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α' corrections of Reissner-Nordström black holes

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Abstract: We study the first-order in α' corrections to non-extremal 4-dimensional dyonic Reissner-Nordström (RN) black holes with equal electric and magnetic charges in the context of Heterotic Superstring effective field theory (HST) compactified on a T^6. The particular embedding of the dyonic RN black hole in HST considered here is not supersymmetric in the extremal limit. We show that, at first order in α', consistency with the equations of motion of the HST demands additional scalar and vector fields become active, and we provide explicit expressions for all of them. We determine analytically the position of the event horizon of the black hole, as well as the corrections to the extremality bound, to the temperature and to the entropy, checking that they are related by the first law of black-hole thermodynamics, so that $\partial S/\partial M = 1/T$. We discuss the implications of our results in the context of the Weak Gravity Conjecture, clarifying that entropy corrections for fixed mass and charge at extremality do not necessarily imply corrections to the extremal charge-to-mass ratio.

Keywords: Black Holes in String Theory, Superstrings and Heterotic Strings, Superstring Vacua

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Contents

1 Introduction

Superstring Theory (ST) is our prime candidate for a consistent theory of quantum gravity. One of the main applications of such a theory would be the study of black holes and their quantum behaviour. Thus, it is hardly surprising that one of the main areas of research in ST is black-hole physics: construction of black-hole solutions, calculation of their Hawking temperature and Bekenstein-Hawking entropy, the microscopic interpretation of the latter, etc.

A large part of all this research relies on the effective field theory actions that describe the low-energy behavior of ST and which are (if the vacua chosen preserve any supersymmetries) standard supergravity theories plus terms of higher order in the Regge slope parameter α' (and, correspondingly, in curvatures) and in the string coupling constant.[1] The terms of higher order in α' are important from the ST point of view because they represent genuine stringy departures of matter-coupled General Relativity due to the non-vanishing string's length ℓ_s, with $\alpha' = \ell_s^2$.

Most of the stringy black-hole solutions constructed in the literature, though, only solve the zeroth-order limit of these effective field theories. This means that they can

[1]The addition of these terms still leaves us with (less standard) supergravity theories, with terms of higher order in curvatures, but supergravity theories nevertheless, since supersymmetry should be preserved. Often, the $\alpha' = 0$ limit is improperly referred to as the "supergravity limit", though.

only be considered good ST solutions if one can prove that taking into account the terms of higher order in α' only introduces small corrections in the solutions. Because of the technical complications involved in dealing with higher-order actions, only an estimation of the size of these corrections based on the values of curvature scalars evaluated over the zeroth-order solution are typically made. Quite frequently, it is possible to minimize these scalars by constraining the relative values of the black hole parameters, hoping that any possible stringy effects are also minimized.

However, we are learning that the introduction of higher-curvature terms can have important physical consequences which one cannot make disappear by simply decreasing the curvature scalars. In a perturbative approach, these corrections can be interpreted as introducing delocalized sources in the equations of motion, which may contribute to the global charges and energy of the system [1–4]. Moreover, as it was shown in refs. [2, 3] using the results of ref. [5] and by direct computation of the α' corrections of some supersymmetric black-hole solutions, the curvature scalars do not capture all the possible non-vanishing terms than can occur in the equations of motion at higher orders in α'. While the higher-curvature terms are relevant in any configuration, it has been shown that, in some special situations, taking them into account is just fundamental — see [6, 7]. The inevitable conclusion is that very relevant information can be acquired by performing explicit calculations of the α' corrections to the zeroth-order black-hole solutions. Our goal in this paper is to extend the results found in refs. [2, 3, 8] to non-supersymmetric and non-extremal black holes and compute explicitly their first-order α' corrections in some consistent ST effective action framework.

Corrections to non-extremal, uncharged, rotating black holes (non-extremal Kerr black holes) have been studied long ago, in refs. [9–11], where it was shown that, at first order in α', stringy fields different from the metric are activated (the dilaton and the Kalb-Ramond 2-form).[2] In this work we address this problem for 4-dimensional, charged, non-rotating, non-extremal, (Reissner-Nordström (RN)) black holes embedded in ST.

One of the lessons learned in refs. [2, 3] is that α' corrections to 4- and 5-dimensional systems can be conveniently computed directly in $d = 10$, without having to make any assumptions or approximations, especially in the framework of the Heterotic Superstring Theory (HST) effective field theory. One just needs to find the 10-dimensional solution whose dimensional reduction gives rise to the black hole (or other) solution under consideration, if it exists. Otherwise, the lower-dimensional solution is not a ST theory solution and computing its α' corrections is meaningless.

Since RN black holes are not purely gravitational (there is, at least, one vector field active), there is more than one embedding of the 4-dimensional RN black hole in 10-dimensional HST, corresponding to the many ways in which the vector field can be obtained from the 10-dimensional fields: as Kaluza-Klein or winding vector fields, from 10-dimensional vector fields etc. An important difference between the possible embeddings is the amount of unbroken supersymmetries of their extremal limits. For instance, the em-

[2]On the other hand, the backreaction of those fields onto the metric appear at order α'^2, and such corrections have been more recently computed in the stringy-inspired Einstein-dilaton-Gauss-Bonnet and dynamical Chern-Simons theories — see e.g. [12–16].

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bedding of the extremal RN black hole considered in ref. [3] preserved half of the possible supersymmetries unbroken. Here we are going to consider an embedding which breaks all supersymmetries in the extremal limit, where it will coincide, up to T-duality transformations, with the embedding found in ref. [17]. This embedding is described in section 2. In section 3 we will describe the first-order α' corrections for the 10-dimensional solution that gives rise to this 4-dimensional RN black hole and we will dimensionally reduce the 10-dimensional configuration to recover the 4-dimensional α'-corrected fields (the calculations are described in the appendices). Then, we determine the position of the event horizon of the corrected solution in section 4, its temperature in section 5 and its Wald entropy in section 6. We discuss our results and describe their relation with the WGC in section 7.

2 A non-supersymmetric dyonic Reissner-Nordström black hole

Our starting point is a zeroth-order in α' solution of the 10-dimensional Heterotic Superstring effective field theory (HST)[3] given by the following 10-dimensional fields, which we distinguish from the 4-dimensional ones by the hats:[4]

$$\begin{aligned} d\hat{s}^2 &= a^2 dt^2 - \frac{dr^2}{a^2} - r^2[d\theta^2 + \sin^2\theta d\phi^2] - dz^2 - d\vec{y}^{\,2}_{(5)}\,, \\ \hat{\phi} &= \hat{\phi}_\infty\,, \\ \hat{H} &= e(dt \wedge dr + r^2 \sin\theta d\theta \wedge d\phi) \wedge dz\,. \end{aligned} \tag{2.1}$$

The functions $a(r)$ and $e(r)$ are given by

$$a^2 = 1 - \frac{2M}{r} + \frac{p^2/2}{r^2}\,, \qquad\qquad e = \frac{p}{r^2}\,, \tag{2.2}$$

and ϕ_∞, p, M are physical constants.

The above 10-dimensional metric is the direct product of that of the 4-dimensional, non-extremal RN black hole of mass M and that of a flat T^6. A trivial dimensional reduction on that T^6 (with coordinates $z, y^1, \cdots, y^5$), gives the 4-dimensional metric of that black hole and no additional, active, Kaluza-Klein vector or scalar fields.

The function a that characterized the RN black hole metric can be rewritten in the form

$$a^2 = \frac{(r-r_+)(r-r_-)}{r^2}\,, \tag{2.3}$$

where, as usual,

$$r_\pm = M \pm \sqrt{M^2 - p^2/2}\,, \tag{2.4}$$

are the values of r at which the outer $(+)$ and inner $(-)$ horizons are placed, if they are real, as we are going to assume here.

On the other hand, the dimensional reduction of the Kalb-Ramond 2-form only gives a dyonic vector field B_μ with equal (up to signs) electric and magnetic charges (proportional

[3] The action and equations of motion of this theory are described in appendix A.

[4] Using the components of the Ricci tensor etc. computed in appendix C, it takes little time to check that it satisfies eqs. (A.11)–(A.13) at zeroth order in α'.

to the constant p in the solution), whose field strength $F(B)_{\mu\nu}$ squares to zero.[5] It is this property that allows us to have a constant Kaluza-Klein scalar in the z direction, since the equation of motion of that scalar would be $\nabla^2 k \sim F^2$. The dilaton field is also constant in $d = 4$.

As we have mentioned in the introduction, in the extremal limit $M = |p|/\sqrt{2}$ the 10-dimensional solution is T-dual in the z direction to the non-supersymmetric, purely gravitational solution found in ref. [17] and, therefore, it is not supersymmetric. Being related by T-duality, these two 10-dimensional solutions give rise to the same 4-dimensional RN black hole.

3 α' corrections

In order to find the α' corrections to this solution, we have to use an ansatz that can accommodate both the above solution and the potential α' corrections, which may activate other components of the 10-dimensional metric or the Kalb-Ramond field or the dilaton [9–11]. If the ansatz is not general enough, it will not be possible to solve all the equations of motion and it will be necessary to add to it further active components to be determined.

After several trials, we have arrived, for the zeroth-order solution that we are considering, to the following ansatz:

$$\begin{aligned} d\hat{s}^2 &= A^2 dt^2 - B^2 dr^2 - r^2[d\theta^2 + \sin^2\theta d\phi^2] - C^2[dz + F dt]^2 - d\vec{y}_5^{\,2}\,, \\ \hat{\phi} &= \hat{\phi}_\infty + \alpha' \delta_\phi\,, \\ \hat{H} &= D\,\hat{e}^{\,0} \wedge \hat{e}^{\,1} \wedge \hat{e}^{\,4} + E\,\hat{e}^{\,2} \wedge \hat{e}^{\,3} \wedge \hat{e}^{\,4} + G\,\hat{e}^{\,0} \wedge \hat{e}^{\,2} \wedge \hat{e}^{\,3}\,, \end{aligned} \tag{3.1}$$

where the Zehnbein 1-forms $\hat{e}^{\,a}$ are defined in eq. (C.3) and where A, B, C, D, E, F, G and δ_ϕ are functions of the coordinate r. The expansion of the 7 functions A, B, C, D, E, F, G in powers of α' is assumed to be of the form

$$\begin{aligned} &A \sim a + \alpha'\delta_A\,, \quad B \sim a^{-1} + \alpha'\delta_B\,, \quad C \sim 1 + \alpha'\delta_C\,, \quad F \sim \alpha'\delta_F\,, \\ &D \sim e + \alpha'\delta_D\,, \quad E \sim e + \alpha'\delta_E\,, \qquad G \sim \alpha'\delta_G\,, \end{aligned} \tag{3.2}$$

where the functions a and e are those present in the zeroth-order solution eq. (2.2).

Thus, setting $\alpha' = 0$ in the above configuration eqs. (3.1) we recover the RN solution eq. (2.1) and the 8 functions δ_A, δ_B, δ_C, δ_D, δ_E, δ_F, δ_G and δ_ϕ describe the first-order α' corrections to that solution.

The details of the procedure we have followed to find these corrections can be found in appendix B. Here we are just going to quote the results in 4-dimensional language (unhatted fields), stressing that we have determined the new integration constants by demanding that the mass is not renormalized by the α' corrections[6] and that the fields are regular at the outer (event) horizon at r_+ since it is not possible to keep them regular at both r_+ and r_-

[5] This property follows trivially from $\hat{H}^2 = 0$.

[6] This is equivalent to considering the M that appears in the corrected solutions as the renormalized mass.

(which are assumed to be different in this calculation) simultaneously. The singularity of the scalar fields at r_- is clearly related to the instability of the Cauchy horizon.

First of all, observe that, once the dilaton and the Kaluza-Klein scalar measuring the size of the S^1 parametrized by the coordinate z, $k \equiv |\hat{g}_{zz}|^{1/2}$, are activated, the 4-dimensional metric in the Einstein frame will be given by

$$\begin{aligned} ds^2 &= Ce^{-2(\hat{\phi}-\hat{\phi}_\infty)}\left[A^2dt^2 - B^2dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right] \\ &\equiv N^2 f dt^2 - \frac{d\rho^2}{f} - \rho^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \end{aligned} \tag{3.3}$$

where we have defined the new coordinate ρ by

$$\rho = rC^{1/2}e^{-(\hat{\phi}-\hat{\phi}_\infty)}\,, \tag{3.4}$$

and two new functions N and f in terms of which the equations (and the solutions) take a surprisingly far simpler form:

$$N^2 = 1 + \alpha'\frac{p^2/8}{\rho^4}\,, \tag{3.5}$$

$$f = 1 - \frac{2M}{\rho} + \frac{p^2/2}{\rho^2} - \alpha'\frac{p^2/4}{\rho^4}\left[1 - \frac{3M/2}{\rho} + \frac{11p^2/40}{\rho^2}\right]\,. \tag{3.6}$$

Observe that the two radial coordinates coincide at zeroth order:

$$r = \rho + \mathcal{O}(\alpha')\,, \tag{3.7}$$

and that the α' corrections vanish asymptotically, quite fast. As we will discuss later on, the corrections start becoming dominant for very small values of the radial coordinate, typically well inside the inner horizon. In figure 1 we show the profile of $g_{tt} = N^2 f$ for the solution with $M = p/\sqrt{2}$, corresponding to the extremal case at zeroth order. We observe that α'-corrections take this solution away from extremality, a fact that we will study in more detail in the next sections.

The rest of the 4-dimensional fields which are active include, apart from the dilaton $e^{-\phi}$ and the Kaluza Klein scalar k that we have mentioned above, a Kaluza-Klein vector field A_μ, a winding[7] vector field B_μ and a Kalb-Ramond 2-form $B_{\mu\nu}$ that, in 4-dimensions, can be traded by an axion field that we are going to denote by χ. They take much more complicated forms than the metric, with logarithmic divergences at $r = r_-$. In order to describe them, we first write them in terms of a minimal number of functions and corrections whose value can be found in appendix B.

First of all, using the relation (B.2), $2\delta_\phi = r^2\delta_D/p$, the 4-dimensional Kaluza-Klein scalar and dilaton fields are given by

$$k = 1 + \alpha'\delta_C\,, \tag{3.8}$$

$$e^{-\phi} = e^{-\hat{\phi}_\infty}\left[1 - \frac{\alpha'}{2p}(r^2\delta_D - p\delta_C)\right]\,. \tag{3.9}$$

[7] This is a vector field that is part of the 10-dimensional Kalb-Ramond 2-form, while the Kaluza-Klein vectors are part of the 10-dimensional metric.

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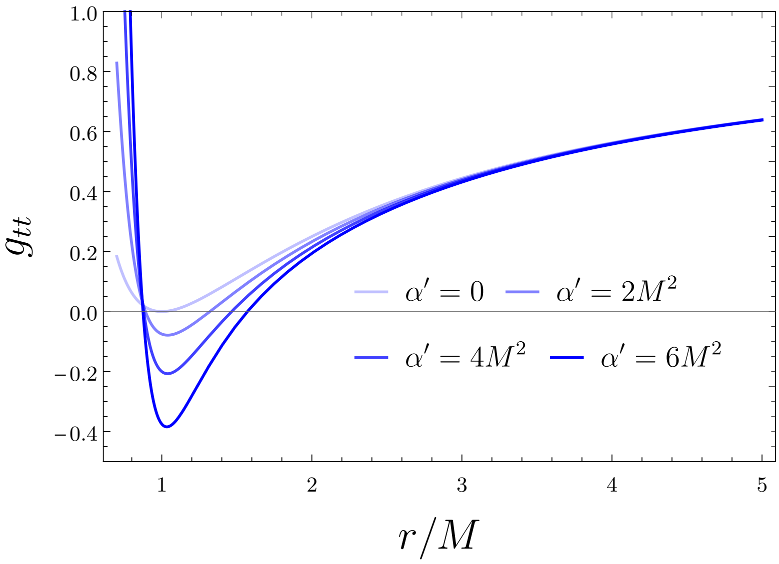

Figure 1. Profile of $g_{tt}(r)$ for the α'-corrected Reissner-Norström black hole corresponding to the case $M = p/\sqrt{2}$. α'-corrections take the black hole away from extremality and increase the size of the outer horizon. The values of α'/M^2 chosen are somewhat exaggerated for illustration purposes.

The field strengths of the Kaluza-Klein vector field (A) and of the vector field that originates in the 10-dimensional Kalb-Ramond 2-form (B) are given by

$$F(A) = -\alpha'\delta_F' dt \wedge d\rho \tag{3.10}$$

$$\begin{aligned} F(B) &= e\left[1 + \alpha'(\delta_N + \delta_D/e)\right] dt \wedge d\rho \\ &\quad + e\rho^2 \left[1 + \alpha'\delta_E/e\right] \sin\theta d\theta \wedge d\phi \,. \end{aligned} \tag{3.11}$$

Finally, the 4-dimensional 3-form field strength is given by

$$H = \alpha' a \delta_G r^2 \sin\theta dt \wedge d\theta \wedge d\phi \,. \tag{3.12}$$

As we have already mentioned, in 4 dimensions, the Kalb-Ramond 2-form can be traded by the axion field χ that, in this case, would only depend on the coordinate r:

$$d\chi = -\alpha' \frac{\delta_G}{a} d\rho \,. \tag{3.13}$$

The expressions for all these fields are quite involved and exhibit logarithmic divergencies at $r = r_-$, but we can compute their charges, defined asymptotically ($\rho \to \infty$) by

$$F(A, B) \sim \frac{Q_{A,B}}{\rho^2} dt \wedge d\rho + P_{A,B} \sin\theta d\theta \wedge d\phi \,, \tag{3.14}$$

$$e^{\phi} \sim e^{\phi_\infty} \left[1 + \frac{Q_\phi}{\rho}\right] , \tag{3.15}$$

$$k \sim 1 + \frac{Q_k}{\rho} , \tag{3.16}$$

$$d\chi \sim -\frac{Q_\chi}{\rho^2} d\rho \,. \tag{3.17}$$

We readily get

$$Q_k = \frac{\alpha' r_- \left(140r_+^3 - 154r_+^2 r_- + 35r_+ r_-^2 - 9r_-^3\right)}{140r_+^3 \left(r_+^2 + 4r_+ r_- + r_-^2\right)} + \mathcal{O}(\alpha'^2)\,, \tag{3.18}$$

$$Q_\phi = \alpha' \frac{16r_-^4 - 77r_-^3 r_+ - 49r_-^2 r_+^2 + 70r_+^3(r_- + r_+)}{280r_+^3(r_-^2 + 4r_- r_+ + r_+^2)} + \mathcal{O}(\alpha'^2)\,, \tag{3.19}$$

$$Q_\chi = 0 + \mathcal{O}(\alpha'^2)\,, \tag{3.20}$$

$$Q_A = \frac{\alpha' r_-(r_+ - r_-)^2}{4\sqrt{2r_+ r_- r_+^3}} + \mathcal{O}(\alpha'^2)\,, \tag{3.21}$$

$$P_A = 0\,, \tag{3.22}$$

$$Q_B = p\,, \tag{3.23}$$

$$P_B = p\,. \tag{3.24}$$

A few comments are in order. First, we note that there are no new independent charges, as all of them are completely determined by M and p. In the uncharged limit $p \to 0$ we see that the metric reduces to the Schwarzschild one, and the only nonvanishing charge is Q_ϕ:

$$Q_k = \frac{\alpha'}{2M}\left\{(p/M)^2 + \mathcal{O}\left((p/M)^3\right)\right\}\,, \tag{3.25}$$

$$Q_\phi = \frac{\alpha'}{8M}\left\{1 + \frac{1}{4}(p/M)^2 + \mathcal{O}\left((p/M)^4\right)\right\}\,, \tag{3.26}$$

$$Q_A = \frac{\alpha'}{32M}\left\{(p/M) + \mathcal{O}\left((p/M)^2\right)\right\}\,. \tag{3.27}$$

This is in agreement with previous computations of corrections in uncharged solutions [9–11, 13, 14].

4 Horizons

Let us now study the horizons of the α'-corrected metric determined by eqs. (3.3), (3.5) and (3.6). The horizons of the metric are determined by the zeroes of the function $f(\rho)$, which, using the definitions

$$x \equiv \rho/M\,, \qquad q \equiv p/(\sqrt{2}M)\,, \qquad \alpha \equiv \alpha'/M^2\,, \tag{4.1}$$

can be written in the form

$$f(x) = 1 - \frac{2}{x} + \frac{q^2}{x^2} - \frac{\alpha q^2}{2x^4} + \frac{3\alpha q^2}{4x^5} - \frac{11\alpha q^4}{40x^6}\,. \tag{4.2}$$

As expected, the α' corrections become dominant for small values of x, well inside the inner (Cauchy) horizon. In the uncorrected RN black hole, $f(x)$ is positive inside this horizon and diverges when x approaches the singularity at $x = 0$. In the α'-corrected RN black hole, when we move towards $x = 0$ from the Cauchy horizon, $f(x)$ is positive, but it always reaches a maximum and starts decreasing so that $\lim_{x\to 0+} f(x) = -\infty$. Therefore, a generic feature of the corrected black holes is that they have a third horizon inside the

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Cauchy horizon,[8] although it is always placed close to the region at which the curvature becomes so large that higher corrections in α' can no longer be ignored.

In order to find the corrections to the positions of the horizons in the α'-corrected RN black holes, we can study the zeroes of the 6^{th}-order polynomial $P(x) \equiv x^6 f(x)$

$$P(x) = x^6 - 2x^5 + q^2x^4 - \frac{1}{2}\alpha q^2 x^2 + \frac{3}{4}\alpha q^2 x - \frac{11}{40}\alpha q^4 \,, \tag{4.3}$$

to first order in α.

Let us start with the extremal case in which

$$q = 1 + a\alpha \,, \tag{4.4}$$

for some numerical constant a. Notice that the charge-to-mass ratio q can differ from 1 in the extremal limit once higher-curvature interactions are incorporated [8, 18–20], which has been connected to the weak gravity conjecture [21]. At first order in α we have

$$P(x) = x^6 - 2x^5 + (1+2a\alpha)x^4 - \frac{1}{2}\alpha x^2 + \frac{3}{4}\alpha x - \frac{11}{40}\alpha + \mathcal{O}(\alpha^2) \,. \tag{4.5}$$

The numerical results obtained for several values of q suggest that, in the non-superextremal cases, there is a complex pole and its conjugate plus a pole at a negative value of x, so that, in the extremal limit, it should be possible to factorize $P(x)$ as follows

$$P(x) = |x - (b\alpha + ic\alpha^{1/4})|^2 (x + d\alpha^{1/4})(x - e\alpha^{1/4})[x - (1+f\alpha)]^2 + \mathcal{O}(\alpha^2) \,, \tag{4.6}$$

for constants b, c, d, e, f, g to be determined by comparison with eq. (4.5). We readily find the values of the two constants which determine the corrections to the extremality relation between the charge and mass and to the position of the horizon

$$a = 1/80 \,, \qquad f = 3/40 \,, \tag{4.7}$$

so that,

$$M_{\text{ext}} = (p/\sqrt{2}) \left[1 - \frac{\alpha'}{80} \frac{1}{(p/\sqrt{2})^2}\right] + \mathcal{O}(\alpha'^2) \,, \tag{4.8}$$

$$\rho_{\text{h ext}} = M_{\text{ext}} \left(1 + \frac{3\alpha'}{40 M_{\text{ext}}^2}\right) + \mathcal{O}(\alpha'^2) = (p/\sqrt{2}) \left[1 + \frac{\alpha'}{16} \frac{1}{(p/\sqrt{2})^2}\right] + \mathcal{O}(\alpha'^2) \,. \tag{4.9}$$

This expression for the correction to the mass in the extremal limit was anticipated in [8].

Far from extremality, it should be possible to factorize the polynomial in eq. (4.3) as follows:

$$P(x) = |x - (b\alpha + ic\alpha^{1/4})|^2 (x + d\alpha^{1/4})(x - e\alpha^{1/4})[x - (x_+ + f\alpha)][x - (x_- + g\alpha)] + \mathcal{O}(\alpha^2) \,, \tag{4.10}$$

with $x_\pm = 1 \pm \sqrt{1-q^2}$. Again, comparing with eq. (4.3) we find

$$2f = -\frac{1}{\sqrt{1-q^2}} \left[\frac{9}{40} - \frac{21}{20q^2} + \frac{4}{5q^4}\right] - \frac{13}{20q^2} + \frac{4}{5q^4} + \mathcal{O}(\alpha) \,. \tag{4.11}$$

[8] Or a second horizon inside the event horizon in the extremal case in which the two outermost horizons coincide.

We have checked numerically that

$$\rho_{\rm h} = r_+ + \frac{\alpha'}{M}\left\{-\frac{13}{20}(M/p)^2+\frac{8}{5}(M/p)^4-\frac{1}{\sqrt{1-\frac{1}{2}(p/M)^2}}\left[\frac{9}{80}-\frac{21}{20}(M/p)^2+\frac{8}{5}(M/p)^4\right]\right\}, \tag{4.12}$$

gives a very good approximation to the position of the event horizon to first order in α', even close to extremality. The position of the inner horizon is given by

$$\rho_- = r_- + \frac{\alpha'}{M}\left\{-\frac{13}{20}(M/p)^2+\frac{8}{5}(M/p)^4+\frac{1}{\sqrt{1-\frac{1}{2}(p/M)^2}}\left[\frac{9}{80}-\frac{21}{20}(M/p)^2+\frac{8}{5}(M/p)^4\right]\right\}, \tag{4.13}$$

but it is only good for small (but larger than 1) values of $\sqrt{2}(M/p)$.

In the near-extremality regime the square root term becomes imaginary before extremality is reached. Therefore, we must make a different ansatz for the polynomial eq. (4.3):

$$P(x) = |x-(b\alpha+ic\alpha^{1/4})|^2(x+d\alpha^{1/4})(x-e\alpha^{1/4})[x^2-2(1+h\alpha)x+(1+j\alpha)q^2]+\mathcal{O}(\alpha^2)\,, \tag{4.14}$$

where h and j are two additional real constants which are found to have the values

$$h = -\frac{13}{40q^2}+\frac{2}{5q^4}\,, \tag{4.15a}$$

$$j = \frac{9}{40q^2}-\frac{17}{10q^4}+\frac{8}{5q^6}\,. \tag{4.15b}$$

The two roots corresponding to the horizons are

$$\tilde{x}_\pm = 1+h\alpha \pm \sqrt{1-q^2+(2h-q^2j)\alpha}\,, \tag{4.16}$$

and, parametrizing q near extremality by $q = 1+\delta\alpha$ and replacing h and j by their values, given above, we get

$$\rho_\pm = M \pm \sqrt{2\alpha'}\sqrt{\frac{1}{80}-\delta}+\frac{3}{40M}\alpha'+\mathcal{O}(\alpha'^{3/2})\,. \tag{4.17}$$

The extremal limit is $\delta = 1/80$, and, there, we have

$$\rho_{\rm h\,ext} = M\left(1+\frac{3}{40}\frac{\alpha'}{M^2}\right)\,. \tag{4.18}$$

Close to that limit, replacing $\delta\alpha'$ by its value $M(p/\sqrt{2}-M)$ in eq. (4.17), we find that the horizon is placed at

$$\rho_{\rm h\,nearext} = M\left\{1+\sqrt{2}\sqrt{1+\frac{1}{80}\frac{\alpha'}{M^2}-\frac{p}{\sqrt{2}M}}+\frac{3}{40}\frac{\alpha'}{M^2}\right\}\,. \tag{4.19}$$

Nevertheless, it is useful to rewrite this formula explicitly in terms of the small quantity $M-M_{\rm ext} = M-\frac{p}{\sqrt{2}}+\frac{\sqrt{2}\alpha'}{80p}$, in whose case it reads

$$\rho_{\rm h\,nearext} = \frac{p}{\sqrt{2}}+\frac{\sqrt{2}\alpha'}{16p}+2^{1/4}p^{1/2}\sqrt{M-M_{\rm ext}}+(M-M_{\rm ext})+\ldots \tag{4.20}$$

5 Temperature

The Hawking temperature is related to the surface gravity by the famous formula

$$T = \frac{\kappa}{2\pi}\,, \tag{5.1}$$

while the surface gravity of the event horizon of a static, spherically symmetric black hole, located at $\rho = \rho_{\rm h}$, is given by

$$\kappa = \frac{1}{2}\left[\frac{1}{\sqrt{-g_{tt}g_{\rho\rho}}}\frac{dg_{tt}}{d\rho}\right]_{\rho_{\rm h}}\,. \tag{5.2}$$

In terms of the variable $x \equiv \rho/M$ and taking into account that the event horizon corresponds to the outermost first-order zero of the function f in eq. (3.6), we find the following expression for the surface gravity

$$T = \frac{1}{4\pi M}\frac{N(x_{\rm h})}{x_{\rm h}^6}\left[\frac{P(x)}{x-x_{\rm h}}\right]_{x_{\rm h}}\,, \tag{5.3}$$

where $P(x)$ is the polynomial defined in eq. (4.3) if we are far from the extremal limit. In that regime, the polynomial can be written in the form eq. (4.10) and $x_{\rm h} = x_+ + f\alpha$, and, therefore, we just have to evaluate at $x = x_{\rm h}$, to first order in α, the fifth-order polynomial

$$\begin{aligned}\frac{P(x)}{x-x_{\rm h}} &= |x-(b\alpha+ic\alpha^{1/4})|^2(x+d\alpha^{1/4})(x-e\alpha^{1/4})[x-(x_-+g\alpha)]+\mathcal{O}(\alpha^2)\\ &= \left[x^4-\left(\frac{13}{20q^2}-\frac{4}{5q^4}\right)\alpha x^3-\left(\frac{9}{40}-\frac{2}{5q^2}\right)\alpha x^2+\frac{1}{5}\alpha x-\frac{11}{40}\alpha q^2\right][x-(x_-+g\alpha)]\\ &\quad+\mathcal{O}(\alpha^2)\,.\end{aligned} \tag{5.4}$$

At first order, recognizing

$$f+g = -\frac{13}{20q^2}+\frac{4}{5q^4}\,, \tag{5.5}$$

we get

$$\begin{aligned}\left[\frac{P(x)}{x-x_{\rm h}}\right]_{x_{\rm h}} &= x_+^4(x_+-x_-)+\alpha\left\{x_+^4(f-g)+(x_+-x_-)\left[(5f+g)x_+^3\right.\right.\\ &\quad\left.\left.-\left(\frac{9}{40}-\frac{2}{5q^2}\right)x_+^2+\frac{1}{5}x_+-\frac{11}{40}q^2\right]\right\}+\mathcal{O}(\alpha^2)\,.\end{aligned} \tag{5.6}$$

Plugging this result into eq. (5.3) and operating we can write the temperature in the form

$$T = T^{(0)}\left\{1-\frac{\alpha}{x_+^4}\left[\frac{(g-f)}{x_+-x_-}q^2x_+^2+\left(\frac{9}{40}-\frac{2}{5q^2}\right)x_+^2-\frac{1}{5}x_++\frac{3}{20}q^2\right]\right\}+\mathcal{O}(\alpha^2)\,, \tag{5.7}$$

where

$$T^{(0)} = \frac{(x_+-x_-)}{4\pi M x_+^2} = \frac{\sqrt{M^2-\frac{p^2}{2}}}{2\pi\left(M+\sqrt{M^2-\frac{p^2}{2}}\right)^2}\,, \tag{5.8}$$

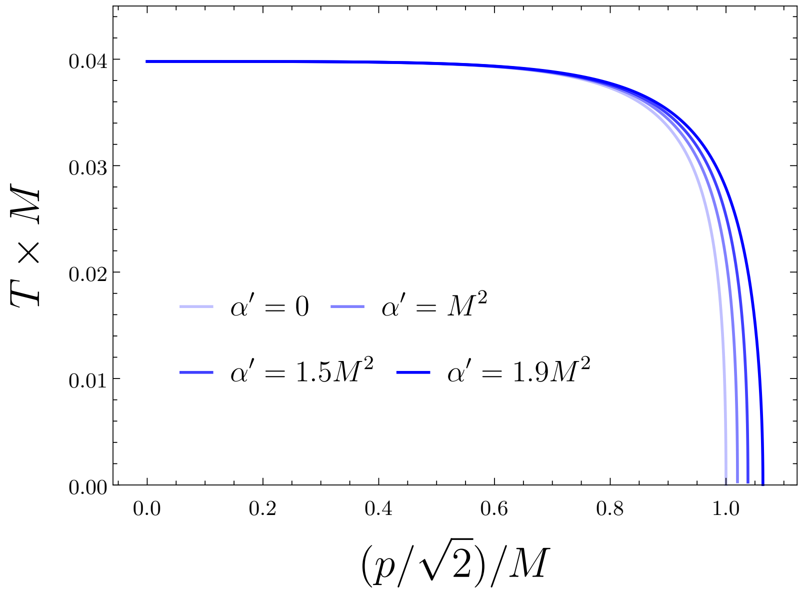

Figure 2. Temperature of the α'-corrected Reissner-Nordströn black holes. We show T as a function of the charge rescaling all quantities in terms of the mass M. The corrections are more relevant for large charges and we see that extremality is reached for $p/\sqrt{2} > M$.

is the temperature of the uncorrected RN black hole. Operating with the actual values of f and g, and making use of the definitions

$$q = p/(\sqrt{2}M)\,, \qquad x_\pm = 1 \pm \sqrt{1-q^2}\,, \qquad \alpha = \alpha'/M^2\,, \tag{5.9}$$

we obtain our final expression for the temperature,

$$T = \frac{\sqrt{M^2 - \frac{p^2}{2}}}{2\pi\left(M + \sqrt{M^2-\frac{p^2}{2}}\right)^2} + \frac{\alpha'\left(M + 3\sqrt{M^2-\frac{p^2}{2}}\right)\left(M - \sqrt{M^2-\frac{p^2}{2}}\right)^2}{160\pi\sqrt{M^2-\frac{p^2}{2}}\left(M+\sqrt{M^2-\frac{p^2}{2}}\right)^5} + \mathcal{O}(\alpha'^2)\,. \tag{5.10}$$

This expression diverges for $M \to p/\sqrt{2}$, but this is simply indicating that the approximation implied in (5.4) is no longer valid. Instead, in the near-extremal limit it is straightforward to obtain

$$T = \frac{1}{\pi p^2}\left[2^{1/4}p^{1/2}\sqrt{M - M_{\rm ext}} + 4(M - M_{\rm ext}) + \dots\right]\,, \tag{5.11}$$

where we recall that $M_{\rm ext} = \frac{p}{\sqrt{2}} - \frac{\sqrt{2}\alpha'}{80p} + \mathcal{O}(\alpha'^2)$. Thus the temperature vanishes in the limit $M \to M_{\rm ext}$, as it should. We note that in the expression above all the corrections enter implicitly through the shift in the extremal mass, for formally the expansion in powers of $M - M_{\rm ext}$ is the same as in the RN black hole. We observe that, near extremality, the corrections to the temperature of a solution of fixed mass are of order $\alpha'^{1/2}$. In particular, the solution with $M = p/\sqrt{2}$, which at zeroth order corresponds to the extremal case, possesses a non-vanishing temperature $T = \alpha'^{1/2}/(4\pi\sqrt{10}M^2)$. The complete profile of the temperature as a function of the charge is shown in figure 2 for a few values of α'/M^2.

6 Entropy

In order to compute the entropy of this black hole, it is necessary to take into account the presence of higher-curvature terms in the action. Wald's entropy formula [22, 23] takes into account the possible presence of these terms and yields an entropy that satisfies the first law of black-hole thermodynamics. However, this formula was derived under the assumption that all the fields in the theory are tensors. This is very restrictive, as the only physical fields in our current description of Nature which are tensors, apart from the metric, are scalars. Therefore, strictly speaking, it has not been proven that Wald's formula can be applied even to the Einstein-Maxwell theory, since the Maxwell field is not a tensor field, but a connection. It is also unclear whether Wald's formula can be applied to theories with fields with any kind of gauge freedom, either. This is true even for General Relativity itself when it is formulated in terms of a Vierbein! Fortunately, Jacobson and Moh showed in ref. [24] that, once the subtleties associated to the ("induced" or "compensating") local Lorentz transformations that the Vierbein suffers when one acts on it with a diffeomorphism are taking into account, Wald's formula can be applied essentially unchanged.

The Heterotic Superstring effective action, reviewed in appendix A, is a much more complicated beast, though. To start with, it has to be formulated, necessarily, in terms of a Zehnbein, in order to include spinor fields. One can deal with both of them in the same way as Jacobson and Moh dealt with the Vierbein in 4 dimensions: using the Lie-Lorentz derivative.[9] Then, (most likely) one can prove that the black-hole entropy is given by Wald's formula once again. However, the action also includes Yang-Mills gauge fields which do not just occur via the gauge-covariant Yang-Mills field strength but also via the Chern-Simons 3-form eq. (A.6), which transforms in a completely different way. Actually, the same happens to the Zehnbein: it also occurs in the action via the Chern-Simons 3-form of the spin connection 1-form eq. (A.4). This does not mean that the action is not gauge- or Lorentz-invariant, because these terms only occur via the 3-form field strength in eq. (A.7) which is gauge- and local-Lorentz invariant thanks to the very special way in which the Kalb-Ramond 2-form behaves under gauge and local-Lorentz transformations (the so-called Nicolai-Townsend transformations). Taking all this into account, it has been shown in ref. [26] that Wald's formula also applies to the Heterotic Superstring effective action,[10] justifying the results obtained in refs. [2, 3].

Wald's formula for the black-hole entropy can be written in the form

$$S = -2\pi \int_\Sigma d^2x \sqrt{|h|} \frac{\partial \mathcal{L}}{\partial R_{abcd}} \epsilon_{ab}\epsilon_{cd} \,, \tag{6.1}$$

where $|h|$ is the absolute value of the determinant of the metric induced over the event horizon, ϵ^{ab} is the event horizon's binormal normalized so that $\epsilon_{ab}\epsilon^{ab} = -2$ and R_{abcd} is

[9] See, for instance, ref. [25] and references therein.

[10] Ref. [26] deals with a family of actions which is, in certain respects, more general than the Heterotic Superstring's but which do not include Yang-Mills gauge fields. However, there is no real difference between the behavior of gauge fields and local-Lorentz tensors or spinors and it is clear that the results obtained can be extended to include Yang-Mills fields straightforwardly. On the other hand, in ref. [26] it is assumed (but not directly proven) that a generalization of the Lie-Lorentz derivative can be constructed. This point clearly deserves further investigation.

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the Riemann tensor. We will work in the ("modified") Einstein-frame metric. Then, it can be shown [27] that the partial derivative of the Heterotic Superstring effective action compactified on a trivial T^5 and then on S^1 with respect to the Riemann tensor of that conformal frame is given in terms of 4-dimensional objects by

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial R_{abcd}} = \frac{1}{16\pi G_N^{(4)}} \Big\{ & g^{ab,\,cd} - \frac{\alpha'}{8}\Big[e^{-4(\phi-\phi_\infty)} H^{(0)\,abg}\left(\omega_g{}^{cd} + 2\Sigma_g{}^{cd}\right) \\ & + e^{-2(\phi-\phi_\infty)}\left(-2\tilde{R}^{(0)\,abcd}_{(-)} + K^{(-)\,[a|c}K^{(-)\,|b]d} + K^{(+)\,ab}K^{(+)\,cd}\right)\Big]\Big\}\,,\end{aligned} \tag{6.2}$$

where

$$H^{(0)}{}_{\mu\nu\rho} \equiv 3\partial_{[\mu}B^{(0)}{}_{\nu\rho]} - \frac{3}{2}A_{[\mu}G^{(0)}{}_{\nu\rho]} - \frac{3}{2}B^{(0)}{}_{[\mu}F_{\nu\rho]}\,, \tag{6.3}$$

$$K^{(\pm)}{}_{\mu\nu} \equiv kF_{\mu\nu} \pm k^{-1}G^{(0)}{}_{\mu\nu}\,, \tag{6.4}$$

$$\Sigma_\mu{}^a{}_b \equiv \Delta_\mu{}^a{}_b - \frac{1}{2}H^{(0)}{}_\mu{}^a{}_b\,, \tag{6.5}$$

$$\Delta_{\mu\,ab} \equiv -\partial_{[\mu}\phi\eta_{b]c} + e_{[c|\,\mu}e_{|b]}{}^\nu\partial_\nu\phi - e_{[c|}{}^\nu e_{b\,|\mu]}\partial_\nu\phi\,, \tag{6.6}$$

$$\tilde{\Omega}^{(0)}_{(-)\,\mu}{}^a{}_b \equiv \omega_\mu{}^a{}_b + \Sigma_\mu{}^a{}_b\,, \tag{6.7}$$

$$g^{ab,\,cd} \equiv \frac{1}{2}\left(g^{ac}g^{bd} - g^{ad}g^{bc}\right)\,, \tag{6.8}$$

and where $g^{ab} = \eta^{ab}$. Here A_μ is the KK vector field and $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ its field strength, $B^{(0)}{}_\mu$ is the winding vector field at zeroth-order in α' and $G^{(0)}{}_{\mu\nu} = 2\partial_{[\mu}B^{(0)}{}_{\nu]}$ its field strength, $B^{(0)}{}_{\mu\nu}$ is the Kalb-Ramod 2-form at zeroth order in α' and $H^{(0)}{}_{\mu\nu\rho}$ its gauge-invariant field strength expressed in a manifestly T-duality-invariant form. Furthermore, $\tilde{R}^{(0)}_{(-)\,\mu\nu}{}^a{}_b$ is the curvature 2-form of the connection $\tilde{\Omega}^{(0)}_{(-)\,\mu}{}^a{}_b$, which differs from the usual torsionful spin connection $\Omega^{(0)}_{(-)\,\mu}{}^a{}_b$ by the dilaton-dependent $\Delta_\mu{}^a{}_b$ contribution which arises in the Weyl rescaling from the string to the Einstein frame.

The uncorrected RN black hole has $k = 1, F_{\mu\nu} = H^{(0)}{}_{\mu\nu\rho} = 0$ and $\phi = \phi_\infty$ at zeroth order in α', which means that $\tilde{R}^{(0)}_{(-)\,\mu\nu}{}^a{}_b = R^{(0)}_{(-)\,\mu\nu}{}^a{}_b = R^{(0)a}_{\mu\nu}{}_b$, the Riemann curvature of the original, uncorrected, RN black hole. Wald's formula in $G_N^{(4)} = 1$ units takes the form

$$\begin{aligned} S &= -\frac{1}{8}\int_\Sigma d^2x\sqrt{|h|}\epsilon_{ab}\epsilon_{cd}\Big\{g^{ab,\,cd} \\ &\qquad -\frac{\alpha'}{8}\left[-2R^{(0)\,abcd}_{(-)} + G^{(0)\,[a|c}G^{(0)\,|b]d} + G^{(0)\,ab}G^{(0)\,cd}\right]\Big\} \\ &= -\frac{1}{8}\int_\Sigma d^2x\sqrt{|h|}\left\{-2 + \alpha'\left[R^{(0)\,0101} - \frac{3}{4}\left(G^{(0)\,01}\right)^2\right]\right\} \\ &= \frac{A_{\rm h}}{4} - \alpha'\left[\frac{1}{2}(a^2)''_{\rm h} - \frac{3p^2}{4\rho_{\rm h}^4}\right]\frac{A^{(0)}_{\rm h}}{8} \\ &= \pi\rho_{\rm h}^2\left\{1 + \alpha'\left[\frac{M}{\rho_{\rm h}^3} - \frac{3p^2}{8\rho_{\rm h}^4}\right]\right\}\,, \end{aligned} \tag{6.9}$$

where $\rho_{\rm h}$ is the radius of the event horizon, and $A_{\rm h}$ is the area of the event horizon, $4\pi\rho_{\rm h}^2$.

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This formula, which we rewrite here for the sake of convenience,

$$S = \pi\rho_{\rm h}^2\left\{1+\alpha'\left[\frac{M}{\rho_{\rm h}^3}-\frac{3p^2}{8\rho_{\rm h}^4}\right]\right\}\,, \tag{6.10}$$

is one of the main results of this paper, but we must test it against the temperature computed in section 5.

Far from the extremal limit we can use the value of the radius of the horizon ρ_h given in eq. (4.12), and after some simplifications we arrive at the following result for the entropy,

$$S=\pi\left[\left(M+\sqrt{M^2-\frac{p^2}{2}}\right)^2+\frac{\alpha'\left(18M\sqrt{M^2-\frac{p^2}{2}}+21\left(M^2-\frac{p^2}{2}\right)+M^2\right)}{40\sqrt{M^2-\frac{p^2}{2}}\left(\sqrt{M^2-\frac{p^2}{2}}+M\right)}\right]+\mathcal{O}(\alpha'^2)\,. \tag{6.11}$$

Now, in a thermodynamic system with energy M and entropy S, the temperature is defined through the standard relation

$$\frac{\partial S}{\partial M}=\frac{1}{T}\,, \tag{6.12}$$

and in the case of black holes this temperature should coincide with the Hawking's one on account of the first law of black hole mechanics. By deriving eq. (6.11) with respect to M, it is easy to check that the thermodynamic temperature agrees with Hawking's temperature in eq. (5.10) at first order in α', which constitutes a strong consistency test of our computations.

In the near-extremal limit, according to eq. (4.20), which we reproduce here for the sake of convenience, we have

$$\rho_{\rm h\,near ext}=\frac{p}{\sqrt{2}}+\frac{\sqrt{2}\alpha'}{16p}+2^{1/4}p^{1/2}\sqrt{M-M_{\rm ext}}+(M-M_{\rm ext})+\ldots \tag{6.13}$$

and, substituting this value in eq. (6.10) we get

$$S/\pi=\frac{p^2}{2}+\frac{3\alpha'}{8}+2^{3/4}p^{3/2}\sqrt{M-M_{\rm ext}}+\sqrt{8}p(M-M_{\rm ext})+\ldots\,. \tag{6.14}$$

It is straightforward to check that the entropy and temperature in the near-extremal regime, given by eqs. (6.14) and (5.11) also satisfy the thermodynamic relation $\partial S/\partial M=T^{-1}$. If we take the extremal limit in this expression, $M\to M_{\rm ext}$, we observe that the entropy gets an $\mathcal{O}(\alpha')$ correction

$$S=S_{\rm ext}^{(0)}+\frac{3\pi}{8}\alpha'\,,\qquad \text{where}\quad S_{\rm ext}^{(0)}=\pi p^2/2\,. \tag{6.15}$$

However, this expression should not be trusted due to the presence of logarithmic divergences of some of the fields (which are generically found at the Cauchy horizon) at the event horizon. Indeed, from (6.2) it is manifest that the dilaton divergence would produce an infinite correction to the entropy, which is meaningless. Hence, the analysis presented here is only valid for non-extremal configurations. In the extremal limit, it seems the black

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hole becomes singular after the higher-curvature corrections are incorporated so it makes no sense to attribute a value to its entropy.

Near-extremality, the corrections to the entropy are of order $\alpha'^{1/2}$ as in ref. [20]. In particular, for the solution with $M = p/\sqrt{2} \equiv M^{(0)}_{\rm ext}$ we find

$$S\Big|_{M=M^{(0)}_{\rm ext}} = \pi\left[\frac{p^2}{2} + \frac{p\alpha'^{1/2}}{2\sqrt{5}} + \mathcal{O}(\alpha')\right]. \tag{6.16}$$

7 Discussion

In this paper we have computed the first-order in α' corrections to a dyonic Reissner-Nordström black hole explicitly embedded in the Heterotic String Theory. To the best of our knowledge, this is the first explicit example of a non-extremal Reissner-Nordström solution containing all of the α'-corrections. In the extremal limit, we have seen that the charge-to-mass ratio of the solution is positively corrected

$$\frac{p/\sqrt{2}}{M}\bigg|_{\rm ext} = 1 + \frac{\alpha'}{80M^2} + \mathcal{O}(\alpha'^2), \tag{7.1}$$

in agreement with the mild form of the weak gravity conjecture. Nevertheless, since it seems that the black hole becomes singular in that limit, this only provides some sort of indirect signal in favour of the conjecture. In the context of HST, the first example of regular extremal black hole with corrections to the charge-to-mass ratio in agreement with the WGC was recently given in ref. [8].[11]

We have also computed the corrections to the temperature and to the entropy of these black holes — see (5.10) and (6.11). The temperature is straightforwardly computed from the surface gravity of the horizon, but the calculation of the entropy through the evaluation of Wald's formula presents some subtleties associated to the presence of Chern-Simons terms in the 10-dimensional action. Those subtleties can be handled with the methods used in refs. [2, 3] but it is most reassuring to see that they completely disappear for these black-hole solutions when the 10-dimensional action is compactified as in ref. [27] and Wald's formula takes the explicitly gauge-invariant form eq. (6.9).[12] Another possibility to compute the entropy would be to rewrite the HST action in a gauge-invariant manner without performing the dimensional reduction as in [4]. We have checked that the application of this method produces the same result (6.11). As a highly non-trivial check of our computations, we have shown that the thermodynamic relation $\partial S/\partial M = 1/T$ holds at order α'.

We have found that the entropy shift is always positive for this family of solutions. In previous works in the literature, it has been claimed that the positivity of the corrections to the entropy imply a positive correction to the charge-to-mass ratio at extremality [31–33].

[11]The GHS solution [28–30], whose corrections where obtained in ref. [19], does not describe a black hole in the extremal limit.

[12]For more general solutions one has to use eq. (6.2), though. This expression contains explicit contributions from the spin connection which are not manifestly invariant under local Lorentz transformations and, at this point, it is not clear if they give non-trivial contributions to the entropy.

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On the contrary, this claim has been disputed by the counterexample presented in ref. [8], in which $\Delta S > 0$ but $\Delta(q/M) = 0$. It is interesting to ask what the situation is here, since we have a non-extremal solution at our disposal and we can perform a more detailed analysis. In ref. [33] the shifts are claimed to satisfy a universal relation,

$$\Delta M_{\rm ext} = -T_0(M,\vec{Q})\Delta S(M,\vec{Q})|_{M\approx M^{(0)}_{\rm ext}}\,. \tag{7.2}$$

Here, $\Delta M_{\rm ext}$ is the change in the energy of the solution at zero temperature, while $T_0(M,\vec{Q})$ and $\Delta S(M,\vec{Q})$ are, respectively, the unperturbed temperature and the shift in the entropy for fixed values of the mass and charges. As $T_0(M,\vec{Q})$ is parametrically small for $M \to M^{(0)}_{\rm ext}$, one sees that, whenever $\Delta M_{\rm ext} \neq 0$, the expression for $\Delta S(M,\vec{Q})$ that must be used in (7.2) becomes divergent as $M \to M^{(0)}_{\rm ext}$, so it cannot really correspond to the correction to the entropy for this value of the mass, which should be finite. For this reason, according to the prescription given in ref. [33], the right hand side of (7.2) must be evaluated taking M to be slightly larger than the unperturbed extremal limit, which is denoted as $M \approx M^{(0)}_{\rm ext}$, defined such that the corrections to the temperature at fixed mass and charges are subdominant. In the particular case we study in this article, this could be expressed as follows,

$$\alpha'/p \ll M - \frac{p}{\sqrt{2}} \ll p\,. \tag{7.3}$$

In this regime, the right hand side of (7.2), computed using the perturbative correction to the entropy given in expression (6.11) and the uncorrected temperature in (5.8), yields the right value of $\Delta M_{\rm ext}$ for our solution at the order we are working.

However, it would be convenient to have an expression similar to (7.2) in which the ambiguity in the value of evaluation of M is eliminated. For $M - \frac{p}{\sqrt{2}} \sim \mathcal{O}(\alpha'/p)$, eq. (7.2) cannot be correct because, as we said, it would require the entropy to be divergent. Nevertheless, in our solution the correction to the entropy remains finite in that regime, which we might call the "very near-extremal" regime. In particular, by explicit evaluation we find the following relation for our solution:

$$\Delta M_{\rm ext} = -\frac{1}{2}T(M^{(0)}_{\rm ext},\vec{Q})\Delta S(M^{(0)}_{\rm ext},\vec{Q})\,, \tag{7.4}$$

where now $T(M^{(0)}_{\rm ext},\vec{Q})$ is the actual (corrected) value of the temperature — see (5.11) — for the solution with $M = M^{(0)}_{\rm ext}$, while $\Delta S(M^{(0)}_{\rm ext},\vec{Q})$ is the correction to the entropy of the extremal black hole for fixed mass and charges, which is of order $\alpha'^{1/2}$ as shown in (6.16). We can see, through a very simple argument, that this formula probably holds in general. Near-extremality, the entropy will generically have the following expansion as a function of the mass (keeping the charges constant),

$$S = S_{\rm ext}(\vec{Q}) + k(\vec{Q})\sqrt{M - M_{\rm ext}} + \ldots\,, \tag{7.5}$$

for some function of the charges $k(\vec{Q})$ and where $S_{\rm ext}(\vec{Q})$ is the entropy at extremality (containing the corresponding corrections). The fact that the first term comes with a fractional power of $(M - M_{\rm ext})$ is consequence of the first law of thermodynamics, as

$\partial S/\partial M = T^{-1}$ diverges in the zero temperature limit. Then, taking the derivative of (7.5), using the first law and evaluating at $M = M^{(0)}_{\text{ext}}$ (which is consistent only if $M^{(0)}_{\text{ext}} \geq M_{\text{ext}}$), it is straightforward to get

$$\Delta M_{\text{ext}} = -\frac{1}{2} T(M^{(0)}_{\text{ext}}) \left[S(M^{(0)}_{\text{ext}}, \vec{Q}) - S_{\text{ext}}(\vec{Q}) \right] , \tag{7.6}$$

and then it is easy to note that, to leading order $S(M^{(0)}_{\text{ext}}, \vec{Q}) - S_{\text{ext}}(\vec{Q}) = \Delta S(M^{(0)}_{\text{ext}}, \vec{Q})$, since the leading corrections to the entropy come from the term $\sqrt{M - M_{\text{ext}}}$ and are of order $\alpha'^{1/2}$. On the other hand, the corrections to the extremal entropy generically appear at first order in α' and they play no role in the relation (7.4).

Equation (7.6) clarifies the relation between the perturbations to the entropy at fixed mass and charges and the shift to the charge-to-mass ratio at extremality.[13] If $\Delta M_{\text{ext}} \neq 0$, a positive value of $\Delta S(M^{(0)}_{\text{ext}}, \vec{Q})$ implies a positive correction to the charge-to-mass ratio. However, it is also possible to have $\Delta S(M^{(0)}_{\text{ext}}, \vec{Q}) > 0$ and $\Delta M_{\text{ext}} = 0$, since in that case the relation (7.6) is trivially satisfied because no correction to the extremal mass implies $T(M^{(0)}_{\text{ext}}) = 0$. Hence, we conclude that the fact that the perturbation to the entropy at fixed mass is positive[14] does not imply the mild version of the Weak Gravity Conjecture. This observation clarifies the counterexample found in ref. [8].

One of the most important lessons we extract from the results we presented here is that String Theory requires the activation of many additional fields when higher-derivative corrections are taken into account. Thus, our staring point was a dyonic Reissner-Nordström black hole, which is a solution of Einstein-Maxwell theory. However, when that solution is embedded in the HST, not only we get corrections to the metric and to the Maxwell field, but also new fields acquire a non-trivial profile. In the case at hands, we activate three scalars: the dilaton, an axion and a Kaluza-Klein scalar, and a Kaluza-Klein vector field.

The exploration of constraints on the higher-derivative corrections to simple models such as Einstein-Maxwell or Einstein-Maxwell-dilaton (EMD)[15] theories inspired by *quantum black hole* physics is currently attracting much attention [19, 20, 31–38]. A recurrent assumption in these explorations is that no additional degrees of freedom are activated at higher orders. In the light of the results presented here and in previous literature [8–11], it is reasonable to wonder whether this assumption can have significant consequences. The activation of additional fields due to higher-derivative terms seems to be quite a generic feature of String Theory, and truncating the new fields might be inconsistent in this context.

[13] A subsequently modified version of ref. [33] has reproduced our equation (7.6).

[14] The positivity of this variable is to be expected when the perturbation is due to the inclusion of higher-derivative operators motivated by the UV-completion of the effective theory — see [31].

[15] This model arises from the effective theory of the Heterotic Superstring compactified on T^6 ($\mathcal{N} = 4$, $d = 4$ supergravity) after several truncations are made [28]. The consistency of those truncations is ensured by the fact that they are performed in the equations of motion. The action of the EMD model leads to the truncated equations of motion. The black-hole solutions of this model were first found in ref. [28] and rederived later on in refs. [29, 30]. Further truncation to the Einstein-Maxwell model can be achieved by constraining the form of the Maxwell field, which has to be dyonic with equal electric and magnetic charges (or one has to introduce several Abelian vector fields with electric and magnetic charges). Thus, not all the solutions of the Einstein-Maxwell theory can be embedded in the Heterotic Superstring effective action because unequal electric and magnetic charges always generate a non-trivial scalar field.

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In our current analysis, the additional fields acquire a non-trivial profile of order $\mathcal{O}(\alpha')$, which implies that they will backreact on the geometry at order $\mathcal{O}(\alpha'^2)$. Thus, the additional degrees of freedom do not play a role in the corrections to the entropy or to the extremality bound at leading order in α', but they sure will do so at $\mathcal{O}(\alpha'^2)$ and higher orders. Thus, the presence of new degrees of freedom cannot be ignored in order to analyze, for instance, the positivity of the corrections to the entropy beyond first order in the perturbative expansion. In fact, it would be interesting to obtain the $\mathcal{O}(\alpha'^2)$ corrections to the solution we have studied, or to the ones presented in [8]. This is perhaps a less challenging task than it would appear, since no α'^2 terms occur explicitly in the HST effective action (they only appear implicitly through the iterative definition of the 3-form field strength $\hat{H}$). Work in this direction is alredy in progress [39].

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A The Heterotic Superstring effective action to $\mathcal{O}(\alpha')$

In order to describe the Heterotic Superstring effective action to $\mathcal{O}(\alpha')$ as given in ref. [40] (but in the string frame), we start by defining the zeroth-order 3-form field strength of the Kalb-Ramond 2-form B:

$$H^{(0)} \equiv dB\,, \tag{A.1}$$

and constructing with it the zeroth-order torsionful spin connections

$$\Omega^{(0)}_{(\pm)}{}^{a}{}_{b} = \omega^{a}{}_{b} \pm \frac{1}{2} H^{(0)}_{\mu}{}^{a}{}_{b} dx^{\mu}\,, \tag{A.2}$$

where $\omega^{a}{}_{b}$ is the Levi-Civita spin connection 1-form.[16] With them we define the zeroth-order Lorentz curvature 2-form and Chern-Simons 3-forms

$$R^{(0)}_{(\pm)}{}^{a}{}_{b} = d\Omega^{(0)}_{(\pm)}{}^{a}{}_{b} - \Omega^{(0)}_{(\pm)}{}^{a}{}_{c} \wedge \Omega^{(0)}_{(\pm)}{}^{c}{}_{b}\,, \tag{A.3}$$

$$\omega^{\mathrm{L}\,(0)}_{(\pm)} = d\Omega^{(0)}_{(\pm)}{}^{a}{}_{b} \wedge \Omega^{(0)}_{(\pm)}{}^{b}{}_{a} - \frac{2}{3} \Omega^{(0)}_{(\pm)}{}^{a}{}_{b} \wedge \Omega^{(0)}_{(\pm)}{}^{b}{}_{c} \wedge \Omega^{(0)}_{(\pm)}{}^{c}{}_{a}\,. \tag{A.4}$$

Next, we introduce the gauge fields. We will only activate a $\mathrm{SU}(2) \times \mathrm{SU}(2)$ subgroup of the full gauge group of the Heterotic Theory and we will denote by $A^{A_{1,2}}$ $(A_{1,2} = 1, 2, 3)$

[16] We follow the conventions of ref. [41] for the spin connection and the curvature.

the components. The gauge field strength and the Chern-Simons 3-form of each SU(2) factor are defined by

$$F^A = dA^A + \frac{1}{2}\epsilon^{ABC} A^B \wedge A^C \,, \tag{A.5}$$

$$\omega^{\rm YM} = dA^A \wedge A^A + \frac{1}{3}\epsilon^{ABC} A^A \wedge A^B \wedge A^C \,. \tag{A.6}$$

Then, we are ready to define recursively

$$\begin{aligned}
H^{(1)} &= dB + \frac{\alpha'}{4}\left(\omega^{\rm YM} + \omega^{\rm L\,(0)}_{(-)}\right) , \\
\Omega^{(1)}_{(\pm)}{}^{a}{}_{b} &= \omega^{a}{}_{b} \pm \frac{1}{2} H^{(1)}_{\mu}{}^{a}{}_{b} dx^{\mu} , \\
R^{(1)}_{(\pm)}{}^{a}{}_{b} &= d\Omega^{(1)}_{(\pm)}{}^{a}{}_{b} - \Omega^{(1)}_{(\pm)}{}^{a}{}_{c} \wedge \Omega^{(1)}_{(\pm)}{}^{c}{}_{b} , \\
\omega^{\rm L\,(1)}_{(\pm)} &= d\Omega^{(1)}_{(\pm)}{}^{a}{}_{b} \wedge \Omega^{(1)}_{(\pm)}{}^{b}{}_{a} - \frac{2}{3}\Omega^{(1)}_{(\pm)}{}^{a}{}_{b} \wedge \Omega^{(1)}_{(\pm)}{}^{b}{}_{c} \wedge \Omega^{(1)}_{(\pm)}{}^{c}{}_{a} . \\
H^{(2)} &= dB + \frac{\alpha'}{4}\left(\omega^{\rm YM} + \omega^{\rm L\,(1)}_{(-)}\right) ,
\end{aligned} \tag{A.7}$$

and so on.

In practice only $\Omega^{(0)}_{(\pm)}, R^{(0)}_{(\pm)}, \omega^{\rm L\,(0)}_{(\pm)}, H^{(1)}$ will occur to the order we want to work at, but, often, it is more convenient to work with the higher-order objects ignoring the terms of higher order in α' when necessary. Thus we will suppress the (n) upper indices from now on.

Finally, we define three "T-tensors" associated to the α' corrections

$$\begin{aligned}
T^{(4)} &\equiv \frac{3\alpha'}{4}\left[F^A \wedge F^A + R_{(-)}{}^{a}{}_{b} \wedge R_{(-)}{}^{b}{}_{a}\right] , \\
T^{(2)}{}_{\mu\nu} &\equiv \frac{\alpha'}{4}\left[F^A{}_{\mu\rho} F^A{}_{\nu}{}^{\rho} + R_{(-)\,\mu\rho}{}^{a}{}_{b} R_{(-)\,\nu}{}^{\rho\, b}{}_{a}\right] , \\
T^{(0)} &\equiv T^{(2)\,\mu}{}_{\mu} .
\end{aligned} \tag{A.8}$$

In terms of all these objects, the Heterotic Superstring effective action in the string frame and to first-order in α' can be written as

$$S = \frac{g_s^2}{16\pi G_N^{(10)}} \int d^{10}x \sqrt{|g|}\, e^{-2\phi}\left\{R - 4(\partial\phi)^2 + \frac{1}{2\cdot 3!}H^2 - \frac{1}{2}T^{(0)}\right\} , \tag{A.9}$$

where $G_N^{(10)}$ is the 10-dimensional Newton constant, ϕ is the dilaton field and the vacuum expectation value of e^{ϕ} is the Heterotic Superstring coupling constant g_s. R is the Ricci scalar of the string-frame metric $g_{\mu\nu}$.

The derivation of the complete equations of motion is quite a complicated challenge. Following ref. [42], we separate the variations with respect to each field into those corresponding to occurrences via $\Omega_{(-)}{}^{a}{}_{b}$, that we will call *implicit*, and the rest, that we will call *explicit*:

$$
\begin{aligned}
\delta S &= \frac{\delta S}{\delta g_{\mu\nu}}\delta g_{\mu\nu} + \frac{\delta S}{\delta B_{\mu\nu}}\delta B_{\mu\nu} + \frac{\delta S}{\delta A^{A_i}{}_{\mu}}\delta A^{A_i}{}_{\mu} + \frac{\delta S}{\delta\phi}\delta\phi \\
&= \left.\frac{\delta S}{\delta g_{\mu\nu}}\right|_{\text{exp.}}\delta g_{\mu\nu} + \left.\frac{\delta S}{\delta B_{\mu\nu}}\right|_{\text{exp.}}\delta B_{\mu\nu} + \left.\frac{\delta S}{\delta A^{A_i}{}_{\mu}}\right|_{\text{exp.}}\delta A^{A_i}{}_{\mu} + \frac{\delta S}{\delta\phi}\delta\phi \\
&\quad + \frac{\delta S}{\delta\Omega_{(-)}{}^{a}{}_{b}}\left(\frac{\delta\Omega_{(-)}{}^{a}{}_{b}}{\delta g_{\mu\nu}}\delta g_{\mu\nu} + \frac{\delta\Omega_{(-)}{}^{a}{}_{b}}{\delta B_{\mu\nu}}\delta B_{\mu\nu} + \frac{\delta\Omega_{(-)}{}^{a}{}_{b}}{\delta A^{A_i}{}_{\mu}}\delta A^{A_i}{}_{\mu}\right). \qquad \text{(A.10)}
\end{aligned}
$$

We can then apply a lemma proven in ref. [40]: $\delta S/\delta\Omega_{(-)}{}^{a}{}_{b}$ is proportional to α' and to the zeroth-order equations of motion of $g_{\mu\nu}, B_{\mu\nu}$ and ϕ plus terms of higher order in α'.

The upshot is that, if we consider field configurations which solve the zeroth-order equations of motion[17] up to terms of order α', the contributions to the equations of motion associated to the implicit variations are at least of second order in α' and we can safely ignore them here.

If we restrict ourselves to this kind of field configurations, the equations of motion reduce to

$$R_{\mu\nu} - 2\nabla_{\mu}\partial_{\nu}\phi + \frac{1}{4}H_{\mu\rho\sigma}H_{\nu}{}^{\rho\sigma} - T^{(2)}{}_{\mu\nu} = 0\,, \qquad \text{(A.11)}$$

$$(\partial\phi)^2 - \frac{1}{2}\nabla^2\phi - \frac{1}{4\cdot 3!}H^2 + \frac{1}{8}T^{(0)} = 0\,, \qquad \text{(A.12)}$$

$$d\left(e^{-2\phi}\star H\right) = 0\,, \qquad \text{(A.13)}$$

$$\alpha' e^{2\phi}\mathfrak{D}_{(+)}\left(e^{-2\phi}\star F^{A_i}\right) = 0\,, \qquad \text{(A.14)}$$

where $\mathfrak{D}_{(+)}$ stands for the exterior derivative covariant with respect to each SU(2) subgroup and with respect to the torsionful connection $\Omega_{(+)}$: suppressing the subindices $1, 2$ that distinguish the two subgroups, it takes the explicit form

$$e^{2\phi}d\left(e^{-2\phi}\star F^{A}\right) + \epsilon^{ABC}A^{B}\wedge\star F^{C} + \star H\wedge F^{A} = 0\,. \qquad \text{(A.15)}$$

If the ansatz is given in terms of the 3-form field strength, we also need to solve the Bianchi identity

$$dH - \frac{1}{3}T^{(4)} = 0\,, \qquad \text{(A.16)}$$

as well.

B Solution of the equations for the corrections

In this appendix we are going to show how we have solved the equations of motion of the Heterotic Superstring effective field theory to first order in α' using the ansatz eqs. (3.1) and (3.2), which describes corrections to the zeroth-order solution in eqs. (2.1) and (2.2) codified in the functions δ_X with $X = A, B, C\cdots$.

[17] These can be obtained from eqs. (A.11)–(A.14) by setting $\alpha' = 0$. This eliminates the Yang-Mills fields, the T-tensors and the Chern-Simons terms in H.

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Because of this formulation of our anstaz, we can apply the lemma of ref. [40] and, therefore, we only need to solve eqs. (A.11)–(A.13), since we are not going to introduce 10-dimensional Yang-Mills fields. In all computations we will ignore all terms of second order in α' and higher. We will denote by k_i the integration constants and the Einstein equations will be denoted by $\mathcal{E}_{ab}$. The components of the Kalb-Ramond 3-form field strength, the spin connection, the torsionful spin connection and their curvatures, which are necessary to write the equations for our ansatz, can be found in appendix C.

It is convenient to start by studying the equation of motion of the Kalb-Ramond field eq. (A.13). Substitution of the ansatz gives

$$e^{-2\phi} = \frac{k_1}{Dr^2}\,, \tag{B.1}$$

and, expanding the function D and comparing with the expansion of e^{ϕ}, both in eqs. (3.2), we find that

$$k_1 = e^{-2\phi_\infty}/p\,, \qquad 2\delta_\phi = \delta_D r^2/p\,. \tag{B.2}$$

Next, we consider the Bianchi identity of the Kalb-Ramond 3-form field strength eq. (A.16). Substituting the ansatz, we obtain a relation between δ_E and δ_C and a relation between δ_F and δ_D

$$\delta_E = -\frac{p}{r^2}\delta_C + \frac{p}{2}\frac{1-a^2}{r^4} - \frac{p^3/8}{r^6} + \frac{k_2}{r^2}\,, \tag{B.3}$$

$$\delta_F = -\frac{pa^2}{2r^3} - \frac{r^2 a}{p}\delta_G + k_3\,. \tag{B.4}$$

The integration constant k_2 corrects the value of the electric and magnetic charges. Therefore, we will simply set $k_2 = 0$. As a general rule, we will adjust the integration constants so that there are no α' corrections of the fields at infinity. Thus, we also set $k_3 = 0$.

The dilaton equation (A.12) gives the following relation between δ_D and δ_E:

$$\left[r^2 a^2 (r^2\delta_D)'\right]' = 2p^2(\delta_D - \delta_E) + \frac{p}{16r^6}\left(25p^4 - 96Mp^2 r + 96M^2 r^2\right)\,, \tag{B.5}$$

and from the Einstein equations we get the following relations:

- $\mathcal{E}_{04}$

$$\delta_G = \frac{a}{8}\left[\frac{4}{r}(a^2)' - \frac{p^2}{r^4} - \frac{r^2}{p}\delta_F'\right]'\,, \tag{B.6}$$

- $\mathcal{E}_{44}$

$$\left(r^2 a^2 \delta_C'\right)' = p(\delta_D - \delta_E) + \frac{p^4}{8r^6}\,, \tag{B.7}$$

- $\mathcal{E}_{00} + \mathcal{E}_{11}$

$$\left(\frac{\delta_A}{a} + a\delta_B\right)' = \frac{r}{2p}(p\delta_C - r^2\delta_D)'' - \frac{p^2}{4r^5}\,, \tag{B.8}$$

- $\mathcal{E}_{22}$ and $\mathcal{E}_{33}$

$$\frac{(r^2a^5\delta_B)'}{a^2} = r^2a^2\left(\frac{\delta_A}{a} - r^2\frac{\delta_D}{p} + \delta_C\right)' + pr\delta_E - \frac{1}{4r^5}\left[(2p^2+12M^2)r^2 - 14Mp^2 r + 3p^4\right]\,, \tag{B.9}$$

- $\mathcal{E}_{00}$

$$\frac{1}{ar^2}\left[r^2a^3\left(\frac{\delta_A}{a}\right)'\right]' = \frac{(a^2)'}{2p}(r^2\delta_D - p\delta_C)' + \frac{p}{r^2}\delta_D + \frac{1}{2(a^2)'r^4}\left\{\left[r^2(a^2)'\right]^2 a\delta_B\right\}' + \frac{1}{8r^8}\left[6(4M^2-p^2)r^2 - 16Mp^2r + 5p^4\right] . \tag{B.10}$$

These equations can be easily decoupled. Substituting eq. (B.4) in eq. (B.6) gives a second order equation for δ_G. Using the standard definition of r_+ and r_- eq. (2.4) with $0 < r_- < r_+ < r$, imposing reality and regularity on r_+ (it is not possible to have regularity both on r_+ and r_-) and the above condition on the vanishing of the corrections to the fields at infinity, we find

$$\delta_G = \frac{a}{24r^5r_-^2r_+^3(r-r_-)}\Big\{r_-\left[-6r^3\left(r_-^4+3r_-^2r_+^2+2r_-r_+^3+2r_+^4\right)+6r^2r_-r_+^2\left(3r_-^2+r_+^2\right) + 4rr_-^2r_+^3(3r_--7r_+)+40r_-^3r_+^4\right] - 12r^3r_+(r-r_-)(r_-+r_+)\left(r_-^2+r_+^2\right)\log\left(1-\frac{r_-}{r}\right)\Big\} + \frac{k_{(4)}}{r^2} . \tag{B.11}$$

Then, using this result in eq. (B.4) with $k_3 = 0$ we get δ_F and we see that we must also set $k_{(4)} = 0$.

Combining eq. (B.5) and eq. (B.7) gives

$$\left[r^2a^2(r^2\delta_D - 2p\delta_C)'\right]' = \frac{p}{16r^6}\left(21p^4 - 96Mp^2r + 96M^2r^2\right) , \tag{B.12}$$

which can be integrated, giving

$$r^2\delta_D - 2p\delta_C = +\frac{1}{40\sqrt{2}r^4(r_-r_+)^{5/2}}\Big\{4r^4\left(r_-^4 - 9r_-^3r_+ + r_-^2r_+^2 - 9r_-r_+^3 + r_+^4\right)\log\left(1-\frac{r_-}{r}\right) + \frac{r_-r_+}{4}\left[2r^3(r_-+r_+)\left(r_-^2 - 10r_-r_+ + r_+^2\right) + r^2r_-r_+\left(r_-^2 - 19r_-r_+ + r_+^2\right) - 6rr_-^2r_+^2(r_-+r_+) + 21r_-^3r_+^3\right]\Big\} . \tag{B.13}$$

Using this relation to express δ_D in terms of δ_C in eq. (B.7), and using eq. (B.3) to express δ_E in terms of δ_C in eq. (B.7), we obtain a second order equation for δ_C solved by

$$\delta_C = \frac{1}{140r^2r_-^3r_+^3}\Big[r^2\left(9r_-^4+74r_-^3r_++51r_-^2r_+^2+74r_-r_+^3+9r_+^4\right) +2r_-r_+\left(34r_-^2+23r_-r_++34r_+^2\right)(r_-r_+-r(r_-+r_+))\Big]\log\left(1-\frac{r_-}{r}\right) + \frac{1}{560r^4}\left\{\frac{580r^3r_-}{r_+^2} + \frac{12r^2\left(595rr_-+143rr_++226r_-^2+83r_-r_+\right)}{r_-^2+4r_-r_++r_+^2} - \frac{2r^2(756r+535r_-)}{r_+} + 4r\left(\frac{74r^2}{r_-}-77r+37r_-\right)+r_+\left(\frac{36r^3}{r_-^2}-\frac{254r^2}{r_-}+148r+101r_-\right)\right\} . \tag{B.14}$$

Using this result in eq. (B.3) with $k_2 = 0$ we get δ_E and using it in eq. (B.13) we get δ_D. The latter gives us δ_ϕ via eq. (B.2).

Only δ_A and δ_B remain to be determined. We could integrate eq. (B.8) to get δ_A in terms of δ_B, and, substituting everything in eq. (B.9), we could obtain a first order equation for δ_B which could also be immediately integrated. However, given that the dilaton and Kaluza-Klein scalars become non-trivial when the α' corrections are taken into account and given that the Einstein metric includes certain powers of them, it is more convenient to use variables different from A and B to describe the metric. As a matter of fact, some of the equations take a much simpler form in terms of those variables.

We define two new variables N and f and a new radial coordinate ρ from the 4-dimensional Einstein metric, given by

$$ds^2_{(4)} = Ce^{-2(\phi-\phi_\infty)}\left[A^2dt^2 - B^2dr^2 - r^2d\Omega^2_{(2)}\right] \equiv N^2 f dt^2 - \frac{d\rho^2}{f} - \rho^2 d\Omega^2_{(2)}\,, \tag{B.15}$$

and we define the α' corrections to N and f by

$$N^2 = 1 + \alpha'\delta_N\,, \qquad f = a^2(r) + \alpha'\tilde{\delta}_f = a^2(\rho) + \alpha'\delta_f\,. \tag{B.16}$$

The corrections δ_N, δ_f and $\tilde{\delta}_f$ are related to the other corrections defined before by

$$\delta_N = \left(\delta_C - r^2\frac{\delta_D}{p}\right) - r\left(\delta_C - r^2\frac{\delta_D}{p}\right)' + 2\left(\frac{\delta_A}{a} + a\delta_B\right)\,, \tag{B.17}$$

$$\tilde{\delta}_f = a^2\left[r\left(\delta_C - r^2\frac{\delta_D}{p}\right)' - 2a\delta_B\right]\,. \tag{B.18}$$

$$\delta_f = \tilde{\delta}_f - \frac{r}{2}(a^2)'\left(\delta_C - r^2\frac{\delta_D}{p}\right)\,. \tag{B.19}$$

As we have advanced, some of the above equations simplify when expressed in terms of δ_N and δ_f, namely:

- $\mathcal{E}_{00} + \mathcal{E}_{11}$

$$\delta_N' + \frac{p^2}{2r^5} = 0\,, \tag{B.20}$$

- $\mathcal{E}_{00} + \mathcal{E}_{22}$

$$\delta_f'' - \frac{2}{r^2}\delta_f = -\frac{p}{4r^8}\left(11p^3 - 42Mpr + 18pr^2\right)\,, \tag{B.21}$$

  where we substituted the expression (B.3) for δ_E with $k_2 = 0$.

These equations can be easily integrated to give

$$\delta_N = \frac{p^2/8}{r^4}\,, \tag{B.22}$$

$$\delta_f = -\frac{p^2/4}{r^4}\left(1 - \frac{3M/2}{r} + \frac{11p^2/40}{r^2}\right) + r^2k_5 + \frac{k_6}{r}\,. \tag{B.23}$$

We can set $k_6 = 0$ because that integration constant simply renormalizes the mass. As for k_5, substituting the expressions we have found for the δs in eq. (B.9) (or equivalently (B.10)), one finds that $k_5 = 0$.

Observe that, since the new radial coordinate $\rho = r + \alpha'\delta_\rho$, r can be replaced by ρ in all the α'-correction functions δ_X except for δ_E and δ_D.

C Connections and curvatures

Our ansatz for the metric is

$$ds^2 = A^2 dt^2 - B^2 dr^2 - r^2[d\theta^2 + \sin^2\theta d\phi^2] - C^2[dz + F dt]^2 - -d\vec{y}_5^{\,2}\,, \tag{C.1}$$

where A, B, C, F are functions of the coordinate r. The expansions of these functions in powers of α' are assumed to be of the form

$$A \sim a + \alpha'\delta_A\,, \quad B \sim a^{-1} + \alpha'\delta_B\,, \quad C \sim 1 + \alpha'\delta_C\,, \quad F \sim \alpha'\delta_F\,, \tag{C.2}$$

and, since we are only interested in keeping terms of zeroth and first orders in α', at some point we will discard terms such as $C'F, F^2$ etc.

In the obvious Vielbein basis

$$e^0 = A dt\,, \quad e^1 = B dr\,, \quad e^2 = r d\theta\,, \quad e^3 = r\sin\theta d\phi\,, \quad e^4 = C[dz + F dt]\,, \quad e^i = dy^i\,, \tag{C.3}$$

the only non-vanishing components of the spin connection ($de^a = \omega^a{}_b \wedge e^b$) are

$$\begin{aligned}
\omega^0{}_1 &= -\frac{A'}{AB}e^0 + \frac{CF'}{2AB}e^4\,, & \omega^0{}_4 &= \frac{CF'}{2AB}e^1\,, & \omega^1{}_2 &= \frac{1}{Br}e^2\,,\\
\omega^1{}_3 &= \frac{1}{Br}e^3\,, & \omega^1{}_4 &= \frac{CF'}{2AB}e^0 + \frac{C'}{BC}e^4\,, & \omega^2{}_3 &= \frac{\cot\theta}{r}e^3\,,
\end{aligned} \tag{C.4}$$

or

$$\begin{aligned}
\omega^0{}_1 &= \left(-\frac{A'}{B} + \frac{C^2FF'}{2AB}\right)dt + \frac{C^2F'}{2AB}dz\,, & \omega^0{}_4 &= \frac{CF'}{2A}dr\,,\\
\omega^1{}_2 &= \frac{1}{B}d\theta\,, & \omega^1{}_3 &= \frac{\sin\theta}{B}d\phi\,,\\
\omega^1{}_4 &= \left(\frac{CF'}{2B} + \frac{C'F}{B}\right)dt + \frac{C'}{B}dz\,, & \omega^2{}_3 &= \cos\theta d\phi\,.
\end{aligned} \tag{C.5}$$

Taking into account the above expansions in α' and keeping only terms of up to first order in α', we can already simplify some terms:

$$FF' \sim C'F \sim C'F' \sim 0 + \mathcal{O}(\alpha'^2)\,, \qquad CF' \sim (CF)' \sim F' + \mathcal{O}(\alpha'^2)\,, \tag{C.6}$$

and, to this order, we can replace the above components of the spin connection 1-form by

$$\begin{aligned}
\omega^0{}_1 &= -\frac{A'}{B}dt + \frac{F'}{2AB}dz\,, & \omega^0{}_4 &= \frac{F'}{2A}dr\,,\\
\omega^1{}_2 &= \frac{1}{B}d\theta\,, & \omega^1{}_3 &= \frac{\sin\theta}{B}d\phi\,,\\
\omega^1{}_4 &= \frac{F'}{2B}dt + \frac{C'}{B}dz\,, & \omega^2{}_3 &= \cos\theta d\phi\,.
\end{aligned} \tag{C.7}$$

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Using these components, the non-vanishing components of the curvature 2-form ($R^a{}_a = d\omega^a{}_b - \omega^a{}_c \wedge \omega^c{}_b$) can be readily calculated:

$$
\begin{aligned}
R_{01} &= \left(\frac{A'}{B}\right)' dt \wedge dr + \left(\frac{F'}{2AB}\right)' dr \wedge dz \\
&= \left(\frac{A'}{B}\right)' \frac{1}{AB}\, e^0 \wedge e^1 + \left(\frac{F'}{2AB}\right)' \frac{1}{B}\, e^1 \wedge e^4 \,, \\
R_{02} &= \frac{A'}{B^2} dt \wedge d\theta + \frac{F'}{2AB^2} d\theta \wedge dz \\
&= \frac{A'}{AB^2 r}\, e^0 \wedge e^2 + \frac{F'}{2AB^2 r}\, e^2 \wedge e^4 \,, \\
R_{03} &= \frac{A' \sin\theta}{B^2} dt \wedge d\phi + \frac{F' \sin\theta}{2AB^2} d\phi \wedge dz \\
&= \frac{A'}{AB^2 r}\, e^0 \wedge e^3 + \frac{F'}{2AB^2 r}\, e^3 \wedge e^4 \,, \\
R_{04} &= \frac{A'C'}{B^2} dt \wedge dz = \frac{A'C'}{AB^2} e^0 \wedge e^4 \,, \\
R_{12} &= \frac{B'}{B^2} dr \wedge d\theta = \frac{B'}{B^3 r}\, e^1 \wedge e^2 \\
R_{13} &= \frac{B' \sin\theta}{B^2} dr \wedge d\phi = \frac{B'}{B^3 r}\, e^1 \wedge e^3 \,, \\
R_{14} &= -\left(\frac{C'}{B}\right)' dr \wedge dz + \left(\frac{F'}{2AB}\right)' A\, dt \wedge dr \\
&= -\left(\frac{C'}{B}\right)' \frac{1}{B}\, e^1 \wedge e^4 + \left(\frac{F'}{2AB}\right)' \frac{1}{B}\, e^0 \wedge e^1 \,, \\
R_{23} &= \frac{B^2 - 1}{B^2} \sin\theta d\theta \wedge d\phi = \frac{B^2-1}{B^2 r^2}\, e^2 \wedge e^3 \,, \\
R_{24} &= \frac{F'}{2B^2} dt \wedge d\theta - \frac{C'}{B^2} d\theta \wedge dz = \frac{F'}{2AB^2 r}\, e^0 \wedge e^2 - \frac{C'}{B^2 r}\, e^2 \wedge e^4 \,, \\
R_{34} &= \frac{F' \sin\theta}{2B^2} dt \wedge d\phi - \frac{C' \sin\theta}{B^2} d\phi \wedge dz = \frac{F'}{2AB^2 r}\, e^0 \wedge e^3 - \frac{C'}{B^2 r}\, e^3 \wedge e^4 \,.
\end{aligned}
\tag{C.8}
$$

The (flat) non-vanishing components of the Ricci tensor are

$$
\begin{aligned}
R_{00} &= -\frac{1}{ABCr^2} \left(\frac{A'Cr^2}{B}\right)' \,, \\
R_{04} &= \frac{1}{2Br^2} \left(\frac{F'r^2}{AB}\right)' \,, \\
R_{11} &= \frac{1}{AB} \left(\frac{A'}{B}\right)' + \frac{2}{Br} \left(\frac{1}{B}\right)' + \frac{1}{B} \left(\frac{C'}{B}\right)' \,, \\
R_{22} = R_{33} &= \frac{1}{ABCr^2} \left(\frac{ACr}{B}\right)' - \frac{1}{r^2} \,, \\
R_{44} &= \frac{1}{ABr^2} \left(\frac{AC'r^2}{B}\right)' \,,
\end{aligned}
\tag{C.9}
$$

and their expansion in α' takes the form

$$
\begin{aligned}
R_{00} &= -\frac{1}{2r^2}\left[(a^2)'r^2\right]' + \frac{\alpha'}{2r^2}\left\{\left[(a^2)'r^2\right]'\left(\frac{\delta_A}{a}+a\delta_B+\delta_C\right)-2\left[ar^2(\delta_A'+a'\delta_C-aa'\delta_B)\right]'\right\},\\
R_{04} &= \frac{\alpha' a}{2r^2}\left(\delta_F' r^2\right)',\\
R_{11} &= \frac{1}{2r^2}\left[(a^2)'r^2\right]' - \alpha'\left\{\frac{1}{2}(a^2)''\left(\frac{\delta_A}{a}+a\delta_B\right)-\left[a(\delta_A'-a'a\delta_B)\right]'\right.\\
&\quad\left. + \frac{2a^2}{r}(3a'\delta_A+a\delta_B')-a\left(a\delta_C'\right)'\right\},\\
R_{22} &= R_{33} = \frac{1}{r^2}\left(a^2 r\right)' - \frac{1}{r^2} + \alpha'\left\{-\frac{2a}{r^2}\left(a^2r\right)'\delta_B + \frac{a^2}{r}\left(\frac{\delta_A}{a}-a\delta_B+\delta_C\right)'\right\},\\
R_{44} &= \frac{\alpha'}{r^2}\left(a^2r^2\delta_C'\right)'. \qquad\qquad (C.10)
\end{aligned}
$$

It is trivial to see that $a^2 = 1 + k/r$ (the Schwarzschild solution) satisfies the Einstein equations in vacuum $R_{ab} = 0$ at zeroth order in α'.

Our ansatz for the Kalb-Ramond 3-form field strength H is

$$
H = De^0 \wedge e^1 \wedge e^4 + Ee^2 \wedge e^3 \wedge e^4 + Ge^0 \wedge e^2 \wedge e^3 , \qquad (C.11)
$$

so

$$
H^2 = 6(D^2 + G^2 - E^2) . \qquad (C.12)
$$

The expansions of D, E, G in powers of α' are assumed to be of the form

$$
D \sim e + \alpha'\delta_D , \quad E \sim e + \alpha'\delta_E , \quad G \sim \alpha'\delta_G , \qquad (C.13)
$$

and, therefore,

$$
H^2 = 12e\alpha'(\delta_D - \delta_E) . \qquad (C.14)
$$

We only need to compute the $\Omega_{(-)\,ab} = \omega_{ab} - \frac{1}{2}H_{cab}e^c$ connection to zeroth order in α'. The non-vanishing components are

$$
\begin{aligned}
&\Omega_{(-)\,01} = -a'e^0 - \frac{1}{2}e\,e^4 , \quad && \Omega_{(-)\,04} = \frac{1}{2}e\,e^1 , \quad && \Omega_{(-)\,12} = -\frac{a}{r}\,e^2 ,\\
&\Omega_{(-)\,13} = -\frac{a}{r}\,e^3 , && \Omega_{(-)\,14} = -\frac{1}{2}e\,e^0 , && \Omega_{(-)\,23} = -\frac{\cot\theta}{r}\,e^3 - \frac{1}{2}e\,e^4 , \qquad (C.15)\\
&\Omega_{(-)\,24} = \frac{1}{2}e\,e^3 , && \Omega_{(-)\,34} = -\frac{1}{2}e\,e^2 .
\end{aligned}
$$

and the non-vanishing components of its curvature 2-form are

$$
\begin{aligned}
R_{(-)}{}^{0}{}_{1} &\sim \left[\frac{1}{2}(a^2)'' - \frac{p^2}{4r^4}\right] e^0\wedge e^1 + \frac{pa}{r^3} e^1\wedge e^4\,,\\
R_{(-)}{}^{0}{}_{2} &\sim -\frac{p^2}{4r^4} e^1\wedge e^3 - \frac{pa}{2r^3} e^2\wedge e^4 + \frac{(a^2)'}{2r} e^0\wedge e^2\,,\\
R_{(-)}{}^{0}{}_{3} &\sim \frac{p^2}{4r^4} e^1\wedge e^2 - \frac{pa}{2r^3} e^3\wedge e^4 + \frac{(a^2)'}{2r} e^0\wedge e^3\,,\\
R_{(-)}{}^{0}{}_{4} &\sim -\frac{p^2}{4r^4} e^0\wedge e^4\,,\\
R_{(-)}{}^{1}{}_{2} &\sim -\frac{p^2}{4r^4} e^0\wedge e^3 + \frac{pa}{2r^3} e^3\wedge e^4 + \frac{(a^2)'}{2r} e^1\wedge e^2\,,\\
R_{(-)}{}^{1}{}_{3} &\sim \frac{p^2}{4r^4} e^0\wedge e^2 - \frac{pa}{2r^3} e^2\wedge e^4 + \frac{(a^2)'}{2r} e^1\wedge e^3\,,\\
R_{(-)}{}^{1}{}_{4} &\sim -\frac{p^2}{4r^4} e^1\wedge e^4 + \frac{pa}{r^3} e^0\wedge e^1 + \frac{pa}{r^3} e^2\wedge e^3\,,\\
R_{(-)}{}^{2}{}_{3} &\sim \left(\frac{a^2}{r^2} - \frac{1}{r^2} + \frac{p^2}{4r^4}\right) e^2\wedge e^3 - \frac{pa}{r^3} e^1\wedge e^4\,,\\
R_{(-)}{}^{2}{}_{4} &\sim \frac{p^2}{4r^4} e^2\wedge e^4 - \frac{pa}{2r^3} e^0\wedge e^2 + \frac{pa}{2r^3} e^1\wedge e^3\,,\\
R_{(-)}{}^{3}{}_{4} &\sim \frac{p^2}{4r^4} e^3\wedge e^4 - \frac{pa}{2r^3} e^0\wedge e^3 - \frac{pa}{2r^3} e^1\wedge e^2\,,
\end{aligned}
\tag{C.16}
$$

or

$$
\begin{aligned}
&R_{(-)\,0101} = \frac{1}{2}(a^2)'' - \frac{p^2}{4r^4}\,, && R_{(-)\,0114} = -\frac{pa}{r^3}\,, && R_{(-)\,0202} = \frac{(a^2)'}{2r}\,,\\
&R_{(-)\,0213} = -\frac{p^2}{4r^4}\,, && R_{(-)\,0224} = \frac{pa}{2r^3}\,, && R_{(-)\,0303} = \frac{(a^2)'}{2r}\,,\\
&R_{(-)\,0312} = \frac{p^2}{4r^4}\,, && R_{(-)\,0334} = \frac{pa}{2r^3}\,, && R_{(-)\,0404} = -\frac{p^2}{4r^4}\,,\\
&R_{(-)\,1203} = \frac{p^2}{4r^4}\,, && R_{(-)\,1212} = -\frac{(a^2)'}{2r}\,, && R_{(-)\,1234} = \frac{pa}{2r^3}\,,\\
&R_{(-)\,1302} = -\frac{p^2}{4r^4}\,, && R_{(-)\,1313} = -\frac{(a^2)'}{2r}\,, && R_{(-)\,1324} = -\frac{pa}{2r^3}\,,\\
&R_{(-)\,1401} = \frac{pa}{r^3}\,, && R_{(-)\,1414} = \frac{p^2}{4r^4}\,, && R_{(-)\,1423} = \frac{pa}{r^3}\,,\\
&R_{(-)\,2314} = -\frac{pa}{r^3}\,, && R_{(-)\,2323} = \frac{1-a^2}{r^2} - \frac{p^2}{4r^4}\,, && R_{(-)\,2402} = -\frac{pa}{2r^3}\,,\\
&R_{(-)\,2413} = \frac{pa}{2r^3}\,, && R_{(-)\,2424} = -\frac{p^2}{4r^4}\,, && R_{(-)\,3403} = -\frac{pa}{2r^3}\,,\\
&R_{(-)\,3412} = -\frac{pa}{2r^3}\,, && R_{(-)\,3434} = -\frac{p^2}{4r^4}\,.
\end{aligned}
\tag{C.17}
$$

Then, for a^2 as given in eq. (2.2)

$$R_{(-)abcd}R_{(-)}{}^{abcd} \sim 4\left[\left(-\frac{p^2}{4r^4}+\frac{1}{2}(A^2)''\right)^2+\left(\frac{(A^2)'}{r}\right)^2+\left(\frac{p^2}{4r^4}-\frac{1}{r^2}+\frac{A^2}{r^2}\right)^2\right]$$
$$= \frac{25p^4/2}{r^8}-\frac{48p^2M}{r^7}+\frac{48M^2}{r^6}\,. \tag{C.18}$$

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