

Polynomial upper bound on interior Steklov nodal sets

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Abstract. We study solutions of uniformly elliptic PDE with Lipschitz leading coefficients and bounded lower order coefficients. We extend previous results of A. Logunov ([9]) concerning nodal sets of harmonic functions and, in particular, prove polynomial upper bounds on interior nodal sets of Steklov eigenfunctions in terms of the corresponding eigenvalue λ .

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1. Introduction

This paper considers non trivial solutions u to the following general second order elliptic equation

$$Lu := \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(a^{ij}(x) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^N b^i(x) \frac{\partial u}{\partial x_j} + c(x)u = 0 \quad (1)$$

in some smooth bounded domain $\Omega \subset \mathbb{R}^n$. We make the following assumptions on the coefficients of L :

(1) L is uniformly elliptic, that is for a fixed $\eta > 0$

$$a^{ij}(x) \xi_i \xi_j \geq \eta |\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^n, x \in \Omega; \quad (2)$$

(2) the coefficients of L are bounded,

$$\sum_{i,j} |a^{ij}(x)| + \sum_i |b^i(x)| + |c(x)| \leq \Lambda, \quad x \in \Omega; \quad (3)$$

(3) the leading coefficients are Lipschitz,

$$\sum_{ij} |a^{ij}(x) - a^{ij}(y)| \leq \Gamma |x - y|. \quad (4)$$

We focus our interest on the relation between the zero set and the local growth properties of a solution u . In particular, we will address nodal sets of functions whose leading order coefficients $A(x) = \{a^{ij}\}_{i,j=1}^n$ are derived from the Laplace operator. Using normal coordinates we will hence assume that

$$A(0) = I,$$

where I denotes the $n \times n$ -identity matrix. This assumption allows us to reduce the amount of technicalities when we discuss the generalized frequency function below.

1.1. Doubling indices and nodal set. Given a fixed ball B such that $2B \subset \Omega$, the *doubling index* $N(B)$ is a measure of the local growth of u on B defined by

$$N(B) = \log_2 \frac{\sup_{2B} |u|}{\sup_B |u|}$$

Here and for the rest of the paper rB is the ball concentric to B and scaled by a factor $r > 0$. As the following simple example shows, the doubling index can be seen as a local generalization of the degree of a polynomial for continuous functions. Letting $u = x^n$ and $B = [-r, r]$, we have

$$N(B) = \log_2 \frac{\sup_{[-2r, 2r]} |x|^n}{\sup_{[-r, r]} |x|^n} = \log_2 \frac{(2r)^n}{r^n} = n.$$

Thus, the doubling index indeed recovers the degree up to a constant. We will often write $N(x, r)$ for the doubling index of u on the ball $B(x, r)$.

The *nodal set* of u is simply its zero set

$$Z_u = \{u^{-1}(0)\}.$$

Sparked by the famous conjecture of Yau [15, 16] on nodal sets of Laplace eigenfunctions, it is a celebrated problem to try to estimate the Hausdorff measure $\mathcal{H}^{n-1}(Z_u)$ of the nodal set of solutions to various partial differential equations. By the work [7] of Hardt-Simon, it is known that $\mathcal{H}^{n-1}(Z_u)$ is finite.

The seminal papers by Donnelly and Fefferman [2] (see also the more recent work [11, 12] of the second author) highlight how the doubling index can be used to obtain bounds on the size of the nodal set. Our main result is along these lines and extends the work of Logunov [9] for Laplace eigenfunctions to solutions u of equation (1). More precisely, we show that the size of the nodal set of such solutions is controlled by the doubling index in the following way.

Theorem 1.1. *There exist positive numbers $r_0 = r_0(M, g)$, $C = C(M, g)$ and $\alpha = \alpha(n)$ such that for any solution u of equation (1) in a domain Ω satisfying the conditions (2)–(4), we have*

$$\mathcal{H}^{n-1}(Z_u \cap Q) \leq C \operatorname{diam}^{n-1}(Q) N^\alpha(Q),$$

where $Q \subset B(p, r_0)$ is an arbitrary cube in Ω .

Here, $N(Q)$ is the uniform doubling index of u on a cube Q as defined by

$$N(Q) := \sup_{x \in Q, r \in (0, \operatorname{diam}(Q))} N(x, r).$$

The proof adapts the machinery developed by Logunov to solutions of more general elliptic equations. In the present note we will follow the work [9] indicating the appropriate modifications when one deals with such equations. From our standpoint, these modifications include an adaptation of the doubling scaling, a propagation of smallness estimate and an accumulation of growth (referred to as Simplex lemma in [9]).

In Section 2, we build a toolbox consisting mostly of elliptic estimates and almost monotonicity of a generalized frequency function – see equation (7). In contrast to [9], the generalized frequency function needs to be more carefully estimated, but it turns out that a similar scaling property holds (cf. Lemma 2.2).

Onwards, in Section 3, the toolbox is used to verify our generalized versions of the crucial simplex and hyperplane lemmata. Here one also needs to introduce appropriate gradient estimates and propagation of smallness for equations with rougher coefficients, whereas [9] exploits bounds pertinent to harmonic functions.

Now, the obtained two lemmata work together to investigate the additivity properties of the frequency. The underlying principal idea can be roughly summarized as follows: if the frequency of u on a big cube Q is high, then it cannot be high in too many disjoint subcubes $q_i \subset Q$. For the rest of the discussion, we essentially refer to [9], as the statements follow directly.

1.2. Application: interior nodal sets of Steklov eigenfunctions. Let M be a smooth, connected and compact manifold of dimension $n \geq 2$ with non-empty smooth boundary ∂M and denote by $\Delta = \Delta_g$ the Laplace–Beltrami operator on M . The Steklov eigenfunctions on M are solutions to

$$\begin{cases} \Delta\phi = 0 & \text{in } M, \\ \partial_\nu\phi = \lambda\phi & \text{on } \partial M. \end{cases} \tag{5}$$

In this setting, the spectrum is discrete and is composed of the eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \nearrow \infty.$$

Given a Steklov eigenfunction $u = u_\lambda$, we distinguish the codimension 1 *interior nodal set*

$$Z_\lambda = \{x \in M : \phi(x) = 0\}$$

and the codimension 2 *boundary nodal set*

$$N_\lambda = \{p \in \partial M : \phi(p) = 0\}.$$

As mentioned earlier, we are interested in measuring the size of these nodal sets. It is expected (see [5]) that their size is controlled by the Steklov eigenvalue. More precisely, it is conjectured that

$$c_1\lambda \leq \mathcal{H}^{n-1}(Z_\lambda) \leq c_2\lambda$$

and

$$c_3\lambda \leq \mathcal{H}^{n-2}(N_\lambda) \leq c_4\lambda,$$

where \mathcal{H}^n is the n -dimensional Hausdorff measure. In the above, the c_i are positive constants that may only depend on the geometry of the manifold M . These conjectures are similar to the famous Yau conjecture for nodal sets of eigenfunctions of the Laplace operator. We now briefly present the current best results present in the literature, starting with the interior nodal set:

Table 1. Current best bounds for $\mathcal{H}^{n-1}(Z_\lambda)$

Regularity and dimension	Current Best Lower Bound	Current Best Upper Bound
$C^\omega, n = 2$	$c\lambda$ [10] ✓	$c\lambda$ [10, 19] ✓
$C^\omega, n \geq 3$		$c\lambda$ [19] ✓
$C^\infty, n = 2$	c [13]	$c\lambda^{\frac{3}{2}}$ [20]
$C^\infty, n \geq 3$	$c\lambda^{\frac{2-n}{2}}$ [13]	

In the case of the boundary nodal set, we have

Table 2. Current best bounds for $\mathcal{H}^{n-2}(N_\lambda)$

Regularity and dimension	Current Best Lower Bound	Current Best Upper Bound
$C^\omega, n \geq 2$		$c\lambda$ [17] ✓
$C^\infty, n = 2$	$c\lambda$ [14]	
$C^\infty, n \geq 3$	$c\lambda^{\frac{4-n}{2}}$ [14]	

We use Theorem 1.1 to provide a polynomial upper bound for interior nodal sets in the smooth case in any dimension $n \geq 2$.

Theorem 1.2. *Let M be a smooth, connected and compact manifold of dimension $n \geq 2$ with non-empty smooth boundary ∂M . Let ϕ_λ be a Steklov eigenfunction on M corresponding to the eigenvalue λ . Then*

$$\mathcal{H}^{n-1}(Z_\lambda) \leq c\lambda^\alpha,$$

where $c = c(M, g)$ and $\alpha = \alpha(n)$.

The proof is based on a gluing procedure that transforms M into a compact manifold without boundary. Doing so and working locally then allows to transfer the study of the nodal set of ϕ to that of a solution u to the elliptic equation 1. The details are presented in Section 5.

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2. Tool box

2.1. Elliptic estimates. We recall Theorem 8.17 (cf. also Theorem 8.24) of [4] for operators of the type L as above. For any weak solution $u \in W^{1,2}$ of $Lu = 0$ and $\epsilon > 0$, we have the following elliptic estimate

$$\sup_{B(x,\rho)} |u|^2 \leq c_1 \int_{B(x,(1+\epsilon)\rho)} u^2, \tag{6}$$

where $c_1 = c_1(n, L, \epsilon)$. On the other hand, for every continuous function

$$\int_{B(x,\rho)} u^2 \leq \sup_{B(x,\rho)} |u|^2.$$

2.2. Properties of the frequency function. The frequency function associated to a harmonic function, due to Almgren, is a well-known object which allows one to study growth properties and doubling conditions. It possesses several useful properties, among which – a monotonicity property.

In the spirit of [3] and [1], we would like to address a certain construction of a frequency function adopted to a more general situation, i.e. a solution to an elliptic PDE of the form L .

Let $w \in W_{loc}^{1,2}(B_1)$. We define

$$H(r) := \int_{\partial B_r} w^2 d\sigma, \quad D(r) := \int_{B_r} |\nabla w|^2 dx,$$

$$I(r) := \int_{B_r} (|\nabla w|^2 + w(\mathbf{b} \cdot \nabla w) + cw^2) dx.$$

Here $d\sigma$ is an abbreviation for the standard surface measure on the sphere ∂B_r , i.e. $dV_{\partial B_r}$. In particular, $d\sigma$ does not refer to the normalized surface measure. The generalized frequency $\beta(r)$ is then defined as

$$\beta(r) := \frac{rI(r)}{H(r)}. \tag{7}$$

Let us now pause to briefly comment the above construction. First, in contrast to the case of harmonic functions, the sign of $\beta(r)$ is no longer clearly determined. Furthermore, it is not a priori clear whether $H(r)$ is not vanishing for a large set of radii r .

However, it turns out that for sufficiently small r (depending only on n and the bounds on L), the quantity $H(r)$ is positive (cf. Lemma 9, [1] and Lemma 2.2, [3]). Moreover, for such small r one can arrange that

$$\frac{\beta(r)}{r} \geq -C, \tag{8}$$

where C is a positive number, which depends only on n and the bounds on L (cf. Corollary 10, [1]).

Second, our definition of $\beta(r)$ follows the one in [1] (cf. Notation 8 therein). In particular, as already pointed out, in our definition we implicitly utilize the fact that $A(0) = I$, which allows us to reduce the amount of technical details. One can also define and work with $\beta(r)$ in the general case via a certain diagonalization procedure, which we briefly discuss below (see Theorem 2.1 below).

Now, as mentioned the frequency β enjoys a certain almost monotonicity property:

Theorem 2.1. *There exist positive numbers R, c_2, c_3 , depending only on n and the bounds on the operator L , i.e. (2)–(4), such that*

$$\beta(r_1) \leq c_3\beta(r_2) + c_2,$$

for any choice of positive numbers r_1, r_2 satisfying

$$0 < r_1 < r_2 < R.$$

Moreover, if one chooses an arbitrarily small positive number ϵ , then for a sufficiently small r_2 , one may take

$$c_3 = 1 + \epsilon.$$

Sketch of proof. The proof is to some extent technical. However, the needed statements is directly obtained by inspecting and following the lines of Theorem 3.2.1, [6] and Theorem 2.1, [3] where the authors consider also the case of an unbounded lower order coefficient $c(x)$. Roughly speaking, the proof can be divided in three steps and for convenience we now briefly outline the strategy. For further details, we refer to Theorem 2.1, [3].

First, one diagonalizes the operator L by introducing an appropriate metric g adapted to the matrix $A(x) = \{a^{ij}\}_{i,j=1}^n$ of leading order coefficients. This method appears in several other unique continuation results, notably in the works of Aronszajn and we refer to [3] as well as the references therein. Afterwards, one works in normal coordinates with respect to the metric g where the leading order matrix $A(x)$ is actually diagonalized, namely of the form $\mu(x)I$, where $\mu(x)$ is a positive Lipschitz function and where I denotes the identity matrix. We also point out that by construction, $\mu(x)$ is close to the constant 1, provided $A(x)$ is close to the identity.

Further on, one introduces the generalized frequency function $\beta_g(r)$ with slightly modified $H_g(r)$, $D_g(r)$ and $I_g(r)$ that are now encapsulating the function $\mu(x)$ – see the formulas (2.8)–(2.10) in [3]. An advantage of the use of β_g instead of β is that integration by parts is simpler and reduces several further technical complications.

Second, one would like to estimate the derivative of $\beta_g(r)$ with respect to r . To this end, one needs to obtain expressions for the derivatives of $H_g(r)$ and $I_g(r)$. While the derivative $\frac{d}{dr}H_g(r)$ is somewhat straightforward to obtain, some care needs to be taken for $\frac{d}{dr}I_g(r)$. In [3] the authors utilize a variational argument: they study the kinetic energy functional $D_g(r)$ along a certain variational family $u^t_{t \in J}$, defined over an appropriate interval J . In fact the family u^t is obtained via rescaling. Then, a computation of the variation of the kinetic energy yields a suitable expression for the derivative of $D_g(r)$, which also implies a suitable expression for the derivative of $I_g(r)$.

Third, one truncates the frequency function, i.e. for a sufficiently small r_0 one considers the set

$$\Omega_{r_0} = \{r \leq r_0: \beta_g(r) > \max(1, \beta_g(r_0))\}.$$

As β_g is absolutely continuous, this set is actually a countable collection of intervals $\bigcup_j (a_j, b_j)$. We observe that on the complement of Ω_{r_0} one already has

bounds on $\beta_g(r)$. Now, using the expressions for the derivatives of $I_g(r)$, $H_g(r)$ and assuming that r is sufficiently small (depending only on bounds on L) one obtains the bound

$$\frac{\beta'_g(r)}{\beta_g(r)} \geq -b, \quad \text{for all } r \in \Omega_{r_0},$$

where b is a positive number that depends on n and the bounds on L . This implies that if $r \in (a_{j_0}, b_{j_0})$ for some index j_0 , then after integration over the interval (r, b_{j_0}) one obtains

$$\beta_g(r) \leq \beta_g(r) \exp(b(b_{j_0} - r)) \leq \exp(b(b_{j_0} - r)) \max(1, \beta_g(r_0)).$$

Now, we can select

$$c_3 := e^{b(r_0-r)}, \quad c_2 := (1 + Cr\epsilon)e^{b(r_0-r)},$$

where C is the constant from (8). It follows that

$$\beta_g(r) \leq c_3\beta_g(r_0) + c_2,$$

for any r in the interval $(0, r_0)$, where we have also used the bound. The above arguments could be repeated if r_0 is replaced by any smaller number r_2 from the interval $(0, r_0)$, which would also lead to replacing the set Ω_{r_0} by Ω_{r_2} . This yields the almost monotonicity statement for the generalized frequency function $\beta_g(r)$. The corresponding statement for $\beta(r)$ follows by comparability of the corresponding quantities $I_g(r)$, $I(r)$ and $H_g(r)$, $H(r)$, as we assume that $A(x)$ is close to the identity for points x near the origin. \square

We also have the following derivation formula (cf. Corollary 3.2.8 in [6]; Proposition 11, [1]; formula (2.16), [3])

$$\frac{d}{dr} \left(\log \frac{H(r)}{r^{n-1}} \right) = O(1) + 2 \frac{\beta(r)}{r} \geq -c_4,$$

where $c_4 = c_4(n) > 0$. As a consequence, we get

Lemma 2.1. *The function $e^{c_4 r} H(r)$ is increasing for $r \in (0, r_0)$.*

Now let $0 < R_1 < R_2 < r_0$. An integration yields

$$H(R_2) = H(R_1) \left(\frac{R_2}{R_1} \right)^{n-1} \exp \left(O(1)(R_2 - R_1) + 2 \int_{R_1}^{R_2} \frac{\beta(r)}{r} dr \right).$$

Using the almost monotonicity of the frequency, we estimate the integral on the right hand side by

$$\log\left(\frac{R_2}{R_1}\right)2c_3^{-1}(\beta(R_1) - C_1) \leq \int_{R_1}^{R_2} \frac{\beta(r)}{r} dr \leq \log\left(\frac{R_2}{R_1}\right)(c_2 + c_3\beta(R_2)),$$

which, after absorbing the dimensional constants, yields

$$c_5\left(\frac{R_2}{R_1}\right)^{2c_3^{-1}(\beta(R_1)-c_2)} \leq \frac{H(R_2)}{H(R_1)} \leq c_5\left(\frac{R_2}{R_1}\right)^{2(c_3\beta(R_2)+c_2)}. \tag{9}$$

2.3. Doubling numbers and scaling. The main technical tool we need the following lemma.

Lemma 2.2. *Let $\epsilon \in (0, 1)$. There exist positive constants $c_6 = c_6(\epsilon)$ and $r_0 = r_0(\epsilon)$, such that for any $u \in W^{1,2}(B)$ with $Lu = 0$, we have*

$$t^{N(x,\rho)(1-\epsilon)-c_6} \leq \frac{\sup_{B(x,t\rho)} |u|}{\sup_{B(x,\rho)} |u|} \leq t^{N(x,t\rho)(1+\epsilon)+c_6},$$

where $x \in r_0B, \rho > 0, t > 2$ satisfy the condition $B(x, t\rho) \subset r_0B$. Furthermore, there is a threshold $N_0 = N_0(\epsilon)$, such that if $N(x, \rho) > N_0$, then the constant c_6 can be dropped in the above estimate and one has

$$t^{N(x,\rho)(1-\epsilon)} \leq \frac{\sup_{B(x,t\rho)} |u|}{\sup_{B(x,\rho)} |u|} \leq t^{N(x,t\rho)(1+\epsilon)}.$$

Proof. The argument goes along the lines of the Appendix in [9] with appropriate modifications. For completeness we provide the technical details. We prove the following claim.

Claim 2.1. *Suppose $\epsilon > 0$ is a sufficiently small number. Then there exists a positive number r_1 depending only on ϵ and the bounds on the operator L , such that*

$$\beta(p, r(1 + \epsilon))(1 - 100\epsilon) - c_7 \leq N(p, r) \leq \beta(p, 2r(1 + \epsilon))(1 + 100\epsilon) + c_7,$$

for every number r in the interval $(0, r_1)$ and every point p in the ball r_1B .

Using the elliptic estimate (6) and Lemma 2.1, there exists $\epsilon > 0$ such that

$$\sup_{B(p,r)} |u|^2 \leq c_8 H((1 + 2\epsilon)r) / r^{n-1}.$$

Using Lemma 2.1, there holds $H((1 - \epsilon)2r) \leq e^{2c_4\epsilon r} H(2r)$ so that

$$\begin{aligned} \sup_{B(p,2r)} |u|^2 &\geq \frac{1}{\omega_n(2r)^n} \int_{B(p,2r)} u^2 \\ &\geq \frac{1}{\omega_n(2r)^n} \int_{2r(1-\epsilon)}^{2r} H(\rho) d\rho \\ &\geq c_2 \frac{H(2r(1-\epsilon))}{r^{n-1}}, \end{aligned} \tag{10}$$

where we have used the fact that $H(2\rho) \leq e^{2c_r(\rho'-\rho)} H(2\rho')$ for $\rho < \rho'$ and where $c_2(\epsilon, n) = \frac{\epsilon}{\omega_n 2^{n-1} e^{2c_4 r_0}}$. Using the latter, we estimate the doubling indices as follows

$$N(p, r) := \log_2 \frac{\sup_{B(p,2r)} |u|}{\sup_{B(p,r)} |u|} \geq \frac{1}{2} \log_2 \frac{H(2r(1-\epsilon))}{H(r(1+\epsilon))} + c_9,$$

where $c_9 = \log_2 \frac{c_2}{c_8}$. The last quotient is controlled via the generalized frequency as given in (9). Further, assume that r_0 is sufficiently small, so that $c_3 = 1 + \epsilon$. Then, we have

$$\begin{aligned} \frac{1}{2} \log_2 \frac{H(2r(1-\epsilon))}{H(r(1+\epsilon))} &\geq \frac{1}{2} \log_2 \left[c_5 \left(\frac{2(1-\epsilon)}{1+\epsilon} \right)^{\frac{2\beta(r(1+\epsilon))-c_2}{1+\epsilon}} \right] \\ &\geq \frac{\beta(r(1+\epsilon)) - c_2}{1+\epsilon} \log_2 \left(\frac{2(1-\epsilon)}{1+\epsilon} \right) + \frac{1}{2} \log_2(c_5) \\ &\geq \frac{\beta(r(1+\epsilon))}{1+\epsilon} \log_2 \left[\frac{2(1-\epsilon)}{1+\epsilon} \right] - c_{10}. \end{aligned}$$

Now, we recall that for small r , the frequency function is ‘‘almost non-negative’’ in the sense of (8). Thus, for a sufficiently small $\epsilon > 0$ we get

$$\frac{\beta(r(1+\epsilon))}{1+\epsilon} \log_2 \left[\frac{2(1-\epsilon)}{(1+\epsilon)} \right] - c_9 \geq \beta(r(1+\epsilon))(1-20\epsilon) - c_7.$$

Hence, we obtain

$$N(p, r) \geq \beta(p, r(1+\epsilon))(1-100\epsilon) - c_7.$$

Similarly, one sees

$$N(p, r) \leq \beta(p, 2r(1+\epsilon))(1+100\epsilon) + c_7,$$

provided that ϵ and r_0 are sufficiently small. Indeed, this time, the quotient appearing in the definition of $N(p, r)$ is estimated in a reversed way: one obtains

an upper bound for the numerator and a lower bound on the denominator. This is achieved by using the elliptic estimate (6) and Lemma 2.1 as before. Thus, one obtains an upper bound on $N(p, r)$ in terms of similar quotient of the type $H(\rho')/H(\rho)$. The latter is further estimated from above in the terms of the right estimate in (9). This finishes the proof of the claim.

We now proceed showing the lower bound in Lemma 2.2. First, we can assume that t is bounded away from 2. Indeed, if $t \leq 2^{1+\epsilon}$, then as $t > 2$ we have $t^{N(x,\rho)(1-\epsilon)} \leq 2^{N(x,\rho)}$. Hence

$$\sup_{B(x,t\rho)} |u| \geq \sup_{B(x,2\rho)} |u| \geq c_{11} 2^{N(x,\rho)} \sup_{B(x,\rho)} |u| \geq t^{N(x,\rho)(1-\epsilon)} \sup_{B(x,\rho)} |u|,$$

which gives the lower bound and the additional statement as well. Thus, we assume that $t > 2^{1+\epsilon}$. Let us also set $\tilde{\epsilon} := \epsilon/1000$, so that $(1 - \tilde{\epsilon})t > 2(1 + \tilde{\epsilon})$. Using the estimate (10) and definition of the doubling index, the frequency scaling (9), we have

$$\frac{\sup_{B(x,t\rho)} |u|^2}{\sup_{B(x,\rho)} |u|^2} \geq \frac{c_2(t\rho)^{1-n} H((1 - \tilde{\epsilon})t\rho)}{2^{-2N(x,\rho)} \sup_{B(x,2\rho)} u^2}.$$

To bound the numerator from below we use the estimate (9) over balls with radii $(2\rho(1 + \tilde{\epsilon}))$ and $(t\rho(1 - \tilde{\epsilon}))$, followed by an application of Claim 2.1. This way we can also absorb the term t^{1-n} in the constants. To bound the denominator we use the elliptic estimate (6). Hence, we have

$$\begin{aligned} \frac{\sup_{B(x,t\rho)} |u|^2}{\sup_{B(x,\rho)} |u|^2} &\geq c_{11} \frac{\left(\frac{(1-\tilde{\epsilon})t}{2(1+\tilde{\epsilon})}\right)^{(2N(x,\rho)/(1+100\tilde{\epsilon})(1+\tilde{\epsilon})) - c_9} H(2\rho(1 + \tilde{\epsilon}))}{2^{-2N(x,\rho)} H(2\rho(1 + \tilde{\epsilon}))} \\ &= c_{11} 2^{2N(x,\rho)} \left(\frac{(1 - \tilde{\epsilon})t}{2(1 + \tilde{\epsilon})}\right)^{(2N(x,\rho)/(1+100\tilde{\epsilon})(1+\tilde{\epsilon})) - c_9}. \end{aligned}$$

Now, we can absorb the powers of 2 from the numerator and denominator, further adjusting the constants c_{11}, c_9 if needed. The latter is thus bounded from below by

$$c_{12} \left(\frac{(1 - \tilde{\epsilon})t}{(1 + \tilde{\epsilon})}\right)^{(2N(x,\rho)/(1+100\tilde{\epsilon})(1+\tilde{\epsilon})) - c_{13}} \geq c_{14} t^{2N(x,\rho)(1-\epsilon) - c_6},$$

where we have absorbed the quotient $(1 - \tilde{\epsilon})/(1 + \tilde{\epsilon})$ by using the smallness of $\tilde{\epsilon}$ and further adjusting the participating constants. In particular, we reduce the power of t by a small multiple of $N(x, \rho)$.

This concludes the proof of the lower bound. The upper bound in Lemma 2.2 follows similarly. To show the additional statements in the Lemma, it suffices to take $\epsilon/2$ instead of ϵ and require that

$$N(x, \rho) > \frac{2}{\epsilon} c_6(\epsilon/2) =: N_0(\epsilon/2). \quad \square$$

We will also need the following comparison for doubling numbers at nearby points (cf. Lemma 7.4, [9]).

Lemma 2.3. *There exists a radius $r_0 > 0$ and a threshold N_0 such that, for $x_1, x_2 \in B(p, r)$ and $\rho > 0$ such that $d(x_1, x_2) < \rho < r_0$, $N(x_1, \rho) > N_0$, there exists a constant $C > 0$ such that*

$$N(x_2, C\rho) > \frac{99}{100} N(x_1, \rho).$$

Proof. The proof proceeds exactly as in Lemma 7.4, [9], using Lemma 2.2 above. □

3. Additivity of frequency

The main motivation behind the results in [9] seems to be the investigation of additivity of the frequency function. The two principal ideas are encapsulated within the simplex and hyperplane lemmata.

3.1. Barycenter accumulation. Roughly speaking, we will assert the following: suppose that the doubling exponents at the vertices $\{x_1, \dots, x_{n+1}\}$ of a simplex are large (i.e. bounded below by a fixed $N_0 > 0$). Then, the doubling exponent at the barycenter of the simplex $x_0 := \frac{1}{n} \sum_{i=1}^{n+1} x_i$ is bounded below by $(1 + c)N_0$, where $c > 0$ is a fixed constant. Heuristically, the growth “accumulates” at the barycenter. The proof proceeds via direct use of the frequency properties discussed in Section 2.

Definition 3.1. Given a simplex $S := \{x_1, \dots, x_{n+1}\}$, we define the relative width $w(S)$ of S as

$$w(S) := \frac{\text{width}(S)}{\text{diam}(S)},$$

where $\text{diam}(S)$ is the diameter of S and $\text{width}(S)$ is the smallest possible distance between two parallel hyperplanes, containing S in the region between them.

Further on, we will consider simplices S whose relative width is bounded below as $w(S) \geq w_0 := w_0(n) > 0$ – the specific bound w_0 will be specified later.

Now, in order to apply the scaling of frequency we will need the following covering lemma.

Lemma 3.1. *Let $S := \{x_1, \dots, x_{n+1}\}$ be an arbitrary simplex satisfying $w(S) \geq w_0$. There exist a constant $\alpha := \alpha(n, w_0) > 0$ and a number (ratio) $K := K(n, w_0) \geq \frac{2}{w_0}$, so that if one takes a radius $\rho := K \operatorname{diam}(S)$, then one has*

$$B(x_0, (1 + \alpha)\rho) \subset \bigcup_{i=1}^{n+1} B(x_i, \rho).$$

Moreover, for $t > 2$ there exists $\delta(t) \in (0, 1)$ with $\delta(t) \rightarrow 0$ as $t \rightarrow \infty$, so that

$$B(x_i, t\rho) \subset B(x_0, (1 + \delta)t\rho).$$

The main result of this subsection is the following proposition.

Proposition 3.1. Let $\{B_i\}_{i=1}^{n+1}$ be a collection of balls centered at the vertices $\{x_i\}_{i=1}^{n+1}$ of the simplex S and radii not exceeding $\frac{\rho}{2}$, where $\rho = \rho(n, w_0)$ comes from Lemma 3.1. Then, there exist the positive constants $c := c(n, w_0)$, $C := C(n, w_0) \geq K$, $r := r(w_0, L)$ and $N_0 := N_0(w_0, L)$ with the following property:

If $S \subset B(p, r)$ and if $N(B_i) > N > N_0, i = 1, \dots, n + 1$, then

$$N(x_0, C \operatorname{diam} S) > (1 + c)N.$$

Proof. First, Lemma 2.2 shows that by taking larger balls, the doubling exponents essentially increase, so we can assume that all balls B_i have the radius ρ .

Let us set

$$M := \sup_{\bigcup_{i=1}^{n+1} B(x_i, \rho)} |u|,$$

and let us suppose that M is achieved on the ball $B(x_{i_0}, \rho)$ for a fixed index i_0 .

In particular, by Lemma 3.1 we have

$$\sup_{B(x_0, (1+\alpha)\rho)} \leq M.$$

Further, let us introduce parameters $t > 2, \epsilon > 0$ to be specified below and assume that the second statement in Lemma 2.2 holds for the ball $B(x_{i_0}, t\rho)$, by which we see

$$\sup_{B(x_{i_0}, t\rho)} |u| \geq Mt^{N(1-\epsilon)}.$$

Moreover, assuming that the scaling in Lemma 2.2 is functional at the barycenter x_0 and recalling Lemma 3.1, we conclude

$$\begin{aligned} \left(\frac{t(1+\delta)}{1+\alpha}\right)^{N(x_0,t(1+\delta)\rho)(1+\epsilon)+c_6} &\geq \frac{\sup_{B(x_0,t(1+\delta)\rho)} |u|}{\sup_{B(x_0,(1+\alpha)\rho)} |u|} \\ &\geq \frac{\sup_{B(x_{i_0},t\rho)} |u|}{\sup_{B(x_0,(1+\alpha)\rho)} |u|} \\ &\geq \frac{Mt^{N(1-\epsilon)}}{M} = t^{N(1-\epsilon)}. \end{aligned}$$

Specifying the parameters, we select $t > 2$ large enough to ensure $\delta(t) \leq \frac{\alpha}{2}$, and hence

$$\frac{t(1+\delta)}{1+\alpha} \leq t^{1-\gamma},$$

for some $\gamma = \gamma(t, \alpha) \in (0, 1)$. Thus, putting the last estimates together we see

$$t^{(1-\gamma)N(x_0,t(1+\delta)\rho)(1+\epsilon)+c_6} \geq t^{N(1-\epsilon)}$$

and therefore

$$N(x_0, t(1+\delta)\rho) \geq \frac{1-\epsilon}{(1+\epsilon)(1-\gamma)}N - c_3.$$

Selecting an $\epsilon = \epsilon(\gamma) > 0$ we can arrange that

$$\frac{1-\epsilon}{(1+\epsilon)(1-\gamma)} > 1 + 2c,$$

for some $c := c(\gamma) > 0$. Hence, we conclude

$$N(x_0, t(1+\delta)\rho) \geq N(1+2c) - c_3 \geq (1+c)N + (cN_0 - c_3) > (1+c)N,$$

provided that N_0 is sufficiently big. □

3.2. Propagation of smallness. We use propagation of smallness to derive estimate on the doubling exponents. The main auxiliary result in this discussion is the propagation of smallness for Cauchy data. In terms of exposition, we follow the discussion in [9] with appropriate changes whenever we need to address operators with rough coefficients and lower regularity instead of the standard Laplacian and smooth coefficients.

Lemma 3.2 (cf. Lemma 4.3, [8]). *Let u be a solution of (1) in the half-ball B_1^+ where the conditions (2), (3), (4) are satisfied. Let us set*

$$F := \left\{ (x', 0) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1}, |x'| < \frac{3}{4} \right\}.$$

If the Cauchy conditions

$$\|u\|_{H^1(F)} + \|\partial_n u\|_{L^2(F)} \leq \epsilon < 1 \quad \text{and} \quad \|u\|_{L^2(B_1^+)} \leq 1.$$

are satisfied, then

$$\|u\|_{L^2(\frac{1}{2}B_1^+)} \leq C\epsilon^\beta,$$

where the constants C, β depend on n, η, C_1, C_2 .

It is convenient to introduce the following doubling index.

Definition 3.2. The doubling index $N(Q)$ of a cube Q is defined as

$$N(Q) := \sup_{x \in Q, r \in (0, \text{diam}(Q))} N(x, r).$$

An immediate observation is that

$$N(q) \leq N(Q), \quad \text{if } q \subseteq Q,$$

and if $Q \subseteq \bigcup_i Q_i$ with $\text{diam}(Q) \leq \text{diam}(Q_i)$, then there exists an index i_0 such that

$$N(Q) \leq N(Q_{i_0}).$$

Proposition 3.2 (cf. Lemma 4.1, [9]). Let Q be a cube $[-R, R]^n$ in \mathbb{R}^n and let us divide Q into $(2A + 1)^n$ equal sub-cubes q_i with side-length $\frac{2R}{2A+1}$. Let $\{q_{i,0}\}$ be the collection of sub-cubes which intersect the hyperplane $\{x_n = 0\}$ and suppose that there exist centers $x_i \in q_{i,0}$ and radii $r_i < 10 \text{diam}(q_{i,0})$ so that $N(x_i, r_i) > N$ where N is fixed. Then there exist constants $A_0 = A_0(n), R_0 = R_0(L), N_0 = N_0(L)$ (here we mean dependence on the bounds of the operator L) with the following property: if $A > A_0, N > N_0, R < R_0$, then

$$N(Q) > 2N.$$

Proof. We assume that R_0 is small enough, so that Lemma 2.2 holds with $\epsilon = \frac{1}{2}$ and the equation (1) at this scale is satisfied along with the conditions (2), (3), (4). Moreover, at this scale we can also use Lemma 3.2.

To ease notation, without loss of generality by scaling we may assume that $R = \frac{1}{2}, R_0 \geq \frac{1}{2}$. Let B be the unit ball centered at 0. We consider the half ball $\frac{1}{32}B^+ \subset \frac{1}{8}B$ and wish to apply the propagation of smallness for Cauchy data problems. To this end, we need to bound u and ∇u on $F := \frac{1}{32}B^+ \cap \{x_n = 0\}$.

Step 1: Bound on u . First, let us set

$$M := \sup_{\frac{1}{8}B} |u|,$$

by which we have

$$\sup_{B(x_i, \frac{1}{32})} |u| \leq M, \quad \text{for all } x_i \in \frac{1}{16}B.$$

Hence, for $x_i \in \frac{1}{16}B$ Lemma 2.2 and the assumption that $N(x_i, r_i) > N$ imply

$$\sup_{8q_{i,0}} |u| \leq \sup_{B(x_i, \frac{16\sqrt{n}}{2A+1})} |u| \leq C \left(\frac{512\sqrt{n}}{2A+1} \right)^{\frac{N}{2}} \sup_{B(x_i, \frac{1}{32})} |u| \leq e^{-c_1 N \log A} M,$$

where $c_1 = c_1(n) > 0$ and we have assumed in the last step that N, A are sufficiently large.

Step 2: Bound on ∇u . Further, we wish to bound the gradient $|\nabla u|$. We recall the following facts.

Lemma 3.3. *Let u be a solution of equation (1) in a domain Ω satisfying the conditions (2)–(4). Then, if $\Omega' \subset\subset \Omega$, we have*

$$\|u\|_{W^{2,2}(\Omega')} \leq C \|u\|_{L^2(\Omega)},$$

where C depends on the parameters in (2)–(4) and $d(\Omega', \partial\Omega)$.

For a proof of Lemma 3.3 we refer to Theorem 8.8, the remark thereafter and Problem 8.2, [4].

Lemma 3.4. *Let $u \in W^{2,2}(\mathbb{R}^n)$ and let us consider the trace of u onto the hyperplane $\{x_n = 0\} \cong \mathbb{R}^{n-1}$ which, abusing of notation, we also denote by u . Then*

$$\|\nabla u\|_{L^2(\mathbb{R}^{n-1})} \leq C(\|u\|_{W^{2,2}(\mathbb{R}^n)} + \|u\|_{L^2(\mathbb{R}^{n-1})}),$$

where $C = C(n)$.

For a proof of Lemma 3.4 we refer to Lemma 23, [1]. Using Lemma 3.4 for functions of the form χu , where χ is a standard smooth cut-off function and $u \in W^{2,2}$ we see that

$$\|\nabla u\|_{L^2(\mathbb{R}^{n-1} \cap B_r)} \leq C(\|u\|_{W^{2,2}(B_{2r})} + \|u\|_{L^2(\mathbb{R}^{n-1} \cap B_{2r})}),$$

where χ is supported in B_{2r} .

Using these last lemmas along with the standard Sobolev trace estimate, we have

$$\begin{aligned} \|\nabla u\|_{L^2(F \cap q_{i,0})} &\leq C(\|u\|_{W^{2,2}(2q_{i,0})} + \|u\|_{L^2(F \cap 2q_{i,0})}) \\ &\leq C_1 \|u\|_{W^{2,2}(4q_{i,0})} \leq C_2 \|u\|_{L^2(8q_{i,0})}. \end{aligned}$$

Again using the trace estimate, this shows that

$$\begin{aligned} \|u\|_{W^{1,2}(F \cap q_{i,0})} + \|\partial_n u\|_{L^2(F \cap q_{i,0})} &\leq C_3 \|u\|_{W^{2,2}(4q_{i,0})} + \|\nabla u\|_{L^2(F \cap q_{i,0})} \\ &\leq C_4 \|u\|_{L^2(8q_{i,0})} \\ &\leq \frac{C_5}{(2A+1)^n} \sup_{8q_{i,0}} |u|. \end{aligned}$$

Summing up over the cubes $q_{i,0}$ and using the bound in the first step, we get

$$\|u\|_{W^{1,2}(F)} + \|\partial_n u\|_{L^2(F)} \leq \frac{C_5}{(2A+1)^{n-1}} \sup_{8q_{i,0}} |u| \leq e^{-c_2 N \log A} M.$$

Step 3: Propagation of smallness. Let us observe that

$$\|u\|_{L^2(\frac{1}{32}B^+)} \leq C_6 M.$$

and set

$$v := \frac{u}{C_6 M},$$

by which we have

$$\|v\|_{L^2(\frac{1}{32}B^+)} \leq 1.$$

Hence, by the bounds in Steps 1 and 2 and propagation of smallness from Lemma 3.2 we get

$$\|v\|_{L^2(\frac{1}{64}B^+)} \leq \epsilon^\beta,$$

where $\epsilon = e^{-c_3 N \log A}$. Let us now select a ball $B(p, \frac{1}{256}) \subset \frac{1}{64}B^+$ and observe that by (6)

$$\sup_{B(p, \frac{1}{256})} |v| \leq \epsilon^\beta,$$

which implies

$$\sup_{B(p, \frac{1}{256})} |u| \leq e^{-c_4 \beta N \log A} M.$$

Moreover, as $\frac{1}{8}B \subset B(p, \frac{1}{2})$, we have by definition $\sup_{B(p, \frac{1}{2})} |u| \geq M$. This implies

$$\frac{\sup_{B(p, \frac{1}{2})} |u|}{\sup_{B(p, \frac{1}{256})} |u|} \geq e^{c_4 \beta N \log A}.$$

Finally, applying the doubling scaling Lemma 2.2 we have

$$\frac{\sup_{B(p, \frac{1}{2})} |u|}{\sup_{B(p, \frac{1}{256})} |u|} \leq (128)^{\tilde{N}/2},$$

where \tilde{N} is the doubling index for $B(p, \frac{1}{2})$. Therefore,

$$\tilde{N} \geq c_5 N \log A \geq 2N,$$

where A is assumed to be sufficiently large. □

4. Counting Good/Bad cubes and application to nodal geometry

Using the results of Section 3, one can deduce the following result.

Theorem 4.1. *There exist constants $c > 0$, an integer A depending on the dimension d only and positive numbers $N_0 = N_0(M, g), r = r(M, g)$ such that for any cube $Q \in B(p, r)$ the following holds:*

If Q is partitioned into A^n equal sub-cubes q_i , then

$$\#\{q_i : N(q_i) \geq \max\left(\frac{N(Q)}{1+c}, N_0\right)\} \leq \frac{A^{n-1}}{2}.$$

The proof is combinatorial in nature and we refer to Theorem 5.1, [9] for complete details. As an application of the previous theorem, we also have our main theorem

Theorem 4.2. *There exist positive numbers $r_0 = r_0(M, g), C = C(M, g)$ and $\alpha = \alpha(n)$ such that for any solution u of equation (1) in a domain Ω satisfying the conditions (2)–(4), we have*

$$\mathcal{H}^{n-1}(\{u = 0\} \cap Q) \leq C \text{diam}^{n-1}(Q) N^\alpha(Q),$$

where $Q \subset B(p, r_0)$ is an arbitrary cube in Ω .

For details, we refer to Theorem 6.1, [9].

5. Application to Steklov eigenfunctions

Our goal is to transform a solution ϕ_λ to the Steklov problem (5) on a manifold M into a solution u to equation (1) on some domain $\Omega \subset \mathbb{R}^n$.

5.1. Getting rid of the boundary. There exists a procedure (see [1, 20, 19]) to transform M into a compact manifold without boundary, which we highlight here. We first let $d(x) := \text{dist}(x, \partial M)$ be the distance between a point $x \in M$ and the boundary. We then define

$$\delta(x) = \begin{cases} d(x) & x \in M_\rho, \\ l(x) & x \in M \setminus M_\rho, \end{cases}$$

where $\rho = \rho(M) > 0$ is such that $d(x)$ is smooth in a ρ neighborhood M_ρ of ∂M in M . We choose $l \in C^\infty(M \setminus M_\rho)$ in such a way that makes δ smooth on M . It now follows that

$$v(x) := e^{\lambda\delta(x)}\phi_\lambda(x),$$

identifies with ϕ_λ on M and satisfies a Neumann boundary condition. More precisely, v solves

$$\begin{cases} \Delta_g v + b(x) \cdot \nabla_g v + q(x)v = 0 & \text{in } M, \\ \partial_\nu v = 0 & \text{on } \partial M, \end{cases}$$

where $\nu = -\nabla\delta$ is the unit outward normal and with

$$\begin{cases} b(x) = -2\lambda\nabla_g\delta(x), \\ q(x) = \lambda^2|\nabla\delta(x)|^2 - \lambda\Delta_g\delta(x). \end{cases}$$

The fact that v satisfies a Neumann boundary condition now allows us to get rid of the boundary by gluing to copies of M together along the boundary and extend v in the natural way. Denote by $\bar{M} = M \cup M$ the compact boundaryless manifold obtained by doing so. We remark that the induced metric \bar{g}_{ij} on \bar{M} is Lipschitz on ∂M . Using the canonical isometric involution that interchanges the two copies M of \bar{M} , we can then extend v, b and q to \bar{M} . Abusing notation and writing v for the extension, we obtain that v satisfies the elliptic equation

$$\Delta_{\bar{g}} v + \bar{b}(x) \cdot \nabla_{\bar{g}} v + \bar{q}(x)v = 0$$

in \bar{M} and we have the following bounds

$$\begin{cases} \|\bar{b}\|_{W^{1,\infty}(N)} \leq C\lambda, \\ \|\bar{q}\|_{W^{1,\infty}(N)} \leq C\lambda^2. \end{cases}$$

Fix a point O in \bar{M} . In local coordinates around O , we have

$$\Delta_{\bar{g}} f = \frac{1}{\sqrt{|\bar{g}|}} \partial_i (\sqrt{|\bar{g}|} \bar{g}^{ij} \partial_j f), \quad (\nabla_{\bar{g}} f)^i = \bar{g}^{ij} \partial_j f.$$

where $\sqrt{|\bar{g}|}$ is the determinant of the extended metric tensor \bar{g} . Since the extended metric is Lipschitz and recalling the boundedness of \bar{b} and \bar{q} , it then follows that v is a solution of equation (1) with L satisfying the conditions (2, 3, 4).

In order to get uniform control over the coefficients, we now work at wavelength scale and consider the ball $B(x_0, 1/\lambda) \subset \bar{M}$. We introduce

$$v_{x_0, \lambda}(x) := v\left(x_0 + \frac{x}{\lambda}\right),$$

for $x \in B(0, 1)$. Then, $v_{x_0, \lambda}$ satisfies equation (1) where the coefficients (a^{ij}) , b^i and c are uniformly bounded in L^∞ by a constant not depending on λ . Moreover, the ellipticity constant of the (a^{ij}) does not change and the Lipschitz constant Γ can only improve. It is clear that the family of $v_{x_0, \lambda}$ solves equation 1 and satisfies the conditions (2), (3), (4) without any dependence on λ . In what follows, we will thus be able to apply Theorem 4.2 uniformly on this family. For more details on the above, we refer the reader to Section 3.2 of [1].

5.2. Upper bound for the nodal set

Remark 5.1. Many of the results we collect in this subsection work only within a small enough scale $r < r_0$. Since we work locally at wavelength scale $r = \frac{1}{\lambda}$, all those results hold for λ big enough.

We now fix a point x_0 in \bar{M} , let $r_0 = \lambda^{-1}$ and choose normal coordinates in a geodesic ball $B_{\bar{g}}(x_0, r_0)$. Without loss of generality, we assume r_0 is smaller than the injectivity radius of \bar{M} . For x, y in $B_{\bar{g}}(x_0, r_0)$, we respectively denote the Euclidean and Riemannian distance by $d(x, y)$ and $d_{\bar{g}}(x, y)$. For λ big enough, we have

$$d_{\bar{g}}(x, y) \leq 2d(x, y) \tag{11}$$

for any two distinct points $x, y \in B_{\bar{g}}(x_0, r_0)$. By construction, the nodal sets of the eigenfunction ϕ_λ and its extension v coincide in M . Combining this observation with equation (11) allows to compare the size of the corresponding nodal sets on small balls. Indeed, for any $r < r_0/2$, one has

$$\mathcal{H}^{n-1}(Z_{\phi_\lambda} \cap B_{\bar{g}}(O, r)) \leq \mathcal{H}^{n-1}(Z_v \cap B(x, 2r)). \tag{12}$$

Denoting by $Z_{v_{x_0, \lambda}}$ the nodal set of $v_{x_0, \lambda}$, we then remark that

$$\mathcal{H}^{n-1}(Z_v \cap B(x, 2r)) \leq \lambda^{1-n} \mathcal{H}^{n-1}(Z_{v_{x_0, \lambda}}).$$

Also, by Proposition 1 in [19], there exists $c_1 > 0$ such that the doubling index of $N_{x_0, \lambda}(x, r)$ of $v_{x_0, \lambda}$ on the ball $B(x, r) \subset B(0, 1)$ satisfies

$$N_{x_0, \lambda}(x, r) \leq c_1 \lambda$$

for any $r < r_0$. We choose $r < r_0/4$ and let Q be the cube centered at origin and of side length r so that the above now implies

$$N_{x_0, \lambda}(Q) = \sup_{x \in Q, r \in (0, \text{diam}(Q))} N_{x_0, \lambda}(x, r) \leq c_1 \lambda.$$

Collecting all of the above, noticing that $B(0, 2r) \subset Q$ and using Theorem 4.2, we finally get that

$$\begin{aligned} \mathcal{H}^{n-1}(Z(\phi_\lambda) \cap B_{\bar{g}}(x_0, r)) &\leq \lambda^{1-n} \mathcal{H}^{n-1}(Z_{v_{x_0, \lambda}} \cap Q) \\ &\leq c_1(n) \lambda^{1-n} N^\alpha(Q) \\ &\leq c_2(n) \lambda^{\alpha-n+1}. \end{aligned}$$

Covering M with $\sim \lambda^n$ balls $B(x_0, r)$ of radius $r = \frac{1}{4\lambda}$ finally yields

$$\mathcal{H}^{n-1}(Z_\lambda) \leq c \lambda^{\alpha+1}$$

and thus concludes the proof of Theorem 1.2.

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