LINEAR PHASE SPACE DEFORMATIONS WITH ANGULAR MOMENTUM SYMMETRY

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ABSTRACT. Motivated by the work of Leznov–Mostovoy [LM03], we classify the linear deformations of standard $2n$-dimensional phase space that preserve the obvious symplectic $\mathfrak{o}(n)$-symmetry. As a consequence, we describe standard phase space, as well as T^*S^n and $T^*\mathbb{H}^n$ with their standard symplectic forms, as degenerations of a 3-dimensional family of coadjoint orbits, which in a generic regime are identified with the Grassmannian of oriented 2-planes in $\mathbb{R}^{n+2}$.

Keywords: Coadjoint orbits; Lie algebra deformations; momentum maps.

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

The notions of momentum maps and symplectic reduction provide a very convenient formulation of integrability in classical mechanics [GS84, GS06]. A standard example is the integrability of central potential Hamiltonians such as the n-dimensional Kepler and harmonic oscillator problems, which can be understood in terms of dynamical (i.e. symplectic) symmetries for the groups $\mathrm{SO}(n+1)$ and $\mathrm{SU}(n)$ [Len24, Fra65]. These principles can be exploited in more general contexts, such as the analogous integrability of the Kepler and harmonic oscillator problems on the round n-sphere studied by Higgs [Hig79]. The standard formulation of these problems can then be recovered from a limiting process, if we interpret the sectional curvature of the n-sphere as a deformation parameter yielding a commutative linear deformation of the Poisson structure in standard phase space.

We will address a natural generalization of the previous idea. A precise formulation relies on three facts (two of which are proved in appendix A):

(i) $(\mathbb{R}^n \oplus \mathbb{R}^n, \omega = \sum_{i=1}^n dx_i \wedge dp_i)$ is symplectomorphic to a connected component $\mathcal{O}_{2n}^+$ of a coadjoint orbit $\mathcal{O}_{2n}$ of $\mathrm{G}_n = \mathrm{O}(n) \ltimes \mathrm{H}_n$, where H_n is the $2n+1$-dimensional Heisenberg group. The action of $\mathrm{O}(n)$ on H_n is the standard one on its $\mathbb{R}^n \oplus \mathbb{R}^n$ subgroup and trivial on the central extension element.

(ii) $\mathfrak{g}_n = \mathfrak{o}(n) \ltimes \mathfrak{h}_n$ is the orthogonal Lie algebra associated to a quadratic form Q_0 on $\mathbb{R}^{n+2}$ of isotropy index 2 and signature $(n, 0)$, and $\mathcal{O}_{2n}$ is

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identified with a Zariski open set in $\widetilde{\mathrm{Gr}}_2\left(\mathbb{R}^{n+2}\right)$, the Grassmannian of oriented 2-planes in $\mathbb{R}^{n+2}$.

(iii) (T^*S^n, ω) and $(T^*\mathbb{H}^n, \omega)$ with their standard symplectic forms are symplectomorphic to coadjoint orbits (a connected component in the latter case) for *deformations* of $\mathfrak{g}_n$ that are respectively isomorphic to $\mathfrak{e}(n+1) = \mathfrak{o}(n+1) \ltimes \mathbb{R}^{n+1}$ and $\mathfrak{e}(n,1) = \mathfrak{o}(n,1) \ltimes \mathbb{R}^{n+1}$. The deformation parameter is interpreted as the sectional curvature of an underlying configuration space (remark 8).

The problem that we pose is the following: *to classify all deformations of the Lie algebra $\mathfrak{g}_n$ and the subsequent coadjoint orbits that specialize to the previous examples.*[1]

At a technical level, the problem is equivalent to the understanding of the Lie algebra cohomology $H^2(\mathfrak{g}_n, \mathfrak{g}_n)$. In virtue of (ii), the problem is related to the study of equivalence classes of deformations of a degenerate quadratic form in $\mathbb{R}^{n+2}$ of isotropy index 2 and signature $(n, 0)$. Our treatment emphasizes the physical and geometric features of the problem while solving it, and is motivated by the work of Leznov-Mostovoy [LM03]. They studied the Kepler problem on a 3-dimensional family of deformations of standard phase space in the special case $n = 3$, interpreted as commutative deformations of the standard Poisson structure on $C^\infty\left(\mathbb{R}^3 \oplus \mathbb{R}^3\right)$. The main result of this article is a proof that the generators of the Leznov-Mostovoy deformations determine three cocycles spanning $H^2(\mathfrak{g}_n, \mathfrak{g}_n)$ for arbitrary $n \geq 3$.

Theorem 1. *Let $n \geq 3$. The space of infinitesimal deformations of $\mathfrak{g}_n^{\mathbb{C}}$ is three-dimensional. Every infinitesimal deformation is integrable, and the generic deformation is isomorphic to $\mathfrak{o}(n+2, \mathbb{C})$.*

However, the induced linear deformations of $\mathfrak{g}_n^{\mathbb{C}}$ are not independent (corollary 1), marking a subtle difference between the infinitesimal and global pictures. The effective family of linear deformations of $\mathfrak{g}_n^{\mathbb{C}}$ turns out to be at most two-dimensional.

The phase space deformations that we will study are natural generalizations of the phase space of a manifold of constant sectional curvature, and contain the latter as particular cases (remark 8). Such deformations can be understood geometrically in terms of the Grassmannian of oriented planes in $\mathbb{R}^{n+2}$, relative to a family of deformations of a degenerate quadratic form of signature $(n, 0)$ and isotropy index 2. In particular, there is a generic regime of deformation parameters for which the induced symplectic manifold is *compact*. This leads to the possibility of extending the study of classical and quantum integrable systems on spaces of constant curvature to the most general phase space deformations that preserve a notion of angular momentum symmetry, by means of the study of the geometry of suitable momentum maps. We plan to address such a problem in the future.

[1] The study of deformations and contractions of Lie algebras in physics originates in [IW53]. The reader can find a leisurely exposition of such ideas in [GS84, GS06].

The work is organized as follows. Section 2 is dedicated to presenting a proof of theorem 1 following an application of the Hochschild-Serre spectral sequence. An argument for an alternative proof in terms of the geometry of quadratic forms is given in remark 1. The rest of the article is a series of applications of theorem 1. Section 3 describes the general family $\mathcal{O}_{2n}(\boldsymbol{\varepsilon})$ of coadjoint orbits, induced by deformations of $\mathfrak{g}_n$, that correspond to the deformations of standard phase space (corollary 3). Section 4 describes some geometric structures in $\mathcal{O}_{2n}(\boldsymbol{\varepsilon})$ corresponding to the induced deformations of the position and momentum polarizations in phase space, and the Euclidean group momentum map corresponding to the free-motion Hamiltonian.

2. Deformations of the Lie algebra $\mathfrak{o}(n) \ltimes \mathfrak{h}_n$

It will be convenient to work with complex Lie algebras, since the real deformations of $\mathfrak{g}_n$ can be regarded as all possible real forms of a complex deformation of its complexification. The infinitesimal deformations of a complex Lie algebra $\mathfrak{g}$ are described by the Lie algebra cohomology space $H^2(\mathfrak{g},\mathfrak{g})$ with respect to the adjoint representation [CE48, Ger64, LN67, Fia85]. We are interested in the case when $\mathfrak{g}$ is a semidirect product of a *semisimple* Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ and an ideal $\mathfrak{h} \subset \mathfrak{g}$, i.e. $\mathfrak{g} = \mathfrak{k} \ltimes \mathfrak{h}$. Hence $\mathfrak{g}$ is also a module for $\mathfrak{k}$ and $\mathfrak{h}$. Let $E_2^{p,q} = H^p\left(\mathfrak{k}, H^q(\mathfrak{h},\mathfrak{g})\right)$. It follows from Whitehead's lemma that $E_2^{1,1} = 0$ and $E_2^{2,0} = 0$. The Hochschild-Serre spectral sequence [HS53, Wei95] collapses from the E_2-term for $p+q=2$. Hence, restriction induces the isomorphism

$$H^2(\mathfrak{g},\mathfrak{g}) \cong E_2^{0,2} \cong H^2(\mathfrak{h},\mathfrak{g})^{\mathfrak{k}} \tag{2.1}$$

(cf. [HS53, theorem 13]). In order to describe the deformations of $\mathfrak{g}_n^{\mathbb{C}}$, we will first consider the case $\mathfrak{k} = \mathfrak{o}(n,\mathbb{C})$, $\mathfrak{h} = \mathbb{C}^n$ so $\mathfrak{g} = \mathfrak{e}_n^{\mathbb{C}} = \mathfrak{o}(n,\mathbb{C}) \ltimes \mathbb{C}^n$, the complexification of the Euclidean Lie algebra in dimension $n \geq 3$. In terms of a canonical basis $\{\mathbf{e}_i, \mathbf{l}_{jk}\}$, $\mathfrak{e}_n^{\mathbb{C}}$ is defined by the commutation relations

$$[\mathbf{e}_i,\mathbf{e}_j] = 0, \quad [\mathbf{l}_{ij},\mathbf{e}_k] = \delta_{ik}\mathbf{e}_j - \delta_{jk}\mathbf{e}_i, \quad [\mathbf{l}_{ij},\mathbf{l}_{kl}] = \delta_{ik}\mathbf{l}_{jl} + \delta_{jl}\mathbf{l}_{ik} - \delta_{il}\mathbf{l}_{jk} - \delta_{jk}\mathbf{l}_{il}.$$

We will now describe the space $H^2\left(\mathbb{C}^n, \mathfrak{e}_n^{\mathbb{C}}\right)^{\mathfrak{o}(n,\mathbb{C})}$ explicitly. By definition, a 2-cocycle is a linear map $f : \bigwedge^2 \mathbb{C}^n \to \mathfrak{e}_n^{\mathbb{C}}$ satisfying

$$[\mathbf{e}_i, f(\mathbf{e}_j,\mathbf{e}_k)] - [\mathbf{e}_j, f(\mathbf{e}_i,\mathbf{e}_k)] + [\mathbf{e}_k, f(\mathbf{e}_i,\mathbf{e}_j)] = 0 \quad \forall i,j,k.$$

A 2-coboundary is a linear map of the form

$$f(\mathbf{e}_i,\mathbf{e}_j) = [l(\mathbf{e}_i),\mathbf{e}_j] + [\mathbf{e}_i, l(\mathbf{e}_j)]$$

for some linear map $l : \mathbb{C}^n \to \mathfrak{e}_n^{\mathbb{C}}$. A 2-cocycle f is called *invariant* if $\forall g \in \mathfrak{o}(n,\mathbb{C})$,

$$(g \cdot f)(\cdot,\cdot) = [g, f(\cdot,\cdot)] - f([g,\cdot],\cdot) - f(\cdot,[g,\cdot]) = \text{a coboundary}. \tag{2.2}$$

Lemma 1. *The space of infinitesimal deformations of the complexification of the Euclidean Lie algebra* $\mathfrak{e}_n^{\mathbb{C}}$*,* $n \geq 3$*, is one-dimensional and generated by the invariant 2-cocycle*

$$f(\mathbf{e}_i, \mathbf{e}_j) = \mathbf{l}_{ij}.$$

Every infinitesimal deformation is integrable. Together, they determine a one-dimensional family of Lie algebras $\mathfrak{e}_n^{\mathbb{C}}(\varepsilon)$*, where* $\mathfrak{e}_n^{\mathbb{C}}(\varepsilon) \cong \mathfrak{o}(n+1, \mathbb{C})$ *for* $\varepsilon \neq 0$ *(cf.* [GS84]*).*

Proof. Any 2-coboundary necessarily takes values in $\mathbb{C}^n$. Hence, any two invariant 2-cocycles taking values in $\mathfrak{o}(n, \mathbb{C})$ are cohomologous if and only if they are equal.

That f is an invariant element in $Z^2\left(\mathbb{C}^n, \mathfrak{e}_n^{\mathbb{C}}\right)$ under the $\mathfrak{o}(n, \mathbb{C})$-action is a routine computation. By the previous remark, the cohomology class of f is nontrivial.

It remains to show that up to a constant, this is the only possibility for f. First, let us assume that $\forall i, j,\ f(\mathbf{e}_i, \mathbf{e}_j) \in \mathfrak{o}(n, \mathbb{C})$, then the invariance condition (2.2) implies that for any $g \in \mathfrak{o}(n, \mathbb{C})$ such that $[g, \mathbf{e}_i] = [g, \mathbf{e}_j] = 0$, $[g, f(\mathbf{e}_i, \mathbf{e}_j)]$ is identically 0. From the structure of $\mathfrak{o}(n, \mathbb{C})$, it follows that

$$\left\{\ker\left(\mathrm{ad}_{\mathbf{e}_i}|_{\mathfrak{o}(n,\mathbb{C})}\right) \cap \ker\left(\mathrm{ad}_{\mathbf{e}_j}|_{\mathfrak{o}(n,\mathbb{C})}\right)\right\}^{\perp} = \mathbb{C} \cdot \mathbf{l}_{ij},$$

where the left hand side denotes the subalgebra of $\mathfrak{o}(n, \mathbb{C})$ annihilated by

$$\ker\left(\mathrm{ad}_{\mathbf{e}_i}|_{\mathfrak{o}(n,\mathbb{C})}\right) \cap \ker\left(\mathrm{ad}_{\mathbf{e}_j}|_{\mathfrak{o}(n,\mathbb{C})}\right).$$

Thus, $f(\mathbf{e}_i, \mathbf{e}_j) = c\mathbf{l}_{ij}$ for some $c \in \mathbb{C}$ is the only invariant cocycle with image lying in $\mathfrak{o}(n, \mathbb{C})$.

Now, let us assume that f is an invariant cocycle with $f(\mathbf{e}_i, \mathbf{e}_j) \in \mathbb{C}^n$, for some i, j. We claim that the same holds for any other value of f. Indeed, for any $k \neq i, j$, the invariance of f under $\mathbf{l}_{ik}$ implies that $[\mathbf{l}_{ki}, f(\mathbf{e}_i, \mathbf{e}_j)] - f(\mathbf{e}_k, \mathbf{e}_j)$ is equal to a 2-coboundary evaluated at $\mathbf{e}_i \wedge \mathbf{e}_j$, and therefore $f(\mathbf{e}_k, \mathbf{e}_j) \in \mathbb{C}^n$. Moreover, for any $l \neq i, j, k$, a similar argument on the invariance of f under $\mathbf{l}_{jl}$ shows that $f(\mathbf{e}_k, \mathbf{e}_l) \in \mathbb{C}^n$. In conclusion, the image of an invariant 2-cocycle either lies fully in $\mathfrak{o}(n, \mathbb{C})$ or in $\mathbb{C}^n$.

To conclude the classification, let us assume that $f : \bigwedge^2 \mathbb{C}^n \to \mathbb{C}^n$ is an arbitrary linear map. We claim that f is a 2-coboundary. To see this, observe that the linear space of such maps and the space of linear maps $l : \mathbb{C}^n \to \mathfrak{o}(n, \mathbb{C})$ are equidimensional, and the coboundary map $l \mapsto [l(\cdot), \cdot] + [\cdot, l(\cdot)]$ is a linear map between these spaces. A basis for the space of linear maps $l : \mathbb{C}^n \to \mathfrak{o}(n, \mathbb{C})$ is given by the set $\{l_{ijk}(\mathbf{e}_m) = \delta_{im}\mathbf{l}_{jk}\}_{j<k}$. Since

$$(dl_{ijk})(\mathbf{e}_m, \mathbf{e}_n) = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\mathbf{e}_k + (\delta_{in}\delta_{km} - \delta_{im}\delta_{kn})\mathbf{e}_j,$$

and the collection of the latter maps is obviously linearly independent, we conclude that the coboundary map is an isomorphism. Therefore, all linear maps $f : \bigwedge^2 \mathbb{C}^n \to \mathbb{C}^n$ are 2-coboundaries, and any invariant 2-cocycle taking values in $\mathbb{C}^n$ is necessarily trivial in cohomology.

To conclude the proof, it is a routine computation to verify that the bracket deformation of $\mathfrak{e}_n^{\mathbb{C}}$ determined by any choice of nontrivial invariant 2-cocycle,

$$[\mathbf{e}_i, \mathbf{e}_j]_\varepsilon = \varepsilon \mathbf{l}_{ij}, \quad \varepsilon \in \mathbb{C},$$

and with the rest of brackets kept the same, satisfies the Jacobi identity and defines a Lie algebra $\mathfrak{e}_n^{\mathbb{C}}(\varepsilon)$ that is isomorphic to $\mathfrak{o}(n+1, \mathbb{C})$ if $\varepsilon \neq 0$. In particular, the specialization $\varepsilon \in \mathbb{R}$ leads to the real forms $\mathfrak{o}(n+1)$ for $\varepsilon > 0$, and $\mathfrak{o}(n, 1)$ for $\varepsilon < 0$. □

Proof of theorem 1. The Heisenberg Lie algebra is defined as a central extension of the abelian Lie algebra $\mathbb{C}^{2n}$, and $\mathfrak{o}(n, \mathbb{C})$ acts trivially in the extension term. This implies that $\mathfrak{g}_n^{\mathbb{C}} = \mathfrak{o}(n, \mathbb{C}) \ltimes \mathfrak{h}_n^{\mathbb{C}}$ is a central extension of the Lie algebra $\mathfrak{o}(n, \mathbb{C}) \ltimes \mathbb{C}^{2n}$, and moreover, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{h}_n^{\mathbb{C}} & \longrightarrow & \mathfrak{g}_n^{\mathbb{C}} & \longrightarrow & \mathfrak{o}(n, \mathbb{C}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{C}^{2n} & \longrightarrow & \mathfrak{o}(n, \mathbb{C}) \ltimes \mathbb{C}^{2n} & \longrightarrow & \mathfrak{o}(n, \mathbb{C}) & \longrightarrow & 0
\end{array}$$

We claim that there is a $1-1$ correspondence

$$\left\{ \begin{array}{c} \text{Infinitesimal} \\ \text{deformations of } \mathfrak{g}_n^{\mathbb{C}} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Infinitesimal} \\ \text{deformations of} \\ \mathfrak{o}(n, \mathbb{C}) \ltimes \mathbb{C}^{2n} \end{array} \right\}$$

In other words, there is an isomorphism

$$H^2 \left(\mathfrak{h}_n^{\mathbb{C}}, \mathfrak{g}_n^{\mathbb{C}} \right)^{\mathfrak{o}(n, \mathbb{C})} \cong H^2 \left(\mathbb{C}^{2n}, \mathfrak{o}(n, \mathbb{C}) \ltimes \mathbb{C}^{2n} \right)^{\mathfrak{o}(n, \mathbb{C})}.$$

It will be convenient to consider a canonical basis for $\mathfrak{h}_n^{\mathbb{C}}$, $\{\mathbf{e}_1, \ldots, \mathbf{e}_{2n}, \mathbf{I}\}$, with commutation relations

$$[\mathbf{e}_i, \mathbf{e}_{n+j}] = \delta_{ij} \mathbf{I}, \quad i, j = 1, \ldots, n.$$

Any invariant cocycle $f : \bigwedge^2 \mathfrak{h}_n^{\mathbb{C}} \to \mathfrak{g}_n^{\mathbb{C}}$ is cohomologous to a cocycle f' satisfying $f(\mathbf{e}_i, \mathbf{I}) = 0 \ \forall i$. To see this, notice that the cocycle condition implies that $[\mathbf{e}_j, f(\mathbf{e}_i, \mathbf{I})] = 0 \ \forall i, j$, therefore $f(\mathbf{e}_i, \mathbf{I}) = c_i \mathbf{I}$. The linear map $l : \mathfrak{h}_n^{\mathbb{C}} \to \mathfrak{g}_n^{\mathbb{C}}$ defined as

$$l(\mathbf{e}_j) = 0, \qquad l(\mathbf{I}) = \sum_{i=1}^{n} (c_i \mathbf{e}_{n+i} - c_{n+i} \mathbf{e}_i),$$

satisfies $(dl)(\mathbf{e}_i, \mathbf{I}) = f(\mathbf{e}_i, \mathbf{I})$, and the claim follows. Moreover, a similar argument shows that any invariant cocycle is cohomologous to a cocycle with image in $\mathfrak{o}(n, \mathbb{C}) \ltimes \mathbb{C}^{2n}$. Indeed, assume that $f(\mathbf{e}_i, \mathbf{e}_j) = c_{ij} \mathbf{I}$. Then the linear map $l : \mathfrak{h}_n^{\mathbb{C}} \to \mathfrak{g}_n^{\mathbb{C}}$ given by $l(\mathbf{e}_k) = \delta_{ik} c_{ij} \mathbf{e}_{j-n}$ if $j > n$ (or $= -\delta_{ik} c_{ij} \mathbf{e}_{j+n}$ if $j \leq n$) satisfies $(dl)(\mathbf{e}_i, \mathbf{e}_j) = c_{ij} \mathbf{I}$ and zero otherwise. This concludes the proof of the isomorphism in cohomology. Thus, it is enough to understand the infinitesimal deformations of $\mathfrak{o}(n, \mathbb{C}) \ltimes \mathbb{C}^{2n}$.

Lemma 1 can be used in the classification of the independent invariant 2-cocycles $f : \bigwedge \mathbb{C}^{2n} \to \mathfrak{o}(n,\mathbb{C}) \ltimes \mathbb{C}^{2n}$. A similar argument as in the proof of lemma 1 shows that an invariant 2-cocycle which is not a 2-coboundary necessarily has image in $\mathfrak{o}(n,\mathbb{C})$. When thought of as a $\mathfrak{o}(n,\mathbb{C})$-module, the subalgebra $\mathbb{C}^{2n}$ splits as a direct sum $\mathbb{C}^n_1 \oplus \mathbb{C}^n_2$ of invariant $\mathfrak{o}(n,\mathbb{C})$-subspaces. There is an induced splitting $\bigwedge^2 \mathbb{C}^{2n} = \left(\bigwedge^2 \mathbb{C}^n_1\right) \oplus (\mathbb{C}^n_1 \otimes \mathbb{C}^n_2) \oplus \left(\bigwedge^2 \mathbb{C}^n_2\right)$, and any invariant 2-cocycle can be decomposed into three different components. The classification problem is then reduced to the classification of invariant 2-cocycles on each component. It follows from lemma 1 that the restriction to $\mathbb{C}^n_1$ and $\mathbb{C}^n_2$ determines two nontrivial one-dimensional spaces of invariant 2-cocycles, spanned by

$$f_1(\mathbf{e}_i, \mathbf{e}_j) = \mathbf{l}_{ij} \quad \text{for } i, j \le n, \text{ and zero otherwise,} \tag{2.3}$$

$$f_2(\mathbf{e}_{n+i}, \mathbf{e}_{n+j}) = \mathbf{l}_{ij} \quad \text{for } i, j \le n, \text{ and zero otherwise.} \tag{2.4}$$

These are the only possibilities that are supported in the invariant subspaces $\bigwedge^2 \mathbb{C}^n_1$ and $\bigwedge^2 \mathbb{C}^n_2$. The remaining possibility would consist of an invariant 2-cocycle supported in $\mathbb{C}^n_1 \otimes \mathbb{C}^n_2$. There is an obvious choice, namely

$$f_3(\mathbf{e}_i, \mathbf{e}_{n+j}) = \mathbf{l}_{ij} \quad \text{for } i, j \le n, \text{ and zero otherwise.} \tag{2.5}$$

A similar argument as in the proof of lemma 1 shows that any other invariant 2-cocycle supported in $\mathbb{C}^n_1 \otimes \mathbb{C}^n_2$ must be a multiple of f_3. Therefore, any nontrivial invariant 2-cocycle is a linear combination of f_1, f_2 and f_3.

Let $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{C}$. The lift to $\mathfrak{g}_n$ of the Lie bracket deformations of $\mathfrak{o}(n,\mathbb{C}) \ltimes \mathbb{C}^{2n}$ induced by the previous cocycles is determined by

$$[\mathbf{e}_i, \mathbf{e}_j]_{\varepsilon_1} = \varepsilon_1 \mathbf{l}_{ij}, \quad [\mathbf{e}_{n+i}, \mathbf{e}_{n+j}]_{\varepsilon_2} = \varepsilon_2 \mathbf{l}_{ij}, \quad [\mathbf{e}_i, \mathbf{e}_{n+j}]_{\varepsilon_3} = \delta_{ij}\mathbf{I} + \varepsilon_3 \mathbf{l}_{ij}.$$

and additionally,

$$[\mathbf{e}_i, \mathbf{I}] = \varepsilon_3 \mathbf{e}_i - \varepsilon_1 \mathbf{e}_{n+i}, \qquad [\mathbf{e}_{n+i}, \mathbf{I}] = \varepsilon_2 \mathbf{e}_i - \varepsilon_3 \mathbf{e}_{n+i},$$

while the remaining basis elements' Lie brackets are unchanged. It is a routine computation to verify that these Lie bracket deformations satisfy the Jacobi identity. Hence they integrate to a three-dimensional family of deformations $\mathfrak{g}^{\mathbb{C}}_n(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. When $\varepsilon_1, \varepsilon_2 \neq 0$, and $\varepsilon_3^2 \neq \varepsilon_1\varepsilon_2$, there is an isomorphism $\mathfrak{g}^{\mathbb{C}}_n(\varepsilon_1, \varepsilon_2, \varepsilon_3) \cong \mathfrak{g}^{\mathbb{C}}_n(1,1,0) = \mathfrak{o}(n+2,\mathbb{C})$. The details on this isomorphism and the full classification of deformations are described in proposition 1. □

Remark 1. There is yet another way to describe the linear deformations of $\mathfrak{g}^{\mathbb{C}}_n$, and in particular the invariant 2-cocycles generating $H^2\left(\mathfrak{g}^{\mathbb{C}}_n, \mathfrak{g}^{\mathbb{C}}_n\right)$, in terms of the geometry of the of the quadratic space $\left(\mathbb{C}^{n+2}, Q^{\mathbb{C}}_0\right)$. Any deformation $Q^{\mathbb{C}}_{\boldsymbol{\varepsilon}}$ of the quadratic form $Q^{\mathbb{C}}_0$ induces a deformation of the orthogonal Lie algebra $\mathfrak{o}\left(\mathbb{C}^{n+2}, Q^{\mathbb{C}}_0\right) \cong \mathfrak{g}^{\mathbb{C}}_n$, in such a way that if a new quadratic form $\tilde{Q}^{\mathbb{C}}_{\boldsymbol{\varepsilon}}$ is induced by an orthogonal transformation of $Q^{\mathbb{C}}_{\boldsymbol{\varepsilon}}$, the corresponding deformations $\mathfrak{g}^{\mathbb{C}}_n(\boldsymbol{\varepsilon})$ and $\tilde{\mathfrak{g}}^{\mathbb{C}}_n(\boldsymbol{\varepsilon})$ are isomorphic.

Consider the canonical basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n+2}\}$ of $\mathbb{C}^{n+2}$, together with the correspondence

$$\mathbf{v}_i \wedge \mathbf{v}_j \mapsto \mathbf{l}_{ij}, \quad \mathbf{v}_i \wedge \mathbf{v}_{n+1} \mapsto \mathbf{e}_i, \quad \mathbf{v}_i \wedge \mathbf{v}_{n+2} \mapsto \mathbf{e}_{n+i}, \quad \mathbf{v}_{n+1} \wedge \mathbf{v}_{n+1} \mapsto \mathbf{I}$$

where $1 \le i < j \le n$. It is straightforward to verify that the deformations of the corresponding bilinear form of Q_0 along the totally isotropic plane $W = \mathrm{Span}\{\mathbf{v}_{n+1}, \mathbf{v}_{n+2}\}$, parametrized as

$$(\mathbf{v}_{n+1}, \mathbf{v}_{n+1}) = \varepsilon_1, \quad (\mathbf{v}_{n+2}, \mathbf{v}_{n+2}) = \varepsilon_2, \quad (\mathbf{v}_{n+1}, \mathbf{v}_{n+2}) = \varepsilon_3$$

induce the deformation $\mathfrak{g}_n^{\mathbb{C}}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. Therefore, we conclude *a posteriori* that the map

$$\{\text{Deformations of } Q_0 \text{ along } W\} \to \left\{ Z^2\left(\mathfrak{h}_n^{\mathbb{C}}, \mathfrak{o}(n, \mathbb{C})\right)^{\mathfrak{o}(n,\mathbb{C})} \cong H^2(\mathfrak{g}_n^{\mathbb{C}}, \mathfrak{g}_n^{\mathbb{C}}) \right\}$$

is a bijection.

Remark 2. The special cases $n = 1, 2$ were excluded from the proof since $\mathfrak{g}_1 = \mathfrak{h}_1$ and $\mathfrak{o}(2)$ is abelian. However, it follows from remark 1 that the spaces $H^2(\mathfrak{g}_n, \mathfrak{g}_n)$ are still three-dimensional when $n = 1, 2$.

3. Deformation of special coadjoint orbits

Let us assume that $n \ge 3$. The three-parameter family of deformations of $\mathfrak{g}_n$ can be conveniently prescribed as a deformation of the Lie-Poisson structure on a basis for $\mathfrak{g}_n^\vee$. Let us consider the dual coordinates $\{l_{ij}\}_{1 \le i < j \le n}$ in $\mathfrak{o}(n)^\vee$ with canonical commutation relations

$$\{l_{ij}, l_{kl}\} = \delta_{ik} l_{jl} - \delta_{il} l_{jk} + \delta_{jl} l_{ik} - \delta_{jk} l_{il} \tag{3.1}$$

together with the Darboux coordinates $\{x_i, p_i\}_{i=1}^n$, and the central extension coordinate I in $\mathfrak{h}_n^\vee$, on which the coordinates l_{ij} act as

$$\{l_{ij}, x_k\} = \delta_{ik} x_j - \delta_{jk} x_i, \quad \{l_{ij}, p_k\} = \delta_{ik} p_j - \delta_{jk} p_i, \quad \{l_{ij}, I\} = 0. \tag{3.2}$$

The commutation relations (3.1)–(3.2) do not admit nontrivial deformations as a consequence of the simplicity of $\mathfrak{o}(n)$, and in particular, they are not affected by the integration of the cocycles (2.3)–(2.5). On the other hand, the linear deformations of the induced Lie–Poisson bracket of the chosen basis for $\mathfrak{g}_n^\vee$ manifest in the remaining commutation relations. Let ε_1, ε_2 and ε_3 be complex parameters corresponding to the cocycles f_1, f_2 and f_3 in $H^2\left(\mathfrak{g}_n^{\mathbb{C}}, \mathfrak{g}_n^{\mathbb{C}}\right)$. The Lie–Poisson bracket deformations in $\left(\mathfrak{g}_n^{\mathbb{C}}\right)^\vee$ take the explicit form

$$\begin{aligned} &\{x_i, x_j\} = \varepsilon_1 l_{ij}, \qquad \{p_i, p_j\} = \varepsilon_2 l_{ij}, \\ &\{x_i, p_j\} = \delta_{ij} I + \varepsilon_3 l_{ij}, \\ &\{x_i, I\} = \varepsilon_3 x_i - \varepsilon_1 p_i, \quad \{p_i, I\} = \varepsilon_2 x_i - \varepsilon_3 p_i, \end{aligned} \tag{3.3}$$

Remark 3. Let $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$. For every $\lambda \in \mathbb{C}^*$, the nonzero triples $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and $\boldsymbol{\varepsilon}' = \lambda\cdot(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ define isomorphic Lie algebras under scaling of generators. Therefore, in order to describe the different isomorphism classes of nontrivial deformations of $\mathfrak{g}_n^{\mathbb{C}}$, it is sufficient to consider them in terms of a stratification of the projective plane $\mathbb{P}(\boldsymbol{\varepsilon})$.

Remark 4. Different special values of nonzero triples $\boldsymbol{\varepsilon}$ determine special Lie algebra deformations. By definition, $\mathfrak{g}_n^{\mathbb{C}}(1,1,0) = \mathfrak{o}(n+2,\mathbb{C})$, which is seen under the relabeling $x_i = l_{in+1}$, $p_i = l_{in+2}$, $I = l_{n+1n+2}$. Moreover, the Lie algebras $\mathfrak{g}_n^{\mathbb{C}}(1,0,0)$ and $\mathfrak{g}_n^{\mathbb{C}}(0,1,0)$ are isomorphic to $\mathfrak{o}(n+1,\mathbb{C}) \ltimes \mathbb{C}^{n+1}$ under the respective relabelings $p_i = l_{in+1}$ and $x_i = l_{in+1}$. Finally, let $\mathfrak{d}_n$ denote the deformation of $\mathfrak{h}_n$ corresponding to the triple $(0,0,1)$. Then $\mathfrak{d}_n^{\mathbb{C}}$ is completely characterized by its ideals $\mathrm{Span}\{I, x_i, p_i\} \cong \mathfrak{sl}(2,\mathbb{C})$, $i = 1,\ldots,n$, and $\mathfrak{g}_n^{\mathbb{C}}(0,0,1) = \mathfrak{o}(n,\mathbb{C}) \ltimes \mathfrak{d}_n^{\mathbb{C}}$. Let $\mathscr{C} \subset \mathbb{P}(\boldsymbol{\varepsilon})$ be the flat conic defined by the equation

$$\varepsilon_3^2 = \varepsilon_1\varepsilon_2,$$

and for $i = 1,2,3$, let

$$\mathscr{L}_i = \{\varepsilon_i = 0\} \subset \mathbb{P}(\boldsymbol{\varepsilon}).$$

Then we have that $\mathscr{C} \cap \mathscr{L}_1 = [0:1:0]$, $\mathscr{C} \cap \mathscr{L}_2 = [1:0:0]$, and $\mathscr{L}_1 \cap \mathscr{L}_2 = [0:0:1]$. With the exception of the latter, all such special points belong to $\mathscr{L}_3$.

Proposition 1. *The isomorphism classes of nontrivial deformations $\mathfrak{g}_n^{\mathbb{C}}(\boldsymbol{\varepsilon})$ are stratified in the projective plane $\mathbb{P}(\boldsymbol{\varepsilon})$ as follows:*

(i) $\mathfrak{g}_n^{\mathbb{C}}(\boldsymbol{\varepsilon}) \cong \mathfrak{o}(n+2,\mathbb{C})$, *if $[\boldsymbol{\varepsilon}] \in \mathscr{U}$, where $\mathscr{U}$ denotes the Zariski open locus*

$$\mathscr{U} = \mathbb{P}(\boldsymbol{\varepsilon}) \setminus \{\mathscr{C} \cup \mathscr{L}_1 \cup \mathscr{L}_2\}.$$

(ii) $\mathfrak{g}_n^{\mathbb{C}}(\boldsymbol{\varepsilon}) \cong \mathfrak{o}(n+1,\mathbb{C}) \ltimes \mathbb{C}^{n+1}$ *if $[\boldsymbol{\varepsilon}] \in \mathscr{C}$.*

(iii) $\mathfrak{g}_n^{\mathbb{C}}(\boldsymbol{\varepsilon}) \cong \mathfrak{o}(n,\mathbb{C}) \ltimes \mathfrak{d}_n^{\mathbb{C}}$ *if $[\boldsymbol{\varepsilon}] \in (\mathscr{L}_1 \cup \mathscr{L}_2) \setminus \mathscr{C}$. Notice that*

$$(\mathscr{L}_1 \cup \mathscr{L}_2) \setminus \mathscr{C} = (\mathscr{L}_1 \setminus [0:1:0]) \cup (\mathscr{L}_2 \setminus [1:0:0]).$$

Proof. The proof follows after a systematic implementation of the following fundamental principle: at a special value of $\boldsymbol{\varepsilon}$, all cocycles that haven't been integrated to a deformation become trivial in cohomology, and hence, the remaining deformations become equivalent to of a linear transformation of the basis elements.

(i) Let $\pi_3 : \mathbb{P}(\boldsymbol{\varepsilon}) \setminus \mathscr{L}_1 \cap \mathscr{L}_2 \to \mathscr{L}_3$ be the projection $\pi_3(\varepsilon_1,\varepsilon_2,\varepsilon_3) = (\varepsilon_1,\varepsilon_2,0)$. If $\boldsymbol{\varepsilon} \in \mathscr{U}$, we have that $\varepsilon_1 \neq 0$, $\varepsilon_2 \neq 0$, and $\varepsilon_3^2/\varepsilon_1\varepsilon_2 \neq 1$. Then, there is an isomorphism $\mathfrak{g}_n^{\mathbb{C}}(\pi_3(\boldsymbol{\varepsilon})) \cong \mathfrak{g}_n^{\mathbb{C}}(\boldsymbol{\varepsilon})$ induced by the linear transformation defined by

$$x_i \mapsto x_i + \frac{\lambda\varepsilon_3}{2\varepsilon_2}p_i, \quad p_i \mapsto p_i + \frac{\lambda\varepsilon_3}{2\varepsilon_1}x_i, \quad I \mapsto \left(1 - \frac{\lambda^2\varepsilon_3^2}{4\varepsilon_1\varepsilon_2}\right)I, \quad l_{ij} \mapsto \lambda l_{ij},$$

where

$$\lambda = \frac{2\varepsilon_1\varepsilon_2}{\epsilon_3^2}\left(1 - \sqrt{1 - \frac{\varepsilon_3^2}{\varepsilon_1\varepsilon_2}}\right) = 1 + O\left(\frac{\varepsilon_3^2}{\varepsilon_1\varepsilon_2}\right).$$

In order to show that $\mathfrak{g}_n^{\mathbb{C}}(\pi_3(\boldsymbol{\varepsilon})) \cong \mathfrak{g}_n^{\mathbb{C}}(1,1,0) \cong \mathfrak{o}(n+2,\mathbb{C})$, let

$$x_i = \sqrt{\varepsilon_1} l_{in+1}, \qquad p_i = \sqrt{\varepsilon_2} l_{in+2}, \qquad I = \sqrt{\varepsilon_1\varepsilon_2} l_{n+1n+2}.$$

(ii) Assume $[\boldsymbol{\varepsilon}] \in \mathscr{C} \setminus \{[1:0:0],[0:1:0]\}$. An isomorphism $\mathfrak{g}_n^{\mathbb{C}}(\varepsilon_1,0,0) \cong \mathfrak{g}_n^{\mathbb{C}}(\boldsymbol{\varepsilon})$ is defined by the linear transformation acting as the identity on x_i, I and l_{ij}, and mapping

$$p_i \mapsto p_i + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} x_i$$

An analogous isomorphism can be constructed to show that $\mathfrak{g}_n^{\mathbb{C}}(0,1,0) \cong \mathfrak{g}_n^{\mathbb{C}}(\boldsymbol{\varepsilon})$.
(iii) Assume that $\boldsymbol{\varepsilon} \in \mathscr{L}_1 \setminus [0:1:0]$. The linear transformation defined by

$$p_i \mapsto p_i - \frac{\varepsilon_2}{2\varepsilon_3} x_i,$$

and acting as the identity on x_i, I, and l_{ij} defines the isomorphism $\mathfrak{g}_n^{\mathbb{C}}(\boldsymbol{\varepsilon}) \cong \mathfrak{g}_n^{\mathbb{C}}(0,0,1) = \mathfrak{o}(n,\mathbb{C}) \ltimes \mathfrak{d}_n^{\mathbb{C}}$. An analogous argument implies the result for any $\boldsymbol{\varepsilon} \in \mathscr{L}_2 \setminus [1:0:0]$. □

Corollary 1. *Any deformation $\mathfrak{g}_n^{\mathbb{C}}(\boldsymbol{\varepsilon})$ with $[\boldsymbol{\varepsilon}] \in \mathscr{U}$ depends only on the two effective parameters $\varepsilon_1, \varepsilon_2$. Any deformation $\mathfrak{g}_n^{\mathbb{C}}(\boldsymbol{\varepsilon})$ with $[\boldsymbol{\varepsilon}] \in \mathscr{C}$ depends only on one effective parameter (either ε_1 or ε_2). Any deformation $\mathfrak{g}_n^{\mathbb{C}}(\boldsymbol{\varepsilon})$ with $[\boldsymbol{\varepsilon}] \in (\mathscr{L}_1 \cup \mathscr{L}_2) \setminus \mathscr{C}$ depends only on the effective parameter ε_3.*

Remark 5. From now on, we will assume that the deformation parameters $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ are real, unless otherwise stated.

In order to describe the different real forms of the deformations $\mathfrak{g}_n(\boldsymbol{\varepsilon})$ that arise by restriction to $\mathbb{R}$, it is necessary to consider instead a stratification of $\mathbb{R}^3 \setminus \{(0,0,0)\}$. The lift $\mathrm{pr}^{-1}(\mathscr{C}|_{\mathbb{R}}) \subset \mathbb{R}^3 \setminus \{(0,0,0)\}$ has two connected components $\mathscr{C}_+$, $\mathscr{C}_-$, depending on whether $\varepsilon_1, \varepsilon_2 \geq 0$ or $\varepsilon_1, \varepsilon_2 \leq 0$. $\mathrm{pr}^{-1}(((\mathscr{L}_1 \cup \mathscr{L}_2) \setminus \mathscr{C})|_{\mathbb{R}})$ will be denoted by $\mathscr{L}$ (although it possesses two connected components, the corresponding real forms are isomorphic). The set $\mathbb{R}^3 \setminus \{(\mathscr{C}_+ \cup \mathscr{C}_- \cup \mathscr{L}\}$ can be decomposed as

$$\mathscr{R}_{++} \cup \mathscr{R}_{--} \cup \mathscr{R}_{+-},$$

with the regions $\mathscr{R}_{++}$ and $\mathscr{R}_{--}$ characterized by the conditions $\varepsilon_1, \varepsilon_2 > 0$ and $\varepsilon_1, \varepsilon_1 < 0$, respectively (each region consisting of 3 connected components). The remaining region $\mathscr{R}_{+-}$ consists of all triples $\boldsymbol{\varepsilon}$ for which either $\varepsilon_1 > 0$, $\varepsilon_2 < 0$ or $\varepsilon_1 < 0$, $\varepsilon_2 > 0$.

Corollary 2. *In terms of the previous stratification of* $\mathbb{R}^3 \setminus \{(0,0,0)\}$*, the isomorphism type of the real forms* $\mathfrak{g}_n(\boldsymbol{\varepsilon})$ *for* $\boldsymbol{\varepsilon} \neq (0,0,0)$ *is*

$$\begin{cases} \mathfrak{o}(n+2) & \text{if} \quad \boldsymbol{\varepsilon} \in \mathscr{R}_{++}, \\ \mathfrak{o}(n+1,1) & \text{if} \quad \boldsymbol{\varepsilon} \in \mathscr{R}_{+-}, \\ \mathfrak{o}(n,2) & \text{if} \quad \boldsymbol{\varepsilon} \in \mathscr{R}_{--}, \\ \mathfrak{o}(n+1) \ltimes \mathbb{R}^{n+1} & \text{if} \quad \boldsymbol{\varepsilon} \in \mathscr{C}_{+}, \\ \mathfrak{o}(n,1) \ltimes \mathbb{R}^{n+1} & \text{if} \quad \boldsymbol{\varepsilon} \in \mathscr{C}_{-}, \\ \mathfrak{o}(n) \ltimes \mathfrak{d}_n|_{\mathbb{R}} & \text{if} \quad \boldsymbol{\varepsilon} \in \mathscr{L}. \end{cases}$$

3.1. **Special coadjoint orbits.** The rank of a semi-simple Lie algebra $\mathfrak{g}$ is equal to the dimension of the center of its universal enveloping algebra $U(\mathfrak{g})$—a space generated by the so-called *Casimir invariants.* $Z(U(\mathfrak{g}))$ can be equivalently described in terms of the Lie-Poisson structure in $C^\infty(\mathfrak{g}^\vee)$. For $\mathfrak{o}(n+2,\mathbb{C})$ and the dual basis $\{l_{ij}\}_{1\leq i<j\leq n+2}$ with Poisson brackets (3.1), the Casimir invariants can be determined explicitly as the homogeneous polynomials

$$C_{2k} = \operatorname{tr}\left(\mathrm{L}^{2k}\right), \quad k = 1, \ldots, \lfloor n/2 \rfloor + 1,$$

where $\mathrm{L} = (l_{ij})$. The choice of values for the Casimir invariants determines all the coadjoint orbits of maximal dimension in the orbit stratification of $\mathfrak{o}(n+2,\mathbb{C})^\vee$, isomorphic to the quotient of $\mathrm{O}(n+2,\mathbb{C})$ by a maximal torus. There is an analogous description of the coadjoint orbits in $\mathfrak{o}(n+2,\mathbb{C})^\vee$ isomorphic to the homogeneous space $\mathrm{O}(n+2,\mathbb{C})/\mathrm{SO}(2,\mathbb{C}) \times \mathrm{O}(n,\mathbb{C})$, and which are the *minimal* nontrivial orbits when $n \neq 2, 4$ [Wol78]. The next result is described in [BS97].

Lemma 2 ([BS97])**.** *The* $2n$*-dimensional coadjoint orbits in* $\mathfrak{o}(n+2)^\vee$ *are isomorphic to the homogeneous space* $\mathrm{SO}(n+2)/\mathrm{SO}(2) \times \mathrm{SO}(n)$ *and form a 1-dimensional algebraic family determined by the collection of quadratic equations*

$$C_2 = -2r^2, \tag{3.4}$$

$$l_{i_1 i_2} l_{i_3 i_4} = l_{i_1 i_3} l_{i_2 i_4} - l_{i_1 i_4} l_{i_2 i_3}, \quad 1 \leq i_1 < i_2 < i_3 < i_4 \leq n+2. \tag{3.5}$$

Remark 6. The set of quadratic equations (3.4)–(3.5) identify the given coadjoint orbits with the Grassmannian $\widetilde{\mathrm{Gr}}_2\left(\mathbb{R}^{n+2}\right)$ of oriented 2-planes in $\mathbb{R}^{n+2}$, as they can be understood as a 2 : 1 lift of the classical Plücker embedding. The Plücker relations indicate that equations (3.5) are overdetermined, and can be generated by any subcollection of $\binom{n}{2}$ equations containing a given fixed element l_{ij}, i.e. l_{n+1n+2}.

Applying corollary 2 and lemma 2 to the generic deformations $\mathfrak{g}_n(\boldsymbol{\varepsilon})$, $\boldsymbol{\varepsilon} \in \mathbb{R}^3$, and letting $x_i = l_{in+1}$, $p_i = l_{in+2}$, and $I = l_{n+1n+2}$, equation (3.4) becomes

$$C_2 = -2\left(I^2 + \varepsilon_1 x^2 + \varepsilon_2 p^2 - 2\varepsilon_3 xp - (\varepsilon_3^2 - \varepsilon_1\varepsilon_2)l^2\right), \tag{3.6}$$

where

$$x^2 = \sum_{i=1}^n x_i^2, \qquad p^2 = \sum_{i=1}^n p_i^2, \qquad xp = \sum_{i=1}^n x_i p_i, \qquad l^2 = \sum_{1\le i<j\le n} l_{ij}^2.$$

The remaining equations do not depend on the deformation parameters. We emphasize the ones containing $\{I, x_i, p_j\}$,

$$Il_{ij} = x_i p_j - x_j p_i, \tag{3.7}$$

$$l_{ij}x_k - l_{ik}x_j + l_{jk}x_i = 0, \qquad l_{ij}p_k - l_{ik}p_j + l_{jk}p_i = 0. \tag{3.8}$$

Notice that the subcollection (3.7) generalizes the usual definition of angular momentum and generate (3.5), while equations (3.8) generalize the vector analysis relations $\mathbf{l}\cdot\mathbf{x} = \mathbf{l}\cdot\mathbf{p} = 0$.

Definition 1. The coadjoint orbits $\mathcal{O}_{2n}(\boldsymbol{\varepsilon}) \subset \mathfrak{g}_n(\boldsymbol{\varepsilon})^\vee$ are the special $2n$-dimensional orbits defined by the choice of value $C_2 = -2$ in equation (3.6).

Remark 7. It follows from remarks 1 and 6 that over the open set $\mathscr{R}_{++}$, (3.6)–(3.7) correspond to the equations that determine a $2:1$ lift of the Plücker embedding, identifying $\mathcal{O}_{2n}(\boldsymbol{\varepsilon})$ with the Grassmannian of oriented planes in $\mathbb{R}^{n+2}$. The different solutions of the quadratic equation (3.6) correspond to the different choices of orientation of a given 2-plane in $\mathbb{R}^{n+2}$. If we consider the degeneration $\boldsymbol{\varepsilon} \to 0$, the limiting equations

$$I = \pm 1 \qquad \text{and} \qquad \pm\, l_{ij} = x_i p_j - x_j p_i, \tag{3.9}$$

define two disjoint orbits $\mathcal{O}_{2n}^+$ and $\mathcal{O}_{2n}^-$ in $\mathfrak{g}_n^\vee$, each isomorphic to $\mathbb{R}^n \oplus \mathbb{R}^n$. In turn, the degeneration of the canonical symplectic structure determined by the Kirillov–Konstant–Souriau symplectic form [RSTS94, Kir04] on $\mathcal{O}_{2n}(\boldsymbol{\varepsilon})$ corresponds to the symplectic structure on standard phase space

$$\left(\mathbb{R}^n \oplus \mathbb{R}^n, \sum_{i=1}^n dx_i \wedge dp_i\right)$$

for each of the two orbits in $\mathfrak{g}_n^\vee$. The existence of two limiting connected components corresponds to the limiting degenerations in the work of Higgs [Hig79] for the cotangent bundles T^*S^n as the sectional curvature is allowed to vanish (cf. remark 8).

Corollary 3. *For any $\boldsymbol{\varepsilon} \in \mathbb{R}^3 \setminus \{(0,0,0)\}$, the special coadjoint orbits $\mathcal{O}_{2n}(\boldsymbol{\varepsilon})$ define a family of deformations of standard phase space, and carry a canonical "angular momentum" representation of $\mathfrak{o}(n)$ in $C^\infty(\mathcal{O}_{2n}(\boldsymbol{\varepsilon}))$ (cf. [LM03]). These orbits are diffeomorphic to the Grassmannian of oriented 2-planes $\widetilde{\mathrm{Gr}}_2\left(\mathbb{R}^{n+2}\right)$ over the open region $\mathscr{R}_{++}$.*

Remark 8. Over the lift of the flat conic $\mathrm{pr}^{-1}(\mathscr{C}_{\mathbb{R}}) = \mathscr{C}_+ \cup \mathscr{C}_-$ (and in particular, in the lines $\varepsilon_2 = \varepsilon_3 = 0$ and $\varepsilon_1 = \varepsilon_3 = 0$), the coadjoint orbits $\mathcal{O}_{2n}(\boldsymbol{\varepsilon})$ degenerate to a manifold isomorphic to T^*S^n if $\boldsymbol{\varepsilon} \in \mathscr{C}_+$ and two copies of $T^*\mathbb{H}^n$ if $\boldsymbol{\varepsilon} \in \mathscr{C}_-$. A proof of this fact is given in proposition 2. Although the symplectic structure inherited in T^*S^n (resp. $T^*\mathbb{H}^n$) is the standard one (see remark 10), the variables x_i and p_j do not define Darboux coordinates. Instead, their commutation relations resemble physically the result of adding an external magnetic field in standard phase space [Nov82, Per90]. From a physical point of view, the study of dynamical problems over the complete family of coadjoint orbits $\mathcal{O}_{2n}(\boldsymbol{\varepsilon})$ can also be interpreted as the study of deformations of dynamical systems on n-manifolds of constant sectional curvature (which is equal to ε_2 when $\varepsilon_1 = \varepsilon_3 = 0$).

Proposition 2. *There is an induced isomorphism $\mathcal{O}_{2n}(\boldsymbol{\varepsilon}) \cong (T^*S^n, \omega)$ over the locus $\mathscr{C}_+$, where ω denotes the corresponding standard symplectic form. Over the locus $\mathscr{C}_-$, $\mathcal{O}_{2n}(\boldsymbol{\varepsilon})$ is a disjoint union of two connected components $\mathcal{O}^+_{2n}(\boldsymbol{\varepsilon})$ and $\mathcal{O}^-_{2n}(\boldsymbol{\varepsilon})$, each symplectomorphic to $(T^*\mathbb{H}^n, \omega)$, corresponding to the values $I > 0$ and $I < 0$ respectively.*

Proof. It is enough to corroborate this in the case $\varepsilon_1 = \varepsilon_3 = 0$; when $\varepsilon_2 > 0$ (resp. $\varepsilon_2 < 0$). Then, equation (3.6) determines an n-sphere homogeneous space model in the affine variables I, x_i (resp. a two-sheeted n-hyperboloid model, with connected components corresponding to the values $I > 0$ and $I < 0$). Moreover, the orbit $\mathcal{O}_{2n}(\boldsymbol{\varepsilon})$ of the $\mathrm{SO}(n+1) \ltimes \mathbb{R}^{n+1}$-action (resp. the $\mathrm{SO}(n,1) \ltimes \mathbb{R}^{n+1}$-action) on $(\mathfrak{o}(n+1) \ltimes \mathbb{R}^{n+1})^*$ (resp. on $(\mathfrak{o}(n,1) \ltimes \mathbb{R}^{n+1})^*$) determined by equations (3.6)-(3.7) has the structure of a rank-n subbundle $E \to S^n$ (resp. two bundles $E^+ \to \mathbb{H}^n$ and $E^- \to \mathbb{H}^n$) of the trivial vector bundle $\mathfrak{o}(n+1)^* \times S^n$ (resp. two copies of $\mathfrak{o}(n,1)^* \times \mathbb{H}^n$), with fiber at a point $(I, x_1, \dots, x_n)$ given by the kernel of the map $L_{(I,x_1,\dots,x_n)} : \mathfrak{o}(n+1)^* \to \mathfrak{o}(n)^*$ (resp. $\mathfrak{o}(n,1)^*$), defined as

$$\left(L_{(I,x_1,\dots,x_n)}(l,p)\right)_{ij} = I l_{ij} - x_i p_j + x_j p_i + \sum_{k=1}^{n} \left(l_{ij} x_k - l_{ik} x_j + l_{jk} x_i\right).$$

By construction, the bundle of orthonormal frames of E (resp. E^+ and E^-) is isomorphic to $\mathrm{SO}(n+1) \to S^n$ (resp. $\mathrm{SO}(n,1) \to \mathbb{H}^n$), with fibers corresponding to the isotropy groups of points $(I, x_1, \dots, x_n)$ (depending on the values $I > 0$ or $I < 0$ in the second case), which gives the isomorphism $E \cong T^*S^n$ (resp. $E^{\pm} \cong T^*\mathbb{H}^n$). □

Observe that on $\mathscr{C}_-$, only the connected component $\mathcal{O}^+_{2n}(\boldsymbol{\varepsilon})$ is of physical significance, as it degenerates to the component $\mathcal{O}^+_{2n}$ corresponding to the value $I = 1$ when $\boldsymbol{\varepsilon} \to 0$.

Remark 9. For any $\boldsymbol{\varepsilon} \in \mathscr{R}_{++}$, $\mathscr{R}_{+-}$, or $\mathscr{R}_{--}$, the coadjoint orbits $\mathcal{O}_{2n}(\boldsymbol{\varepsilon})$ are also irreducible Hermitian symmetric spaces, acquiring a natural Kähler structure [Bor54]. The tangent space at any point in a given orbit $\mathcal{O}_{2n}(\boldsymbol{\varepsilon})$ is respectively modeled by one of the quotients $\mathfrak{m} = \mathfrak{o}(n+2)/\mathfrak{o}(2) \oplus \mathfrak{o}(n)$,

$\mathfrak{o}(n+1,1)/\mathfrak{o}(2)\oplus\mathfrak{o}(n-1,1)$ or $\mathfrak{o}(n,2)/\mathfrak{o}(2)\oplus\mathfrak{o}(n-2,2)$, and the integrable almost complex structure can be defined as $J=\mathrm{ad}_{\mathbf{I}}$, where $\mathbf{I}$ is a generator of $\mathfrak{o}(2)\subset\mathfrak{o}(2)\oplus\mathfrak{o}(n-i,i)$, $i=0,1,2$. Therefore, it follows that all orbits $\mathcal{O}_{2n}(\boldsymbol{\varepsilon})$ possess a natural Kähler polarization generalizing the standard complex coordinate polarization determined by

$$\left(\mathbb{C}^n,\frac{\sqrt{-1}}{2}\sum_{i=1}^{n}dz_i\wedge d\bar{z}_i\right),\qquad z_i=x_i+\sqrt{-1}p_i.$$

4. Singular real polarizations and free motion

By their very definition, the family of coadjoint orbits $\mathcal{O}_{2n}(\boldsymbol{\varepsilon})$ possess two natural real polarizations, singular over the set $\{I=0\}$, and invariant under a family of groups of symplectomorphisms isomorphic to a deformation of the Euclidean group. These are spanned by the Hamiltonian vector fields corresponding to the collections of functions $\{x_i/I\}$, $\{p_i/I\}$ in involution

$$\{x_i/I,x_j/I\}=0,\qquad\{p_i/I,p_j/I\}=0,\qquad 1\le i,j\le n,$$

and will be called, respectively, the *position* and *momentum* polarizations. Both position and momentum polarizations are invariant under a global $\mathrm{SO}(n)$-action, generalizing the standard rotational action in position and momentum coordinates, and which is characterized infinitesimally by the momentum map

$$\lambda:\mathcal{O}_{2n}(\boldsymbol{\varepsilon})\to\mathfrak{o}(n)^*,\qquad(\lambda)_{ij}=l_{ij}.$$

Let $q_i=x_i/I$, $i=1,\dots,n$. The choice of the position polarization motivates the introduction of a family of functions playing the role of the free-motion Hamiltonians in deformed phase space, namely

$$H_0(\boldsymbol{\varepsilon})=\frac{1}{2}\left(p^2+\varepsilon_2 l^2\right)=\frac{1}{2}\left(p^2+\varepsilon_2\left(p^2q^2-(pq)^2\right)\right),\tag{4.1}$$

and which posses two equivalent geometric interpretations in terms of the dynamical symmetries of a family of $\binom{n+1}{2}$-dimensional Lie groups. They not only coincide with the quadratic Casimir invariants of the Lie subalgebras spanned by the dual elements $\{l_{ij}\}$ and $\{p_k\}$, but also correspond to $|\mu_0(\boldsymbol{\varepsilon})|^2$, the square of the norm of a family of momentum maps

$$\mu_0(\boldsymbol{\varepsilon}):\mathcal{O}_{2n}(\boldsymbol{\varepsilon})\to\mathfrak{k}(\boldsymbol{\varepsilon})^*$$

where

$$\mathfrak{k}(\boldsymbol{\varepsilon})\cong\begin{cases}\mathfrak{o}(n+1) & \text{if}\quad \varepsilon_2>0,\\ \mathfrak{o}(n,1) & \text{if}\quad \varepsilon_2<0,\\ \mathfrak{e}_{n+1} & \text{if}\quad \varepsilon_2=0.\end{cases}$$

Thus, $H_0(\boldsymbol{\varepsilon})$ and $\mu_0(\boldsymbol{\varepsilon})$ respectively generalize the standard free-motion Hamiltonian and the corresponding Euclidean group momentum map in

$(\mathbb{R}^n \oplus \mathbb{R}^n, \omega)$. In particular, over the contraction $\varepsilon_1 = \varepsilon_3 = 0$, the coordinates $\{q_i\}$ correspond to the *gnomonic coordinates* over the n-sphere [Hig79] if $\varepsilon_2 > 0$ and hyperbolic n-space if $\varepsilon_2 < 0$ and $I > 0$, while H_0 corresponds to the Hamiltonian inducing geodesic motion.

Remark 10. In the coordinates $\{q_i, p_j\}$, defined over the open set $\{I \neq 0\}$, Kirillov's symplectic form on the family $\mathcal{O}_{2n}(0, \varepsilon_2, 0)$ takes the simple form

$$\omega_{\boldsymbol{\varepsilon}} = -d\theta_{\boldsymbol{\varepsilon}}, \qquad \theta_{\boldsymbol{\varepsilon}} = \sum_{i=1}^{n} \left(p_i - \varepsilon_2 \frac{(q,p) q_i}{1 + \varepsilon_2 q^2} \right) dq_i. \tag{4.2}$$

The analogous expression over the real flat conic $\varepsilon_3^2 = \varepsilon_1 \varepsilon_2$ can then be reconstructed by means of a suitable linear transformation (see proposition 1). In particular, when $\varepsilon_1 = \varepsilon_3 = 0$, $\varepsilon_2 > 0$ (resp. $\varepsilon_2 < 0$), the above explicit expression for the Liouville form $\theta_{\boldsymbol{\varepsilon}}$ provides the standard Darboux coordinates with conjugated momenta

$$p_i - \varepsilon_2 \frac{(q,p) q_i}{1 + \varepsilon_2 q^2}$$

on T^*S^n, (resp. $T^*\mathbb{H}^n$ when either $I > 0$ or $I < 0$).

Appendix A. Standard phase space as a coadjoint orbit

Proposition 3. *There is a symplectomorphism*

$$\left(\mathbb{R}^n \oplus \mathbb{R}^n, \omega = \sum_{i=1}^{n} dx_i \wedge dp_i \right) \cong \mathcal{O}_{2n}^+$$

to a connected component of a $2n$-dimensional coadjoint orbit $\mathcal{O}_{2n}$ of the group $\mathrm{O}(n) \ltimes \mathrm{H}_n$, mapping the standard $\mathrm{O}(n)$-action on $\mathbb{R}^n \oplus \mathbb{R}^n$ to the corresponding coadjoint action on $\mathcal{O}_{2n}$.

Proof. It is convenient to identify a suitable set of generators and relations on the dual space $\mathfrak{g}_n^\vee$. Let $\{x_1, p_1, \ldots, x_n, p_n, I\}$ be a set of standard dual variables for the Heisenberg Lie algebra $\mathfrak{h}_n$, and let $\{l_{ij}\}_{1 \leq i < j \leq n}$ be dual variables for the orthogonal Lie algebra $\mathfrak{o}(n)$. By definition, the symplectic structure of a coadjoint orbit is determined by the Lie-Poisson bracket on $C^\infty(\mathfrak{g}_n^\vee)$. Consider the $2n$-dimensional coadjoint orbit $\mathcal{O}_{2n} \subset \mathfrak{g}_n^\vee$ determined by fixing the values $I = \pm 1$, together with the *angular momentum relations*

$$l_{ij} = x_i p_j - x_j p_i, \qquad 1 \leq i < j \leq n.$$

The canonical commutation relations $\{x_i, p_j\} = \delta_{ij}$ will follow if we restrict to the connected component $\mathcal{O}_{2n}^+$ given by $I = 1$. The correspondence of symplectic $\mathrm{O}(n)$-actions readily follows. ∎

Proposition 4. *Let Q_0 denote the quadratic form in $\mathbb{R}^{n+2}$ prescribed by*

$$Q_0(a_1, \ldots, a_{n+2}) = a_1^2 + \cdots + a_n^2.$$

There is an isomorphism $\mathfrak{g}_n \cong \mathfrak{o}\left(\mathbb{R}^{n+2}, Q_0\right)$. There is an induced diffeomorphism between $\mathcal{O}_{2n}$ and the Zariski open subset $\mathscr{U} \subset \widetilde{\mathrm{Gr}}_2\left(\mathbb{R}^{n+2}\right)$ consisting *of oriented 2-planes* $P \subset \mathbb{R}^{n+2}$ *such that* $a_{n+1} \wedge a_{n+2}|_P \not\equiv 0$*, i.e., whose image under the Plücker embedding lies in the complement of the zero locus*

$$Z(a_{n+1} \wedge a_{n+2}) \subset \mathbb{P}\left(\bigwedge^2 \mathbb{R}^{n+2}\right).$$

Proof. Recall that any bilinear form $(\cdot,\cdot)$ on a vector space V induces a Lie algebra structure on $\bigwedge^2 V$ in terms of the orthogonal endomorphisms

$$v \wedge w \mapsto L_{v\wedge w}(u) := (v,u)w - (w,u)v,$$

[GS84, GS06]. A direct computation shows that the Lie algebra structure on $\bigwedge^2 \mathbb{R}^{n+2}$ induced by the quadratic form $Q_0(a_1,\ldots,a_{n+2}) = a_1^2 + \cdots + a_n^2$ is isomorphic to $\mathfrak{g}_n$ under the dual correspondence

$$a_i \wedge a_j \mapsto l_{ij} \quad 1 \le i < j \le n,$$

$$a_i \wedge a_{n+1} \mapsto x_i, \quad a_i \wedge a_{n+2} \mapsto p_i, \quad 1 \le i \le n, \quad a_{n+1} \wedge a_{n+2} \mapsto I.$$

Such a correspondence identifies the angular momentum relations along the hyperplanes $I = \pm 1$ in $\mathfrak{g}_n$ with the Plücker relations along the hyperplanes $a_{n+1} \wedge a_{n+2} = \pm 1$. Let $\mathrm{pr} : \widetilde{\mathrm{Gr}}_2\left(\mathbb{R}^{n+2}\right) \to \mathrm{Gr}_2\left(\mathbb{R}^{n+2}\right)$ the projection forgetting orientation, $\iota : \mathrm{Gr}_2\left(\mathbb{R}^{n+2}\right) \hookrightarrow \mathbb{P}\left(\bigwedge^2 \mathbb{R}^{n+2}\right)$ the classical Plücker embedding, and let

$$\mathscr{U} = \widetilde{\mathrm{Gr}}_2\left(\mathbb{R}^{n+2}\right) \setminus (\mathrm{pr} \circ \iota)^{-1}\left(Z(a_{n+1} \wedge a_{n+2})\right)$$

Since the level sets $a_{n+1} \wedge a_{n+2} = \pm 1$ in $\bigwedge^2 \mathbb{R}^{n+2}$ determine uniquely a choice of orientation in every 2-plane in $\mathrm{Gr}_2\left(\mathbb{R}^{n+2}\right) \setminus \iota^{-1}\left(Z(a_{n+1} \wedge a_{n+2})\right)$, we conclude that there is an induced diffeomorphism $\mathscr{U} \cong \mathcal{O}_{2n}$. In particular the stabilizer in G_n of the unique totally isotropic 2-plane W in $\mathbb{R}^{n+2}$ is equal to $\mathrm{O}(n) \times \mathbb{R} \subset \mathrm{O}\left(\mathbb{R}^{n+2}, Q_0\right) \cong \mathrm{G}_n$, with the $\mathbb{R}$-factor corresponding to central extension elements (cf. remark 7). ☐

Remark 11. Under the correspondences described in propositions 3 and 4, the Howe pair $(\mathrm{O}(n), \mathrm{SL}(2,\mathbb{R}))$ of $(\mathbb{R}^n \oplus \mathbb{R}^n, \omega)$ [KKS78, How89] is induced by the maximal compact subgroup $G \subset \mathrm{O}\left(\mathbb{R}^{n+2}, Q_0\right) \cong \mathrm{G}_n$ and the group G' of endomorphisms of the unique totally isotropic plane in $\left(\mathbb{R}^{n+2}, Q_0\right)$ preserving the area element $a_{n+1} \wedge a_{n+2} = 1$.

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References

[Bor54] A. Borel, *Kählerian coset spaces of semisimple Lie groups*, Proc. Nat. Acad. Sci. U. S. A. **40** (1954), 1147–1151.

[BS97] O. M. Boyarskyi and T. V. Skrypnik, *Degenerate orbits of adjoint representation of orthogonal and unitary groups regarded as algebraic submanifolds*, Ukrainian Math. J. **49** (1997), no. 7, 1003–1015.

[CE48] C. Chevalley and S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc. **63** (1948), no. 1, 85–124.

[Fia85] A. Fialowski, *Deformations of Lie algebras*, Mat. Sb. (N. S.) **127(169)** (1985), no. 4, 476–482.

[Fra65] D. M. Fradkin, *Three-dimensional isotropic harmonic oscillator and* SU_3, Am. J. Phys. **33** (1965), no. 3, 207–211.

[Ger64] M. Gerstenhaber, *On the deformation of rings and algebras*, Ann. Math. **79** (1964), no. 1, 59–103.

[GS84] V. Guillemin and S. Sternberg, *Symplectic techniques in physics*, Cambridge University Press, 1984.

[GS06] ———, *Variations on a theme by Kepler*, Colloquium Publications Vol. 42, American Mathematical Soc., 2006.

[Hig79] P. W. Higgs, *Dynamical symmetries in a spherical geometry I*, J. Phys. A **12** (1979), no. 3, 309–323.

[How89] R. Howe, *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. **313** (1989), no. 2, 539–570.

[HS53] G. Hochschild and J-P. Serre, *Cohomology of Lie algebras*, Ann. Math. (1953), 591–603.

[IW53] E. Inonu and E. P. Wigner, *On the contraction of groups and their representations*, Proc. Nat. Acad. Sci. U. S. A. **39** (1953), 510–524.

[Kir04] A. A. Kirillov, *Lectures on the orbit method*, Graduate Studies in Mathematics, Vol. 64, American Mathematical Soc., 2004.

[KKS78] D. Kazhdan, B. Kostant, and S. Sternberg, *Hamiltonian group actions and dynamical systems of Calogero type*, Comm. Pure Appl. Math. **31** (1978), no. 4, 481–507.

[Len24] W. Lenz, *Über den Bewegungsverlauf und die Quantenzustände der gestörten Keplerbewegung*, Z. Phys. **24** (1924), no. 1, 197–207.

[LM03] A. Leznov and J. Mostovoy, *Classical dynamics in deformed spaces*, J. Phys. A **36** (2003), no. 5, 1439–1449.

[LN67] M. Levy-Nahas, *Deformation and contraction of Lie algebras*, J. Math. Phys. **8** (1967), no. 6, 1211–1222.

[Nov82] S. P. Novikov, *The Hamiltonian formalism and a multivalued analogue of Morse theory*, Uspekhi Mat. Nauk **37** (1982), no. 5(227), 3–49, 248, English translation in Russian Math. Surveys **37**:5 (1982), 1–56.

[Per90] A. Perelomov, *Integrable systems of classical mechanics and Lie algebras*, Birkhäuser, 1990.

[RSTS94] A. Reyman and M. A. Semenov-Tian-Shansky, *Group-theoretical methods in the theory of finite-dimensional integrable systems*, Dynamical systems VII, Springer Berlin Heidelberg, 1994.

[Wei95] C. A. Weibel, *An introduction to homological algebra*, Studies in Advanced Mathematics, Vol. 38, Cambridge University Press, 1995.

[Wol78] J. Wolf, *Representations associated to minimal co-adjoint orbits*, Differential Geometrical Methods in Mathematical Physics II. Springer Berlin Heidelberg (1978), 329–349.

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