □

Proposition 7.2. *Let $\Lambda = c_1\Lambda_1 + c_2\Lambda_2$ be the highest weight of the algebraic representation (η, E), where $c_1, c_2 \in \mathbb{Z}$, $c_1, c_2 \geq 0$. For the element $w_2w_1w_2 \in W^{P_2}$ of length 3 the highest weight*

$$\mu_{w_2w_1w_2} = w_2w_1w_2(\Lambda + \rho_{P_0}) - \rho_{P_0}$$

of the $(\mathfrak{l}_2, K_\infty \cap L_{2,\infty})$-module $F_{\mu_{w_2w_1w_2}}$ is given by

$$(3c_2 + 2c_1 + 4)\omega_2,$$

where ω_2 denotes the fundamental dominant weight for L_{P_2}.

Proof. With regard to the action of $v_2 := w_2w_1w_2$ in W^{P_2} we have

$$v_2(x\Lambda_2 + y\Lambda_1) = (-2x - y)\Lambda_2 + (3x + 2y)\Lambda_1.$$

Thus, we obtain

$$\mu_{v_2} = v_2(\Lambda + \rho_{P_0}) - \rho_{P_0} = (-2c_2 - c_1 - 4)\Lambda_2 + (3c_2 + 2c_1 + 4)\Lambda_1.$$

We use the parametrization $t : k^* \times k^* \to L_0$ which is adapted to the short root α_1. The fundamental dominant weight for L_2 in this parametrization is given by

$$\omega_2(t(a, b)) = \frac{1}{2}\alpha_1(t(a, b)) = a^{\frac{1}{2}}b^{-\frac{1}{2}}.$$

In terms of this parametrization we see

$$\mu_{v_2}(t(a, b)) = (ab)^{-2c_2-c_1-4} \cdot a^{3c_2+2c_1+4} = a^{c_2+c_1} \cdot b^{-2c_2-c_1-4}.$$

Using the identity $\frac{-c_2-4}{2} = \frac{(c_2+c_1)+(-2c_2-c_1-4)}{2}$, we can write

$$\mu_{v_2}(t(a, b)) = (ab)^{\frac{-c_2-4}{2}} \cdot (a^{\frac{1}{2}}b^{-\frac{1}{2}})^{3c_2+2c_1+4}.$$

Thus, the highest weight of the L_2-module in question is $(3c_2 + 2c_1 + 4)\omega_2$. □

In turn we also have to determine, in terms of the complex parameter $s \in \mathbb{C}$ corresponding to $\nu_s = \widetilde{\rho}_{P_r} \otimes s \in \check{\mathfrak{a}}_{P_r,\mathbb{C}}$, $r = 1, 2$, the point of evaluation for an Eisenstein series which is attached to a cuspidal cohomology class of L_r of type (π, w), $w \in W^{P_r}$, $\ell(w) = 3$. This point is given by $-w(\Lambda + \rho_{P_0}) \in \check{\mathfrak{a}}_{P_r,\mathbb{C}}$, obtained by restriction to $\check{\mathfrak{a}}_{P_r,\mathbb{C}}$. If (η, E) is the trivial representation this computation was done in [45, 6.2].

Proposition 7.3. *The point of evaluation for an Eisenstein series which is attached to a cuspidal cohomology class of L_r of type (π, w), $w \in W^{P_r}$, $\ell(w) = 3$ is given*
(1) *in the case P_1 by*

$$-w_1 w_2 w_1 (\Lambda + \rho_{P_0})_{|\check{\mathfrak{a}}_{P_1,\mathbb{C}}} = \frac{c_1 + 1}{2} \widetilde{\rho}_{P_1},$$

(2) *in the case P_2 by*

$$-w_2 w_1 w_2 (\Lambda + \rho_{P_0})_{|\check{\mathfrak{a}}_{P_2,\mathbb{C}}} = \frac{c_2 + 1}{2} \widetilde{\rho}_{P_2}.$$

Proof. In the case of the maximal parabolic P_1 we obtain

$$-w_1 w_2 w_1 (\Lambda + \rho_{P_0})(t'(a,b)) = (a^2 b)^{-2c_2 - c_1 - 3} \cdot (ab)^{3c_2 + 2c_1 + 5} = a^{-c_2 - 1} \cdot b^{c_2 + c_1 + 2}.$$

Recall that $ab = \Lambda_1(t'(a,b))$ and $\Lambda_1 = \widetilde{\rho}_{P_1}$. By rearranging the last formula, we get

$$-w_1 w_2 w_1 (\Lambda + \rho_{P_0})(t'(a,b)) = (ab)^{\frac{c_1+1}{2}} \cdot a^{-\frac{c_1}{2} - c_2 - \frac{3}{2}} \cdot b^{\frac{c_1}{2} + c_2 + \frac{3}{2}}.$$

It follows that the restriction of $-w_1 w_2 w_1 (\Lambda + \rho_{P_0})$ to $\check{\mathfrak{a}}_{P_1,\mathbb{C}}$ is given by $\frac{c_1+1}{2} \widetilde{\rho}_{P_1}$.

In the case of the maximal parabolic P_2 we obtain

$$-w_2 w_1 w_2 (\Lambda + \rho_{P_0})(t(a,b)) = (ab)^{2c_2 + c_1 + 3} \cdot a^{-3c_2 - 2c_1 - 5} = a^{-c_2 - c_1 - 2} \cdot b^{2c_2 + c_1 + 3}.$$

Recall that $ab = \Lambda_2(t(a,b))$ and $\Lambda_2 = \widetilde{\rho}_{P_2}$. By rearranging the last formula, we get

$$-w_2 w_1 w_2 (\Lambda + \rho_{P_0})(t(a,b)) = (ab)^{\frac{c_2+1}{2}} \cdot a^{-\frac{3c_2}{2} - c_1 - \frac{5}{2}} \cdot b^{\frac{3c_2}{2} + c_1 + \frac{5}{2}}.$$

It follows that the restriction of $-w_2 w_1 w_2 (\Lambda + \rho_{P_0})$ to $\check{\mathfrak{a}}_{P_2,\mathbb{C}}$ is given by $\frac{c_2+1}{2} \widetilde{\rho}_{P_2}$. □

7.3 Residues of Eisenstein series

In our discussion of possible residues of the Eisenstein series attached to cohomological cuspidal automorphic representations of the Levi components of the maximal parabolic k-subgroups of the group G_2 we follow the general outline of Section 4. Firstly, given a maximal parabolic k-subgroup P_r, $r = 1, 2$, of G_2, we are going to describe the adjoint action of the dual group $\mathrm{GL}_2(\mathbb{C})$ of the Levi component L_r, $r = 1, 2$, on ${}^L\mathfrak{n}_r$.

Let ρ_2 denote the two-dimensional standard representation of $\mathrm{GL}_2(\mathbb{C})$. Then the exterior square $\wedge^2 \rho_2$ is the one-dimensional representation of $\mathrm{GL}_2(\mathbb{C})$ given by $\det \rho_2$. Let $\mathrm{Sym}^3 \rho_2$ denote the 3rd symmetric power representation of ρ_2. Then the four-dimensional representation $r^o = \mathrm{Sym}^3 \rho_2 \otimes (\wedge^2 \rho_2)^{-1}$ is called the adjoint cube representation. As determined in [41] resp. [60, p. 268], the adjoint action of the L-group $\mathrm{GL}_2(\mathbb{C})$ of L_1 on ${}^L\mathfrak{n}_1$ decomposes as $r_1 \oplus r_2 = r^o \oplus \wedge^2 \rho_2$ whereas the adjoint action of the L-group $\mathrm{GL}_2(\mathbb{C})$ of L_2 on ${}^L\mathfrak{n}_2$ decomposes as $r_1 \oplus r_2 \oplus r_3 = \rho_2 \oplus \wedge^2 \rho_2 \oplus [\rho_2 \otimes \wedge^2 \rho_2]$.

Thus we obtain for the global normalizing factor in question in the case P_1 the expression

$$r^S(\nu_s, \pi, w_0) = \frac{L^S(s, \pi, r^o)}{L^S(s+1, \pi, r^o)} \cdot \frac{L^S(2s, \pi, \wedge^2 \rho_2)}{L^S(2s+1, \pi, \wedge^2 \rho_2)}.$$

In the case of the maximal parabolic k-subgroup P_2 we obtain the expression

$$r^S(\nu_s, \pi, w_0) = \frac{L^S(s, \pi, \rho_2)}{L^S(s+1, \pi, \rho_2)} \cdot \frac{L^S(2s, \pi, \wedge^2 \rho_2)}{L^S(2s+1, \pi, \wedge^2 \rho_2)} \cdot \frac{L^S(3s, \pi, \rho_2 \otimes \wedge^2 \rho_2)}{L^S(3s+1, \pi, \rho_2 \otimes \wedge^2 \rho_2)}.$$

Here $L(s, \pi, \wedge^2 \rho_2) = L(s, \omega_\pi)$ is the Hecke L-function attached to the central character ω_π, and

$$L(s, \pi, \rho_2 \otimes \wedge^2 \rho_2) = L(s, \pi \otimes \omega_\pi, \rho_2)$$

the principal L-function attached to π twisted by its central character ω_π.

Proposition 7.4. *Let P be a maximal parabolic k-subgroup with Levi decomposition $P = L_P N_P$, i.e., $L_P \cong \mathrm{GL}_2$. Let ϕ be an associate class of cuspidal automorphic representations represented by $\pi \otimes e^{\langle \lambda, H_P(\cdot)\rangle}$ of $L_P(\mathbb{A})$, where π is a unitary cuspidal automorphic representation of $L_P(\mathbb{A})$ and λ in the closure of the positive Weyl chamber associated to P. Let $\mathcal{L}_{E,\{P\},\phi} \subset \mathcal{A}_{E,\{P\},\phi}$ denote the subspace of the space of automorphic forms supported in ϕ which consists of all square-integrable automorphic forms supported in ϕ, i.e., spanned by the residues of the Eisenstein series attached to π at possible poles at λ.*

The space $\mathcal{L}_{E,\{P\},\phi}$ is trivial except possibly if the cuspidal automorphic representation π is selfdual, that is, $w_0(\pi) \cong \pi$.

Proof. The maximal parabolic subgroup P is self-associate, thus, by [48, Section IV 3.12], the Eisenstein series $E_P^G(f, \nu_s)$ attached to $f \in W_\pi$ is holomorphic in the region $\mathrm{Re}(s) > 0$ unless $w_0(\pi) \cong \pi$. Consequently, the space $\mathcal{L}_{E,\{P\},\phi}$ is trivial except possibly if π is selfdual. □

Theorem 7.5 (The case $P = P_1$). *Let ϕ be an associate class of cuspidal automorphic representations represented by $\pi \otimes e^{\langle \lambda, H_{P_1}(\cdot)\rangle}$ of $L_1(\mathbb{A})$, where π is a unitary selfdual cuspidal automorphic representation of $L_1(\mathbb{A})$ and λ in the closure of the positive Weyl chamber associated to P_1. Then the space $\mathcal{L}_{E,\{P_1\},\phi}$ is non-trivial if and only if one of the following two batches of assertions hold:*

(1) (a) $\lambda = \frac{1}{2}\tilde{\rho}_{P_1}$ *i.e.,* $s = \frac{1}{2}$,
 (b) *the central character ω_π of π is trivial,*
 (c) *the L-function $L(s, \pi, r^o)$ attached to π does not vanish at $s = \frac{1}{2}$,*

(2) (a) $\lambda = \tilde{\rho}_{P_1}$ *i.e.,* $s = 1$,
 (b) *the L-function $L(s, \pi, r^o)$ attached to π has a pole at $s = 1$.*

Theorem 7.6 (the case $P = P_2$). *Let ϕ be an associate class of cuspidal automorphic representations represented by $\pi \otimes e^{\langle \lambda, H_P(\cdot)\rangle}$ of $L_2(\mathbb{A})$, where π is a unitary selfdual cuspidal automorphic representation of $L_2(\mathbb{A})$ and λ in the closure of the positive Weyl chamber associated to P_2. Then the space $\mathcal{L}_{E,\{P_2\},\phi}$ is non-trivial if and only if the following assertions hold:*

(a) $\lambda = \frac{1}{2}\tilde{\rho}_{P_2}$, *i.e.,* $s = \frac{1}{2}$,
(b) *the central character ω_π of π is trivial,*
(c) *the principal L-function $L(s, \pi, \rho_2)$ attached to π does not vanish at $s = \frac{1}{2}$.*

Proof. The line of arguments in the proofs of the two theorems are similar. Given a cohomological unitary cuspidal automorphic representations π of $L_r(\mathbb{A})$, and given $f \in W_\pi$, we have to determine the possible poles of the corresponding Eisenstein series $E_{P_r}^G(f, \nu_s)$ in the region $\mathrm{Re}(s) > 0$.

The infinite component of π is a discrete series representation. By the Ramanujan conjecture, as proved in [12], the non-archimedean components π_v of π are tempered representations. Then, by [63, Theorems 5.3 and 5.4.], the local normalized intertwining operator $N(\nu_s, \pi_v, w_0)$ is holomorphic for $\mathrm{Re}(s) \geq 0$, and, using [57, Proposition 3.1], the operator is non-zero. Thus the possible poles of the standard intertwining operator $M(\nu_s, \pi, w_0)$ for $s \geq 0$ coincide with the poles of the global normalizing factor $r(\nu_s, \pi, w_0)$.

The L-function $L(s, \pi, \rho_2)$ converges absolutely for $\mathrm{Re}(s) > 1$ by, for example, [32], and $L(s, \pi, \rho_2)$ is non-zero in the region given by $\mathrm{Re}(s) \geq 1$, [31]. As proved in [18], $L(s, \pi, \rho_2)$ is holomorphic for $\mathrm{Re}(s) = 1$, and in fact entire. It is known, for example, by [64] that the Hecke L-function $L(s, \omega_\pi)$ is entire if ω_π is non-trivial, while it has simple poles at $s = 0$ and $s = 1$ and is holomorphic elsewhere if ω_π is trivial.

In the case of P_2, it follows that the global normalizing factor $r(\nu_s, \pi, w_0)$ does not have a pole at all half-integral arguments s with $\mathrm{Re}(s) \geq 1$. However, the factor $L(2s, \omega_\pi)$ has a pole at $s = \frac{1}{2}$ if and only if ω_π is trivial. This pole can be possibly compensated for by a zero of the first factor in the expression for $r(\nu_s, \pi, w_0)$. Note that in the case P_2 the third factor originating with $L(3s, \pi \otimes \omega_\pi)$ is entire. This proves the second theorem.

In the case P_1, we also require the analytic properties of the L-function $L(s, \pi, r^o)$. It is known from [36] that this L-function is entire if π is not monomial, while if π is monomial it may possibly have simple poles only at $s = 0$ and $s = 1$. This shows part (2) of the first theorem. For part (1) of the first theorem, we argue as above for the second theorem, with the non-vanishing of $L(s, \pi, r^o)$ at $s = \frac{1}{2}$ playing the role of the non-vanishing of $L(s, \pi, \rho_2)$ at $s = \frac{1}{2}$. □

Now we are in a position to construct square-integrable cohomology classes which are represented by residues of Eisenstein series supported in the associate class of the maximal parabolic subgroup $P = P_2$. However, in the other case, that is, $P = P_1$, we cannot show the existence of such classes. First of all, the representation spanned by residues at a possible pole of the Eisenstein series at $\lambda = \tilde{\rho}_{P_1}$, i.e., $s = 1$, would have the Langlands quotient $J(\nu_1, D_m)$ as the archimedean components, where D_m is a discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of certain lowest $\mathrm{O}(2)$-type m. However, as shown in the Appendix, such Langlands quotient is not cohomological and thus there is no contribution to cohomology coming from these residues. On the other hand, the Langlands quotient that would appear as the archimedean component of the representation spanned by the residues at $\lambda = \frac{1}{2}\tilde{\rho}_{P_1}$, i.e., $s = \frac{1}{2}$, is cohomological. However, we cannot prove the existence of a unitary cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A})$ such that the Eisenstein series attached to π has a pole at $s = \frac{1}{2}$. The problem is in the subtle non-vanishing condition for the adjoint cube L-function $L(s, \pi, r^o)$ at $s = \frac{1}{2}$ within the class of unitary cuspidal automorphic representations π with trivial central character.

7.4 Existence of residual Eisenstein cohomology classes

Let k be a totally real algebraic number field of degree $d = [k : \mathbb{Q}]$. We consider the summand

$$\bigoplus_{\phi \in \Phi_{E,\{P_2\}}} H^*_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{E,\{P_2\},\phi} \otimes E)$$

in the square-integrable cohomology $H^*_{(\mathrm{sq})}(G, E)$ of G corresponding to the associate class $\{P_2\}$ of maximal parabolic k-subgroups in G represented by P_2.

Given the irreducible finite-dimensional representation (η, E) of the group $G_\infty = \mathrm{Res}_{k/\mathbb{Q}}(\mathrm{G}_2)(\mathbb{R})$ in a complex vector space, its highest weight can be written as $\Lambda = (\Lambda)_{\iota_v}$, $v \in V_\infty$, where ι_v denotes the embedding $k \to \mathbb{R}$ which corresponds to $v \in V_\infty$. For the sake of simplicity we assume that $\Lambda_{\iota_v} = \Lambda_{\iota_{v'}}$ for all archimedean places $v, v' \in V_\infty$. Recall that this representation originates from an algebraic representation of the algebraic k-group G. We write $\Lambda = c_1\Lambda_1 + c_2\Lambda_2$, c_1, c_2 non-negative integers, where Λ_i, $i = 1, 2$, denote the fundamental dominant weights.

Theorem 7.7. *Suppose that the highest weight Λ of the representation (η, E) of G is of the form $\Lambda = c_1\Lambda_1$, that is, $c_2 = 0$. Then there exists a selfdual unitary cuspidal automorphic representation π of $L_2(\mathbb{A})$ such that for $\phi \in \Phi_{E,\{P_2\}}$ represented by $\pi \otimes e^{\langle \nu_{1/2}, H_{P_2}(\cdot)\rangle}$ we have*

$$H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{L}_{E,\{P_2\},\phi} \otimes E) \neq 0.$$

Moreover, the residual Eisenstein cohomology space $H^*_{(sq)}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{E,\{P_2\},\phi} \otimes E)$ does not vanish. In degree $q = 3d$, these classes represented by residues of Eisenstein series contribute to the total cohomology group $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{E,\{P_2\}} \otimes E) \subset H^*_{\mathrm{Eis}}(G, E)$.

Proof. First, we make sure that there exist suitable non-trivial cohomology classes of type (π, w) with $w = w_2w_1w_2$, that is, we have to analyze the space

$$H^*(\mathfrak{l}_2, K_\infty \cap L_{2,\infty}; V_{\pi_\infty} \otimes F_{\mu_w}).$$

By Proposition 7.2, the highest weight of the $(\mathfrak{l}_2, K_\infty \cap L_{2,\infty})$-module $F_{\mu_{w_2w_1w_2}}$ is given by $(3c_2 + 2c_1 + 4)\omega_2$, where ω_2 denotes the fundamental dominant weight for L_2. Under the assumption on the highest weight Λ of (η, E) the weight $\mu_{w_2w_1w_2}$ takes the form $(2c_1 + 4)\omega_2$. Since the integral coefficient $(2c_1 + 4)$ is even, there exists, using Proposition 6.2, an irreducible cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_k)$ whose central character ω_π is trivial, whose archimedean components π_v in $\pi_\infty = \bigotimes_{v \in V_\infty} \pi_v$ are discrete series representations of $\mathrm{GL}_2(\mathbb{R})$ compatible with $(2c_1 + 4)\omega_2$, and whose corresponding L-function $L(s, \pi, \rho_2)$ does not vanish at $s = \frac{1}{2}$. Note that the corresponding class in $H^*(\mathfrak{l}_2, K_\infty \cap L_{2,\infty}; V_{\pi_\infty} \otimes F_{\mu_{w_2w_1w_2}})$ is non-trivial.

Second, given such a cohomology class of type $(\pi, w_2w_1w_2)$ as just constructed, we consider the corresponding Eisenstein series $E^G_{P_2}(f, \nu_s)$. By Section 7.3, in general, the evaluation point of interest for us is

$$-w_2 w_1 w_2(\Lambda + \rho_{P_0})_{|\mathfrak{a}_{P_2,\mathbb{C}}} = \frac{c_2+1}{2}\widetilde{\rho}_{P_2}.$$

Since by our assumption $c_2 = 0$, this amounts to consider the point $\frac{1}{2}\widetilde{\rho}_{P_2}$. As a consequence of Theorem 7.6, the Eisenstein series $E^G_{P_2}(f, \nu_s)$ in question has a simple pole at $s_0 := \frac{1}{2}$. The map

$$f \cdot e^{\langle \nu_{s_0} + \rho_{P_2}, H_{P_2}(\cdot)\rangle} \mapsto (s - s_0) E^G_{P_2}(f, \nu_s)\big|_{s=s_0}$$

is an intertwining of the induced representation $I(\nu_{s_0}, \pi)$ and the space of automorphic forms on $G(k)\backslash G(\mathbb{A})$. It is non-trivial and consists of square integrable automorphic forms. Recall that the poles of the Eisenstein series coincide with the poles of the intertwining operator which accounts for the second summand in the constant term of $E^G_P(f, \nu_s)$ along P, given by, using the notation $f_s := f_{\nu_s}$,

$$E^G_P(f, \nu_s)_P(g) = f_s(g) + M(\nu_s, \pi, w_0) f_s(g),$$

where $w_0 \in W$ is the unique non-trivial Weyl group element such that $w_0(\Delta \setminus \{\alpha_1\}) \subset \Delta$, while $w_0(\alpha_1)$ is a negative root. One sees that w_0 equals w_{P_2}, the longest element in W^{P_2}. Note that the archimedean components π_v in $\pi_\infty = \bigotimes_{v \in V_\infty} \pi_v$ are discrete series representations, thus, tempered representations. By the very construction of the Langlands quotients within the classification of irreducible representations of real groups (see [44]) it follows that the image of the local operator $M(\nu_{1/2}, \pi_v, w_0)$ coincides[8] with the unique irreducible Langlands quotient of the representation $I(\nu_{1/2}, \pi_v)$. This unique irreducible quotient is usually denoted by $J(\nu_{1/2}, \pi_v)$. By construction π_v, $v \in V_\infty$, is compatible with the weight $(2c_1 + 4)\omega_2$, hence $J(\nu_{1/2}, \pi_v)$ is a representation of the real Lie group G_2 with non-vanishing relative Lie algebra cohomology. More precisely, using the notation in the Appendix, it is (up to infinitesimal equivalence) of the form $A_{\mathfrak{q}_1}(\chi_1)$ for a suitable admissible character χ_1. These non-tempered representations $J(\nu_{1/2}, \pi_v)$ have non-trivial cohomology in degree 3 and 5; it vanishes in other degrees. We obtain as a consequence that

$$H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{L}_{E,\{P_2\},\phi} \otimes E) \neq 0.$$

In particular, it is non-vanishing in the minimal degree $3d$. Finally, using [51, Theorem I.1 = III.1], we can conclude that these non-vanishing square-integrable classes represented by residues of Eisenstein series contribute non-trivially to $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{E,\{P_2\}} \otimes E) \subset H^*_{\mathrm{Eis}}(G, E)$. □

Remark 7.8. By means of the global theta lifting related to the dual reductive pair (H_Q, SL_2), where H_Q denotes a suitable orthogonal group containing G_2 as a subgroup one finds in [45] a construction of cuspidal automorphic representations which give rise to non-vanishing cohomology classes in $H^*_{\mathrm{cusp}}(G, E)$. The archimedean components of these representations are non-tempered and correspond to the irreducible unitary representations $A_{\mathfrak{q}_1}(\chi_1)$ for a suitable character χ_1. The classes so obtained are shadows of the residual cohomology classes constructed above.

8 The symplectic group of *k*-rank two

8.1 Roots, weights and parabolic subgroups

Let k be a totally real algebraic number field of degree $d = [k : \mathbb{Q}]$. We consider the k-split simple simply connected symplectic group $G = \mathrm{Sp}_2$ of k-rank two. Let P_0 be a minimal parabolic k-subgroup, and $P_0 = L_0 N_0$ its Levi decomposition, which are fixed throughout the paper. The maximal split torus L_0 is isomorphic to a product of two copies of $\mathbb{G}_m/k$, and N_0 is the unipotent radical. Let Φ, Φ^+, Δ denote the corresponding sets of roots, positive roots, simple roots, respectively. If e_i is the projection of L_0 to its ith component, then $\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = 2e_2\}$. The fundamental dominant weights are $\Lambda_1 = \alpha_1 + \frac{1}{2}\alpha_2$ and $\Lambda_2 = \alpha_1 + \alpha_2$. Let W be the Weyl group of G with respect to L_0, generated by the reflections w_i associated to the roots α_i, $i = 1, 2$.

8 This holds as well for the image of $N(\nu_{1/2}, \pi_v, w_0)$ because these two operators are proportional for tempered representations π_v.

For $r = 1, 2$, the maximal proper standard parabolic k-subgroup $P_{\Delta\setminus\{\alpha_r\}}$ corresponding to the subset $\Delta \setminus \{\alpha_r\}$ of Δ is denoted in short by P_r, and its Levi decomposition by $P_r = L_r N_r$, where L_r is the Levi factor, and N_r the unipotent radical. For $r = 1$ we have $L_1 \cong \mathrm{GL}_1 \times \mathrm{SL}_2$, and for $r = 2$ we have $L_2 \cong \mathrm{GL}_2$. Observe that the parabolic subgroups P_r are self-associate. However, P_r is conjugate to its opposite parabolic subgroup P_r^{opp} by a representative of the unique non-trivial Weyl group element $w_0 \in W$ with the property that $w_0(\Delta \setminus \{\alpha_r\}) \subset \Delta$.

As in the previous case, we choose as a suitable basis for $\check{\mathfrak{a}}_{P_r,\mathbb{C}} \cong \mathbb{C}$ the element $\widetilde{\rho}_{P_r} = \langle \rho_{P_r}, \check{\alpha}_r\rangle^{-1}\rho_{P_r}$. We obtain, as already observed in [53],

$$\widetilde{\rho}_{P_1} = \frac{1}{2}\rho_{P_1}, \quad \widetilde{\rho}_{P_2} = \frac{2}{3}\rho_{P_2}.$$

We always identify accordingly $s \in \mathbb{C}$ with $\nu_s = \widetilde{\rho}_{P_r} \otimes s \in \check{\mathfrak{a}}_{P_r,\mathbb{C}}$.

8.2 Classes of type (π, w), $w \in W^P$

Given a maximal parabolic k-subgroup P_r, $r = 1, 2$, of Sp_2, the way to analyze which types (π, w), $w \in W^{P_r}$, occur is analogous to the case of the group G_2 dealt with. Based on the computations in [53, 56], we obtain the following results. First, the Lie algebra cohomology $H^*(\mathfrak{n}_r, E)$ of $\mathfrak{n}_r := \mathfrak{n}_{P_r}$ with coefficients in the irreducible representation (η, E) of Sp_2 is given as an $(\mathfrak{l}_r, K_\infty \cap L_{r,\infty})$-module as the sum

$$H^*(\mathfrak{n}_r, E) = \bigoplus_{w \in W^{P_r}} F_{\mu_w},$$

where the sum ranges over w in the set W^{P_r} of the minimal coset representatives for the right cosets of W_{P_r} in W, and F_{μ_w} denotes the irreducible finite-dimensional $(\mathfrak{l}_r, K_\infty \cap L_{r,\infty})$-module of highest weight $\mu_w = w(\Lambda + \rho_{P_0}) - \rho_{P_0}$, where $\Lambda \in \check{\mathfrak{a}}_{P_0,\mathbb{C}}$ is the highest weight of (η, E).

As already proved in [56], the elements of length 2 in W^{P_r} are the ones of interest for us; only in that case a residue of an Eisenstein series is possible. Following [56] one has:

Proposition 8.1. *Let $\Lambda = c_1\Lambda_1 + c_2\Lambda_2$ be the highest weight of the algebraic representation (η, E), where $c_1, c_2 \in \mathbb{Z}$, $c_1, c_2 \geq 0$.*
(1) *For the element $w_1 w_2 \in W^{P_1}$ of length 2 the highest weight $\mu_{w_1w_2}$ of the $(\mathfrak{l}_1, K_\infty \cap L_{1,\infty})$-module $F_{\mu_{w_1w_2}}$ is given by $(c_1 + c_2 + 1)\omega_1$.*
(2) *For the element $w_2 w_1 \in W^{P_2}$ of length 2 the highest weight $\mu_{w_2w_1}$ of the $(\mathfrak{l}_2, K_\infty \cap L_{2,\infty})$-module $F_{\mu_{w_2w_1}}$ is given by $(c_1 + 2c_2 + 2)\omega_2$,*
where ω_r denotes the fundamental dominant weight for L_r, $r = 1, 2$.

Proposition 8.2. *The point of evaluation for an Eisenstein series which is attached to a cuspidal cohomology class of L_r of type (π, w), $w \in W^{P_r}$, $\ell(w) = 2$ is given*
(1) *in the case P_1 by*

$$-w_1w_2(\Lambda + \rho_{P_0})_{|\check{\mathfrak{a}}_{P_1,\mathbb{C}}} = (c_2 + 1)\widetilde{\rho}_{P_1},$$

(2) *in the case P_2 by*

$$-w_2w_1(\Lambda + \rho_{P_0})_{|\check{\mathfrak{a}}_{P_2,\mathbb{C}}} = \frac{c_1 + 1}{2}\widetilde{\rho}_{P_2}.$$

8.3 Residues of Eisenstein series

In the case of the symplectic group Sp_2, the discussion of possible residues of the Eisenstein series attached to cohomological cuspidal automorphic representations of the Levi components of the maximal parabolic k-subgroups was essentially carried through in the case $\mathrm{Sp}_2/\mathbb{Q}$ in [56]. In the general case Sp_2/k, the residual spectrum was determined in [33] resp. [37]. We give the final results and briefly indicate the main points in the line of argument.

Let P be a maximal parabolic k-subgroup with Levi decomposition $P = LN$. Let ϕ be an associate class of cuspidal automorphic representations represented by $\pi \otimes e^{\langle \lambda, H_P(\cdot) \rangle}$ of $L(\mathbb{A})$, where π is a unitary cuspidal automorphic representation of $L(\mathbb{A})$. Let $\mathcal{L}_{E,\{P\},\phi} \subset \mathcal{A}_{E,\{P\},\phi}$ denote the subspace of the space of automorphic forms supported in ϕ which consists of all square-integrable automorphic forms in ϕ, i.e., spanned by the residues of the Eisenstein series attached to π at possible poles at λ.

Theorem 8.3 (The case $P = P_2$). *Let P be a maximal parabolic k-subgroup of type P_2 with Levi decomposition $P_2 = L_2N_2$, i.e., $L_2 \cong \mathrm{GL}_2$. Then the following holds:*
(1) *The space $\mathcal{L}_{E,\{P\},\phi}$ is trivial except possibly if the cuspidal automorphic representation π is selfdual, that is, $w_0(\pi) \cong \pi$.*
(2) *Given a cuspidal automorphic representation $\pi \otimes e^{\langle \lambda, H_P(\cdot) \rangle}$ of $L_2(\mathbb{A})$, the space $\mathcal{L}_{E,\{P\},\phi}$ is non-trivial if and only if the following assertions hold:*
 (a) *π is selfdual, that is, $w_0(\pi) \cong \pi$,*
 (b) *$\lambda = \frac{1}{2}\tilde{\rho}_P$, i.e., $s = \frac{1}{2}$,*
 (c) *the central character of π is trivial,*
 (d) *the principal L-function $L(s, \pi, \rho_2)$ attached to π does not vanish at $s = \frac{1}{2}$, that is, $L(\frac{1}{2}, \pi, \rho_2) \neq 0$.*

Proof. The maximal parabolic group P of type P_2 is self-associate, thus, by [48, Section IV 3.12], the Eisenstein series $E_{P_2}^{\mathrm{Sp}_2}(f, \nu_s)$ attached to $f \in W_\pi$ is holomorphic in the region $\mathrm{Re}(s) > 0$ unless π is selfdual. Consequently, the space $\mathcal{L}_{E,\{P_2\},\phi}$ is trivial except possibly if $w_0(\pi) \cong \pi$. This proves (1).

Given a cuspidal automorphic representations π of $L_2(\mathbb{A})$, and given $f \in W_\pi$, we have to determine the possible poles of the corresponding Eisenstein series $E_{P_2}^G(f, \nu_s)$ in the region $\mathrm{Re}(s) > 0$.

By [11, Theorem 11.1], in the global intertwining operator the local normalized intertwining operator $N(\nu_s, \pi_v, w_0)$ is holomorphic and non-vanishing for $s \geq 0$.[9] Thus the possible poles of the standard intertwining operator $M(\nu_s, \pi, w_0)$ for $s \geq 0$ coincide with the poles of the global normalizing factor $r(\nu_s, \pi, w_0)$.

As determined in [41, case (vi)] the adjoint action of the L-group $\mathrm{GL}_2(\mathbb{C})$ of L_2 on ${}^L\mathfrak{n}_2$ decomposes as $r_1 \oplus r_2 = \rho_2 \oplus \wedge^2\rho_2$, where ρ_2 denotes the two-dimensional standard representation of $\mathrm{GL}_2(\mathbb{C})$.[10] Thus we obtain for the global normalizing factor the expression

$$r(\lambda_s, \pi, w_0) = \frac{L(s, \pi)}{L(1+s, \pi)} \cdot \frac{L(2s, \pi, \wedge^2\rho_2)}{L(1+2s, \pi, \wedge^2\rho_2)},$$

where $L(s, \pi, \wedge^2\rho_2)$ denotes the exterior square L-function. We note that the exterior square $\wedge^2\rho_2$ is the determinant $\det\rho_2$ and hence $L(s, \pi, \wedge^2\rho_2) = L(s, \omega_\pi)$ is the Hecke L-function attached to the central character ω_π of π.

A careful investigation of the analytic properties of this normalizing factor as in [33, Theorem 3.3] resp. [37, Theorem 4.4] yields the assertions regarding the existence of a pole and a corresponding residue of the Eisenstein series $E_{P_2}^G(f, \nu_s)$ in the region $\mathrm{Re}(s) > 0$. □

Next we deal with the case of a maximal parabolic k-subgroup of type P_1. In this case one has the Levi decomposition $P_1 = L_1N_1$ with $L_1 \cong \mathrm{GL}_1 \times \mathrm{SL}_2$.

Given an irreducible unitary cuspidal automorphic representation π of $L_1(\mathbb{A})$ with $\pi = \chi \otimes \sigma$, $\sigma = \bigotimes'_{v\in V} \sigma_v$, a unitary cuspidal automorphic representation of $\mathrm{SL}_2(\mathbb{A})$, $\chi = \bigotimes'_{v\in V} \chi_v$, a unitary Hecke character of k, and given $f \in W_\pi$, we have to determine the possible poles of the corresponding Eisenstein series $E_{P_1}^G(f, \nu_s)$ in the region $\mathrm{Re}(s) > 0$.

In general, following [39, 2.5], given a local component σ_v, $v \in V$, of the cuspidal automorphic representation σ of $\mathrm{SL}_2(\mathbb{A})$, there exists an irreducible unitary representation σ_v^+ of $\mathrm{GL}_2(k_v)$ such that σ_v is contained

9 In the case of interest for us that π is cohomological one can also argue in this way: The infinite component of π is a discrete series representation. By the Ramanujan conjecture as proved in [12] the non-archimedean components π_v of π are tempered representations. Then, by [63, Theorems 5.3 and 5.4], the local intertwining operator is holomorphic for $\mathrm{Re}(s) \geq 0$, and, using [57, Proposition 3.1], the operator is non-zero.
10 See also [59, Section 4].

as a subrepresentation in $\sigma_v^+|_{\mathrm{SL}_2(k_v)}$. One can choose the family $\{\sigma_v^+\}_{v\in V}$ in such a way that $\sigma^+ := \bigotimes'_{v\in V} \sigma_v^+$ is a unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. We denote the local Gelbart–Jacquet lift [17] of σ_v^+ from GL_2 to GL_3 by Σ_v, $v \in V$. This lift depends only on the representation σ_v, it is independent of the choice of σ_v^+. By an analysis of the local L-function $L(s, \pi_v, r_1)$, $v \notin S$, (see [33, Section 4]), one obtains the identity

$$L^S(s, \pi, r_1) = L^S(s, \Sigma \otimes \chi),$$

where S denotes a finite set of places containing V_∞ so that for every $v \notin S$ the group $L_1(k_v)$ and π_v are both unramified at v and where r_1 denotes the adjoint action of the L-group $\mathrm{GL}_1(\mathbb{C}) \times \mathrm{SO}_3(\mathbb{C})$ of L_1 on ${}^L\mathfrak{n}_1$, given by the tensor product of the standard representation ρ_1 of $\mathrm{GL}_1(\mathbb{C})$ and the standard representation τ_0 of $\mathrm{SO}_3(\mathbb{C})$, as determined in [41, case (xx)]. The right-hand side denotes the principal L-function of GL_3 attached to the tensor product of $\Sigma := \bigotimes'_v \Sigma_v$ and χ. This L-function converges absolutely for $\mathrm{Re}(s) > 1$ and is non-zero in that region. By [17, Theorem 9.3], this partial L-function is entire for any χ if σ^+ is not monomial.

By definition, if σ^+ is monomial, there exists a non-trivial unitary Hecke character δ of k such that $\sigma^+ \otimes \delta \cong \sigma^+$. Comparing the central character of both sides we see that $\delta^2 = 1$. We denote by K/k the quadratic extension which corresponds by class field theory to $\delta = \delta_{K/k}$. As shown in [39, Lemma 6.5] there exists a unitary Hecke character θ of K such that $\sigma^+ = \pi(\theta)$, that is, σ^+ is obtained by automorphic induction. Since σ^+ is a cuspidal automorphic representation, we have $\theta \neq \theta^c$, $c \in \mathrm{Gal}(K/k)$, $c \neq 1$, i.e., θ does not factor through the norm map from K to k. See Section 6.4 for more details. The Gelbart–Jacquet lift Σ of σ^+ is described in terms of $\pi(\theta)$ as the induced representation $\mathrm{Ind}_{Q(\mathbb{A})}^{\mathrm{GL}_3(\mathbb{A})}(\pi(\theta(\theta^c)^{-1}) \otimes \delta_{K/k})$, where Q denotes the maximal parabolic subgroup of GL_3 of type $(2, 1)$. In the sequel one has to distinguish the two cases whether $\pi(\theta(\theta^c)^{-1})$ is a cuspidal representation or not. In the former case, one obtains

$$L(s, \Sigma \otimes \chi) = L(s, \pi(\theta(\theta^c)^{-1}) \otimes \chi) \cdot L(s, \chi\delta_{K/k}).$$

Thus, $L(s, \Sigma \otimes \chi)$ has a simple pole at $s = 1$ if $\chi = \delta_{K/k}$.

In the latter case, that is, $\pi(\theta(\theta^c)^{-1})$ is not a cuspidal representation, there are exactly three possible choices $(K/k, \theta)$, $(K'/k, \theta')$, $(K''/k, \theta'')$ for the quadratic extension K/k and the unitary Hecke character θ such that $\sigma^+ = \pi(\theta) = \pi(\theta') = \pi(\theta'')$, see [39, p. 774]. Thus, one has that

$$L(s, \Sigma \otimes \chi) = L(s, \chi\delta_{K/k}) L(s, \chi\delta_{K'/k}) L(s, \chi\delta_{K''/k})$$

and the left-hand side has a pole at $s = 1$ if χ coincides with one of these three possible characters δ.

Finally, following the discussion of the intertwining operator at the places $v \notin S$ in [33, Section 4] resp. [37, Section 5] we arrive at the conclusion:

Theorem 8.4 (The case $P = P_1$). *Let P be a maximal parabolic k-subgroup of type P_1 with Levi decomposition $P_1 = L_1 N_1$, i.e., $L_1 \cong \mathrm{GL}_1 \times \mathrm{SL}_2$. Given a cuspidal automorphic representation $\pi \otimes e^{\langle \lambda, H_{P_1}(\cdot)\rangle}$ of $L_1(\mathbb{A})$ with $\pi = \chi \otimes \sigma$, σ a cuspidal automorphic representation of $\mathrm{SL}_2(\mathbb{A})$, χ a unitary Hecke character of k, the space $\mathcal{L}_{E,\{P_1\},\phi}$ is non-trivial if and only if the following assertions hold:*

(a) *The cuspidal representation σ is monomial, that is, with σ^+ of the form $\pi(\theta)$ for some quadratic extension K/k and some unitary Hecke character θ of K and χ determined by σ either as $\chi = \delta_{K/k}$ or as one of the three choices $\chi \in \{\delta_{K/k}, \delta_{K'/k}, \delta_{K''/k}\}$ in the notation as above.*

(b) *$\lambda = \tilde{\rho}_{P_1}$, i.e., $s = 1$.*

8.4 Existence of residual Eisenstein cohomology classes

Now we suppose that k is a totally real algebraic number field of degree $d = [k : \mathbb{Q}]$. We consider the summand

$$\bigoplus_{\phi \in \Phi_{E,\{P\}}} H^*_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{E,\{P\},\phi} \otimes E)$$

in the square-integrable cohomology corresponding to the associate class $\{P\}$ of maximal parabolic k-subgroups in $G = \mathrm{Sp}_2$ represented by P. We have to distinguish the two cases $P = P_1$ and $P = P_2$. Given the irreducible finite-dimensional representation (η, E) of the group $G_\infty = \mathrm{Res}_{k/\mathbb{Q}}(\mathrm{Sp}_2)(\mathbb{R})$ in a complex vector space, its highest weight can be written as $\Lambda = (\Lambda)_{\iota_v}$, $v \in V_\infty$. For the sake of simplicity we assume that $\Lambda_{\iota_v} = \Lambda_{\iota_{v'}}$ for all archimedean places $v, v' \in V_\infty$.

Theorem 8.5 (The case $P = P_2$). *Suppose that the highest weight Λ of the representation (η, E) of $G = \mathrm{Sp}_2$ is of the form $\Lambda = c_2\Lambda_2$, that is, $c_1 = 0$. Then there exists a selfdual unitary cuspidal automorphic representation π of $L_2(\mathbb{A})$ such that for $\pi \otimes e^{\langle \nu_{1/2}, H_{P_2}(\cdot)\rangle}$ we have*

$$H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{L}_{E,\{P_2\},\phi} \otimes E) \neq 0.$$

Moreover, the residual Eisenstein cohomology space $H^*_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{E,\{P_2\},\phi} \otimes E)$ does not vanish. In degree $q = 2d$, these classes represented by residues of Eisenstein series contribute to the total cohomology group $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{E,\{P_2\}} \otimes E) \subset H^*_{\mathrm{Eis}}(G, E)$.

Proof. The line of argument is similar to the proof of Theorem 7.7, thus we can be brief. Following Proposition 8.1, the highest weight of the $(\mathfrak{l}_2, K_\infty \cap L_{2,\infty})$-module $F_{\mu_{w_2w_1}}$ is given by $(c_1 + 2c_2 + 2)\omega_2$, where ω_2 denotes the fundamental dominant weight for L_2. Under the assumption on the highest weight Λ of (η, E) the weight $\mu_{w_2w_1}$ takes the form $(2c_2 + 2)\omega_2$. Since the integral coefficient $(2c_2 + 2)$ is even, there exists, using Proposition 6.2, an irreducible cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_k)$ whose central character ω_π is trivial, whose archimedean components π_v in $\pi_\infty = \bigotimes_{v\in V_\infty} \pi_v$ are discrete series representations of $\mathrm{GL}_2(\mathbb{R})$ compatible with $(2c_2 + 2)\omega_2$, and whose corresponding L-function $L(s, \pi, \rho_2)$ does not vanish at $s = \frac{1}{2}$. Note that the corresponding class in $H^*(\mathfrak{l}_2, K_\infty \cap L_{2,\infty}; V_{\pi_\infty} \otimes F_{\mu_{w_2w_1}})$ is non-trivial.

Second, given such a cohomology class of type (π, w_2w_1) as constructed, we consider the corresponding Eisenstein series $E^G_{P_2}(f, \nu_s)$. By Proposition 8.2, the evaluation point is

$$-w_2w_1(\Lambda + \rho_{P_0})_{|\check{\mathfrak{a}}_{P_2,\mathbb{C}}} = \frac{c_1 + 1}{2}\widetilde{\rho}_{P_2}.$$

Since by assumption $c_1 = 0$, this amounts to consider the point $\frac{1}{2}\widetilde{\rho}_{P_2}$. By Theorem 8.3, the Eisenstein series $E^G_{P_2}(f, \nu_s)$ in question has a simple pole at $s_0 := \frac{1}{2}$. The map

$$f \cdot e^{\langle \nu_{s_0} + \rho_P, H_P(\cdot)\rangle} \mapsto (s - s_0)E^G_{P_2}(f, \nu_s)\big|_{s=s_0}$$

is an intertwining of the induced representation $I(\nu_{s_0}, \pi)$ and the space of automorphic forms on the group $\mathrm{Sp}_2(k)\backslash\mathrm{Sp}_2(\mathbb{A})$. It is non-trivial and consists of square integrable automorphic forms. Recall that the poles of the Eisenstein series coincide with the poles of the intertwining operator which accounts for the second summand in the constant term of $E^G_{P_2}(f, \nu_s)$ along P_2, given by, using the notation $f_s := f_{\nu_s}$,

$$E^G_{P_2}(f, \nu_s)_{P_2}(g) = f_s(g) + M(\nu_s, \pi, w_0)f_s(g),$$

where $w_0 \in W$ is the unique non-trivial Weyl group element such that $w_0(\Delta \setminus \{\alpha_1\}) \subset \Delta$, while $w_0(\alpha_1)$ is negative. One sees that w_0 equals w_{P_2}, the longest element in W^{P_2}. Note that the archimedean components π_v in $\pi_\infty = \bigotimes_{v\in V_\infty} \pi_v$ are discrete series representations, thus, tempered representations. By the very construction of the Langlands quotients within the classification of irreducible representations of real groups (see [44]) it follows that the image of the local operator $M(\nu_{1/2}, \pi_v, w_0)$ coincides with the unique irreducible Langlands quotient of the representation $I(\nu_{1/2}, \pi_v)$. This unique irreducible quotient is usually denoted by $J(\nu_{1/2}, \pi_v)$. By construction π_v, $v \in V_\infty$, is compatible with the weight $(2c_2 + 2)\omega_2$, hence $J(\nu_{1/2}, \pi_v)$ is a representation of the real Lie group $\mathrm{Sp}_2(\mathbb{R})$ with non-vanishing relative Lie algebra cohomology. More precisely, it is (up to infinitesimal equivalence) of the form $A_\mathfrak{q}(\chi)$ for a suitable admissible character χ, $\mathfrak{q}$ a θ-stable parabolic subalgebra. This non-tempered representation $J(\nu_{1/2}, \pi_v)$ has non-trivial cohomology in degree 2 and 4; it vanishes in other degrees. We obtain as a consequence that

$$H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{L}_{E,\{P_2\},\phi} \otimes E) \neq 0.$$

In particular, it is non-vanishing in the minimal degree $2d$. Finally, using [51, Theorem I.1 = III.1], we can conclude that these non-vanishing square-integrable classes represented by residues of Eisenstein series contribute non-trivially to $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{E,\{P_2\}} \otimes E) \subset H^*_{\mathrm{Eis}}(G, E)$. □

Theorem 8.6 (The case $P = P_1$). *Suppose that the highest weight Λ of the representation (η, E) of $G = \mathrm{Sp}_2$ is of the form $\Lambda = c_1\Lambda_1$, that is, $c_2 = 0$. Then there exists a monomial unitary cuspidal automorphic representation π of $L_1(\mathbb{A})$ such that for the associate class ϕ represented by $\pi \otimes e^{\langle \nu_1, H_{P_1}(\cdot)\rangle}$ we have*

$$H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{L}_{E,\{P_1\},\phi} \otimes E) \neq 0.$$

Moreover, the residual Eisenstein cohomology space $H^*_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{E,\{P_1\},\phi} \otimes E)$ does not vanish. In degree $q = 2d$, these classes represented by residues of Eisenstein series contribute to the total cohomology group $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{E,\{P_1\}} \otimes E) \subset H^*_{\mathrm{Eis}}(G, E)$.

Proof. Though some of the ingredients are different, the line of argument is similar to the proof of Theorem 8.5, thus we can be brief. Following Proposition 8.1, the highest weight of the $(\mathfrak{l}_1, K_\infty \cap L_{1,\infty})$-module $F_{\mu_{w_1 w_2}}$ is given by $(c_1 + c_2 + 1)\omega_1$, where ω_1 denotes the fundamental dominant weight for L_1. Under the assumption on the highest weight Λ of (η, E) the weight $\mu_{w_1 w_2}$ takes the form $(c_1 + 1)\omega_1$. There exists, using Proposition 6.4, an irreducible monomial cuspidal automorphic representation σ^+ of $\mathrm{GL}_2(\mathbb{A}_k)$ of the form $\pi(\theta)$ for some imaginary quadratic extension K/k and some unitary Hecke character θ such that the archimedean components σ_v in $\sigma := \sigma^+|_{\mathrm{SL}_2}$ are discrete series representations of $\mathrm{SL}_2(\mathbb{R})$ compatible with $(c_1 + 1)\omega_1$. Note that the corresponding class attached to the cuspidal representation $\pi = \chi \otimes \sigma$ with χ as indicated in Theorem 8.4 in $H^*(\mathfrak{l}_1, K_\infty \cap L_{1,\infty}; V_{\pi_\infty} \otimes F_{\mu_{w_1 w_2}})$ is non-trivial.

Second, given such a cohomology class of type $(\pi, w_1 w_2)$ as constructed, we consider the corresponding Eisenstein series $E^G_{P_1}(f, \nu_s)$. By Proposition 8.2, the evaluation point is

$$-w_1 w_2(\Lambda + \rho_{P_0})_{|\check{\mathfrak{a}}_{P_1,\mathbb{C}}} = (c_2 + 1)\tilde{\rho}_{P_1}.$$

Since by assumption $c_2 = 0$, this amounts to consider the point $\nu_1 = \tilde{\rho}_{P_1}$. By Theorem 8.4, the Eisenstein series $E^G_{P_1}(f, \nu_s)$ in question has a simple pole at $s_0 := 1$. Thus its residues span the space $\mathcal{L}_{E,\{P_1\},\phi}$.

By looking at the intertwining operator as in the previous proof, the archimedean components of this space are isomorphic to the Langlands quotient $J(\nu_1, \pi_v)$. By construction π_v, $v \in V_\infty$, is compatible with the weight $(c_1 + 1)\omega_1$, hence $J(\nu_1, \pi_v)$ is a representation of the real Lie group $\mathrm{Sp}_2(\mathbb{R})$ with non-vanishing relative Lie algebra cohomology. More precisely, it is (up to infinitesimal equivalence) of the form $A_{\mathfrak{q}}(\xi)$ for a suitable admissible character ξ, $\mathfrak{q}$ a θ-stable parabolic subalgebra. This non-tempered representation $J(\nu_1, \pi_v)$ has non-trivial cohomology in degree 2 and 4; it vanishes in other degrees. We obtain as a consequence that

$$H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{L}_{E,\{P_1\},\phi} \otimes E) \neq 0.$$

In particular, it is non-vanishing in the minimal degree $2d$. Finally, using [51, Theorem I.1 = III.1], we can conclude that these non-vanishing square-integrable classes represented by residues of Eisenstein series contribute non-trivially to $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{E,\{P_1\}} \otimes E) \subset H^*_{\mathrm{Eis}}(G, E)$. □

9 The odd special orthogonal group of *k*-rank two

9.1 Residues of Eisenstein series

As before let k be a totally real algebraic number field. We now consider the k-split odd special orthogonal group $G = \mathrm{SO}_5$ of k-rank two. Note that at the archimedean places $\mathrm{SO}_5(\mathbb{R}) = \mathrm{SO}(3, 2)$, the special orthogonal group of signature $(3, 2)$. The residual spectrum for this group is studied by Kim in [35]. We retain the notation of the previous section with minor adjustments. As before P_0 is a fixed minimal parabolic k-subgroup, with the Levi decomposition $P_0 = L_0 N_0$. Then L_0 is isomorphic to a product of two copies of $\mathbb{G}_m/k$. The set of simple roots is

$$\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2\},$$

where e_i is the projection of L_0 to its ith component. The fundamental weights are given by

$$\Lambda_1 = e_1 = \alpha_1 + \alpha_2 \quad \text{and} \quad \Lambda_2 = \frac{1}{2}(e_1 + e_2) = \frac{1}{2}\alpha_1 + \alpha_2.$$

The Weyl group of SO_5 is

$$W = \{1, w_1, w_2, w_1w_2, w_2w_1, w_1w_2w_1, w_2w_1w_2, w_1w_2w_1w_2\},$$

where w_i is the simple reflection with respect to α_i, $i = 1, 2$.

For $r = 1, 2$, the maximal proper standard parabolic k-subgroup $P_{\Delta\setminus\{\alpha_r\}}$ is denoted by P_r, with Levi decomposition $P_r = L_rN_r$. Then $L_1 \cong \mathrm{GL}_1 \times \mathrm{SO}_3$ and $L_2 \cong \mathrm{GL}_2$. We have

$$\widetilde{\rho}_{P_1} = \frac{2}{3}\rho_{P_1} \quad \text{and} \quad \widetilde{\rho}_{P_2} = \frac{1}{2}\rho_{P_2}.$$

Both maximal parabolic subgroups P_r, $r = 1, 2$, are self-associate. Observe that $\widetilde{\rho}_{P_r}$ coincides with the fundamental weight Λ_r, $r = 1, 2$.

Theorem 9.1. *Let P_1 be the maximal standard parabolic k-subgroup as above, with Levi factor $L_1 \cong \mathrm{GL}_1 \times \mathrm{SO}_3$. Let $\pi = \chi \otimes \sigma$ be a unitary cuspidal automorphic representation of $L_1(\mathbb{A})$, where χ is a unitary Hecke character of $\mathbb{I}_k$, and σ a unitary cuspidal automorphic representation of $\mathrm{SO}_3(\mathbb{A})$. Let ϕ be the associate class of cuspidal automorphic representations represented by $\pi \otimes e^{\langle\lambda, H_{P_1}(\cdot)\rangle}$, where $\lambda \in \mathrm{Re}\, X^G_{P_1}$ is in the positive Weyl chamber determined by P_1. Then the space $\mathcal{L}_{E,\{P_1\},\phi}$ of square-integrable automorphic forms supported in ϕ is non-trivial if and only if the following assertions hold:*

(a) $\lambda = \frac{1}{2}\widetilde{\rho}_{P_1} = \frac{1}{2}e_1$, *i.e.*, $s = \frac{1}{2}$,

(b) χ^2 *is the trivial character of* $\mathbb{I}$,

(c) *the principal L-function $L(s, \chi \otimes \sigma', \rho_2)$ attached to $\chi \otimes \sigma'$ is non-zero at $s = \frac{1}{2}$, where σ' is the representation of $\mathrm{GL}_2(\mathbb{A}_k)$ with trivial central character obtained from σ via identification of SO_3 with PGL_2.*

Proof. The normalizing factor that determines the poles of the Eisenstein series at $s > 0$ is given by

$$r(s, \pi, w_0) = \frac{L(s, \chi \otimes \sigma, \rho_1 \otimes \rho_2)}{L(1 + s, \chi \otimes \sigma, \rho_1 \otimes \rho_2)} \cdot \frac{L(2s, \chi, \mathrm{Sym}^2\, \rho_1)}{L(1 + 2s, \chi, \mathrm{Sym}^2\, \rho_1)}.$$

The latter L-function is $L(2s, \chi, \mathrm{Sym}^2\, \rho_1) = L(2s, \chi^2)$, the Hecke L-function attached to the Hecke character χ^2. Since SO_3 may be identified with PGL_2, the Rankin–Selberg L-function $L(s, \chi \otimes \sigma, \rho_1 \otimes \rho_2)$ is the same as the principal L-function $L(s, \chi \otimes \sigma')$, where σ' is obtained from σ via the identification. Hence, the conditions in the theorem follow from the properties of the principal L-functions for GL_2 and the Hecke L-functions. □

Theorem 9.2. *Let P_2 be the maximal standard parabolic k-subgroup as above, with Levi factor $L_2 \cong \mathrm{GL}_2$. Let π be a unitary cuspidal automorphic representation of $L_2(\mathbb{A}_k)$. Let ϕ be the associate class of cuspidal automorphic representations represented by $\pi \otimes e^{\langle\lambda, H_{P_2}(\cdot)\rangle}$, where $\lambda \in \mathrm{Re}X^G_{P_2}$ is in the positive Weyl chamber determined by P_2. Then the space $\mathcal{L}_{E,\{P_2\},\phi}$ of square-integrable automorphic forms supported in ϕ is non-trivial if and only if the following assertions hold:*

(a) $\lambda = \widetilde{\rho}_{P_2} = \frac{1}{2}(e_1 + e_2)$, *i.e.*, $s = 1$,

(b) *the symmetric square L-function $L(s, \pi, \mathrm{Sym}^2\, \rho_2)$ attached to π has a pole at $s = 1$.*

Proof. The poles of Eisenstein series for $s > 0$ are determined by the normalizing factor

$$r(s, \pi, w_0) = \frac{L(s, \pi, \mathrm{Sym}^2\, \rho_2)}{L(1 + s, \pi, \mathrm{Sym}^2\, \rho_2)}.$$

The properties of the symmetric square L-function $L(s, \pi, \mathrm{Sym}^2\, \rho_2)$ are known from [20]. □

9.2 Existence of residual Eisenstein cohomology classes

As explained before, we now have to analyze which types (π, w), $w \in W^{P_r}$, may possibly contribute to the square-integrable cohomology spaces supported in P_r, $r = 1, 2$. Therefore, given $w \in W^{P_r}$, we need to compute the highest weight μ_w and the evaluation point $\lambda_{[w]}$. Besides μ_w, we provide the infinitesimal character χ_w of the unitary cuspidal automorphic representation π of $L_r(\mathbb{A})$ at every archimedean place. Since π must be cohomological, this forces the archimedean components of π to be the discrete series representations compatible with μ_w, i.e., of infinitesimal character χ_w as explained in Section 5. The results are given in Table 9.1

$w \in W^{P_1}$	$\lambda_{[w]} = ? \cdot \widetilde{\rho}_{P_1}$	$\mu_w = ? \cdot \omega'$	χ_w
1	$-\frac{2c_1+c_2+3}{2}$	c_2	$-\frac{c_2+1}{2}e_2$
w_1	$-\frac{c_2+1}{2}$	$2c_1+c_2+2$	$-\frac{2c_1+c_2+3}{2}e_2$
w_1w_2	$\frac{c_2+1}{2}$	$2c_1+c_2+2$	$\frac{2c_1+c_2+3}{2}e_2$
$w_1w_2w_1$	$\frac{2c_1+c_2+3}{2}$	c_2	$\frac{c_2+1}{2}e_2$

Table 9.1: The evaluation points $\lambda_{[w]}$, highest weights μ_w and infinitesimal characters χ_w for $w \in W^{P_1}$ in the case $G = \mathrm{SO}_5$, where ω' is the fundamental weight for GL_2 under the identification of SO_3 and PGL_2.

$w \in W^{P_2}$	$\lambda_{[w]} = ? \cdot \widetilde{\rho}_{P_2}$	$\mu_w = ? \cdot \omega_2$	χ_w
1	$-(c_1+c_2+2)$	c_1	$-\frac{c_1+1}{2}e_1 + \frac{c_1+1}{2}e_2$
w_2	$-(c_1+1)$	c_1+c_2+1	$-\frac{c_1+c_2+2}{2}e_1 + \frac{c_1+c_2+2}{2}e_2$
w_2w_1	c_1+1	c_1+c_2+1	$-\frac{c_1+c_2+2}{2}e_1 + \frac{c_1+c_2+2}{2}e_2$
$w_2w_1w_2$	c_1+c_2+2	c_1	$-\frac{c_1+1}{2}e_1 + \frac{c_1+1}{2}e_2$

Table 9.2: The evaluation points $\lambda_{[w]}$, highest weights μ_w and infinitesimal characters χ_w for $w \in W^{P_2}$ in the case $G = \mathrm{SO}_5$, where ω_2 is the fundamental weight for the Levi factor $L_2 \cong \mathrm{GL}_2$.

for P_1 and Table 9.2 for P_2. The highest weight of E is written in terms of the fundamental weights Λ_1, Λ_2 as

$$\Lambda = c_1\Lambda_1 + c_2\Lambda_2,$$

where $c_1, c_2 \in \mathbb{Z}$ and $c_1, c_2 \geq 0$. The evaluation point $\lambda_{[w]}$ is given in the basis $\widetilde{\rho}_{P_r}$, and the infinitesimal character χ_w in the basis $\{e_1, e_2\}$.

Theorem 9.3. *Let P_1 be the maximal standard parabolic k-subgroup as above, with Levi factor $L_1 \cong \mathrm{GL}_1 \times \mathrm{SO}_3$. Let $\pi = \chi \otimes \sigma$ be a unitary cuspidal automorphic representation of $L_1(\mathbb{A})$, where χ is a unitary Hecke character of $\mathbb{I}$, and σ a unitary cuspidal automorphic representation of $\mathrm{SO}_3(\mathbb{A})$. Let ϕ be the associate class of cuspidal automorphic representations represented by $\pi \otimes e^{\langle \lambda, H_{P_1}(\cdot)\rangle}$, where $\lambda \in \mathrm{Re}X^G_{P_1}$ is in the positive Weyl chamber determined by P_1. Then the cohomology space $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{L}_{E,\{P_1\},\phi} \otimes E)$ is non-trivial if and only if the following assertions are satisfied:*
(a) *$\lambda = \frac{1}{2}\widetilde{\rho}_{P_1} = \frac{1}{2}e_1$, i.e., $s = \frac{1}{2}$,*
(b) *χ^2 is the trivial character of $\mathbb{I}$,*
(c) *the principal L-function $L(s, \chi \otimes \sigma')$ is non-zero at $s = \frac{1}{2}$, where σ' is a representation of $\mathrm{GL}_2(\mathbb{A})$ with trivial central character obtained from σ via identification of SO_3 with PGL_2,*
(d) *the highest weight Λ of the coefficient system E is of the form $\Lambda = c_1\Lambda_1$ with $c_1 \in \mathbb{Z}_{\geq 0}$, i.e., $c_2 = 0$,*
(e) *the local component of σ' at every archimedean place is the discrete series representation D_{2c_1+4} of $\mathrm{GL}_2(\mathbb{R})$ of lowest $\mathrm{O}(2)$-type $2c_1 + 4$.*

The type (π, w), $w \in W^{P_1}$, giving non-trivial cohomology classes has the minimal coset representative

$$w = w_1w_2 \in W^{P_1}$$

of length two.

Proof. Comparing the tables with the theorems in the previous subsection we identify the possible types (π, w) that may contribute to the square-integrable cohomology as follows. For P_1 the only possibility is that the evaluation point is $\lambda_{[w]} = \frac{1}{2}\widetilde{\rho}_{P_1}$, i.e., $s_w = \frac{1}{2}$, because this is the only point at which the Eisenstein series may have a pole in the positive Weyl chamber. This gives condition (a). From Table 9.1, we see that the conditions for existence of non-trivial cohomology classes imply that this may only happen for the minimal coset representative $w = w_1w_2 \in W^{P_1}$ of length two, provided that $c_2 = 0$ and that the infinitesimal character of σ is $(c_1 + \frac{3}{2})e_2$. The first condition implies the form of Λ in assertion (d). The infinitesimal character of SO_3 corresponds via the identification with PGL_2 to the infinitesimal character $(c_1 + \frac{3}{2}, -c_1 - \frac{3}{2})$ for GL_2. Thus, we obtain condition (e) for the archimedean components of σ'. Furthermore, $\pi \cong \chi \otimes \sigma$ should satisfy the conditions of Theorem 9.3, so that the Eisenstein series has a pole. This gives assertions (b) and (c). □

Corollary 9.4. *In the notation of Theorem 9.3, suppose that the highest weight Λ of the representation (η, E) of $G = \mathrm{SO}_5$ is of the form $\Lambda = c_1\Lambda_1$ with $c_1 \in \mathbb{Z}_{\geq 0}$, i.e., $c_2 = 0$. Then there exists a unitary cuspidal automorphic representation $\pi \cong \chi \otimes \sigma$ of $L_1(\mathbb{A})$ such that for the associate class ϕ represented by $\pi \otimes e^{\langle \nu_{1/2}, H_{P_1}(\cdot)\rangle}$ we have*

$$H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{L}_{E,\{P_1\},\phi} \otimes E) \neq 0.$$

Moreover, the residual cohomology space $H^*_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{E,\{P_1\},\phi} \otimes E)$ does not vanish. In the lowest possible degree $q = 2d$, these classes represented by residues of Eisenstein series contribute to the total cohomology group $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{E,\{P_1\}} \otimes E) \subset H^*_{\mathrm{Eis}}(G, E)$.

Proof. For the first part of the corollary, we must show that there exist a unitary cuspidal automorphic representation $\pi \cong \chi \otimes \sigma$ satisfying assertions (b), (c) and (e) of Theorem 9.3. Note that assertion (a) on the evaluation point and assertion (d) on the highest weight Λ are the assumptions in the corollary.

The representation σ' of $\mathrm{GL}_2(\mathbb{A})$, with trivial central character, satisfying assertion (e) exists, as we may take $\sigma' \in \mathcal{D}(\kappa, \mathfrak{q})$ for $\kappa = (2c_1 + 4, \dots, 2c_1 + 4)$, and any level $\mathfrak{q}$ such that $\mathcal{D}(\kappa, \mathfrak{q})$ is non-empty. See Section 6.3. Taking any such representation σ', the existence of a quadratic Hecke character χ of $\mathbb{I}$ such that $L(\frac{1}{2}, \chi \otimes \sigma') \neq 0$ follows from the work of Waldspurger [69], [50, Theorem A.2], since σ'_v is a discrete series representation at all archimedean places. Thus, for such χ assertions (b) and (c) are satisfied.

Finally, to show that the residual cohomology does not vanish, we observe that the lowest possible degree in which the non-tempered representation $J(\nu_{1/2}, \chi_v \otimes D_{2c_1+4})$ has non-trivial cohomology is degree 2. Hence, $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{L}_{E,\{P_1\},\phi} \otimes E)$ is non-vanishing in degree $2d$. Invoking again [51, Theorem I.1 = III.1], we conclude that these non-vanishing square-integrable classes contribute non-trivially to $H^*_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{E,\{P_1\},\phi} \otimes E)$ in degree $2d$, and thus, to the total cohomology group $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{E,\{P_1\}} \otimes E) \subset H^*_{\mathrm{Eis}}(G, E)$. □

Theorem 9.5. *Let P_2 be the maximal standard parabolic k-subgroup as above, with Levi factor $L_2 \cong \mathrm{GL}_2$. Let π be a unitary cuspidal automorphic representation of $L_2(\mathbb{A}_k)$. Let ϕ be the associate class of cuspidal automorphic representations represented by $\pi \otimes e^{\langle \lambda, H_{P_2}(\cdot)\rangle}$, where $\lambda \in \mathrm{Re}\, X^G_{P_2}$ is in the positive Weyl chamber determined by P_2. Then the cohomology space $H^*(\mathfrak{g}, K_\infty; \mathcal{L}_{E,\{P_2\},\phi} \otimes E)$ is non-trivial if and only if the following assertions are satisfied:*

(a) *$\lambda = \tilde{\rho}_{P_2} = \frac{1}{2}(e_1 + e_2)$, i.e., $s = 1$,*
(b) *the symmetric square L-function $L(s, \pi, \mathrm{Sym}^2 \rho_2)$ has a pole at $s = 1$,*
(c) *the highest weight Λ of the coefficient system E is of the form $\Lambda = c_2\Lambda_2$ with $c_2 \in \mathbb{Z}_{\geq 0}$, i.e., $c_1 = 0$,*
(d) *the local component of π at every archimedean place is the discrete series representation D_{c_2+3} of $\mathrm{GL}_2(\mathbb{R})$ of lowest $\mathrm{O}(2)$-type $c_2 + 3$.*

The type (π, w), $w \in W^{P_2}$, giving non-trivial cohomology classes has the minimal coset representative

$$w = w_2 w_1 \in W^{P_2}$$

of length two.

Proof. We compare Theorem 9.5 with the possible types (π, w) in Table 9.2, and argue in a similar way as in the proof of Theorem 9.3. □

Corollary 9.6. *In the notation of Theorem 9.5, suppose that the highest weight Λ of the representation (η, E) of $G = \mathrm{SO}_5$ is of the form $\Lambda = c_2\Lambda_2$ with $c_2 \in \mathbb{Z}_{\geq 0}$ and c_2 even. Then there exists a unitary cuspidal automorphic representation π of $L_2(\mathbb{A})$ such that for the associate class ϕ represented by $\pi \otimes e^{\langle \nu_1, H_{P_2}(\cdot)\rangle}$ we have*

$$H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{L}_{E,\{P_2\},\phi} \otimes E) \neq 0.$$

Moreover, the residual cohomology space $H^*_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{E,\{P_2\},\phi} \otimes E)$ does not vanish. In the lowest possible degree $q = 2d$ these classes represented by residues of Eisenstein series contribute to the total cohomology group $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{E,\{P_2\}} \otimes E) \subset H^*_{\mathrm{Eis}}(G, E)$.

Proof. As in the proof of Corollary 9.4, we need to show that there exists a unitary cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A})$ such that assertions (b) and (d) of Theorem 9.5 are satisfied. Note that assertion (a) on the evaluation point is among assumptions of the corollary. The highest weight Λ satisfies assertion (c) by the assumption $c_2 \in \mathbb{Z}_{\geq 0}$ and c_2 even in the corollary. Note that there is a wider class of Λ satisfying assertion (c), namely those with c_2 odd. These are not considered in the corollary. See the remark below.

Since c_2 is even, the discrete series D_{c_2+3} is of odd lowest O(2)-type. Hence, the existence of π satisfying assertions (b) and (d) is proved in Proposition 6.5.

Finally, the minimal degree in which the non-tempered representation $J(\nu_1, D_{c_2+3})$ has non-trivial cohomology is degree 2. Hence, it follows that $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{L}_{E,\{P_2\},\phi} \otimes E)$ is non-vanishing in degree $2d$, and invoking again [51, Theorem I.1 = III.1], we conclude that these non-vanishing square-integrable classes contribute non-trivially to $H^*_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{E,\{P_2\},\phi} \otimes E)$ in degree $2d$, and thus, to the total cohomology group $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{E,\{P_2\}} \otimes E) \subset H^*_{\mathrm{Eis}}(G, E)$. □

Remark 9.7. The case of highest weight $\Lambda = c_2\Lambda_2$ such that $c_2 \in \mathbb{Z}_{\geq 0}$ and c_2 odd is not covered by the previous corollary. The point is that in that case there is no π with the required properties, as already explained in Remark 6.6. More precisely, in that case the archimedean components of π should be the discrete series representations D_{c_2+3} of even lowest O(2)-type. However, according to Remark 6.6, there is no π such that $L(s, \pi, \mathrm{Sym}^2 \rho_2)$ has a pole at $s = 1$ with the discrete series of even lowest O(2)-type as archimedean local components.

10 The even special orthogonal group of *k*-rank two

10.1 Residues of Eisenstein series

Finally, we consider the k-split even special orthogonal group $G = \mathrm{SO}_4$ of k-rank two. Note that at the archimedean places $\mathrm{SO}_4(\mathbb{R}) = \mathrm{SO}(2, 2)$, the special orthogonal group of signature $(2, 2)$. The residual spectrum for this group was partially determined in [19]. We retain the notation of the previous section with minor adjustments. As before P_0 is a fixed minimal parabolic k-subgroup, with the Levi decomposition $P_0 = L_0N_0$. Then L_0 is isomorphic to a product of two copies of $\mathbb{G}_m/k$. The set of simple roots is

$$\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_1 + e_2\},$$

where e_i is the projection of L_0 to its ith component. The fundamental weights are given as

$$\Lambda_1 = \frac{1}{2}(e_1 - e_2) \quad \text{and} \quad \Lambda_2 = \frac{1}{2}(e_1 + e_2).$$

For $r = 1, 2$, the maximal proper standard parabolic k-subgroup $P_{\Delta\setminus\{\alpha_r\}}$ is denoted by P_r, with Levi decomposition $P_r = L_rN_r$. We have $L_1 \cong \mathrm{GL}_2$ and $L_2 \cong \mathrm{GL}_2$, but they are not associate. In fact, both P_r are self-associate [62, Lemma 3.4]. We have

$$\tilde{\rho}_{P_1} = \rho_{P_1} \quad \text{and} \quad \tilde{\rho}_{P_2} = \rho_{P_2}.$$

Observe that the $\tilde{\rho}_{P_r}$ coincides with the fundamental weight Λ_r, $r = 1, 2$.

Theorem 10.1. *Let P_r be the maximal standard parabolic k-subgroup as above, with Levi factor $L_r \cong \mathrm{GL}_2$. Let π be a unitary cuspidal automorphic representation of $L_r(\mathbb{A}) \cong \mathrm{GL}_2(\mathbb{A})$. Let ϕ be the associate class of cuspidal automorphic representations represented by $\pi \otimes e^{\langle \lambda, H_{P_r}(\cdot)\rangle}$, where λ in the closure of the positive Weyl chamber associated to P_r. Then the space $\mathcal{L}_{E,\{P_r\},\phi}$ of square-integrable automorphic forms supported in ϕ is non-trivial if and only if the following assertions hold:*

(a) *one has*

$$\lambda = \tilde{\rho}_{P_r} = \begin{cases} \frac{1}{2}(e_1 - e_2) & \text{for } r = 1, \\ \frac{1}{2}(e_1 + e_2) & \text{for } r = 2, \end{cases}$$

i.e., $s = 1$,

(b) *the central character ω_π of π is trivial.*

Proof. In both cases $r = 1$ and $r = 2$, the normalizing factor is of the form

$$r(s, \pi, w_0) = \frac{L(s, \pi, \wedge^2\rho_2)}{L(1 + s, \pi, \wedge^2\rho_2)} = \frac{L(s, \omega_\pi)}{L(1 + s, \omega_\pi)},$$

so the analytic properties of the Eisenstein series follow from the properties of the Hecke L-function $L(s, \omega_\pi)$ attached to the central character ω_π of π. □

10.2 Existence of residual Eisenstein cohomology classes

As in the previous section, we provide the evaluation points, highest weights, and the infinitesimal character for possible types (π, w), $w \in W^{P_r}$, that may contribute to square-integrable cohomology supported in P_r. The Weyl group of SO_4 is

$$W = \{1, w_1, w_2, w_1 w_2\},$$

where w_i is the simple reflection with respect to the simple root α_i, $i = 1, 2$. We write the highest weight of E in terms of fundamental weights as $\Lambda = c_1\Lambda_1 + c_2\Lambda_2$, where $c_1, c_2 \in \mathbb{Z}$ and $c_1, c_2 \geq 0$. The results of the computation are given in Table 10.1 for P_1 and Table 10.2 for P_2.

Theorem 10.2. *Let P_r be the maximal standard parabolic k-subgroup as above, with Levi factor $L_r \cong \mathrm{GL}_2$. Let π be a unitary cuspidal automorphic representation of $L_r(\mathbb{A}) \cong \mathrm{GL}_2(\mathbb{A})$. Let ϕ be the associate class of cuspidal automorphic representations represented by $\pi \otimes e^{\langle \lambda, H_{P_r}(\cdot)\rangle}$, where λ is in the closure of the positive Weyl chamber with respect to P_r. Then the cohomology space $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{L}_{E,\{P_r\},\phi} \otimes E)$ is non-trivial if and only if the following assertions hold:*

(a) *one has*

$$\lambda = \tilde{\rho}_{P_r} = \begin{cases} \frac{1}{2}(e_1 - e_2) & \text{for } r = 1, \\ \frac{1}{2}(e_1 + e_2) & \text{for } r = 2, \end{cases}$$

i.e., $s = 1$,

(b) *the central character ω_π of π is trivial,*

(c) *the highest weight Λ of the coefficient system E is of the form*

$$\Lambda = \begin{cases} c\Lambda_2 & \text{for } r = 1, \\ c\Lambda_1 & \text{for } r = 2, \end{cases}$$

where $c \in \mathbb{Z}_{\geq 0}$,

(d) *the local component of π at every archimedean place is the discrete series representation D_{c+2} of $\mathrm{GL}_2(\mathbb{R})$ of lowest $\mathrm{O}(2)$-type $c + 2$.*

The type (π, w), $w \in W^{P_r}$, giving non-trivial cohomology classes has the minimal coset representative

$$w = w_r \in W^{P_r}$$

of length one.

Proof. In the same way as for $G = SO_5$, this follows comparing Theorem 10.1 with Tables 10.1 and 10.2. □

$w \in W^{P_1}$	$\lambda_{[w]} = ? \cdot \tilde{\rho}_{P_1}$	$\mu_w = ? \cdot \omega_1$	χ_w
1	$-(c_1 + 1)$	c_2	$-\frac{c_2+1}{2} e_1 + \frac{c_2+1}{2}(-e_2)$
w_1	$c_1 + 1$	c_2	$-\frac{c_2+1}{2} e_1 + \frac{c_2+1}{2}(-e_2)$

Table 10.1: The evaluation points $\lambda_{[w]}$, highest weights μ_w and infinitesimal characters χ_w for $w \in W^{P_1}$ in the case $G = SO_4$, where ω_1 is the fundamental weight for the Levi factor $L_1 \cong \mathrm{GL}_2$.

$w \in W^{P_2}$	$\lambda_{[w]} = ? \cdot \tilde{\rho}_{P_2}$	$\mu_w = ? \cdot \omega_2$	χ_w
1	$-(c_2 + 1)$	c_1	$-\frac{c_1+1}{2} e_1 + \frac{c_1+1}{2} e_2$
w_2	$c_2 + 1$	c_1	$-\frac{c_1+1}{2} e_1 + \frac{c_1+1}{2} e_2$

Table 10.2: The evaluation points $\lambda_{[w]}$, highest weights μ_w and infinitesimal characters χ_w for $w \in W^{P_2}$ in the case $G = SO_4$, where ω_2 is the fundamental weight for the Levi factor $L_2 \cong \mathrm{GL}_2$.

Corollary 10.3. *In the notation of Theorem 10.2, suppose that the highest weight Λ of the representation (η, E) of $G = \mathrm{SO}_4$ is of the form*

$$\Lambda = \begin{cases} c\Lambda_2 & \text{for } r = 1, \\ c\Lambda_1 & \text{for } r = 2, \end{cases}$$

with $c \in \mathbb{Z}_{\geq 0}$ and c even. Then there exists a unitary cuspidal automorphic representation π of $L_r(\mathbb{A})$ such that for the associate class ϕ represented by $\pi \otimes e^{\langle \nu_1, H_{P_r}(\cdot)\rangle}$ we have

$$H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{L}_{E,\{P_r\},\phi} \otimes E) \neq 0.$$

Moreover, the residual cohomology space $H^*_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{E,\{P_r\}} \otimes E)$ does not vanish. In the lowest possible degree $q = d$ these classes represented by residues of Eisenstein series contribute to the total cohomology group $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{E,\{P_r\},\phi} \otimes E) \subset H^*_{\mathrm{Eis}}(G, E)$.

Proof. Assertion (a) of Theorem 10.2 is the assumption of the corollary. The form of the highest weight satisfies assertion (c). Hence, it remains to show the existence of π satisfying assertions (b) and (d). That is, the central character ω_π of π should be trivial and the archimedean components of π should be the discrete series D_{c+2} of even lowest O(2)-type. However, such π exist, as we may take $\pi \in \mathcal{D}(\kappa, \mathfrak{q})$ for $\kappa = (c+2, \dots, c+2)$ and any $\mathfrak{q}$ such that $\mathcal{D}(\kappa, \mathfrak{q})$ is non-empty, see Section 6.3. This shows the first claim of the corollary.

To show that the residual cohomology classes contribute non-trivially to the residual cohomology space $H^*_{(\mathrm{sq})}(\mathfrak{g}, K_\infty; \mathcal{A}_{E,\{P_r\},\phi} \otimes E)$, observe that in the lowest possible degree 1 the cohomology of the Langlands quotient $J(\nu_1, D_{c+2})$ is non-trivial. Hence, using [51, Theorem I.1 = III.1], we conclude that these cohomology classes contribute non-trivially to the total cohomology group $H^*(\mathfrak{g}, K_\infty; \mathcal{A}_{E,\{P_r\}} \otimes E) \subset H^*_{\mathrm{Eis}}(G, E)$ in degree d. □

Remark 10.4. In Corollary 10.3, we assume that the highest weight Λ is of the form $\Lambda = c\Lambda_i$ with c even. However, in Theorem 10.2, there is another form of Λ that may possibly contribute to the residual cohomology, namely, the case of c odd. But in that case, there is no π satisfying the required properties for a non-trivial cohomology class in Theorem 10.2. More precisely, the central character ω_π of π should be trivial according to assertion (b), while the archimedean components should at the same time be the discrete series representations D_{c+2} of odd lowest O(2)-type according to assertion (d). But this is impossible because the central character of such discrete series representation is non-trivial.

A Unitary representations with non-zero cohomology

It is a fundamental problem to determine (up to infinitesimal equivalence) all irreducible unitary representations (π, H_π) of a real Lie group G with non-vanishing Lie algebra cohomology. A complete solution to this classification problem was given in a constructive approach by Vogan and Zuckerman [66]. An outgrowth of this is the computation of the relative Lie algebra cohomology groups $H^*(\mathfrak{g}, K, H_{\pi,K} \otimes F)$, where $\mathfrak{g}$ denotes the complexified Lie algebra of the given connected real reductive Lie group, $K \subset G$ a maximal compact subgroup.

Following [45, 68], we briefly review in this appendix the classification in the case where G is the exceptional split real Lie group of type G_2. It is a connected group of $\mathrm{rk}_{\mathbb{R}}\, G = 2$. The Weyl group W_G of G is isomorphic to the dihedral group D_6 of order 12. Let K be a maximal compact subgroup of G; its Lie algebra $\mathfrak{k}_0$ is isomorphic to $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$.

Let θ_K be the corresponding Cartan involution and let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}_0$ of G_2. Given an irreducible unitary representation (π, H_π) of G with non-vanishing cohomology with respect to a finite-dimensional representation space F, there is a θ_K-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$. By definition, $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$ such that $\theta_K \mathfrak{q} = \mathfrak{q}$, and $\overline{\mathfrak{q}} \cap \mathfrak{q} = \mathfrak{l}$ is a Levi subalgebra of $\mathfrak{q}$, where $\overline{\mathfrak{q}}$ refers to the image of $\mathfrak{q}$ under complex conjugation with respect to the real form $\mathfrak{g}_0$ of $\mathfrak{g}$. Write $\mathfrak{u}$ for the nilradical of $\mathfrak{q}$. Then $\mathfrak{l}$ is the complexification of a real subalgebra $\mathfrak{l}_0$ of $\mathfrak{g}_0$. The normalizer of $\mathfrak{q}$ in G is connected since G is, and it coincides with the connected Lie subgroup L of G with Lie algebra $\mathfrak{l}_0$. Then $F/\mathfrak{u}F$ is a one-dimensional unitary representation of L. Write $-\lambda : \mathfrak{l} \to \mathbb{C}$ for its differential. Via cohomological induction, the data $(\mathfrak{q}, \lambda)$ determine a unique irreducible unitary representation $A_{\mathfrak{q}}(\lambda)$ of G so that the Harish-Chandra module of (π, H_π) is equivalent to the one of $A_{\mathfrak{q}}(\lambda)$.

It is worth noting that the Levi subgroup L has the same rank as G, is preserved by the Cartan involution θ_K, and the restriction of θ_K to L is a Cartan involution. Moreover, the group L contains a maximal torus $T \subset K$. This result serves as a guideline to construct all possible θ_K-stable parabolic subalgebras $\mathfrak{q}$ in $\mathfrak{g}$ up to conjugation by K. There are only finitely many K-conjugacy classes of θ_K-stable parabolic subalgebras $\mathfrak{q}$ in $\mathfrak{g}$.

In the given case the construction runs as follows: Fix non-zero elements x, y in $\mathfrak{k}_0$, the first one belonging to the first summand, the second to the second, and let $i\mathfrak{t}$ be the real vector space spanned by ix, iy. Then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}_0$, and $\mathfrak{t} \cong \mathbb{C}^2$. We denote the evaluation in the first and second coordinate by e_1 and e_2, respectively, and we write $\alpha_1 = e_2 - e_1$ and $\alpha_2 = 3e_1 - e_2$. Taking α_i, $i = 1, 2$, as simple roots, the set $\Delta^+(\mathfrak{g}, \mathfrak{t})$ of positive roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$ is given as the set

$$\Delta^+(\mathfrak{g}, \mathfrak{t}) = \Delta^+(\mathfrak{k}, \mathfrak{t}) \cup \Delta^+(\mathfrak{p}, \mathfrak{t}),$$

where

$$\Delta^+(\mathfrak{k}, \mathfrak{t}) = \{\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2\}, \quad \Delta^+(\mathfrak{p}, \mathfrak{t}) = \{\alpha_1, \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}.$$

Note that α_1 is the short simple root, and α_2 is the long simple root. The fundamental dominant weights are $\Lambda_1 := 2\alpha_1 + \alpha_2$ and $\Lambda_2 := 3\alpha_1 + \alpha_2$.

Starting off from an element $z \in \mathfrak{t}$, there is an associated θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}_{\mathbb{C}}$ with Levi decomposition $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}_{\mathbb{C}}$ defined by $\mathfrak{q}_{\mathbb{C}}$ = sum of non-negative eigenspaces of $\mathrm{ad}(z)$, $\mathfrak{l}_{\mathbb{C}}$ = centralizer of z, and $\mathfrak{u}_{\mathbb{C}}$ = sum of positive eigenspaces of $\mathrm{ad}(z)$. Let λ be the differential of a unitary character of L, the connected subgroup of G with Lie algebra $\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{g}$, such that $\langle \alpha, \lambda_{|\mathfrak{t}_{\mathbb{C}}} \rangle \geq 0$ for each root α of $\mathfrak{u}$ with respect to $\mathfrak{t}_{\mathbb{C}}$. One refers to such a one-dimensional representation $\lambda : \mathfrak{l}_{\mathbb{C}} \to \mathbb{C}$ as an admissible character. A pair $(\mathfrak{q}, \lambda)$ of a θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}_{\mathbb{C}}$ and an admissible character λ determines a unique irreducible unitary representation $A_{\mathfrak{q}}(\lambda)$ of G with non-vanishing cohomology with respect to a suitable finite-dimensional representation (ν, F) of G.

Up to infinitesimal equivalence, if $\mathfrak{l}_{\mathbb{C}} \subset \mathfrak{k}_{\mathbb{C}}$, one obtains discrete series representations, and there are exactly three of them up to infinitesimal equivalence having the same infinitesimal character for a given admissible character λ. Recall that this number is generally given as the ratio $|W_G/W_K|$, where W_K denotes the Weyl group of K. The only degree in which these three discrete series representations π_i, $i = 1, 2, 3$, have $H^j(\mathfrak{g}, K, H_{\pi_i} \otimes F) \neq 0$ with a suitable coefficient system is $j = 4$.

The trivial representation of G only matters if the coefficient system F is trivial as well. Note that one has $H^j(\mathfrak{g}, K, \mathbb{C}) = \mathbb{C}$ if $j = 0, 4, 8$ and $H^j(\mathfrak{g}, K, \mathbb{C}) = 0$ otherwise.

The most interesting irreducible unitary representations of $G = \mathrm{G}_2$ are (up to infinitesimal equivalence) the ones originating in the following way: Consider two elements $z_j \in \mathfrak{t}$, $j = 1, 2$, with $\alpha_j(z_j) > 0$ and $\alpha_k(z_j) = 0$ for $k \neq j$. We denote the corresponding θ-stable parabolic subalgebra as constructed by $\mathfrak{q}_j$, $j = 1, 2$. The connected subgroup L_j, $j = 1, 2$, is isomorphic to $\mathrm{SL}_2(\mathbb{R}) \times U(1)$. These two algebras $\mathfrak{q}_j$, $j = 1, 2$, are the only θ-stable parabolic subalgebras of $\mathfrak{g}$ with $R(\mathfrak{q}_j) = 3$. Let $\lambda : \mathfrak{l}_j \to \mathbb{C}$ be an admissible character. Then the corresponding irreducible unitary representation $A_{\mathfrak{q}_j}(\lambda)$ of G is non-tempered. We summarize this classification result in the case of an arbitrary coefficient system, see [45, 67, 68].

Proposition A.1. *Let G be the split simple real Lie group of type G_2, $\mathfrak{g}$ its complexified Lie algebra, and $K \subset G$ a maximal compact subgroup. Let (ν, F) be an irreducible finite-dimensional representation of G with highest weight $\Lambda = c_1\Lambda_1 + c_2\Lambda_2$, c_1, c_2 non-negative integers. Then we have:*

- *Fix the index $j \in \{1, 2\}$. If the integral coefficient $c_i = 0$, $i \neq j$, then there exists an admissible character $\chi_j : \mathfrak{l}_j \to \mathbb{C}$ with regard to $\mathfrak{q}_j$ such that the corresponding irreducible non-tempered representation $A_{\mathfrak{q}_j}(\chi_j)$, as constructed above, occurs with*

$$H^q(\mathfrak{g}, K, A_{\mathfrak{q}_j}(\chi_j) \otimes F) = \begin{cases} \mathbb{C} & \text{if } q = 3, 5, \\ 0 & \text{otherwise.} \end{cases}$$

- *If both integral coefficients $c_1 \neq 0$, $c_2 \neq 0$, then there is no irreducible unitary representation (π, H) of G with $H^q(\mathfrak{g}, K, \pi \otimes F) \neq 0$ for $q = 3, 5$.*

Remark A.2. Observe the shift in indices: This occurs as well if we describe the two non-tempered representation as Langlands quotients of principal series representations (see [45, 7.7.(3)]). We have

$$J(P_2, \sigma, \tfrac{1}{2}\widetilde{\rho_{P_2}}) = A_{\mathfrak{q}_1}(\chi_1), \quad J(P_1, \sigma, \tfrac{1}{2}\widetilde{\rho_{P_1}}) = A_{\mathfrak{q}_2}(\chi_2).$$

Here we use the notation used in Section 7 for the principal series representations.

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