

Poisson-Lie U-duality in Exceptional Field Theory

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Abstract

Poisson-Lie duality provides an algebraic extension of conventional Abelian and non-Abelian target space dualities of string theory and has seen recent applications in constructing quantum group deformations of holography. Here we demonstrate a natural upgrading of Poisson-Lie to the context of M-theory using the tools of exceptional field theory. In particular, we propose how the underlying idea of a Drinfeld double can be generalised to an algebra we call an exceptional Drinfeld algebra. These admit a notion of “maximally isotropic subalgebras” and we show how to define a generalised Scherk-Schwarz truncation on the associated group manifold to such a subalgebra. This allows us to define a notion of Poisson-Lie U-duality. Moreover, the closure conditions of the exceptional Drinfeld algebra define natural analogues of the cocycle and co-Jacobi conditions arising in Drinfeld double. We show that upon making a further coboundary restriction to the cocycle that an M-theoretic extension of Yang-Baxter deformations arise. We remark on the application of this construction as a solution-generating technique within supergravity.

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1 Introduction

Abelian T-duality asserts in its simplest form that closed strings do not distinguish between circular target space geometries of radius R and $\frac{1}{R}$ measured in units of the string length scale. This result is a crucial thread of a wider tapestry of U-dualities that are expected to be displayed M-theory whose target spaces contain toroidal directions. This article will consider some generalised notions of Abelian T-duality and show how they are upgraded to analogous concepts within U-duality.

Non-Abelian T-duality [1] is a proposed dualisation of closed-string non-linear sigma-models (NLSM) whose target space admits the action of non-Abelian isometry group. Whilst the status of Non-Abelian T-duality in terms of the string genus expansion remains unclear, recent evidence [2] at two-loops provides confidence that the duality could remain robust to quantum (α') corrections on the worldsheet. What is absolutely clear is that in the context of holography at large N , where both string genus and α' corrections are suppressed, Non-Abelian T-duality can be a powerful solution generating technique as advocated first in [3–6] (see [7] for a review and further references) .

More radically Poisson-Lie (PL) T-duality [8,9] dispenses with the requirement of isometry of a target space but does assume some underlying algebraic structure given by a Drinfeld double. Here again, though the quantum corrections are far from understood beyond one-loop in α' , there have been a number of significant developments. By exploiting the close connection of PL T-duality and the classical Yang-Baxter equation, this has given rise to wide classes of new integrable NLSMs called η -model or Yang-Baxter sigma-models [10,11]. Most notably when applied to the $AdS_5 \times S^5$ superstring this has led to a marginal deformation that is expected encode a quantum-group deformation, with $q \in \mathbb{R}$, of $\mathcal{N} = 4$ Super-Yang-Mills gauge theory [12,13]. Parallel to this has been the realisation with the λ -model [14,15] of an analogous quantum-group deformation with q a root-of-unity; the η - and λ -models are related (at least in simplest bosonic setting where it has been spelt out explicitly) by a PL duality transformation combined with some analytic continuation [16–18]. Alongside η and λ - integrable deformations, non-commutative β -deformations realised by TsT transformations can also be thought of as a Yang-Baxter sigma-model [19–21].

Critical to us will be that the most natural understanding of Poisson-Lie T-duality and its associated target spacetimes is provided by the tools and techniques of Double Field Theory (DFT) [22] and generalised geometry [23–25]. In essence, Poisson-Lie models, i.e. models where Poisson-Lie duality can act, arise as a generalised Scherk-Schwarz reduction [26] in which all non-trivial coordinate dependence is encoded in a twist matrix, or generalised frame field, $E_A{}^I$ [27,28]. Through the Courant bracket, these generalised frame fields realise the algebra of a Drinfeld double. When inequivalent twist matrices give rise to the same structure constants of a Drinfeld double (equivalently when there are different splittings of the Drinfeld double into maximally isotropic subalgebras), Poisson-Lie T-duality is realised as an $O(d, d)$ duality acting on the Scherk-Schwarz reduced model.

An obvious question is then if these generalised T-dualities can be extended to generalised

notions of U-duality. Though some partial descriptions have been recently suggested in the literature [29], thus far an algebraically robust description has been lacking. It is this that we address in the current article in the context of exceptional field theory (ExFT)/exceptional generalised geometry [30–36], the M-theoretic analogues to Double Field Theory/generalised geometry. Given the importance of generalised Scherk-Schwarz reductions in the DFT realisation of Poisson-Lie T-duality, it is natural to use the analogous ExFT structure to build a notion of Poisson-Lie U-duality. This is our goal in this paper.

In ExFT, a splitting is made of $11 - d$ “external” directions and d “internal” directions such that all field content and gauge symmetries are repackaged into representations of the exceptional Lie algebras $\mathfrak{e}_{d,(d)}$. Our focus will be case of $d = 4$ such that the duality group is $SL(5)$ and we will restrict our attention to the “internal” directions in which the Poisson-Lie U-duality will act.

We begin with a review of Poisson-Lie T-duality and its realisation in Double Field Theory in 2, before introducing the natural analogue of a Drinfeld Double, which for want of a better name we call an Exceptional Drinfeld Algebra in section 3. In section 4 we will show how a generalised frame can be introduced that reproduces the algebra of the Exceptional Drinfeld Algebra, allowing us to define a U-duality notion of Poisson-Lie duality. Finally, in sections 5 and 6 we develop methods for constructing an Exceptional Drinfeld Algebra starting from a Lie Group G , and thereby also suggest a U-dual version of Yang-Baxter deformations. Finally, we present some examples in section 7 and conclude in section 8 with a brief outlook.

Note added: While finalising this manuscript, the paper [37] appeared which proposes a U-duality extension of Drinfeld Doubles and has some overlap with our sections 3 and 4 in the unimodular case.

2 Review of Poisson-Lie T-duality

In this section we provide a brief recap of Poisson-Lie T-duality. We will flip the conventional exposition by starting with algebraic considerations to eventually arrive at an associated NLSM describing the NS sector of a closed string; this will serve as a road map for what follows.

Central to the construction is a (classical) Drinfeld double: an even-dimensional real algebra \mathfrak{d} with generators T_A obeying $[T_A, T_B] = iF_{AB}{}^C T_C$ equipped with a symmetric split-signature ad-invariant pairing $\eta(\cdot, \cdot)$ such that \mathfrak{d} admits at least one decomposition $\mathfrak{d} = \mathfrak{g} \oplus \tilde{\mathfrak{g}}$ with \mathfrak{g} and $\tilde{\mathfrak{g}}$ sub-algebras that are maximally isotropic with respect to η . Letting t_a (\tilde{t}^a) be generators for \mathfrak{g} ($\tilde{\mathfrak{g}}$) we have

$$\begin{aligned} \eta(t_a, t_b) &= 0, & \eta(t_a, \tilde{t}^b) &= \delta_a^b, & \eta(\tilde{t}^a, \tilde{t}^b) &= 0, \\ [t_a, t_b] &= i f_{ab}{}^c t_c, & [t_a, \tilde{t}^b] &= i \tilde{f}^{bc}{}_a t_c - i f_{ac}{}^b \tilde{t}^c, & [\tilde{t}^a, \tilde{t}^b] &= i \tilde{f}^{ab}{}_c \tilde{t}^c. \end{aligned} \tag{2.1}$$

The Jacobi identity for \mathfrak{d} imposes a compatibility constraint,

$$0 = f_{ab}{}^c \tilde{f}^{ec}{}_a - 2f_{d[a}{}^c \tilde{f}^{de}{}_{b]} + 2\tilde{f}^{dc}{}_{[a} f_{b]} df^e, \quad (2.2)$$

which is better understood as demanding that $\delta(t_a) = \tilde{f}^{bc}{}_a t_b \otimes t_c$ defines a one-cocycle on \mathfrak{g} valued in $\mathfrak{g} \wedge \mathfrak{g}$ which, as a trivial consequence of the Jacobi identity for $\tilde{\mathfrak{g}}$, also obeys a co-Jacobi identity. Equivalent we may then speak of (\mathfrak{g}, δ) as defining a Lie bi-algebra.

The exponential of a Lie bi-algebra is a Poisson-Lie group, that is a Lie-group manifold G equipped with a Poisson bi-vector compatible with the group composition law and obeying the Schouten identity. Equivalent to this Poisson bi-vector is a one-cocycle on G valued in $\mathfrak{g} \wedge \mathfrak{g}$ denoted by $\Pi = \Pi_g^{ab} t_a \otimes t_b$ which is constructed from the adjoint action of $g \in G$ on \mathfrak{d} as follows:

$$g \cdot t_a \cdot g^{-1} = (a_g)_a{}^b t_b, \quad g \cdot \tilde{t}^a \cdot g^{-1} = (b_g)^{ab} t_b + (a_{g^{-1}})_b{}^a \tilde{t}^b, \quad \Pi_g^{ab} = (b_g)^{ac} (a_{g^{-1}})_c{}^b. \quad (2.3)$$

As a consequence of this definition, Π enjoys some useful properties including:

$$\begin{aligned} \Pi_g^{ab} &= (b_g)^{ac} (a_{g^{-1}})_c{}^b = -\Pi_g^{ba}, \\ \Pi_{hg} &= \Pi_g + (a_{g^{-1}} \otimes a_{g^{-1}}) \Pi_h, \quad \Pi_e = 0, \\ d\Pi_g^{ab} &= -l^c \tilde{f}^{ab}{}_c - 2l^c f_{cd}{}^{[a} \Pi^{b]d}, \end{aligned} \quad (2.4)$$

in which we have introduced the left-invariant one-forms $l = il^a t_a = g^{-1} dg$. The dual vector fields to l^a will be denoted v_a . It will be useful in what follows to build from Π^{ab} a second set of vector fields $\pi^a \equiv \Pi_g^{ab} v_b$. A modest calculation, appealing to the properties of eq. (2.4), shows that these objects realise the algebra of \mathfrak{d} given in eq. (2.1) under the Lie derivative

$$L_{v_a} v_b = -f_{ab}{}^c v_c, \quad L_{v_a} \pi^b = f_{ac}{}^b \pi^c - \tilde{f}^{bc}{}_a v_c, \quad L_{\pi^a} \pi^b = -\tilde{f}^{ab}{}_c \pi^c. \quad (2.5)$$

We now upgrade this discussion to generalised geometry which concerns the generalised tangent bundle E given locally as $TG \oplus T^*G$. On E there is a generalised Lie (Dorfman) derivative which acts on two sections $U = u^i \partial_i + \mu_i dx^i$ and $V = v^i \partial_i + \nu_i dx^i$ by

$$\mathcal{L}_U V = L_u v + (L_u \nu - \nu_i d\mu^i). \quad (2.6)$$

In particular we can introduce two sets of generalised vectors,

$$E_a = v_a, \quad \tilde{E}^a = \pi^a + l^a, \quad (2.7)$$

which we package together by defining a set of *generalised frame fields* $E_A = (E_a, \tilde{E}^a)$ that under the Dorfman derivative furnish the algebra of the double \mathfrak{d} ,

$$\mathcal{L}_{E_A} E_B = -F_{AB}{}^C E_C. \quad (2.8)$$

We may now perform a generalised Scherk-Schwarz reduction of type II supergravity on G .

For this, introduce a set of d^2 constants $E_0 = G_0 + B_0$ assembled into constant *generalised metric*, ie. a representative of the coset $O(d, d)/O(d) \times O(d)$ given by

$$\mathring{\mathcal{H}}_{AB} = \begin{pmatrix} G_0 - B_0 G_0^{-1} B_0 & -B_0 G_0^{-1} \\ G_0^{-1} B_0 & G_0^{-1} \end{pmatrix}, \quad \mathring{\mathcal{H}}_{AB} = \eta_{AC} \mathring{\mathcal{H}}^{CD} \eta_{DB}, \quad \eta_{AB} = \eta(T_A, T_B). \quad (2.9)$$

We can use the generalised frame field to translate this constant *generalised metric* to one defined on $E \approx TG \oplus T^*G$ according to

$$\mathcal{H} = \mathring{\mathcal{H}}^{AB} E_A \otimes E_B. \quad (2.10)$$

From this “curved” generalised metric, we can extract the metric tensor and B-field $E_{ij} = G_{ij} + B_{ij}$ on G which match the target space geometry of the Poisson-Lie σ -model with action [9]

$$S = \int d^2\sigma \, l_+^a \left[(E_0^{-1} + \Pi_g)^{-1} \right]_{ab} l_-^b, \quad (2.11)$$

in which the left invariant one-forms on G have been pulled back to the worldsheet and \pm signify light-cone coordinates. The would-be-Noether currents, $\mathcal{J}_{\pm a} = v_a^i (G_{ij} \pm B_{ij}) \partial_{\pm} x^i$, associated to the G action generated by the vector fields v_a are not conserved but enjoy a remarkable non-commutative conservation law,

$$d \star \mathcal{J}_a = \tilde{f}^{bc} \star \mathcal{J}_b \wedge \star \mathcal{J}_c, \quad (2.12)$$

that is often called a Poisson-Lie symmetry.

We could of course swap the role of \mathfrak{g} and $\tilde{\mathfrak{g}}$ in the entire discussion above constructing $\tilde{\Pi}_{ab}, \tilde{\pi}_a, \tilde{v}^a, \tilde{l}_a$ as well as generalised frame fields \tilde{E}_A . This results in a dual σ -model, \tilde{S} , defined on $\tilde{G} = \exp \tilde{\mathfrak{g}}$ that is canonically equivalent to the first [9, 38]. A set of Buscher rules for such a dualisation is easily formulated in the generalised geometry by starting with a curved space generalised metric \mathcal{H} , undressing the frame fields E_A to return to the flat space generalised metric $\mathring{\mathcal{H}}$, performing an $O(d, d)$ action that implements the swapping $(T_a, \tilde{T}^a) \leftrightarrow (\tilde{T}^a, T_a)$ and then re-dressing with the the generalised frames \tilde{E}_A .

The RG equations governing the two sigma models S and \tilde{S} are equivalent at one-loop and can be formulated in terms of a renormalisation to $\mathring{\mathcal{H}}$. Should the target space of the original theory define a solution of the appropriate (super)gravity (or part thereof) under normal circumstances so too will the dual, and hence this procedure defines a solution generating technique. This is called Poisson-Lie T-duality.

3 Exceptional Drinfeld Algebra

In the ExFT approach to eleven-dimensional supergravity a split is made into a d dimensional “internal” space M and $11 - d$ “external” directions but importantly no restriction is made on the coordinate dependence of any fields and no truncation is assumed from the outset. This

splitting enables a rewriting of the variables of supergravity in a way that makes manifest the $E_{d(d)}$ U-duality symmetry. In order to get efficiently to the core issues, our focus in this paper will be exclusively on the internal directions and we shall ignore both dependence of internal fields on external coordinates and external fields entirely – these extra modes will be spectators as far as the Poisson-Lie U-duality is concerned. Our approach is essentially a specialisation of the general construction of gauged supergravities via ExFT and the inclusions of these spectator modes is well discussed in the literature.

On the internal space the bosonic field content, namely the components of the metric tensor and three-form and dual six-form potentials, are packaged in to a generalised metric \mathcal{M}_{AB} where the index A runs over a representation R_1 of the duality group. As it is most amenable for direct calculation we will henceforth consider the case of $d = 4$ for which $E_{d(d)} \cong SL(5)$ and $R_1 = \mathbf{10}$ and use the composite index notation $A = [\mathcal{A}\mathcal{B}]$ with $\mathcal{A} = 1 \dots 5$. R_1 is sometimes called the coordinate representations¹ and the sections of its associated fibre bundle, $E \approx TM \oplus \wedge^2 T^*M$, encapsulate diffeomorphisms on M and two-form gauge transformations. The gauge algebra is mediated by a generalised Lie derivative which acts on sections of R_1 as introduced in eq. (2.6) with the replacement of one-forms to two-forms. We will also need to consider the representations $R_3 = \mathbf{5}$ associated to the bundle $\Lambda^0 T^*M \oplus \wedge^3 T^*M$ the sections of which undergo gauge transformations mediated by the generalised Lie derivative

$$\mathcal{L}_U W = L_u W^0 + \left(L_u W^{(3)} + W^{(0)} d\mu \right), \quad (3.1)$$

where $U = u + \mu$, $W = W^0 + W^{(3)}$ and u is a vector field, μ is a 2-form, W^0 is a scalar field and $W^{(3)}$ is a 3-form.

Let us now construct the analogue of a Drinfeld double which, for want of inspiration, we call an ‘‘Exceptional Drinfeld Algebra’’. This is a $(n \leq 10)$ -dimensional subalgebra of $\mathfrak{sl}(5) \oplus \mathbb{R}^+$ which we write in a 10-dimensional $SL(5)$ covariant manner, such that the generators are represented by

$$\mathfrak{d} = \text{span} (T_{AB}), \quad (3.2)$$

with $T_{AB} = T_{[\mathcal{A}\mathcal{B}]}$ and $\mathcal{A}, \mathcal{B} = 1, \dots, 5$. However, as we will make explicit later, typically $n < 10$ so the generators T_{AB} are not all linearly independent. Since \mathfrak{d} is a subalgebra of $\mathfrak{sl}(5) \oplus \mathbb{R}^+$, we can introduce the $\mathfrak{sl}(5)$ -invariant $\epsilon_{ABCD\mathcal{E}}$ which is left invariant up to scalings.

The algebra of \mathfrak{d} is a Leibniz algebra given by

$$[T_{AB}, T_{CD}] = \frac{i}{2} F_{AB,CD}^{\mathcal{E}\mathcal{F}} T_{\mathcal{E}\mathcal{F}}, \quad (3.3)$$

where in analogy to 7-d gauged maximal SUGRA, the structure constants, also called the

¹In ExFT it is sometimes asserted that space is augmented by extra coordinates to form a multiplet X^A in the rep R_1 – however the price paid is that additional constraints are required to reduce the dynamics to depend only on the conventional d coordinates (or fewer). Here we will only consider objects that depend on d coordinates and need not invoke this.

embedding tensor, are given by

$$\begin{aligned} F_{AB,C\mathcal{D}}^{\mathcal{E}\mathcal{F}} &= 4F_{AB,[C}^{\mathcal{E}}\delta_{\mathcal{D}}^{\mathcal{F}]}, \\ F_{AB,C}^{\mathcal{D}} &= \frac{1}{2}\epsilon_{ABCE\mathcal{F}}Z^{\mathcal{E}\mathcal{F},\mathcal{D}} + \frac{1}{2}\delta_{[A}^{\mathcal{D}}S_{B]C} + \frac{1}{3}\delta_{[A}^{\mathcal{D}}\tau_{B]C} + \frac{1}{6}\delta_C^{\mathcal{D}}\tau_{AB}. \end{aligned} \quad (3.4)$$

Here S_{AB} is symmetric, τ_{AB} is antisymmetric and $T_{ABC}^{\mathcal{D}} = \frac{1}{2}\epsilon_{ABCE\mathcal{F}}Z^{\mathcal{E}\mathcal{F},\mathcal{D}}$ is traceless, i.e. $Z^{[AB,C]} = 0$. This is, in general, not a Lie algebra because the structure constants need not be anti-symmetric under the interchange of the two sets of lower indices. Moreover, instead of the Jacobi identity, closure of \mathfrak{g} requires the quadratic constraint (also known as the Leibniz identity)

$$2F_{AB,[C}^{\mathcal{G}}F_{|G|\mathcal{D}],\mathcal{E}}^{\mathcal{F}} - F_{AB,\mathcal{G}}^{\mathcal{F}}F_{C\mathcal{D},\mathcal{E}}^{\mathcal{G}} + F_{AB,\mathcal{E}}^{\mathcal{G}}F_{C\mathcal{D},\mathcal{G}}^{\mathcal{F}} = 0. \quad (3.5)$$

We require that \mathfrak{d} contains a four-dimensional subalgebra \mathfrak{g} (corresponding group G) whose generators satisfy

$$\epsilon^{ABCD\mathcal{E}}T_{AB}T_{C\mathcal{D}} = 0. \quad (3.6)$$

This condition plays the analogue of the maximally isotropic condition of Drinfeld doubles. We can always label these generators as $t_a \equiv T_{a5}$ with $a = 1 \dots 4$ and the remaining 6 generators by $\tilde{t}^{ab} \equiv \frac{1}{2}\epsilon^{abcd}T_{cd}$.

Requiring closure under the adjoint action of $g \in G$ on \mathfrak{d} implies that

$$g \cdot T_{AB} \cdot g^{-1} = \frac{1}{2}(A_g)_{AB}{}^{CD}T_{CD}. \quad (3.7)$$

Moreover, since $\mathfrak{d} \subset \mathfrak{sl}(5) \oplus \mathbb{R}^+$, this adjoint action must lie inside $\mathrm{SL}(5) \times \mathbb{R}^+$, i.e.

$$(A_g)_{AB}{}^{CD} = 2(A_g)_{[A}{}^{[C}(A_g)_{B]}{}^{D]}. \quad (3.8)$$

Finally, since $\mathfrak{g} \subset \mathfrak{d}$ is a subalgebra, we must have

$$(A_g)_A{}^B = \begin{pmatrix} \Xi_g^{-1/3}|ad_g|^{-1/3}(ad_g)_a{}^b & \Xi_g^{-1/3}|ad_g|^{1/3}(\lambda_g)_a \\ 0 & \Xi_g^{-1/3}|ad_g|^{1/3} \end{pmatrix}, \quad (3.9)$$

where

$$g \cdot t_a \cdot g^{-1} = (ad_g)_a{}^b t_b, \quad (3.10)$$

is the adjoint action of G on \mathfrak{g} , and Ξ_g and $(\lambda_g)_a$ are related to the \mathbb{R}^+ action and the action of G on \tilde{t}^{ab} , respectively.

Equations (3.7) and (3.9) imply the following properties for $(\lambda_g)_a$:

- λ_a vanishes at the identity of G

$$(\lambda_e)_a = 0. \quad (3.11)$$

- $(\lambda_g)_a$ inherits a group composition rule from that of the adjoint action

$$(\lambda_{hg})_a = (\Xi_g)^{\frac{2}{3}}|ad_g|^{-\frac{2}{3}}(ad_g)_a{}^b(\lambda_h)_b + (\lambda_g)_a, \quad (3.12)$$

which is reminiscent of the composition law obeyed by the Poisson-Lie bi-vector.

In the following, we will often drop the subscript g on Ξ_g and $(\lambda_g)_a$ for simplicity.

The structure constants of the algebra \mathfrak{d} can now be related to λ_a using

$$d(g T_{AB} g^{-1}) = i l^c g [T_c, T_{AB}] g^{-1}, \quad (3.13)$$

where $l = i l^a t_a = g^{-1} dg$ denotes the Maurer-Cartan one-form as in section 2. We first use that \mathfrak{g} is a subalgebra of \mathfrak{d} with structure constants

$$f_{ab}{}^c = \hat{f}_{ab}{}^c + \frac{2}{3} \delta_{[b}^c I_{a]}, \quad I_a = f_{ab}{}^b, \quad (3.14)$$

to identify

$$S_{a5} = \frac{4}{3} I_a + \frac{2}{3} \tau_{a5}, \quad T_{ab5}{}^c = \hat{f}_{ab}{}^c, \quad S_{55} = 0. \quad (3.15)$$

We then find from (3.13)

$$\begin{aligned} d\Xi &= \frac{1}{2} l^a \tau_{a5} \Xi, \\ \lambda_{(a,b)} &= -\frac{1}{4} S_{ab} + \frac{2}{3} \lambda_{(a} I_{b)} + \frac{1}{3} \lambda_{(a} \tau_{b)5}, \\ \lambda_{[a,b]} &= \frac{1}{6} \tau_{ab} + T_{5ab}{}^5 + \hat{f}_{ab}{}^c \lambda_c + \frac{1}{3} \lambda_{[a} \tau_{b]5}, \end{aligned} \quad (3.16)$$

where we defined the derivative of λ_a as

$$d\lambda_a = \lambda_{a,b} l^b. \quad (3.17)$$

Moreover, we can use invariance of the structure constants of \mathfrak{d} under the adjoint action of G . This implies

$$F_{AB,CD}{}^{\mathcal{GH}} A_G{}^\mathcal{E} A_{\mathcal{H}}{}^\mathcal{F} = A_A{}^{\mathcal{G}} A_B{}^{\mathcal{H}} A_C{}^{\mathcal{I}} A_D{}^{\mathcal{J}} F_{\mathcal{GH},\mathcal{IJ}}{}^{\mathcal{EF}}, \quad (3.18)$$

and thus various relations between the structure constants (3.4), Ξ_g , ad_g and $(\lambda_g)_a$ which will be important for us in the following.

4 The Frame Fields and Embedding Tensor

The next step is to furnish the algebra eq. (3.3) via the generalised Lie derivative acting on a set of generalised frame fields. By analogy with the Poisson-Lie case of eq. (2.7) we will construct the generalised frame fields out of objects on \mathfrak{d} , in particular λ^{abc} , as follows

$$E_a = v_a, \quad \tilde{E}^{ab} = \lambda^{abc} v_c + \alpha l^a \wedge l^b, \quad (4.1)$$

where $\lambda^{abc} = \epsilon^{abcd} \lambda_d$, α is a function on G and v_a and l^a are the left-invariant vector fields and one-forms on G .

Taken together these define generalised frame fields in the $\mathbf{10}$, ie. $E_{AB} = (E_a, \frac{1}{2} \epsilon_{abcd} \tilde{E}^{cd})$,

that should obey

$$\mathcal{L}_{E_{AB}} E_{CD} = -F_{AB,CD}{}^{\mathcal{EF}} E_{\mathcal{EF}}, \quad (4.2)$$

with $F_{AB,CD}{}^{\mathcal{EF}}$ given in (3.4).

In addition the frame fields must define $\mathrm{SL}(5) \times \mathbb{R}^+$ group elements and so can be decomposed in terms of objects in the $\bar{\mathbf{5}}$ representation:

$$E_{AB} = 2\Delta^{-1} \mathcal{E}_{[A} \mathcal{E}_{B]}, \quad (4.3)$$

where under the generalised Lie-derivative Δ has weight 1². It is often easier to work with the \mathcal{E}_a , which are given by

$$\mathcal{E}_a = \frac{1}{3!} \epsilon_{abcd} \left(\alpha^2 l^b \wedge l^c \wedge l^d - \alpha \lambda^{bcd} \right), \quad \mathcal{E}_5 = \alpha, \quad \Delta = \alpha^3 |l|, \quad (4.4)$$

and must obey

$$\mathcal{L}_{E_{AB}} \mathcal{E}_C = \theta_{AB,C}{}^{\mathcal{D}} \mathcal{E}_{\mathcal{D}} + \frac{3}{5} \tau_{AB} \mathcal{E}_C, \quad (4.5)$$

where $\theta_{AB,C}{}^{\mathcal{D}}$ are the structure constants (embedding tensor) in the fundamental representation, given as

$$\theta_{AB,C}{}^{\mathcal{D}} = T_{ABC}{}^{\mathcal{D}} + \frac{1}{2} \delta_{[A}^{\mathcal{D}} S_{B]C} + \frac{1}{15} \delta_C^{\mathcal{D}} \tau_{AB} + \frac{1}{3} \delta_{[A}^{\mathcal{D}} \tau_{B]C}. \quad (4.6)$$

We will now show that, with the λ_a defined in section 3, via equation (3.9), and an appropriate function α that we will specify shortly, the E_{AB} of (4.1) satisfy the differential conditions (4.2), once certain further restrictions are placed on the structure constants (3.4). Firstly, we evaluate (4.5) and find

$$\begin{aligned} S_{a5} &= 2L_{v_a} \ln \alpha + 2I_a, & S_{55} &= 0, & S_{ab} &= 4I_{(a} \lambda_{b)} - 4\lambda_{(a,b)} + 4\lambda_{(a} L_{v_{b)}} \ln \alpha, \\ \tau_{a5} &= 3L_{v_a} \ln \alpha + I_a, & \tau_{ab} &= -2 \left(3\lambda_{[a} L_{v_{b]}} \ln \alpha + \lambda_{[a} I_{b]} + \lambda_{[a,b]} \right) \\ T_{ab5}{}^c &= \hat{f}_{ab}{}^c, & T_{ab5}{}^5 &= -\hat{f}_{ab}{}^c \lambda_c + \frac{4}{3} \lambda_{[a,b]}, \\ T_{abc}{}^d &= 3\hat{f}_{[ab}{}^d \lambda_{c]} - 2\lambda_{[a,b} \delta_{c]}^d, & T_{abc}{}^5 &= -3\hat{f}_{[ab}{}^d \lambda_{c]} \lambda_d + 6\lambda_{[a,b} \lambda_{c]}. \end{aligned} \quad (4.7)$$

Recall from (3.11) that λ_a vanishes at the identity of G . Since the expressions (4.7) define structure constants, we can evaluate them at the identity where $\lambda_a = 0$. This implies the following relation between the structure constants.

$$T_{abc}{}^5 = 0, \quad T_{ab5}{}^5 = -\frac{2}{3} \tau_{ab}, \quad T_{abc}{}^d = \tau_{[ab} \delta_{c]}^d. \quad (4.8)$$

Therefore, to perform a Poisson-Lie U-duality, we must impose the further restriction on the algebra \mathfrak{d} that its structure constants (3.4) must satisfy (4.8).

With the structure constants related by (4.8), we can derive the following three equations

²Here $E_A{}^I$ has the natural weight $\frac{3}{5}$.

from the adjoint-invariance condition of the structure constants (3.18).

$$\begin{aligned}
0 &= f_{ab}{}^c \lambda_c + \lambda_{[a} \tau_{b]5} + \frac{2}{3} \lambda_{[a} K_{b]}, \\
0 &= f_{[ab}{}^d \lambda_{c]} - f_{[ab}{}^e \delta_{c]}^d \lambda_e - \frac{4}{3} \lambda_{[a} K_{b} \delta_{c]}^d, \\
0 &= \tau_{[ab} \lambda_{c]},
\end{aligned} \tag{4.9}$$

where $K_a = I_a + \frac{1}{2} \tau_{a5}$. These relations together with (3.16) and for the structure constants related by (4.8) imply that the λ_a introduced in (3.9) and

$$\alpha_g = \alpha_0 (\Xi_g)^{2/3} |ad_g|^{1/3}, \tag{4.10}$$

for some constant α_0 , satisfy precisely the right differential conditions, i.e.

$$\begin{aligned}
S_{a5} &= 2L_{v_a} \ln \alpha + 2I_a, \quad S_{55} = 0, \quad S_{ab} = 4I_{(a} \lambda_{b)} - 4\lambda_{(a,b)} + 4\lambda_{(a} L_{v_{b)}} \ln \alpha, \\
\tau_{a5} &= 3L_{v_a} \ln \alpha + I_a, \quad \tau_{ab} = -2 \left(3\lambda_{[a} L_{v_{b]}} \ln \alpha + \lambda_{[a} I_{b]} + \lambda_{[a,b]} \right) \\
T_{ab5}{}^c &= \hat{f}_{ab}{}^c, \quad -\frac{2}{3} \tau_{ab} = -\hat{f}_{ab}{}^c \lambda_c + \frac{4}{3} \lambda_{[a,b]}, \\
\tau_{[ab} \delta_{c]}^d &= 3\hat{f}_{[ab}{}^d \lambda_{c]} - 2\lambda_{[a,b} \delta_{c]}^d, \quad 0 = -3\hat{f}_{[ab}{}^d \lambda_{c]} \lambda_d + 6\lambda_{[a,b} \lambda_{c]}.
\end{aligned} \tag{4.11}$$

Therefore, the generalised Lie derivative of E_{AB} given in (4.1) furnish the algebra of the Drinfeld extension (3.4) when the structure constants are related by (4.8).

4.1 Poisson-Lie U-dualities

When an exceptional Drinfeld algebra admits two different subalgebras \mathfrak{g} and $\tilde{\mathfrak{g}}$ satisfying the ‘‘maximal isotropy conditions’’ (3.6), we can perform a Poisson-Lie U-duality as follows. Firstly, note that the generators of \mathfrak{g} and $\tilde{\mathfrak{g}}$ are necessarily related by some $SL(5)$ transformation $\mathcal{T}_A{}^B$. Next, for both \mathfrak{g} and $\tilde{\mathfrak{g}}$, we can introduce Ξ , λ_a and Ξ' , λ'_a , respectively as shown in 3. We then construct the frame fields \mathcal{E}_A and \mathcal{E}'_A realising the algebra of the exceptional Drinfeld algebra based on \mathfrak{g} , $\tilde{\mathfrak{g}}$, respectively using our results in 4. Finally, we can perform a generalised Scherk-Schwarz reduction of 11-dimensional supergravity on G using the Ansatz

$$\mathcal{M}_{AB} = \mathcal{E}_A{}^{\bar{A}} \mathcal{E}_B{}^{\bar{B}} \mathcal{M}_{\bar{A}\bar{B}}, \tag{4.12}$$

where $\mathcal{M}_{\bar{A}\bar{B}}$ are the scalar fields of 7-dimensional maximal gauged supergravity and are thus constant on the internal space. We now perform a $SL(5)$ transformation on the fields of the 7-dimensional supergravity, i.e.

$$\mathcal{M}'_{\bar{A}\bar{B}} = \mathcal{T}_{\bar{A}}{}^{\bar{C}} \mathcal{T}_{\bar{B}}{}^{\bar{D}} \mathcal{M}_{\bar{C}\bar{D}}, \tag{4.13}$$

and lift \mathcal{M}' to 11-dimensional supergravity using the frame fields $\mathcal{E}'_{\bar{A}}{}^{\bar{B}}$. Thus the dual background is described by the fields encoded in the generalised metric

$$\mathcal{M}_{AB} = \mathcal{E}'_{\bar{A}}{}^{\bar{A}} \mathcal{E}'_{\bar{B}}{}^{\bar{B}} \mathcal{M}'_{\bar{A}\bar{B}}. \quad (4.14)$$

5 The quadratic constraint

If we are given a Drinfeld extension, as defined in section 3 with structure constants related by 4.8, we can use the results of the previous section to immediately construct the frame fields associated to it, and thus to perform a Poisson-Lie U-duality. However, in practice, we typically want to know when a given algebra \mathfrak{g} can be enlarged into a Drinfeld extension, and, in particular, how to define λ_a and α and thus the frame fields (4.1) given \mathfrak{g} , so that we can perform a Poisson-Lie U-duality. To answer this question, we must first study the closure conditions of \mathfrak{d} , i.e. the quadratic constraints (3.5), to understand what conditions \mathfrak{g} imposes on the structure constants of \mathfrak{d} .

The quadratic constraints (3.5) imply that the irreducible representations of $F_{AB,C}{}^{\mathcal{D}}$ must satisfy

$$\begin{aligned} F_{AB,[C}{}^{\mathcal{E}} \tau_{\mathcal{D}]\mathcal{E}} &= 0, \\ F_{AB,(C}{}^{\mathcal{E}} S_{\mathcal{D})\mathcal{E}} &= 0, \\ 2F_{AB,\mathcal{F}}{}^{[C} Z^{|\mathcal{F}|\mathcal{D}]\mathcal{E}} + F_{AB,\mathcal{F}}{}^{\mathcal{E}} Z^{CD,\mathcal{F}} - \frac{1}{2}\tau_{AB} Z^{CD,\mathcal{E}} &= 0. \end{aligned} \quad (5.1)$$

These live in the $\mathbf{10} \otimes (\mathbf{10} \oplus \mathbf{15} \oplus \overline{\mathbf{40}})$. We now use (3.15) and (4.8) to simplify the quadratic constraints and find:

- The conventional Jacobi identity for \mathfrak{g} : $f_{[ab}{}^e f_{c]e}{}^d = 0$, as well as

$$0 = f_{ab}{}^c \tau_{c5}. \quad (5.2)$$

- The compatibility conditions involving the \mathfrak{g} action on the remaining embedding tensor components, S_{ab} and τ_{ab} :

$$\begin{aligned} 0 &= f_{ab}{}^d S_{cd} - 2f_{c[a}{}^d S_{b]d} - \frac{4}{3}K_{[a} S_{b]c} - \frac{4}{3}K_c \tau_{ab}, \\ 0 &= \frac{1}{2}I_{[a} \tau_{bc]} + \tau_{[ab} \tau_{c]5}, \\ 0 &= f_{c[a}{}^d \tau_{b]d} + \frac{1}{3}K_c \tau_{ab} - \frac{1}{2}\tau_{c[a} \tau_{b]5} + \frac{1}{4}S_{c[a} \tau_{b]5}, \end{aligned} \quad (5.3)$$

in which we recall $K_a = I_a + \frac{1}{2}\tau_{a5}$.

- A dual Jacobi condition, involving only S_{ab} and τ_{ab} and not the structure constants of \mathfrak{g} :

$$\begin{aligned} 0 &= \tau_{[ab}\tau_{cd]}, \\ 0 &= S_{a[b}\tau_{cd]}. \end{aligned} \tag{5.4}$$

5.1 Dual structure constants

It is useful to introduce the “dual structure constants”

$$\tilde{f}^{abc}{}_d = \epsilon^{abce} (S_{de} - 2\tau_{de}). \tag{5.5}$$

These define a map: $\tilde{f} : \mathfrak{g} \rightarrow \Lambda^3 \mathfrak{g}$, given by

$$\tilde{f}(x) = x^a \tilde{f}^{bcd}{}_a t_b \wedge t_c \wedge t_d, \quad \forall x = x^a t_a \in \mathfrak{g}. \tag{5.6}$$

Viewed this way \tilde{f} defines a $\Lambda^3 \mathfrak{g}^*$ -valued Lie algebra 1-cochain.

Now the compatibility conditions (5.3) are equivalent to

$$\begin{aligned} 6f_{f[a} [{}^c \tilde{f}^{de}] f_{b]} + f_{ab} f \tilde{f}^{cde}{}_f + \frac{2}{3} \tilde{f}^{cde}{}_{[a} (\tau_{b]5} - I_{b])} &= 0, \\ \tilde{f}^{abc}{}_c (I_b + 2\tau_{b5}) &= 0, \\ f_{de} [{}^a \tilde{f}^{b]de}{}_c + \frac{2}{3} \tilde{f}^{abd}{}_c (\tau_{d5} - I_d) &= 0. \end{aligned} \tag{5.7}$$

Finally, the dual Jacobi conditions (5.4) are equivalent to

$$\tilde{f}^{abg}{}_c \tilde{f}^{def}{}_g - 3\tilde{f}g^{[de}{}_c \tilde{f}f]^{ab}{}_g = 0. \tag{5.8}$$

5.2 Cocycle condition

The first equation in (5.7)

$$6f_{f[a} [{}^c \tilde{f}^{de}] f_{b]} + f_{ab} f \tilde{f}^{cde}{}_f + \frac{2}{3} \tilde{f}^{cde}{}_{[a} (\tau_{b]5} - I_{b])} = 0, \tag{5.9}$$

is particularly interesting. Let us first focus on the case $I_a = \tau_{a5}$. Then (5.9) implies that $\tilde{f}^{abc}{}_d$ must be a Lie-algebra 1-cocycle, i.e. \tilde{f} must be closed under the Lie algebra differential

$$d\tilde{f}(x, y) \equiv ad_x \tilde{f}(y) - ad_y \tilde{f}(x) - \tilde{f}([x, y]) = 0, \quad \forall x, y \in \mathfrak{g}. \tag{5.10}$$

The usual Lie algebra differential is nilpotent $d^2 = 0$, and thus the cocycle condition is solved by a coboundary

$$\tilde{f} = dm, \tag{5.11}$$

for some $m \in \Lambda^3 \mathfrak{g}^*$, where

$$dm(x) \equiv ad_x m, \quad \forall x \in \mathfrak{g}. \tag{5.12}$$

Now let us turn to $I_a \neq \tau_{a5}$. Now, the modified cocycle condition (5.9) can be expressed in terms of the operator

$$d' \tilde{f}(x, y) \equiv ad_x \tilde{f}(y) - ad_y \tilde{f}(x) - \tilde{f}([x, y]) - \frac{1}{3} \left(\tilde{f}(x)L(y) - \tilde{f}(y)L(x) \right) = 0, \forall x, y \in \mathfrak{g}, \quad (5.13)$$

where

$$L(x) \equiv (\tau_{a5} - I_a) x^a, \forall x = x^a t_a \in \mathfrak{g}. \quad (5.14)$$

However, the operator d' is still nilpotent. Let

$$d' m(x) \equiv ad_x m + \frac{1}{3} L(x) m, \forall x \in \mathfrak{g}, \quad (5.15)$$

for some $m \in \Lambda^3 \mathfrak{g}^*$. Then it is easy to show that $d'^2 = 0$. Therefore, the modified cocycle condition (5.9) can be solved by a coboundary

$$\tilde{f} = d' m, \quad (5.16)$$

for any $m \in \Lambda^3 \mathfrak{g}^*$.

6 Coboundary Ansatz

As discussed above, the quadratic constraints lead to compatibility conditions between the structure constants of \mathfrak{g} and \tilde{f}^{abc}_d (5.7), which include the modified cocycle condition (5.9). Let us now focus on coboundary solutions to these equations, i.e. solutions of the form (5.16). In the case of Poisson-Lie T-duality, the analogous coboundary case leads to Yang-Baxter deformations. Therefore, when \tilde{f} is a modified coboundary as above this should lead to the U-duality analogue of Yang-Baxter deformations. Moreover, Whitehead's lemma states that for semisimple Lie algebras $H^1(\mathfrak{g}) = 0$, i.e. for semisimple Lie algebras every 1-cocycle is a coboundary, and thus the coboundary Ansatz is the only solution for semisimple Lie algebras with $I_a = \tau_{a5}$.

Writing (5.16) out explicitly, it becomes

$$\tilde{f}^{abc}_d = 12 f_{de} [{}^a m^{bc}]^e + \frac{4}{3} m^{abc} (\tau_{d5} - I_d). \quad (6.1)$$

In terms of the irreps τ_{ab} and S_{ab} , the Ansatz (6.1) is

$$S_{ab} = \frac{8}{3} m_{(a} K_{b)}, \quad \tau_{ab} = \frac{4}{3} m_{[a} K_{b]} + 2 f_{ab}{}^c m_c. \quad (6.2)$$

Now the first equation of (5.7) is automatically solved and the remaining compatibility conditions impose the following restrictions on m_a :

$$\begin{aligned} f_{ab}{}^d f_{cd}{}^e m_e + 2 f_{c[a}{}^d K_{b]} m_d + f_{c[a}{}^d \tau_{b]5} m_d - \frac{4}{9} K_c m_{[a} K_{b]} - \frac{2}{3} K_c m_{[a} \tau_{b]5} &= 0 \\ f_{[ab}{}^d K_{c]} m_d &= 0. \end{aligned} \quad (6.3)$$

Finally, the dual Jacobi conditions imply

$$\begin{aligned} f_{[ab}{}^e f_{cd]}{}^f m_e m_f &= 0, \\ K_a f_{[bc}{}^e m_{d]} m_e &= 0. \end{aligned} \tag{6.4}$$

Now let us construct the object λ_a satisfying the first-order constraints (4.11) with S_{ab} and τ_{ab} given as in (6.2), as well as the group composition properties (3.11) and (3.12). It is given by

$$(\lambda_g)_a = m_a - (\Xi_g)^{\frac{2}{3}} |ad_g|^{-\frac{2}{3}} (ad_g)_a{}^b m_b, \tag{6.5}$$

which clearly satisfies (3.11) and (3.12). Moreover, we can readily calculate the Lie derivative

$$L_{v_b} \lambda_a \equiv \lambda_{a,b} = \frac{2}{3} K_b (\lambda - m)_a + f_{ab}{}^c (\lambda - m)_c, \tag{6.6}$$

and find that at the identity where $\lambda_a = 0$

$$S_{ab} = \frac{8}{3} m_{(a} K_{b)}, \quad \tau_{ab} = \frac{4}{3} m_{[a} K_{b]} + 2 f_{ab}{}^c m_c, \tag{6.7}$$

as required by (6.2).

Now we note that (4.11) is satisfied precisely when (4.9) holds. However, as shown in section 3, these equations follow from adjoint-invariance of the structure constants, and thus from integrating the quadratic constraints (3.5) over G . Since we have already satisfied the quadratic constraints (3.5) and ensured that λ_a transforms the right way on G , our Ansatz (6.5) satisfies (4.9) and thus (4.11).

Finally, for a given group G , we can integrate $d\Xi = \frac{1}{2} l^\alpha \tau_{\alpha 5} \Xi$ to find the function Ξ appearing via α , see equation (4.10), in the frame fields (4.1).

Given the analogy with Poisson-Lie T-duality, it is tempting to speculate that the matrix m^{abc} defined above is related to an M-theoretic analogue of Yang-Baxter deformations. In particular, one may want to view $m : \mathfrak{g} \rightarrow \Lambda^2 \mathfrak{g}^*$ and defining an associated 3-bracket. The appropriate notion of closure of the 3-bracket should then be (5.8). There should then be a simple condition involving two powers of m and one of the structure constants of \mathfrak{g} that ensures closure. This would be the analogue of the classical Yang-Baxter equation. We leave a detailed study of this question for future work.

7 Some examples

Let us now consider some examples of the co-boundary case.

The most obvious example here is $\mathfrak{g} = \mathfrak{su}(2) + \mathfrak{u}(1)$ in which we choose T_4 be the $\mathfrak{u}(1)$ generator and $T_i = \tau_i$ be the Pauli matrices for $i = 1 \dots 3$. This however is rather trivial, immediately from $f_{ab}{}^c \tau_{c5} = 0$ we have that $K_i = 0$ for $i = 1, 2, 3$. Then from the constraint $f_{[ab}{}^d K_{c]} m_d = 0$ one concludes that $m_i = 0$ for $i = 1, 2, 3$. From the constancy condition one is left with $0 = f_{[ab}{}^d \lambda_{c]}$ but picking say $a = 1, b = 2, c = 4, d = 3$ one finds that the only remaining

component m_4 is forced to be zero. Of course, since there is a $U(1)$ factor here there can still be solutions for λ not of co-boundary type.

Let us consider $\mathfrak{g} = (II) + \mathfrak{u}(1)$ where (II) is the Bianchi_{II} or Heisenberg-Weyl algebra for which the only non-vanishing commutator is

$$[T_2, T_3] = iT_1 .$$

We can choose a parameterisation of (II) as

$$T_1 = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad T_2 = i \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad T_3 = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and a corresponding group element (in which the overall phase accounts for the $U(1)$ factor)

$$g = e^{i\frac{\theta}{3}} \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} .$$

	K_a	m_a	τ_{a5}	τ_{ab}	S_{ab}	\tilde{f}	
1	$\{0, 0, 0, 0\}$	$\{m_1, m_2, m_3, m_4\}$	-	×	-	×	
2	$\{0, 0, 0, 0\}$	$\{0, m_2, m_3, m_4\}$	-	-	-	-	
3	$\{0, k_2, k_3, 0\}$	$\{0, m_2, \frac{k_3 m_2}{k_2}, 0\}$	×	-	×	×	
4	$\{0, k_2, k_3, k_4\}$	$\{0, 0, 0, 0\}$	×	-	-	-	
5	$\{0, 0, k_3, 0\}$	$\{0, 0, m_3, 0\}$	×	-	×	×	
6	$\{0, 0, k_3, k_4\}$	$\{0, 0, 0, 0\}$	×	-	-	-	
7	$\{0, k_2, 0, 0\}$	$\{0, m_2, 0, 0\}$	×	-	×	×	
8	$\{0, k_2, 0, k_4\}$	$\{0, 0, 0, 0\}$	×	-	-	-	
9	$\{0, 0, 0, k_4\}$	$\{0, 0, 0, 0\}$	×	-	-	-	(7.1)

Table 1: Solutions for $\mathfrak{g} = (II) + \mathfrak{u}(1)$ in which \times indicates a non-zero embedding tensor component

The function $\Xi_g = e^{\frac{1}{3}\theta k_4 - x k_2 - z k_3}$ which enters into the co-boundary ansatz for λ is compatible with the group multiplication law and the differential constraint $d\Xi_g = \frac{1}{2}l^c \tau_{a5} \Xi_g$. The one forms and dual vectors are given as

$$\begin{aligned} l^1 &= -dy + xdz , & l^2 &= -dx , & l^3 &= -dz , & l^4 &= \frac{1}{3}d\theta \\ v_1 &= -\partial_y , & v_2 &= -\partial_x , & v_3 &= -x\partial_y - \partial_z , & v_4 &= 3\partial_\theta . \end{aligned} \tag{7.2}$$

The various constraint equations discussed above admit a set of solutions, illustrated in table 1, leading to constant embedding tensor components.

8 Discussion and outlook

This work opens a number of interesting lines.

As indicated in section 4, by a procedure of undressing, $SL(5)$ action and redressing, one has a map between two generalised metrics giving rise to same lower dimensional supergravity theory and one should like to exploit this as a solution generating technique. An exciting opportunity is to explore pragmatic usages of this technique in constructing new holographic supergravity solutions, a programme which becomes richer as the dimensionality of the internal space is increased.

A immediate mechanical task is to scan through the four-dimensional Lie-algebras (see e.g. [39]) and classify all the Exceptional Drinfeld Algebras $\mathfrak{d} \subset SL(5) \times \mathbb{R}^+$ thereby providing an exceptional analogue of the classification of six-dimensional Drinfeld doubles [40]. The framework proposed above should admit a ready generalisation to the other exceptional generalised geometries based on $\mathfrak{e}_{n,(n)}$, at least for $n \leq 6$, though the details should be worked out. New features occur at larger n : at $n = 6$ a new object λ_{abcdef} would enter and modify the structure; at $n = 7$ a mixed symmetry object $\lambda^{a_1 \dots a_7, a_8}$ must be considered; at $n = 8$ the generalised Lie derivative alone doesn't close and compensating additional shift symmetries must be incorporated. Beyond $n = 8$ one is dealing with infinite dimensional algebras which would be exciting to investigate in this context.

Here we have taken a target space perspective and one might wonder about the implications of this construction on the world-volume of membranes or five-branes in M-theory. Recall the worldsheet currents \mathcal{J}_a associated to the G action generated by the vector fields v_a obey a modified conservation law in the case of Poisson-Lie NLSMs. For the case of $SL(5)$ exceptional generalised geometry we need only think of membranes where a natural expectation is a modified conservation law of the form

$$d \star \mathcal{J}_a = \tilde{f}^{bcd} \mathcal{J}_b \wedge \mathcal{J}_c \wedge \mathcal{J}_d. \quad (8.1)$$

Beyond $SL(5)$, when five-branes should be considered this is likely to be more involved.

The Poisson-Lie scenario described in section 2 can be generalised in two important ways. First, the assumption of a Drinfeld double can be relaxed to only require a single isotropic subgroup, this set-up captures WZW models as well as their integrable λ -deformations. Second, one can take a reduction of the Poisson-Lie model to “dressing cosets” which allows for target spaces that are cosets rather than groups (required to define e.g. integrable Yang-Baxter type deformations of $AdS_5 \times S^5$). One would hope to situate both of these generalisations within the exception generalised geometry setting.

Here we have seen that how the components of the embedding tensor are realised as encoding structure constants of an exceptional Drinfeld algebra. One can thus reduce on the geometry we construct to a lower dimensional gauged supergravity theory. Here the resultant theories obtained after Scherk–Schwartz reduction will be maximally supersymmetric, and so a natural question related to the preceding paragraph, is to obtain the half-maximal or lower supersymmetric analogue construction exploiting the ideas in [41, 42]. The converse question is

interesting; under what circumstances can a lower dimensional gauged supergravity theory be uplifted to an exceptional Drinfeld algebra?

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A Conventions

We consider an algebra \mathfrak{g} (group G) with Hermitian generators t_a obeying $[t_a, t_b] = i f_{ab}{}^c t_c$. We define the trace of the structure constants as $I_a = f_{ab}{}^b$ such that $f_{ab}{}^c = \hat{f}_{ab}{}^c + \frac{2}{3} \delta_{[b}^c I_{a]}$ with $\hat{f}_{ab}{}^b = 0$. When $\dim \mathfrak{g} = 4$ we can make use of the identity $\hat{f}_{ef}{}^{[a} \epsilon^{bc]ef} = 0$.

We denote by l^a the left-invariant Maurer-Cartan one-forms on G , given by $g^{-1}dg = i l^a t_a$, and by v_a their dual vector fields that generate right translations. They satisfy

$$dl^a = \frac{1}{2} f_{bc}{}^a l^b \wedge l^c, \quad \iota_{v_a} l^b = \delta_a{}^b, \quad L_{v_a} v_b = -f_{ab}{}^c v_c, \quad L_{v_a} l^b = +f_{ac}{}^b l^c, \quad (\text{A.1})$$

in which L denotes the Lie-derivative that we recall acts on forms according to $L = dt + \iota d$. We define the adjoint action of G on \mathfrak{g} as

$$\text{ad}_g : t_a \mapsto g t_a g^{-1} = (a_g)_a{}^b t_b. \quad (\text{A.2})$$

Certain manipulations require the identities

$$d(\text{ad}_g)_a{}^b = -l^c f_{ca}{}^d (\text{ad}_g)_d{}^b, \quad d|a_g| = -l^a I_a |a_g|, \quad (\text{ad}_g)_a{}^d (\text{ad}_g)_b{}^e f_{de}{}^c = f_{ab}{}^d (\text{ad}_g)_d{}^c. \quad (\text{A.3})$$

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