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A critical look at β-function singularities at large N

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We propose a self-consistency equation for the β-function for theories with a large number of flavours, N, that exploits all the available information in the critical exponent, ω, truncated at a fixed order in $1/N$. We show that singularities appearing in critical exponents do not necessarily imply singularities in the β-function. We apply our method to (non-)abelian gauge theory, where ω features a negative singularity. The singularities in the β-function and in the fermion mass anomalous dimension are simultaneously removed providing no hint for a UV fixed point in the large-N limit.

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I. INTRODUCTION

There are indications that perturbative series in quantum field theory are, in general, asymptotic series with zero radius of convergence. In theories with a large number of flavour-like degrees of freedom, N, a re-organization of the perturbative expansion in powers of $1/N$ is convenient. It can be shown that at fixed order in $1/N$ expansion, the number of diagrams contributing grows only polynomially rather than factorially: convergent series are obtained that can be summed up within their radius of convergence.

There is a vast literature on resummed results corresponding to the first few orders in $1/N$ expansion, mainly for RG functions obtained via direct diagram resummation or critical-point methods, see e.g. Refs [1–22].

Since the perturbative series at fixed order in $1/N$ are convergent, singularities in the (generically complex) coupling are expected. Appearance of such singularities on the real-coupling axis seems to be true for all the $d = 4$ theories analyzed so far, thereby having a dramatic effect on RG flows. In particular, the appearance of singularities in the coefficients of the $1/N$ expansion for gauge and Yukawa β-functions have inspired speculations of a possible UV fixed point [23–29].

In matter-dominated theories where asymptotic freedom is lost, a non-trivial zero of the β-function can be envisaged if the large-N resummation produces a contribution to β functions such that $\lim_{g\to r}\beta^{1/N}(g) = -\infty$, where r is the radius of convergence of the $1/N$ series. Near the singularity, the $\mathcal{O}(1/N)$ contribution exceeds the leading-order result, and it is clear that a zero must emerge. Unfortunately, close to the radius of convergence the perturbation expansion in $1/N$ order is broken, and higher-order contributions are expected to play a major role.

Moreover, further shadow on the existence of the fixed point as a consistent conformal field theory is cast by studying anomalous dimensions of other operators in the vicinity of the β-function singularity: in the case of large-N QED truncated at $\mathcal{O}(1/N)$, the anomalous dimension of the fermion mass diverges at the β-function singularity [1, 2], and it was recently pointed out that in the large-N QCD the anomalous dimension of the glueball operator breaks the unitarity bound near the singularity [30].

In this letter we provide evidence that these poles are an artifact of the expansion around $N = \infty$. This is corroborated by the study of higher-order contributions in $1/N$ obtained from the $\mathcal{O}(1/N)$ critical exponents. This is made possible by the remarkable fact that a fixed-order truncation in the critical exponents is *not* equivalent to the same-order truncation in β-functions, see also Ref. [31]. We show that a negative singularity at a fixed order in $1/N$ results in singularities in alternating signs at higher orders signaling an instability. Remarkably, such contributions can be re-resummed obtaining a result that is valid for large but finite N close to the would-be singularity. The limit $N \to \infty$ of such a result is finite signaling an inconsistency in the limiting procedure.

We show how to take into account these contributions self-consistently up to a given order in the $1/N$ expansion of the critical exponent. We apply the method concretely for four-dimensional gauge β-function and Gross–Neveu (GN) model in two dimensions. In the case of large-N QED the singularity of the anomalous dimension of the fermion mass is simultaneously removed.

II. β-FUNCTION FROM THE CRITICAL EXPONENTS

In this section following Ref. [31], we review the general form for the β-function in the large-N limit written in terms of the critical exponent, ω. This critical exponent gives the slope of the β-function at the Wilson–Fisher (WF) fixed point, g_c,

$$\beta'(g_c) = \omega(d) \equiv \sum_{n=0}^{\infty} \frac{\omega^{(n)}(d)}{N^n}\,, \qquad (1)$$

where d is the dimension of spacetime[1]. The large-N expansion of the β-function can be incorporated by using

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[1] In the literature this equation is often found as $\omega = -\beta'/2$. We omit this factor for notational convenience.

the following ansatz:

$$\beta(g) = (d-d_c)g + g^2\left(bN + c + \sum_{n=1}^{\infty}\frac{F_n(gN)}{N^{n-1}}\right), \quad (2)$$

where d_c is the critical dimension of the coupling g, b and c are model-dependent one-loop coefficients, and the functions F_n satisfying $F_n(0)=0$ are all-order in gN.

The critical coupling, g_c, can then be self-consistently solved in terms of F_n,

$$g_c = -\frac{d-d_c}{bN + c + \sum_{n=1}^{\infty}\frac{F_n(g_cN)}{N^{n-1}}}, \quad (3)$$

and the slope of the β-function at the WF fixed point can be expanded in $1/N$:

$$\beta'(g_c) = -(d-d_c) + \frac{(d-d_c)^2}{b^2}\sum_{m=1}^{\infty}\frac{F'_m(g_cN)}{N^m} \times \sum_{k=0}^{\infty}(-b)^{-k}(k+1)\left(\frac{c}{N} + \sum_{n=1}^{\infty}\frac{F_n(g_cN)}{N^n}\right)^k. \quad (4)$$

Using Eq. (4), the unknown functions F_n can be related to the critical exponents order by order in $1/N$.

In Ref. [31], we noticed that the critical exponent $\omega^{(1)}$ contributes to the β-function also beyond $\mathcal{O}(1/N)$. Same holds for each $\omega^{(j)}$: it contributes to all F_n with $n \geq j$. In the following, we denote the contribution of $\omega^{(1)},\ldots,\omega^{(j)}$ to F_n, $n\geq j$, by $F_n^{(j)}$.

Since $\omega^{(1)}$, or equivalently F_1, is known, all the $F_n^{(1)}$ can in principle be computed. Explicitly up to $n=3$, one has:

$$\begin{aligned}
F_1^{(1)}(K) &= F_1(K) = \int_0^K \frac{\omega^{(1)}(d_c - bt)}{t^2}\mathrm{d}t,\\
F_2^{(1)}(K) &= \int_0^K \frac{c+F_1(t)}{b}(2F_1'(t) + tF_1''(t))\mathrm{d}t, \qquad (5)\\
F_3^{(1)}(K) &= \int_0^K \frac{1}{2b^2}\Big\{[2(c+F_1(t))^2 + 4bF_2^{(1)}(t)]F_1'(t)\\
&\quad + [4t(c+F_1(t))^2 + 2btF_2^{(1)}(t)]F_1''(t)\\
&\quad + t^2(c+F_1(t))^2F_1'''(t)\Big\}\mathrm{d}t.
\end{aligned}$$

We notice that, if $\omega^{(1)}$ features a negative singularity, this results into sequence of singularities of alternating signs in $F_n^{(1)}$. A concrete example is given by QED: we show $F_1^{(1)}$, $F_2^{(1)}$ and $F_3^{(1)}$ in Fig. 1.

This suggests that the negative pole in F_1 is not guaranteed to persist when all the $F_n^{(1)}$ are taken into account. In the next section, we show that all the $F_n^{(1)}$ can be actually resummed, and the final result features no singularity.

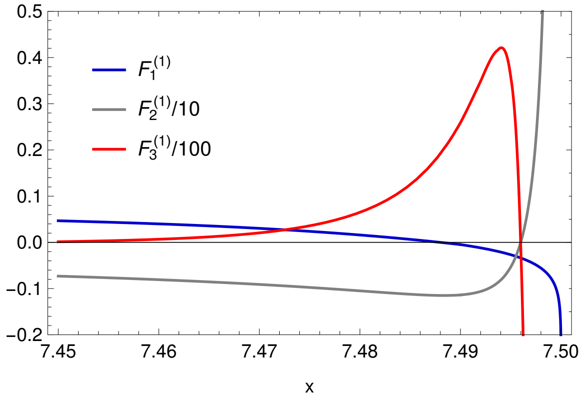


FIG. 1. The functions $F_{1,2,3}^{(1)}$ in the case of QED.

III. SELF-CONSISTENCY EQUATION

Assuming the knowledge of the critical exponent ω for a one-coupling system up to an order $\mathcal{O}(1/N^j)$, we can ask what is the maximum information we can extract about the corresponding β-function. Since a direct resummation of these terms, $F_n^{(j)}$, is not straightforward, we will then employ a different approach. Denoting

$$\mathcal{F}(x,N) \equiv \sum_{n=1}^{\infty}\frac{F_n(x)}{N^{n-1}} \quad (6)$$

the relation $\beta'(g_c) = \omega(d)$ is rewritten as

$$-(d-d_c) + g_c^2 N\mathcal{F}'(x_c,N) = \omega(d), \quad (7)$$

where the dimension and the critical coupling are related via (cf. Eq. (3))

$$d = d_c - g_c(bN + c + \mathcal{F}(x_c,N)). \quad (8)$$

Equation (7) would provide an exact solution, if ω were known to all orders. However, in practice this is not the case, but rather we have access to the contributions induced by $\omega^{(1)},\ldots,\omega^{(j)}$ only. Nonetheless, a consistent solution to Eq. (7) incorporating all known coefficients can be achieved by truncating the critical exponent to

$$\omega(d) = -(d-d_c) + \sum_{n=1}^{j}\frac{1}{N^n}\omega^{(n)}(d), \quad (9)$$

which corresponds to truncating F_n to $F_n^{(j)}$ in $\mathcal{F}(x,N)$, Eq. (6):

$$\mathcal{F}(x,N) \to \mathcal{F}^{(j)}(x,N) \equiv \sum_{n=1}^{\infty}\frac{F_n^{(j)}(x)}{N^{n-1}}, \quad (10)$$

such that $\mathcal{F} \equiv \mathcal{F}^{(\infty)}$.

Let us now concentrate on the simplest case $j=1$, where the truncation leads to the following differential equation for $\mathcal{F}^{(1)}$:

$$\begin{aligned}\partial_x\mathcal{F}^{(1)}(x,N) =& \frac{1}{x^2}\omega^{(1)}(d)\\ =&\frac{1}{x^2}\omega^{(1)}\left[d_c - x\left(b+\frac{c+\mathcal{F}^{(1)}(x,N)}{N}\right)\right] ,\end{aligned} \tag{11}$$

where we have used Eq. (8). If the critical exponent as a function of space-time dimension is known, this is a non-linear first-order differential equation for $\mathcal{F}^{(1)}$. Traditionally, this has been solved order by order in the $1/N$ expansion giving Eq. (5). Indeed, neglecting the back-reaction of $\mathcal{F}^{(1)}$ on the right-hand side of Eq. (11) gives the standard solution $\mathcal{F}^{(1)}(x,\infty)\equiv F_1(x)$. The advantage now is that we can solve Eq. (11) as it is and only afterwards take the large-N limit, which turns out to be finite. This is equivalent to resumming all the $F_n^{(1)}$, given explicitly in Eq. (5) up to $n=3$.

Of particular interest is the case where the critical exponent, $\omega^{(1)}$, has a singularity for some real value of d; e.g. in QED the first singularity of $\omega^{(1)}_{\text{QED}}$ occurs at $d=-1$ translating to a singularity in the $1/N$-perturbative solution for $\beta^{1/N}(x)$ at $x=7.5$. The $\mathcal{F}^{(1)}$ dependence on the right-hand side of Eq. (11) tells that the singularity in the β-function could actually be avoided by a back-reaction of $\mathcal{F}^{(1)}$. In general this requires that the original singularity and b are of opposite signs; in the same-sign case the higher-order terms induced by $\omega^{(1)}$ would just enhance the singularity. This kind of non-resummable singularity is found e.g. in super-QED at $\mathcal{O}(1/N)$ [32] and in O(N) model at $\mathcal{O}(1/N^2)$ [33].

If the singularity and b are of opposite sign, Eq. (11) allows for a smooth $\mathcal{F}^{(1)}$ which, close to the would-be-singularity at, say, $x=x_s$, approaches a *scaling solution* of the form:

$$\mathcal{F}^{(1)}(x,N) = N\left(\frac{a}{x}-b\right)-c, \quad x\gtrsim x_s, \tag{12}$$

where

$$\begin{aligned}&x_s\left(b+\frac{c+\mathcal{F}^{(1)}(x_s,N)}{N}\right)=a,\\ &aN=-\omega^{(1)}(d_c-a).\end{aligned} \tag{13}$$

The second line of Eq. (13) implicitly defines a, which is typically $\mathcal{O}(1)$, and the first line defines x_s. Moreover, from Eq. (12) we see that the singularity one encounters in the β-function due to F_1 is an artifact of taking the large-N limit too early: recalling that $x=gN$, the β function is always well behaved,

$$\beta(g)=\frac{ax}{N}=ag \quad g\gtrsim g_s, \tag{14}$$

where we have used Eq. (12). Equation (14) has no non-trivial zeros: a fixed point can be realized only before entering the scaling solution. For a positive one-loop coeffient, $b>0$, such a zero could be proven if $a<0$. This, however, turns out to be incompatible with the boundary condition $\mathcal{F}(0,N)=0$ and $\omega(d_c)=0$, which require a and b to be of the same sign.

When the RG flow enters the scaling regime, the running coupling can be solved:

$$g(E)=g_0\left(\frac{E}{E_0}\right)^a, \tag{15}$$

where $g_0\equiv g(E_0)\geq g_s$. This corresponds to the classical trajectory for a coupling with dimension $-a$ and shows that the Landau pole could be avoided, since the coupling is finite for $E<\infty$.

When the $\mathcal{O}(1/N^2)$ term in ω, $\omega^{(2)}$, is included, there are two possibilites:

1. the closest singularity at $x=x_s^{(2)}$ is positive,
2. the closest singularity at $x=x_s^{(2)}$ is negative.

In the first case, the β-function clearly grows faster than before close to $x_s^{(2)}$, so that no zero appears. If the new singularity is closer, then this rather implies a Landau pole. As for all the regular points before the first singularity, the contribution of ω_2 is negligible for large enough N. An example of this kind of behaviour is given by the O(N) model [33].

In the second case, we can just apply the same resummation and obtain the asymptotic scaling in Eq. (14) with a modified coefficient a,

$$a=-\frac{1}{N}\omega^{(1)}(d_c-a)-\frac{1}{N^2}\omega^{(2)}(d_c-a), \tag{16}$$

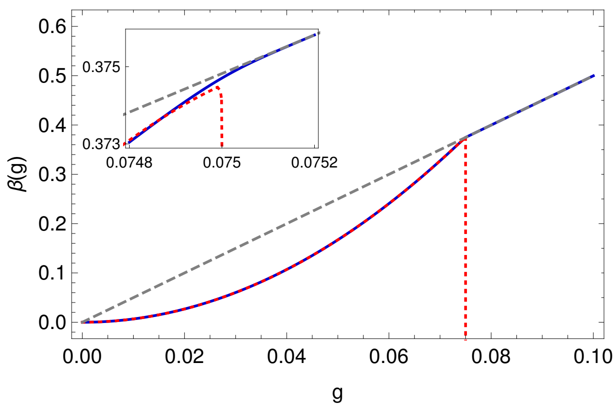
starting at $x\approx\min(x_s,x_s^{(2)})$. Since the one-loop coefficient, b, and a must have the same sign, no fixed point can emerge.

To summarize, the singularities in the critical exponents, or equivalently the singularities in the fixed-order F_n, can not drive the β-function to zero at large N, and therefore there is no hint for asymptotic safety in one-coupling theories in this limit.

We conclude this section by remarking that the resummation we have employed is relevant also beyond the case when the β-function features singularities on the positive real axis: in the next section we will show that the wild oscillations in the β-function of the Gross–Neveu (GN) model can be resummed in the same way.

IV. CONCRETE EXAMPLES

We consider here two prime examples: four-dimensional gauge β-function and Gross-Neveu (GN) model in two dimensions. For the large-N gauge theories, the β-function is known up to $\mathcal{O}(1/N)$, and it features a negative singularity at the rescaled coupling value $x=7.5$ ($x=3$) for (non-)abelian case. Conversely, the GN β-function has no singularities on the positive-coupling axis, but rather features a wildly oscillatory behaviour at large coupling values. However, singularities appear at the negative-coupling axis resulting again in a finite radius of convergence. We will show that the similar back-reaction removing the poles in the gauge β-functions tames the wild oscillations in GN case. Furthermore, the GN critical exponent is known up to $\mathcal{O}(1/N^2)$ allowing us to study the effect of higher-order corrections.



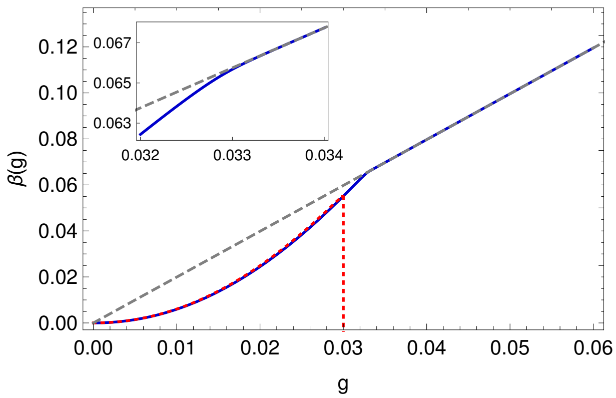


FIG. 2. The β-function for QED (upper panel) and QCD (lower panel) for $N = 100$ computed numerically according to Eq. (11). Dashed lines indicate the scaling solution. The dotted lines show the singular solution one would encounter neglecting the back-reaction in Eq. (11). As we can see, the singularity is removed and the β-function approaches the linear scaling.

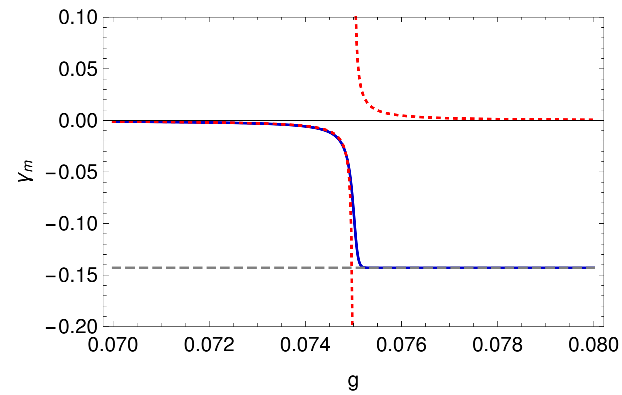


FIG. 3. The quark mass anomalous dimension for QED with $N = 100$. Dotted red line: $\mathcal{O}(1/N)$ result without resummation. Solid line: the solution using Eq. (19). Dashed gray line: $\tilde{\gamma}_m$.

A. QED & QCD

The critical exponent for a general gauge β-function is known up to $\mathcal{O}(1/N)$ and is given in $d = 2\mu$ by [11]

$$\omega^{(1)}(2\mu) = -\frac{\eta^{(1)}(2\mu)}{T_F}\left((2\mu-3)(\mu-3)C_F - \frac{(4\mu^4 - 18\mu^3 + 44\mu^2 - 45\mu + 14)C_A}{4(2\mu-1)(\mu)}\right), \tag{17}$$

where T_F and C_F are the index and quadratic Casimir of the fermion representation, resp., C_A is the Casimir of the adjoint representation, and $\eta^{(1)}$ reads

$$\eta^{(1)}(2\mu) = \frac{(2\mu-1)(\mu-2)\Gamma(2\mu)}{4\Gamma(\mu)^2\Gamma(\mu+1)\Gamma(2-\mu)}. \tag{18}$$

For the abelian case, the first singularity occurs at $\mu = -1/2$, while the non-abelian system has a singularity already at $\mu = 1$.

We compute the β-function by solving Eq. (11) numerically for a benchmark value $N = 100$. In the notation of Eq. (11), QED corresponds to $b = 2/3$, $c = 0$, while QCD is characterised by $b = 2/3, c = -11$. The scaling solutions are given by $a_{\mathrm{QED}} \approx 4.995$, $a_{\mathrm{QCD}} \approx 1.985$. In Fig. 2 we show the numerical solutions to Eq. (11) for QED and QCD with $N = 100$.

In the QED case, the fermion mass anomalous dimension has a singularity at the same coupling value as the first singularity of $\omega^{(1)}$, $x = 7.5$. A fixed point in this coupling region would have the operator $\bar{\psi}\psi$ violating the unitarity bound. Similarly as the critical exponent, ω, we truncate the fermion mass anomalous dimension to $\mathcal{O}(1/N)$:

$$\gamma_m = \frac{\gamma_m^{(1)}(d)}{N} = \frac{\gamma_m^{(1)}\left[d_c - x\left(b + \frac{c+\mathcal{F}^{(1)}(x,N)}{N}\right)\right]}{N}, \tag{19}$$

where the $\mathcal{O}(1/N)$ result is given by $\gamma_m^{(1)}(2\mu) = -2\eta^{(1)}(2\mu)/(\mu-2)$ [10]. Evaluating Eq. (19) with the solution for $\mathcal{F}^{(1)}$, we obtain γ_m in the same truncation as the β-function. We find that the singularity in γ_m is also removed, and the anomalous dimension reaches a constant value above $x = 7.5$ given by

$$\tilde{\gamma}_m = \frac{1}{N}\gamma_m^{(1)}(d_c - a). \tag{20}$$

We show the $\mathcal{O}(1/N)$ result along with the solution according to Eq. (19) in Fig. 3.

B. Gross–Neveu model

The critical exponent, $\lambda(d) = \beta'(g_c)$, for the GN model is currently known up to $\mathcal{O}(1/N^2)$ [18]. The $\mathcal{O}(1/N)$ coefficient is explicitly given by

$$\lambda^{(1)}(2\mu) = \frac{4(\mu-1)^2\Gamma(2\mu)}{\Gamma(2-\mu)\Gamma(\mu)^2\Gamma(\mu+1)}, \tag{21}$$

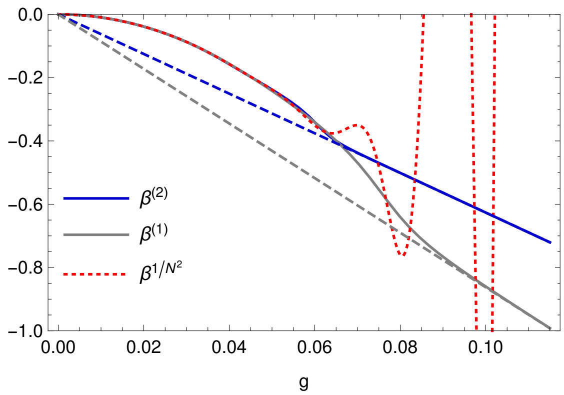


FIG. 4. The solid lines show the GN β-function $\beta^{(2)}$ ($\beta^{(1)}$) for $N = 100$ computed numerically according to Eq. (11) using $\mathcal{F}^{(2)}$ ($\mathcal{F}^{(1)}$), and the dashed lines indicate the corresponding scaling solutions. The dotted red line depicts the $\mathcal{O}(1/N^2)$ β-function without resummation. The scaling solution using $\lambda^{(1)}$ only is given by $a^{(1)} \approx -8.6$, while including $\lambda^{(2)}$ modifies this to $a^{(2)} \approx -6.3$.

while the expression for $\lambda_2(d)$ can be explicitly found in Ref. [18].

In the notations of Eq. (2), the GN model is characterized by $d_c = 2$, $b = -1$ and $c = 2$. We solve again Eq. (11) numerically for benchmark value $N = 100$ both using only the $\mathcal{O}(1/N)$ and $\mathcal{O}(1/N^2)$ critical exponent, λ. We show the resulting β-functions in Fig. 4 along with the β-function computed directly up to $\mathcal{O}(1/N^2)$ using Eq. (2). The scaling solution using only $\lambda^{(1)}$ is given by $a^{(1)} \approx -8.6$, while including $\lambda^{(2)}$ modifies this to $a^{(2)} \approx -6.3$.

V. CONCLUSIONS

We have shown that singularities in a fixed-order large-N critical exponent do not necessarily imply singularities in the β-function approaching from finite N. This is due to the fact that a fixed-order critical exponent generates contributions to every subsequent order in $1/N$ in the β-function. We proposed a self-consistency equation to properly include these contributions.

In the case of negative singularities that have inspired speculations of UV fixed points, it turns out that the same singularity appears with alternating sign at higher-order terms, and resumming these contributions yields an asymptotic linear growth of the β-function rather than a UV zero. As concrete examples we showed this scaling behavior in the case of QED, QCD and the GN model. For QED and QCD, the singularities are removed and in the GN model the wild oscillations tamed. For QED, this procedure simultaneously cures the singularity of the fermion mass anomalous dimension.

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