Identification of Port-Hamiltonian Systems from Frequency Response Data

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Abstract

In this paper, we study the identification problem of strictly passive systems from frequency response data. We present a simple construction approach based on the Mayo-Antoulas generalized realization theory that automatically yields a port-Hamiltonian realization for every strictly passive system with simple spectral zeros. Furthermore, we discuss the construction of a frequency-limited port-Hamiltonian realization. We illustrate the proposed method by means of several examples.

Keywords: Passive systems, port-Hamiltonian system, identification, tangential interpolation

1. Introduction

In this paper, we study the problem of identifying linear finite-dimensional dynamical systems that are *strictly passive*. We are interested in port-Hamiltonian (pH) realizations of such systems which not only inherently encode the underlying conservation laws and physical principles of the process but also have several spectral and robustness properties, see, e.g., [1, 2]. Moreover, pH realizations inherently arise from, e.g., energy-based modeling via bond graphs [3, 4]. Since pH realizations have several intrinsic properties, we seek to identify an underlying pH realization using data. System identification allows to build models from data, see, e.g., [5, 6, 7]. However, we are not aware of any identification method that directly builds a pH model from either time domain or frequency domain data. In this paper, we are interested in frequency response data for a particular reason. That is, there are many ways how to obtain frequency response data directly in an experimental set-up, for example, using *scattering-parameters*, see, e.g., [8], and vibrational analysis, see, e.g., [9]. Moreover, there exists a rich literature, where frequency response data is generated to infer models using time-domain data, see, e.g., [10, 11, 12].

Let us consider continuous-time systems in standard state-space form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = 0, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

where $u : \mathbb{R} \to \mathbb{R}^m$, $x : \mathbb{R} \to \mathbb{R}^n$, and $y : \mathbb{R} \to \mathbb{R}^m$ are vector-valued functions denoting the *input*, *state*, and *output* of the system, respectively. The coefficient matrices satisfy $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times m}$. Furthermore, for passive systems, the input and output dimensions are equal to m since we aim to interpolate with (square) passive transfer functions.

The identification of compact reduced-order models for large-scale passive systems has been an active research area, see, e. g., [13, 14, 15, 16, 17, 18, 19]. However, this requires the availability of system matrices in an explicit or matrix-vector form, which may not be available, especially when the necessary parameters to model a dynamical process are not known. If this is the case, we can obtain frequency response data for a process and infer an underlying pH realization directly from the data. In this direction, the authors in [20] have proposed a two-step approach – the first step is to identify a state-space model using the Loewner approach [21], and in the second step, a non-convex optimization problem is formulated, aiming at finding the closest pH realization. However, the proposed non-convex optimization problem is hard to solve in practice, and often, we may not even get an exact pH presentation of the obtained state-space realization, see [20] for details.

In this paper, we propose a simple construction based on the Loewner approach [21] to infer an underlying pH realization. Precisely, we show that if we choose the interpolation data of the transfer function at the spectral zeros along the corresponding zero directions, then the direct use of the Loewner approach [21] yields a pH realization. Moreover, we discuss an approach for estimating the spectral zeros and zero directions using the frequency response data since direct measurements at spectral zeros and zero directions may not be possible to obtain experimentally.

The structure of the paper is as follows. In Section 2, we briefly recall some important properties of passive systems. Then, in Section 3, we discuss state-space representations and properties of pH realizations. This is followed in the next section by a discussion of degrees of freedom of a pH system in order to have an understanding of how many parameters are needed to describe a minimal pH system. In Section 5, we propose a variant of the Loewner-based approach, realizing the system in pH form when data are available at spectral zeros along with

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zero directions. Furthermore, we discuss the estimation of the dominant spectral zeros and zero directions using frequency response data in Section 6. In Section 7, we illustrate the proposed identification approach by means of a couple of numerical examples, which is followed by a short summary.

In the rest of the paper, we make use of the following notation:

- The Hermitian (or conjugate) transpose of a vector or matrix V is denoted by V^{H} $\left(V^{\mathsf{T}}\right)$ and the identity matrix is denoted by I_n, or I if the dimension is clear.
- We denote real and complex n-vectors ($n \times m$ matrices) by $\mathbb{R}^n$, $\mathbb{C}^n$ ($\mathbb{R}^{n\times m}$, $\mathbb{C}^{n\times m}$), respectively.
- We denote the set of symmetric matrices in $\mathbb{R}^{n\times n}$ by $\mathbb{S}_n$. Positive definiteness (semi-definiteness) of $M \in \mathbb{S}_n$ is denoted by $M > 0$ ($M \geq 0$).

2. Passive Systems

Passive systems and their relationships with *positive-real functions and stability conditions* are well studied. We briefly recall some important properties, which can be found in [22, 23]. We consider continuous-time systems with a real rational transfer matrix $Z(s)$ and define the associated spectral density function:

$$\Phi(s) := Z^{\mathsf{T}}(-s) + Z(s), \tag{2}$$

which coincides with twice the Hermitian part of $Z(s)$ on the $\imath\omega$ axis:

$$\Phi(\imath\omega) = [Z(\imath\omega)]^{\mathsf{H}} + Z(\imath\omega).$$

Definition 2.1 (e.g., [23]). *The rational transfer function $Z(s)$ is called* strictly positive-real *if $\Phi(\imath\omega) > 0$ for all $\omega \in \mathbb{R}$ and it is called* positive-real *if $\Phi(\imath\omega) \geq 0$ for all $\omega \in \mathbb{R}$.*

Definition 2.2 (e.g., [24]). *The transfer function $Z(s)$ is called* asymptotically stable *if the poles of the transfer function are in the open left half-plane, and it is called* stable *if all the poles are in the closed left half-plane, with any pole occurring on the imaginary axis being first-order.*

Definition 2.3 (e.g., [23]). *The transfer function $Z(s)$ is called* strictly passive *if it is strictly positive-real and asymptotically stable and it is called* passive *if it is positive-real and stable.*

Throughout the paper, we will assume that the realizations we deal with are minimal (i. e. controllable and observable), and we will restrict ourselves in this paper to *strictly* passive systems, which implies that the matrix A is invertible and the transfer matrix is proper since poles cannot be on the imaginary axis or at infinity. Moreover, $\Phi(\imath\omega) > 0$ at $\omega = \infty$ implies that we must have $D^{\mathsf{T}} + D > 0$ as well. We will see that this restriction simplifies our discussion significantly. It is also a reasonable restriction because passive systems can be viewed as limiting cases of strictly passive systems.

Since the transfer function is proper, we can represent it in standard state-space form $Z(s) = C(sI_n - A)^{-1}B + D$. For proper transfer functions $Z(s)$ with minimal realization $\mathcal{M} := \{A, B, C, D\}$, there is a necessary and sufficient condition for passivity, known as the Kalman-Yakubovich-Popov linear matrix inequality. An elegant proof of this can be found in [23].

Theorem 2.4 (e.g., [23]). *Let $\mathcal{M} := \{A, B, C, D\}$ be a minimal realization of a proper rational transfer function $Z(s)$ and let*

$$W(X, \mathcal{M}) = \begin{bmatrix} -A^{\mathsf{T}}X - XA & C^{\mathsf{T}} - XB \\ C - B^{\mathsf{T}}X & D + D^{\mathsf{T}} \end{bmatrix}. \tag{3}$$

Then $Z(s)$ is passive if and only if there exists a real symmetric matrix $X \in \mathbb{S}_n$ such that

$$W(X, \mathcal{M}) \geq 0, \quad X > 0, \tag{4}$$

and is strictly passive if and only if there exists a real symmetric matrix $X \in \mathbb{S}_n$ such that

$$W(X, \mathcal{M}) > 0, \quad X > 0. \tag{5}$$

The solutions X of these inequalities are known as *certificates* for the passivity or strict passivity of the system $\mathcal{M}$.

Definition 2.5. *Every solution X of the LMI*

$$\mathbb{X}^{\geq} := \{X \in \mathcal{S} \,|\, W(X, \mathcal{M}) \geq 0,\ X > 0\} \tag{6}$$

is called a certificate for passivity *of the model $\mathcal{M}$ and every solution of the LMI*

$$\mathbb{X}^{>} := \{X \in \mathcal{S} \,|\, W(X, \mathcal{M}) > 0,\ X > 0\} \tag{7}$$

is called a certificate for strict passivity *of the model $\mathcal{M}$.*

3. Port-Hamiltonian Models

In this section, we provide a brief introduction to special realizations of passive systems, known as pH realizations.

Definition 3.1 (e.g., [25]). *A linear time-invariant* pH realization *of a proper transfer function, has the standard state-space form*

$$\begin{aligned} \dot{x}(t) &= (J - R)Qx(t) + (G - P)u(t), \quad x(0) = 0, \\ y(t) &= (G + P)^{\mathsf{T}}Qx(t) + (N + S)u(t), \end{aligned} \tag{8}$$

where the system matrices

$$\mathcal{V} := \begin{bmatrix} -J & -G \\ G^{\mathsf{T}} & N \end{bmatrix}, \quad \mathcal{W} := \begin{bmatrix} R & P \\ P^{\mathsf{T}} & S \end{bmatrix}, \tag{9}$$

satisfy the conditions

$$\mathcal{V} = -\mathcal{V}^{\mathsf{T}}, \quad \mathcal{W} = \mathcal{W}^{\mathsf{T}} \geq 0, \quad Q = Q^{\mathsf{T}} \geq 0.$$

It readily follows from the properties of pH models that when Q and $\mathcal{W}$ are invertible, we can choose $X = Q$ as certificate for the model

$$\mathcal{M} := \left\{(J - R)Q, G - P, (G + P)^{\mathsf{T}}Q, N + S\right\}$$

to show that it satisfies the strict passivity condition (5).

Remark 3.2. *The condition that Q is non-singular is automatically satisfied when the state transition matrix A is non-singular, which is the case for strictly passive systems. We can then also represent the system in generalized state-space form, using $\widehat{x} := Qx$, yielding:*

$$\begin{aligned} Q^{-1}\dot{\widehat{x}} &= (J - R)\widehat{x} + (G - P)u, \\ y &= (G + P)^{\mathsf{T}}\widehat{x} + (N + S)u. \end{aligned} \tag{10}$$

We use such models for representing intermediate results later on. A realization then consists of five matrices,

$$\mathcal{M} := \{A, B, C, D, E\} = \left\{J - R, G - P, (G + P)^T, N + S, Q^{-1}\right\}.$$

Conversely, let $\mathcal{M} := \{A, B, C, D\}$ be a state-space model, satisfying the strict passivity condition (5) with a given certificate $X > 0$. Then, it can always be transformed into pH form, as shown in [26]. We can use a symmetric factorization $X = T^{\mathsf{T}}T$, which implies the invertibility of T, and define a new realization

$$\{A_T, B_T, C_T, D\} := \{TAT^{-1}, TB, CT^{-1}, D\}$$

so that

$$\begin{aligned} &\begin{bmatrix} T^{-\mathsf{T}} & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} -A^{\mathsf{T}}X - XA & C^{\mathsf{T}} - XB \\ C - B^{\mathsf{T}}X & D^{\mathsf{T}} + D \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & I_m \end{bmatrix} \\ &\qquad = \begin{bmatrix} -A_T & -B_T \\ C_T & D \end{bmatrix} + \begin{bmatrix} -A_T^{\mathsf{T}} & C_T^{\mathsf{T}} \\ -B_T^{\mathsf{T}} & D^{\mathsf{T}} \end{bmatrix} > 0. \end{aligned} \tag{11}$$

We can then use the symmetric and skew-symmetric parts of the matrix

$$\mathcal{S} := \begin{bmatrix} -A_T & -B_T \\ C_T & D \end{bmatrix}$$

to define the coefficients of a pH representation via

$$\mathcal{V} := \begin{bmatrix} -J & -G \\ G^{\mathsf{T}} & N \end{bmatrix} := \frac{\mathcal{S} - \mathcal{S}^{\mathsf{T}}}{2}, \; \mathcal{W} := \begin{bmatrix} R & P \\ P^{\mathsf{T}} & S \end{bmatrix} := \frac{\mathcal{S} + \mathcal{S}^{\mathsf{T}}}{2}.$$

This construction yields $\mathcal{W} > 0$ and $Q = I_n$ because of the chosen factorization $X = T^{\mathsf{T}}T$. This is called a *normalized* pH representation. This shows that pH models with strict inequalities $Q > 0$ and $\mathcal{W} > 0$ are nothing but strictly passive systems described in an appropriate coordinate system. On the other hand, it was shown in [27] that normalized pH systems have good robustness properties in terms of their so-called passivity radius, which measures the minimum perturbation that leads to a non-passive system, see [28] for a detailed definition of passivity radius.

4. Degrees of Freedom of a Transfer Function

In the literature, one can find a discussion on the degrees of freedom of a given strictly proper rational transfer function $Z(s)$ with a given McMillan degree n [29]. This corresponds to the minimum number of parameters to describe such a function. Since this literature is quite opaque, we briefly re-derive the basic results using a generic $m \times m$ strictly proper transfer matrix of McMillan degree n without repeated poles. Such a transfer function can be written in its partial fraction expansion as follows:

$$Z(s) = \sum_{k=1}^{n_r} u_k(s - \lambda_k)^{-1}v_k^{\mathsf{T}} + \sum_{k=1}^{(n-n_r)/2} U_k\left(sI_2 - \begin{bmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{bmatrix}\right)^{-1} V_k^{\mathsf{T}},$$

which requires a total of $2(m + 1)n$ real parameters. This can be seen as a state-space model in real Jordan form with 1×1 diagonal elements for the n_r real poles and 2×2 diagonal blocks for the $n_c := n - n_r$ complex conjugate complex poles. But this representation is only unique up to a block diagonal state-space transformation with exactly m degrees of freedom: a scalar t_k for each real pole and a 2×2 block $t_k \begin{bmatrix} c_k & s_k \\ -s_k & c_k \end{bmatrix}$ for each complex conjugate pair, where the real rotation matrix depends on one real parameter. When taking the quotient of the manifold of block-diagonal models with respect to this state space transformation, we are left with the exact number of real degrees of freedom, which is $2mn$ for a strictly proper $m \times m$ transfer function of degree n with real coefficients.

When considering the larger class of real $m \times m$ *proper* rational transfer functions, one has to add the real parameters to *realize* the constant matrix D. If D is constrained to have a particular rank, then we again need to take that into account. A rank r matrix D can be represented by a rank factorization $D = UV^{\mathsf{T}}$ where we can again quotient out the degrees of freedom of an $r \times r$ factor T in an equivalent factorization $D = (UT)(T^{-1}V^{\mathsf{T}})$. Such a factor can thus be represented by $r(2m - r)$ degrees of freedom, which has to be added to those of the strictly proper part of $Z(s)$.

To summarize, a real rational $m \times m$ transfer function $Z(s)$ of McMillan degree n has

- $2mn$ real degrees of freedom when $Z(s)$ is strictly proper,
- $2m(n+r) - r^2$ real degrees of freedom when $Z(s)$ is proper and $Z(\infty)$ has rank r.

This count of the number of degrees of freedom will determine the number of parameters we can assign using tangential interpolation conditions. For a rigorous discussion on these aspects, we refer to [29].

5. Loewner Approach for Identification of a Port-Hamiltonian Realization

In this section, we discuss the identification of a pH realization of a strictly passive transfer function $Z(s)$ of degree n, which is defined via a set of left and right interpolation conditions. Precisely, we seek to infer a pH realization (8) whose transfer function is denoted by $Z(s)$, and satisfies the following interpolation conditions:

$$v_j := \ell_j Z(\mu_j), \;\; w_j := Z(\lambda_j)r_j, \;\; j = 1, \ldots, n, \;\; Z(\infty) = D, \tag{12}$$

where (μ_j, ℓ_j, v_j), and (λ_j, r_j, w_j), $j = 1, \ldots, n$, are sets of self-conjugate left and right interpolation conditions with $\{\ell_j, v_j\} \in$

$\mathbb{C}^{1\times m}$, $\{r_j, w_j\} \in \mathbb{C}^{m\times 1}$, $\{\lambda_j, \mu_j\} \in \mathbb{C}$, and D is the feedthrough term.

Since $Z(s)$ is strictly passive, it is proper and has a standard state-space realization $\{A, B, C, D\}$. Moreover, we have the matrix D to be of full rank and positive-real (i. e. $D + D^{\mathsf{T}} \succ 0$), otherwise the system will not be strictly passive. Then, we recall the so-called *Loewner* and *shifted Loewner* matrices defined in [24]. These matrices have dimensions $n \times n$ and can be given as follows:

$$\mathbb{L} := \begin{bmatrix} \frac{\ell_1 w_1 - v_1 r_1}{\lambda_1 - \mu_1} & \cdots & \frac{\ell_1 w_n - v_1 r_n}{\lambda_n - \mu_1} \\ \vdots & \ddots & \vdots \\ \frac{\ell_n w_1 - v_n r_1}{\lambda_1 - \mu_n} & \cdots & \frac{\ell_n w_n - v_n r_n}{\lambda_n - \mu_n} \end{bmatrix}, \tag{13a}$$

$$\mathbb{L}_\sigma := \begin{bmatrix} \frac{\lambda_1 \ell_1 w_1 - \mu_1 v_1 r_1}{\lambda_1 - \mu_1} & \cdots & \frac{\lambda_n \ell_1 w_n - \mu_1 v_1 r_n}{\lambda_n - \mu_1} \\ \vdots & \ddots & \vdots \\ \frac{\lambda_1 \ell_n w_1 - \mu_n v_n r_1}{\lambda_1 - \mu_n} & \cdots & \frac{\lambda_n \ell_n w_n - \mu_n v_n r_n}{\lambda_n - \mu_n} \end{bmatrix}. \tag{13b}$$

They satisfy the following Sylvester equations:

$$\mathbb{L}\Lambda - M\mathbb{L} = LW - VR, \quad \mathbb{L}_\sigma\Lambda - M\mathbb{L}_\sigma = LW\Lambda - MVR, \tag{14}$$

where we used the definitions

$$L := \begin{bmatrix} \ell_1 \\ \vdots \\ \ell_n \end{bmatrix}, \quad V := \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \quad M := \operatorname{diag}(\mu_1, \ldots, \mu_n), \tag{15}$$

and

$$R := \begin{bmatrix} r_1, \ldots, r_n \end{bmatrix}, \quad W := \begin{bmatrix} w_1, \ldots, w_n \end{bmatrix}, \quad \Lambda := \operatorname{diag}(\lambda_1, \ldots, \lambda_n). \tag{16}$$

The following interpolation result follows from the theory developed in [21] in the special case that the Loewner matrix $\mathbb{L}$ is invertible.

Theorem 5.1 ([21]). *Let $Z(s)$ be a proper transfer function of McMillan degree n. Then the interpolation conditions* (12) *uniquely define $Z(s)$ if the Loewner matrix $\mathbb{L}$ is invertible. Moreover, a minimal generalized state-space realization is then given by*

$$Z(s) = (W - DR)(\mathbb{L}_\sigma - LDR - s\mathbb{L})^{-1}(V - LD) + D$$

and the corresponding system matrix is given by

$$\left[\begin{array}{c|c} A - sE & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} \mathbb{L}_\sigma - s\mathbb{L} & V \\ \hline -W & 0 \end{array}\right] + \left[\begin{array}{c} -L \\ \hline I_m \end{array}\right] D \left[\begin{array}{c|c} R & I_m \end{array}\right].$$

Remark 5.2. *As discussed in [21], the rank of the Loewner matrix $\mathbb{L}$ is related to the McMillan degree of the system, or in other words, the minimal order of the state-space realization. Thus, if the number of interpolation conditions is more than $2n$ and the minimal order of the state-space realization is n, then a compression step is used to determine an underlying minimal realization, see [21].*

As can be noticed, for arbitrary choices of interpolation points and directions, the state-space realization as shown in Theorem 5.1, in general, will not have pH form. Next, let us apply this to the special case where the interpolation points and directions are the so-called spectral zeros and zero directions of $Z(s)$, respectively.

Definition 5.3 (e.g., [23]). *Let $Z(s)$ be a real and strictly passive transfer function of McMillan degree n with associated spectral density function $\Phi(s) := Z^{\mathsf{T}}(-s) + Z(s)$. Then the* spectral zeros and zero directions *of $Z(s)$ are the pairs (s_j, r_j) such that $\Phi(s_j)r_j = 0$.*

Furthermore, we note that the spectral zeros and zero directions of a system can be computed by solving the following generalized eigenvalue problem:

$$\begin{bmatrix} 0 & A & B \\ A^{\mathsf{T}} & 0 & C^{\mathsf{T}} \\ B^{\mathsf{T}} & C & D + D^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} p_j \\ q_j \\ r_j \end{bmatrix} = s_j \begin{bmatrix} 0 & E & 0 \\ -E^{\mathsf{T}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_j \\ q_j \\ r_j \end{bmatrix}. \tag{17}$$

When the zeros are distinct (which is generic), there are exactly n zeros in the open right half-plane and n zeros in the open left half-plane because the spectral density function $\Phi(s)$ has degree $2n$ and has no zeros on the imaginary axis. The definition of the spectral zeros implies

$$\Phi(s_j)r_j = Z^{\mathsf{T}}(-s_j)r_j + Z(s_j)r_j = 0,$$

and hence

$$w_j := Z(s_j)r_j \iff Z^{\mathsf{T}}(-s_j)r_j = -w_j.$$

Since the spectral zeros and zero directions (s_j, r_j) form a self-conjugate set, we can distinguish two cases for these equations, depending on the condition that s_j is a real zero or not. In the real case, we have

$$s_j \in \mathbb{R}: \quad Z(s_j)r_j = w_j \iff r_j^{\mathsf{T}} Z(-s_j) = -w_j^{\mathsf{T}},$$

and in the complex case, we have

$$s_j \notin \mathbb{R}: \quad \begin{cases} Z(s_j)r_j = w_j & \iff r_j^{\mathsf{H}} Z(-\overline{s}_j) = -w_j^{\mathsf{H}}, \\ Z(\overline{s}_j)\overline{r}_j = \overline{w}_j & \iff r_j^{\mathsf{T}} Z(-s_j) = -w_j^{\mathsf{T}}. \end{cases}$$

Therefore, if we define λ_j, $j = 1, \ldots, n$, to be the spectral zeros of $Z(s)$ in the open right half-plane,

$$\Re\left(\lambda_j\right) > 0, \quad Z(\lambda_j)r_j = w_j, \ j = 1, \ldots, n,$$

then the set of right tangential conditions (λ_j, r_j, w_j) is self-conjugate. Moreover, for every right tangential condition $Z(\lambda_j)r_j = w_j$ (and its complex conjugate when λ_j is complex), there is a corresponding left tangential condition

$$r_j^{\mathsf{H}} Z(-\overline{\lambda}_j) = -w_j^{\mathsf{H}}, \ j = 1, \ldots, n.$$

Therefore, we can define left tangential interpolation conditions $\ell_j Z(\mu_j) = v_j$, $j = 1, \ldots, n$ in such a way that

$$M = -\overline{\Lambda} = -\Lambda^{\mathsf{H}}, \qquad L = R^{\mathsf{H}}, \qquad V = -W^{\mathsf{H}},$$

where Λ, L and W are as defined in (16). Using these definitions, the Loewner and shifted Loewner matrices now become

$$\mathbb{L} := \begin{bmatrix} \frac{r_1^{\mathsf{H}} w_1 + w_1^{\mathsf{H}} r_1}{\lambda_1 + \overline{\lambda}_1} & \cdots & \frac{r_1^{\mathsf{H}} w_n + w_1^{\mathsf{H}} r_n}{\lambda_n + \overline{\lambda}_1} \\ \vdots & \ddots & \vdots \\ \frac{r_n^{\mathsf{H}} w_1 + w_n^{\mathsf{H}} r_1}{\lambda_1 + \overline{\lambda}_n} & \cdots & \frac{r_n^{\mathsf{H}} w_n + w_n^{\mathsf{H}} r_n}{\lambda_n + \overline{\lambda}_n} \end{bmatrix}, \tag{18a}$$

$$\mathbb{L}_\sigma := \begin{bmatrix} \frac{\lambda_1 r_1^{\mathsf{H}} w_1 - \overline{\lambda}_1 w_1^{\mathsf{H}} r_1}{\lambda_1 + \overline{\lambda}_1} & \cdots & \frac{\lambda_n r_1^{\mathsf{H}} w_n - \overline{\lambda}_1 w_1^{\mathsf{H}} r_n}{\lambda_n + \overline{\lambda}_1} \\ \vdots & \ddots & \vdots \\ \frac{\lambda_1 r_n^{\mathsf{H}} w_1 - \overline{\lambda}_n w_n^{\mathsf{H}} r_1}{\lambda_1 + \overline{\lambda}_n} & \cdots & \frac{\lambda_n r_n^{\mathsf{H}} w_n - \overline{\lambda}_n w_n^{\mathsf{H}} r_n}{\lambda_n + \overline{\lambda}_n} \end{bmatrix} \tag{18b}$$

and they satisfy the equations

$$\mathbb{L}\Lambda + \Lambda^{\mathsf{H}}\mathbb{L} = R^{\mathsf{H}}W + W^{\mathsf{H}}R, \tag{19a}$$

$$\mathbb{L}_\sigma\Lambda + \Lambda^{\mathsf{H}}\mathbb{L}_\sigma = R^{\mathsf{H}}W\Lambda - \Lambda^{\mathsf{H}}W^{\mathsf{H}}R. \tag{19b}$$

We point out that the matrix $\mathbb{L}$ is Hermitian by construction, while $\mathbb{L}_\sigma$ is skew-Hermitian by construction. For such symmetric conditions, the matrix $\mathbb{L}$ is also called the Pick matrix (see [30, 31]). It follows from Theorem 5.1 that a generalized state-space realization $\{A, B, C, D, E\}$ is given by

$$\left[\begin{array}{c|c} A - sE & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} \mathbb{L}_\sigma - s\mathbb{L} & -W^{\mathsf{H}} \\ \hline -W & 0 \end{array}\right] + \left[\begin{array}{c} -R^{\mathsf{H}} \\ \hline I_m \end{array}\right] D \left[\begin{array}{c|c} R & I_m \end{array}\right]. \tag{20}$$

Notice that the complex matrices and vectors in this section are artificial; in fact, we can transform these matrics and vectors into real ones using a proper unitary transformation. Since the interpolation conditions are self-conjugate, we can transform the construction as follows. Let $v := v_r + \imath v_\imath$ be a complex vector associated with a complex interpolation point $\lambda := \alpha + \imath\beta$, then the unitary transformation $\Pi := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -\imath \\ 1 & \imath \end{bmatrix}$ transforms pairs of complex conjugate data to real data, as can be seen below

$$\begin{bmatrix} v & \overline{v} \end{bmatrix}\Pi = \sqrt{2}\begin{bmatrix} v_r & v_\imath \end{bmatrix}, \quad \Pi^{\mathsf{H}}\begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix}\Pi = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$

If the pairs of complex conjugate vectors and interpolation points have been permuted to be adjacent, then it suffices to apply a block diagonal unitary similarity transformation U with diagonal blocks Π corresponding to each complex conjugate pair $(\lambda, \overline{\lambda})$, to transform (16), (18), and (19) to real equations:

$$\widehat{\mathbb{L}}\Omega + \Omega^{\mathsf{T}}\widehat{\mathbb{L}} = \widehat{R}^{\mathsf{T}}\widehat{W} + \widehat{W}^{\mathsf{T}}\widehat{R}, \text{ and} \tag{21a}$$

$$\widehat{\mathbb{L}}_\sigma\Omega + \Omega^{\mathsf{T}}\widehat{\mathbb{L}}_\sigma = \widehat{R}^{\mathsf{T}}\widehat{W}\Omega - \Omega^{\mathsf{T}}\widehat{W}^{\mathsf{T}}\widehat{R}, \tag{21b}$$

where

$$\widehat{\mathbb{L}} = U^{\mathsf{H}}\mathbb{L}U,\ \widehat{\mathbb{L}}_\sigma = U^{\mathsf{H}}\mathbb{L}_\sigma U,\ \Omega = U^{\mathsf{H}}\Lambda U,\ \widehat{W} = WU,\ \widehat{R} = RU,$$

and Ω is now block diagonal with 2×2 blocks corresponding to each pair of complex conjugate interpolation points. It then also follows from (20) that a *real* generalized state-space realization $\{\widehat{A}, \widehat{B}, \widehat{C}, D, \widehat{E}\}$ is given by

$$\left[\begin{array}{c|c} \widehat{A} - s\widehat{E} & \widehat{B} \\ \hline \widehat{C} & D \end{array}\right] = \left[\begin{array}{c|c} \widehat{\mathbb{L}}_\sigma - s\widehat{\mathbb{L}} & -\widehat{W}^{\mathsf{T}} \\ \hline -\widehat{W} & 0 \end{array}\right] + \left[\begin{array}{c} -\widehat{R}^{\mathsf{T}} \\ \hline I_m \end{array}\right] D \left[\begin{array}{c|c} \widehat{R} & I_m \end{array}\right]. \tag{22}$$

Let us now look at the passivity condition we imposed on the transfer function $Z(s)$. The Loewner matrix $\mathbb{L}$ given in (18) has the structure of a Pick matrix (see, e. g., [30]) since the spectral zeros used for the interpolation are assumed to be distinct. The strict passivity of $Z(s)$ implies that this matrix is positive definite. It follows that $Z(\infty) = D$, and hence that D must be strictly positive-real as well. Since $\mathbb{L}$ is positive definite, so is $\widehat{\mathbb{L}}$ and we can factorize it as $\widehat{\mathbb{L}} = \Gamma^{\mathsf{T}}\Gamma$, where Γ is invertible, by using, for instance, the upper triangular Cholesky factor. Defining

$$\widehat{W}_\Gamma := \widehat{W}\Gamma^{-1}, \quad \widehat{R}_\Gamma := \widehat{R}\Gamma^{-1}, \quad \widehat{\mathbb{L}}_{\sigma\Gamma} := \Gamma^{-\mathsf{T}}\widehat{\mathbb{L}}_\sigma\Gamma^{-1},$$

we obtain an equivalent quadruple for the state-space realization $\{\widehat{A}_\Gamma, \widehat{B}_\Gamma, \widehat{C}_\Gamma, D\} = \{\Gamma^{-\mathsf{T}}\widehat{A}\Gamma^{-1}, \Gamma^{-\mathsf{T}}\widehat{B}, \widehat{C}\Gamma^{-1}, D\}$ of $Z(s)$ as

$$\left[\begin{array}{c|c} \widehat{A}_\Gamma & \widehat{B}_\Gamma \\ \hline \widehat{C}_\Gamma & D \end{array}\right] = \left[\begin{array}{c|c} \widehat{\mathbb{L}}_{\sigma\Gamma} & -\widehat{W}_\Gamma^{\mathsf{T}} \\ \hline -\widehat{W}_\Gamma & 0 \end{array}\right] + \left[\begin{array}{c} -\widehat{R}_\Gamma^{\mathsf{T}} \\ \hline I_m \end{array}\right] D \left[\begin{array}{c|c} \widehat{R}_\Gamma & I_m \end{array}\right]. \tag{23}$$

We then show that this realization is in pH form.

Theorem 5.4. *Construct an $m \times m$ real transfer function $Z(s)$ of McMillan degree n using self-conjugate interpolation conditions as follows:*

$$Z(\infty) = D, \quad Z(\lambda_j)r_j = w_j, \quad r_j^{\mathsf{H}}Z(-\overline{\lambda}_j) = -w_j^{\mathsf{H}}, \quad j = 1, \ldots, n,$$

where $\Re(\lambda_j) > 0$, $D + D^{\mathsf{T}} \succ 0$ and $\widehat{\mathbb{L}} \succ 0$ in which $\Re(\cdot)$ denotes the real part. Then $Z(s)$ is strictly passive and the quadruple $\{\widehat{A}_\Gamma, \widehat{B}_\Gamma, \widehat{C}_\Gamma, D\}$ is in normalized pH form and its spectral zeros and zero directions are given by (λ_j, r_j), $j = 1, \ldots, n$.

Proof. A necessary condition for strict passivity is that the Hermitian part of $Z(s)$ is positive definite on the whole imaginary axis, including infinity, and since $D = Z(\infty)$ is a real matrix, we must have $D + D^{\mathsf{T}} \succ 0$. A necessary and sufficient condition for the passivity of $Z(s)$ with given interpolation data is that the Loewner matrix $\widehat{\mathbb{L}}$ is positive semi-definite, but since we assume $\widehat{\mathbb{L}} \succ 0$, the transfer function is passive. Let us now decompose the real matrix D as $D = N + S$, where S is the symmetric part of D and N is its skew-symmetric part. Then, following the discussion of Section 2, we obtain

$$\mathcal{W} = \mathcal{W}^{\mathsf{T}} = \left[\begin{array}{c} \widehat{R}_\Gamma^{\mathsf{T}} \\ \hline I_m \end{array}\right] S \left[\begin{array}{c|c} \widehat{R}_\Gamma & I_m \end{array}\right] \geq 0,$$

$$\mathcal{V} = -\mathcal{V}^{\mathsf{T}} = \left[\begin{array}{c|c} -\widehat{\mathbb{L}}_{\sigma\Gamma} & \widehat{W}_\Gamma^{\mathsf{T}} \\ \hline -\widehat{W}_\Gamma & 0 \end{array}\right] + \left[\begin{array}{c} \widehat{R}_\Gamma^{\mathsf{T}} \\ \hline I_m \end{array}\right] N \left[\begin{array}{c|c} \widehat{R}_\Gamma & I_m \end{array}\right],$$

which are the conditions for the passivity of a normalized pH system. The standard state-space realization (23) is therefore normalized pH. It follows from the self-conjugacy conditions that

$$\Phi^{\mathsf{T}}(-\lambda_j)r_j = \Phi(\lambda_j)r_j = Z(-\lambda_j)r_j + Z(\lambda_j)r_j = -w_j + w_j = 0,$$

for $j = 1, \ldots, n$, and since $\Phi(s)$ has McMillan degree bounded by $2n$, these are the only zeros of $\Phi(s)$, which implies that $Z(s)$ is then strictly passive. ☐

Algorithm 1 Construction of a pH realization in a normalized form.

Input:

- Spectral zeros λ_j and zero directions r_j, $j = 1, \ldots, n$,
- transfer function measurements, i.e. $w_j = Z(\lambda_j)r_j$, where $Z(s)$ denotes the transfer function,
- the feedthrough term D.

1: Construct the Loewner and shifted Loewner matrices using w_j and r_j as shown in (18).

2: Define $W := \begin{bmatrix} w_1, \ldots, w_n \end{bmatrix}$ and $R := \begin{bmatrix} r_1, \ldots, r_n \end{bmatrix}$.

3: Construct the interpolating realization, ensuring the matching of the transfer function at infinity:
$E = \mathbb{L}$, $A = \mathbb{L}_s - R^{\mathsf{H}}DR$, $B = -W^{\mathsf{H}} - R^{\mathsf{H}}D$, $C = -W + DR$.

4: Perform the unitary transformation to obtain a real realization $(\widehat{E}, \widehat{A}, \widehat{B}, \widehat{C})$.

5: Consider the Choleskey factorization of $\widehat{E} := \Gamma^{\mathsf{T}}\Gamma$.

6: Construct a pH realization in the normalized form as follows:
$\tilde{A} = \Gamma^{-\mathsf{T}}\widehat{A}\Gamma^{-1}$, $\quad \tilde{B} = \Gamma^{-\mathsf{T}}\widehat{B}$, $\quad \tilde{C} = \widehat{C}\Gamma^{-1}$.

Output: A pH realization: $\left(\tilde{A}, \tilde{B}, \tilde{C}, D\right)$.

Remark 5.5. *The conditions that the spectral zeros should be simple can be removed. The construction of the Loewner matrix $\mathbb{L}$ and of the shifted Loewner matrix $\mathbb{L}_\sigma$ then have to be adapted, as explained in [31, 21], but the properties of these matrices are preserved. The tangential interpolation conditions then also involve tangential conditions for the derivatives of $Z(s)$ at the spectral zeros λ_i, but the conclusions remain the same.*

Remark 5.6. *The conditions that we should know the zero directions of the corresponding spectral zeros of $Z(s)$ form a demanding constraint. But this is different in the scalar case since we only need to impose a scalar condition $Z(-\lambda_j) + Z(\lambda_j)$ in each spectral zero. We can then choose $R = \begin{bmatrix} 1, \ldots, 1 \end{bmatrix}$ which implies that $W = \begin{bmatrix} Z(\lambda_1), \ldots, Z(\lambda_n) \end{bmatrix}$.*

Finally, we summarize the construction of a pH realization in the normalized form in Algorithm 1.

6. Estimation of Spectral Zeros and Zero Directions using Frequency Response Data

So far, we have discussed how to construct a pH realization from the transfer function measurements at spectral zeros along with zero directions. However, this may be restrictive as in practice, it is almost impossible to know the spectral zeros and zero directions a priori. Moreover, even if the zeros are known, taking measurements at those points and directions is not straightforward. On the other hand, there exist methods allowing us to obtain the frequency response data of a system which is nothing but the measurements of the transfer function on the imaginary axis. Using these measurements, one can obtain a realization using the classical Loewner approach, proposed in [21], which interpolates the data. However, it is very likely that it will not yield a realization in normalized pH form. But we are interested in a pH realization given the underlying system is strictly passive. To do so, we first propose a strategy sketched in Algorithm 2 to estimate the spectral zeros and directions based on the data on the $j\omega$ axis. Once we have such a data set, we can obtain a passive realization directly using Algorithm 1.

Algorithm 2 Estimation of spectral zeros and directions using frequency response data

Input:

- Samples: frequencies $\{\sigma_i, \mu_i\}$, directions $\{r_i, l_i\}$,
- the transfer function measurements $w_j := H(\sigma_i)r_i, v_j := l_i H(\mu_i)$,
- the direct feedthrough term D.

1: Construct Loewner $\mathbb{L}$ and shifted Loewner $\mathbb{L}_\sigma$ matrices as defined in (13).

2: Build L, V, R, and W as shown in (15) and (16).

3: Construct $\widetilde{\mathbb{L}} = \mathbb{L}$, $\widetilde{\mathbb{L}}_\sigma = \mathbb{L}_\sigma - LDR$, $\widetilde{V} = V - LD$, and $\widetilde{W} = -W + DR$.

4: Perform the SVD of the Loewner matrices:

$$\begin{bmatrix} Y_1 & 0 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{L}} & \widetilde{\mathbb{L}}_\sigma \end{bmatrix},$$

$$\begin{bmatrix} Y_2 & 0 \end{bmatrix} \begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{L}} \\ \widetilde{\mathbb{L}}_\sigma \end{bmatrix}.$$

5: Define $Y_r = Y_1(:, 1 : r)$, $X_r = X_2(:, 1 : r)$, where r is the rank of the matrix Σ_1.

6: Construct a realization $\mathcal{M} := (A, B, C, D, E)$ as follows:

$$E = Y_r^{\mathsf{T}}\widetilde{\mathbb{L}}X_r, \quad A = Y_r^{\mathsf{T}}\widetilde{\mathbb{L}}_\sigma X_r, \quad B = Y_r^{\mathsf{T}}\widetilde{V}, \quad C = \widetilde{W}X_r, \quad D = D.$$

7: Determine spectral zeros and zero directions of the realization $\mathcal{M}$ as defined in Definition 5.3.

8: Estimate transfer function measurements at spectral zeros and zero directions using the realization $\mathcal{M}$.

Output: Spectral zeros, zero directions, and the transfer function measurement estimates at spectral zeros along zero directions.

The main motivation for proposing Algorithm 2 is as follows. As we know, if the transfer functions of two linear systems are the same, then there exists a state-space transformation, allowing us to go from one to the other. Furthermore, it is also known that a minimal realization of order n can be obtained using the Loewner approach using any $2n$ measurements, see [21]. Moreover, if the number of measurements are greater than $2n$, then a minimal realization of order n can be obtained by a compression step [21]. Hence, if there exists a passive realization of the linear system, then such a passive realization can be determined using the realization obtained using the Loewner approach and a state-space transformation. However, a state-space transformation of a linear system does not change the spectral zeros and corresponding directions. Consequently, we can indeed directly use the realization obtained using the Loewner approach to estimate the spectral zeros and corresponding directions and further can evaluate the transfer function at spectral zeros and in the corresponding tangential directions.

Remark 6.1. *One can also construct a reduced-order system as well by truncating singular values of the Loewner matrix at a desired tolerance. This can be followed by determining the spectral zeros and zero directions of the reduced-order system, which can be very different from the original ones; however, the spectral zeros and zero directions of the reduced-order system form a good representative, allowing us to compare the important dynamics of the original system.*

Remark 6.2. *If the transfer function measurements are given in a particular frequency band, then applying Algorithm 2 would yield spectral zeros and zero directions, corresponding to the considered frequency band. If a pH realization in the normalized form is constructed using Algorithm 1, then we obtain the frequency-limited pH realization. This is discussed and illustrated further in the subsequent section.*

Remark 6.3. *Algorithm 1 can also be applied if one aims at finding an underlying pH realization for a given strictly passive system. And such a system can either be given by first principle modeling or can be determined using any system identification method either in time-domain or frequency-domain. One motivational example can be found in [32], where the authors aimed at determining an underlying pH realization of notch filters which then can be interconnected with other pH systems for control-design purposes.*

7. Illustrative and Numerical Examples

In this section, we illustrate the proposed identification approach to construct a passive (pH) realization by means of several examples. All numerical simulations are carried out in MATLAB® version 7.11.0.584 (R2016b) 64-bit on an Intel® Core™i7-6700 CPU @ 3.40GHz, 6MB cache, 8GB RAM, Ubuntu 16.04.6 LTS (x86-64).

7.1. An analytical example

We first consider an analytical example, showing the necessary steps, precisely Algorithm 1, to identify an underlying passive realization whose transfer function is as follows:

$$Z(s) := dI_2 - (sI_2 - A)^{-1} \quad \text{with} \quad A := \begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$

where we take $a = -1$, $b = 1$, $d = 2$ to make the system strictly passive since it is then pH with positive definite matrix $\mathcal{W}$. The poles of $Z(s)$ are the eigenvalues of A and are equal to $-1 \pm \imath$ and hence asymptotically stable. The spectral zeros are the zeros of $\Phi(s) = Z^{\mathsf{T}}(-s) + Z(s)$, which can be determined using the following steps:

$$\begin{aligned}\Pi\Phi(s)\Pi^{\mathsf{H}} &= \Pi Z^{\mathsf{T}}(-s)\Pi^{\mathsf{H}} + \Pi Z(s)\Pi^{\mathsf{H}} \\ &= 2dI_2 - (-sI_2 - \Pi A^{\mathsf{T}}\Pi^{\mathsf{H}})^{-1} - (sI_2 - \Pi A\Pi^{\mathsf{H}})^{-1},\end{aligned}$$

where

$$\Pi = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -\imath \\ 1 & \imath \end{bmatrix}, \qquad \Pi A \Pi^{\mathsf{H}} = \begin{bmatrix} a+\imath b & 0 \\ 0 & a-\imath b \end{bmatrix}.$$

It then turns out that both $\Pi Z(s)\Pi^{\mathsf{H}}$ and $\Pi\Phi(s)\Pi^{\mathsf{H}}$ are diagonal and equal to

$$\Pi Z(s)\Pi^{\mathsf{H}} = \operatorname{diag}\left(2 - \frac{1}{s+1-\imath}, 2 - \frac{1}{s+1+\imath}\right), \tag{24a}$$

$$\Pi\Phi(s)\Pi^{\mathsf{H}} = \operatorname{diag}\left(\frac{6+8\imath s-4s^2}{2+2\imath s-s^2}, \frac{6-8\imath s-4s^2}{2-2\imath s-s^2}\right). \tag{24b}$$

The spectral zeros in the right half-plane are $\lambda = \frac{\sqrt{2}}{2} + \imath$ and $\overline{\lambda} = \frac{\sqrt{2}}{2} - \imath$ and the corresponding zero directions are

$$\Pi\Phi(\lambda)\Pi^{\mathsf{H}}\begin{bmatrix}1\\0\end{bmatrix} = 0 \iff \Phi(\lambda)r_\lambda = 0, \text{ where } r_\lambda = \Pi^{\mathsf{H}}\begin{bmatrix}1\\0\end{bmatrix},$$

and

$$\Pi\Phi\left(\overline{\lambda}\right)\Pi^{\mathsf{H}}\begin{bmatrix}0\\1\end{bmatrix} = 0 \iff \Phi(\lambda)r_{\overline{\lambda}} = 0, \text{ where } r_{\overline{\lambda}} = \Pi^{\mathsf{H}}\begin{bmatrix}0\\1\end{bmatrix}.$$

Hence, we have the following interpolation conditions:

$$Z(\lambda)r_\lambda = w_\lambda, \text{ and } Z(\overline{\lambda})r_{\overline{\lambda}} = \overline{w}_\lambda. \tag{25}$$

Moreover, using (24a), we obtain:

$$\Pi Z(\lambda)\Pi^{\mathsf{H}}\begin{bmatrix}1\\0\end{bmatrix} = \Pi Z(\lambda)r_\lambda = \begin{bmatrix}2 - \frac{1}{\lambda-(a+\imath b)}\\0\end{bmatrix} = \begin{bmatrix}\sqrt{2}\\0\end{bmatrix},$$

implying

$$\Pi w_\lambda = \begin{bmatrix}\sqrt{2}\\0\end{bmatrix}, \text{ and } w_\lambda = \begin{bmatrix}1\\\imath\end{bmatrix}.$$

The Loewner matrix then is obtained from $R = \begin{bmatrix}r_\lambda, r_{\overline{\lambda}}\end{bmatrix} = \Pi^{\mathsf{H}}$, $W = \begin{bmatrix}w_\lambda, \overline{w}_\lambda\end{bmatrix} = \sqrt{2}\Pi^{\mathsf{H}}$ and hence $W^{\mathsf{H}}R = \sqrt{2}I_2$. Using the interpolation conditions (25), we obtain the Loewner and shifted Loewner matrices defined in (21) as follows:

$$\begin{aligned}\mathbb{L} &= \frac{2\sqrt{2}}{\lambda+\overline{\lambda}}I_2 = 2I_2, \\ \mathbb{L}_\sigma &= \begin{bmatrix}\sqrt{2}\frac{\lambda-\overline{\lambda}}{\lambda+\overline{\lambda}} & 0 \\ 0 & -\sqrt{2}\frac{\lambda-\overline{\lambda}}{\lambda+\overline{\lambda}}\end{bmatrix} = 2\begin{bmatrix}\imath & 0\\0 & -\imath\end{bmatrix}.\end{aligned}$$

Hence, the generalized state-space realization (20) becomes

$$\left[\begin{array}{c|c} A-sE & B \\ \hline C & D\end{array}\right] = \left[\begin{array}{c|c} \mathbb{L}_\sigma - s\mathbb{L} & -\sqrt{2}\Pi \\ \hline -\sqrt{2}\Pi^{\mathsf{H}} & 0\end{array}\right] + 2\left[\begin{array}{c} -\Pi \\ \hline I_2\end{array}\right]\left[\begin{array}{c|c}\Pi^{\mathsf{H}} & I_2\end{array}\right].$$

Using the factorization $\mathbb{L} = \Gamma^{\mathsf{H}}\Gamma$ with $\Gamma := \sqrt{2}\Pi^{\mathsf{H}}$, we get $\mathbb{L}_{\sigma\Gamma} = \frac{1}{2}\Pi^{\mathsf{H}}\mathbb{L}_\sigma\Pi$ and

$$\begin{aligned}\left[\begin{array}{c|c} A_\Gamma & B_\Gamma \\ \hline C_\Gamma & D\end{array}\right] &= \left[\begin{array}{c|c} \mathbb{L}_{\sigma\Gamma} & -I_2 \\ \hline -I_2 & 0\end{array}\right] + 2\left[\begin{array}{c} -I_2/\sqrt{2} \\ \hline I_2\end{array}\right]\left[\begin{array}{c|c} I_2/\sqrt{2} & I_2\end{array}\right] \\ &= \left[\begin{array}{cc|cc} -1 & 1 & -c & 0 \\ -1 & -1 & 0 & -c \\ \hline 1/c & 0 & 2 & 0 \\ 0 & 1/c & 0 & 2\end{array}\right]\end{aligned}$$

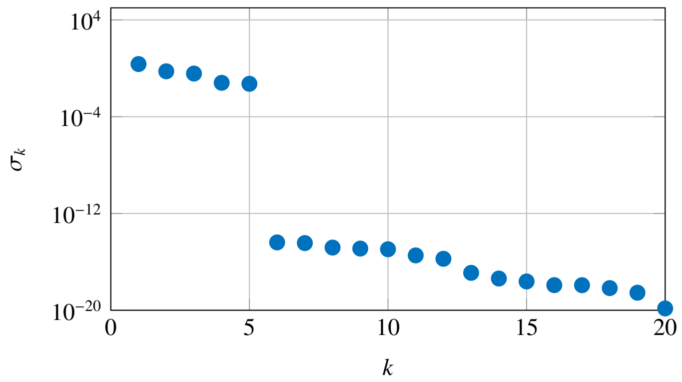

Figure 1: RLC example: The decay of the singular values of the Loewner matrix.

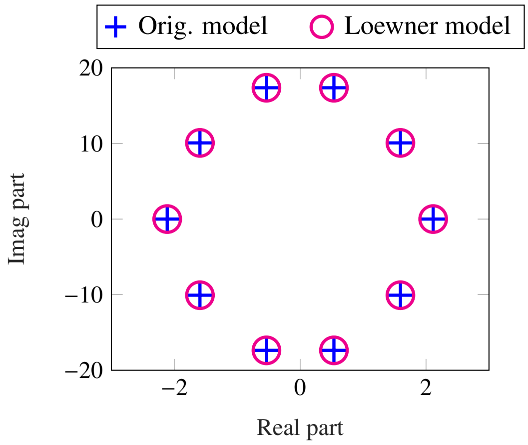

Figure 2: RLC example: Spectral zero of the original and Loewner model.

with $c = \sqrt{2} + 1$. The smallest eigenvalue $\lambda_{\min}(\mathcal{W}_\Gamma)$ of the above model is 0 which is a poor estimate of its passivity radius. But we can apply to this model a similarity scaling with $T = cI_2$, which yields a model $\mathcal{M}_T$ where $A_T = A_\Gamma$ and $D_T = D$ are unchanged but $C_T = -B_T = I_2$. This corresponds to using [27, Lemma 3.2] with the certificate $X = c^{-2}I$, and transforming the model to a new pH form which has a passivity radius equal to $\lambda_{\min}(\mathcal{W}_T) = \frac{1}{2}(3 - \sqrt{5}) \approx 0.382$. However, obtaining a pH realization with maximum passivity radius is out of the scope of the paper.

7.2. *Electric RLC circuit*

As second example, we discuss the electrical circuit example considered in [31]. The system dynamics in state-space form is given as follows:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = 0, \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

where

$$A = \begin{bmatrix} -20 & -10 & 0 & 0 & 0 \\ 10 & 0 & -10 & 0 & 0 \\ 0 & 10 & 0 & -10 & 0 \\ 0 & 0 & 10 & 0 & -10 \\ 0 & 0 & 0 & 10 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 20 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^{\mathsf{T}},$$

and $D = 2$.

To identify the dynamics using data, we assume to have 20 frequency response data between the frequency range $[10^{-1}, 10^3]$. We first employ the Loewner approach [21] to obtain a realization. In Figure 1, we plot the singular values of the Loewner matrix, which allows us to determine the order of a minimal realization. We observe that the singular values after the 5th are at the level of machine precision as one would expect. Hence, we determine a realization of order 5. Next, we show the spectral zeros of the original and Loewner model in Figure 2, indicating that the spectral zeros of both models are the same as expected.

The identified realization is not in the form of a passive pH system. But we can use the spectral zeros and zero directions of the Loewner model, which in this case, are the same as for the original system, and estimate the transfer function at the spectral zeros along with the respective zero directions. Consequently, we apply Algorithm 1 to obtain a realization in the generalized state-space form of a pH system (10), where up to five digits,

$$Q^{-1} = \begin{bmatrix} 0.8795 & 0.0263 & -0.0304 & -0.0511 & 0.0938 \\ 0.0263 & 0.8515 & -0.0770 & -0.1574 & -0.0098 \\ -0.0304 & -0.0770 & 0.2545 & 0.0814 & 0.1136 \\ -0.0511 & -0.1574 & 0.0814 & 0.3560 & 0.0400 \\ 0.0938 & -0.0098 & 0.1136 & 0.0400 & 0.2891 \end{bmatrix},$$

$$J = \begin{bmatrix} 0 & -15.2595 & 0.5921 & 1.7823 & 0.5344 \\ 15.2595 & 0 & -0.4864 & -0.8033 & 1.6342 \\ -0.5921 & 0.4864 & 0 & 0.5204 & -0.5325 \\ -1.7823 & 0.8033 & -0.5204 & 0 & -3.3854 \\ -0.5344 & -1.6342 & 0.5325 & 3.3854 & 0 \end{bmatrix},$$

$$R = \begin{bmatrix} 4.0000 & 0.0000 & -2.8284 & -3.9606 & 0.5598 \\ 0.0000 & 0 & -0.0000 & -0.0000 & 0.0000 \\ -2.8284 & -0.0000 & 2.0000 & 2.8006 & -0.3959 \\ -3.9606 & -0.0000 & 2.8006 & 3.9216 & -0.5543 \\ 0.5598 & 0.0000 & -0.3959 & -0.5543 & 0.0784 \end{bmatrix},$$

$$G = \begin{bmatrix} -0.6563 & 0.3238 & 0.5378 & 0.6924 & -0.2925 \end{bmatrix}^{\mathsf{T}},$$

$$P = \begin{bmatrix} 2.8284 & 0.0000 & -2.0000 & -2.8006 & 0.3959 \end{bmatrix}^{\mathsf{T}},$$

$$N = 0, \quad S = 2.$$

Furthermore, we compare the Bode plots of the original and the identified pH model (3), illustrating that the identified pH realization has the same transfer functions.

7.3. *A large electrical circuit network*

Next, we consider a large RLC circuit network, where 100 electrical capacitances, inductors, and resistances are interconnected. For more details on the circuit topology, we refer to [33]. The modeling of such this circuit leads to a model of order $n = 200$. We assume to have 200 frequency response data points on a log-scale within the frequency range $\left[10^{-1}, 10^3\right]$.

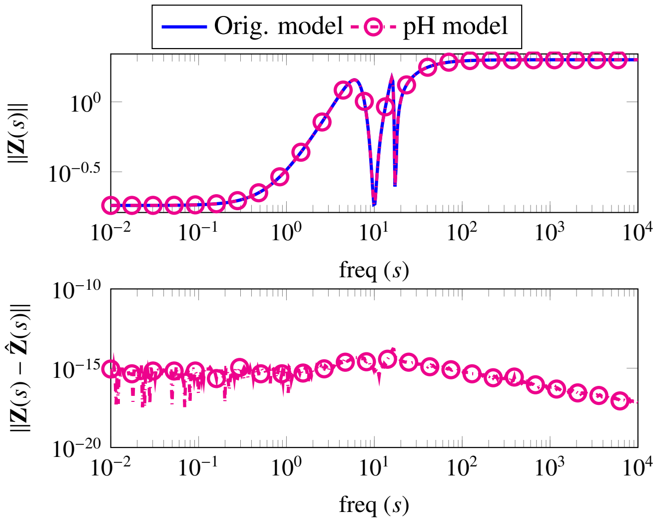

Figure 3: RLC example: Comparison of the Bode plots of the original and pH models.

Towards constructing a pH reduced-order system using the data, we first determine a realization using the classical Loewner method. We plot the decay of the singular values of the Loewner matrix in Figure 4, indicating a sharp decay. Next, we determine two pH realizations by truncating the singular values at $5{\cdot}10^{-2}$ and 10^{-8} (relatively) using Algorithm 1 and Algorithm 2. This leads to pH realizations of order $r = 2$ and $r = 14$. It is expected that the higher-order model captures the original system dynamics much better as compared to the lower-one. Next, we compare the spectral zeros of the original and identified pH models in Figure 5. It is interesting to see how different the spectral zeros of all models are. To compare the quality of the models, we plot the Bode plots of the original and identified pH models in Figure 6, showing the pH models (even order 2) approximate the original model very well.

7.4. *Frequency-limited pH realization*

Lastly, we discuss the construction of a frequency-limited pH realization using the same example as in the previous sub-

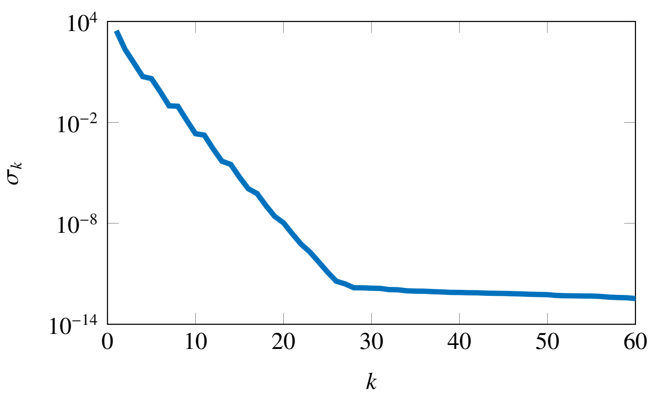

Figure 4: Large-scale RLC circuit: The decay of the singular values of the Loewner matrix.

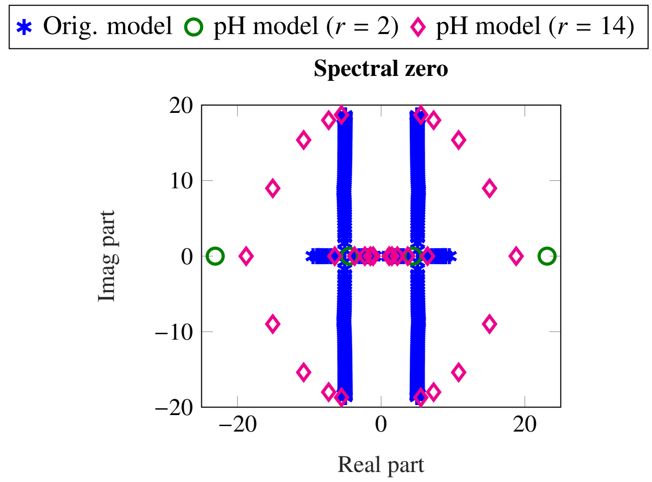

Figure 5: A large RLC circuit network: the spectral zeros of the original and identified pH systems.

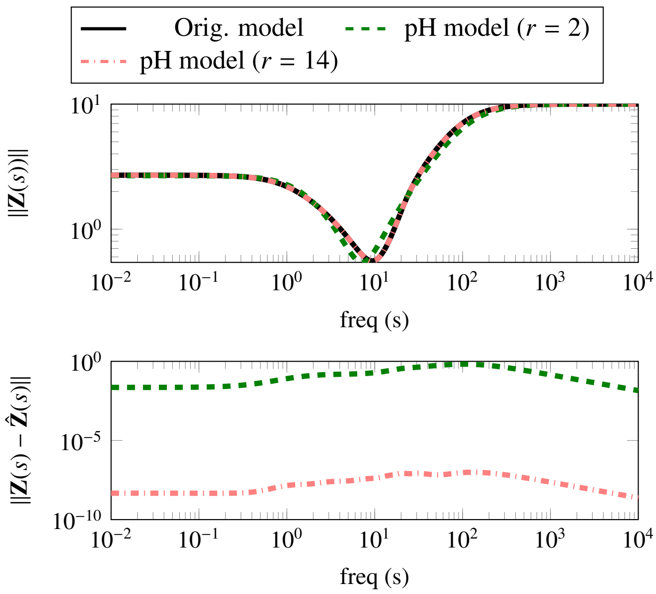

Figure 6: A large RLC circuit network: Comparison of the Bode plots of the original and identified pH models.

section. This means that the transfer function of the pH realization is required to be very accurate in a given frequency band. Let us assume that we are given measurements in the frequency band $\left[5, 15\right]$rad/s. As done in the previous example, we determine two pH models by truncating the singular values at $5 \cdot 10^{-3}$ and 10^{-8}, which gives rise to models of order 2 and 9, respectively. Next, we compare the spectral zeros of original and inferred pH models in Figure 7. It can be observed that the spectral zeros are not only different from the original ones but also from those of the inferred models of order $r = 2, 14$ in the previous example, see Figure 5.

Next, we plot the transfer functions of the original and the identified pH realization in Figure 8. Comparing, in particular, the error plots in Figures 6 and 8, we observe that the identified

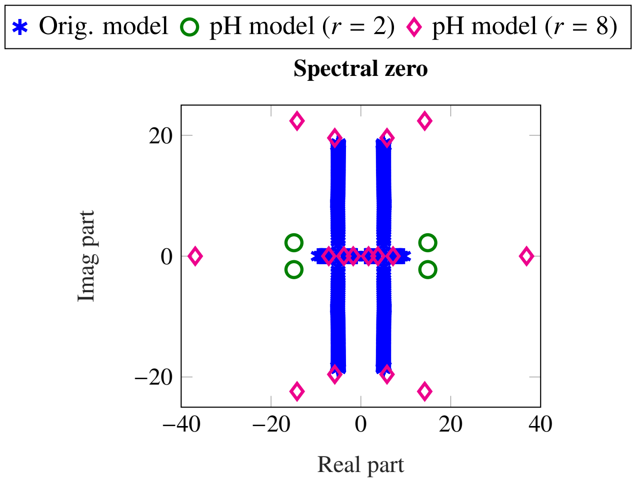

Figure 7: A large RLC circuit network (frequency limited): Comparison of spectral zeros of the original and Loewner model.

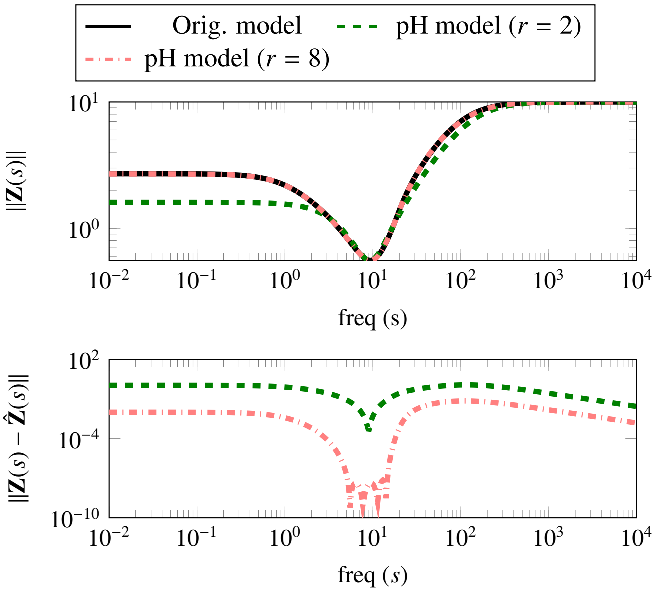

Figure 8: A large RLC circuit network (frequency limited): Comparison of the Bode plots of the original and Loewner model.

pH realization using the data in the frequency band is much more accurate in the considered frequency band than the model identified in the previous subsection, and importantly, it is of a lower dimension if it is truncated at 10^{-8}.

8. Conclusions

In this work, we have studied the identification problem for strictly passive systems. We have proposed a variant of the classical Loewner approach [21], which constructs a realization in pH form. We have also discussed a two-step procedure which allows us to construct a pH realization using data on the imaginary axis. Furthermore, we have investigated the construction of frequency-limited pH realization, which aims at inferring a pH realization in a given frequency band. We have illustrated the proposed methods by means of a couple of variants of electrical circuits. As a future direction, it would be interesting to investigate an identification problem of second-order passive systems by extending the idea proposed in [34].

Code Availability

A MATLAB implementation that generates the results reported in Section 7, can be found at `https://github.com/mpimd-csc/Identify_PortHamiltonian_Realization`.

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