

Perturbations of Extremal Kerr Spacetime: Analytic Framework and Late-time Tails

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(Dated: October 14, 2019)

We develop a complete and systematic analytical approach to field perturbations of extremal Kerr spacetime based on the formalism of Mano, Suzuki and Takasugi (MST) for the Teukolsky equation. Analytical expressions for the radial solutions and frequency-domain Green function in terms of infinite series of special functions are presented. As an application, we compute the leading late-time behavior due to the branch point at zero frequency of scalar, gravitational, and electromagnetic field perturbations on and off the event horizon.

CONTENTS

I. Introduction	2
II. Perturbations of extremal Kerr	4
A. Retarded Green function	4
B. Radial Teukolsky equation	5
C. Transfer function	7
III. MST method	8
A. Series representations for the radial solutions	8
1. Radial series	8
2. Series coefficients and renormalized angular momentum	10
B. Symmetric decomposition of the radial solutions	13
C. Matching radial solutions	14
D. Radial coefficients (scattering amplitudes)	15
1. Scattering amplitudes at infinity	15
2. Scattering amplitudes at the horizon	17
E. Series coefficients and renormalized angular momentum for $\omega \rightarrow 0, m$	17
IV. Analytical properties of the transfer function	18

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V. Branch cut from the origin and tail	20
A. Discontinuity in the up modes	21
B. Discontinuity in the transfer function	22
C. Small- ω asymptotics of radius independent quantities appearing in $\delta\tilde{g}_{\ell m\omega}$	23
D. Late-time $\omega = 0$ tail at finite radii	24
E. Late-time $\omega = 0$ tail at radial infinity	25
F. Late-time $\omega = 0$ tail on the horizon	26
G. Summary of $\omega = 0$ tail results	26
VI. Branch cut from the critical frequency	27
A. Discontinuity in the ingoing modes	28
B. Discontinuity in the transfer function across the critical frequency branch cut	29
VII. Small k asymptotics and link with matched asymptotic expansions	30
A. Radial solutions near the superradiant bound	30
B. Transfer function near the superradiant bound	31
A. Radial Solutions à la Leaver	32
Acknowledgments	33
References	33

I. INTRODUCTION

The prominence of black holes in modern physics and astrophysics makes study of their perturbations of essential importance. Regge, Wheeler, Zerilli, and Moncrief [1–3] pioneered work on linear field perturbations of spherically symmetric (Schwarzschild) black holes. The astrophysically-relevant case, however, is that of rotating black holes, which are described by the Kerr metric. The rotating case was cracked by Teukolsky [4], who derived a master equation for scalar (spin-0), fermion (spin-1/2), electromagnetic (spin-1), and gravitational (spin-2) perturbations of the Kerr metric. While Teukolsky’s equation can be solved numerically (say, in the time domain), it is of interest to develop complimentary analytical techniques. Teukolsky’s master equation is a partial differential equation which separates into a radial and an angular ordinary differential equation by going into the frequency domain. By adopting such a frequency domain approach, Leaver [5] found analytical solutions of the radial equation in terms of infinite series involving special functions. Mano, Suzuki, and Takasugi (MST) [6, 7] cleverly reformulated Leaver’s solutions to produce a practical method of computing observable quantities such as the gravitational waveform from the inspiral of a compact object into a supermassive black hole in the extreme mass-ratio regime. With the advent of computer algebra programs capable of efficiently manipulating and computing special functions, this “MST method” has gained in popularity to become competitive with, and in many ways superior to, direct numerical solution of the linearized perturbation equation. The MST method has been used for calculating the self-force, post-Newtonian coefficients and gauge-invariant quantities, the retarded Green function, the quantum correlator, the renormalized expectation value of the quantum stress-energy tensor and radiation emission in Schwarzschild and Kerr spacetimes in [8–13]; of particular relevance

to this paper, it has also been used to calculate the late-time tail to high-order in Schwarzschild and Kerr spacetimes in [14, 15].

Kerr black holes possess an outer event horizon and an inner Cauchy horizon beyond which the Cauchy value problem is not well-posed. In the case of maximal rotation, called extremal, the Cauchy and event horizons coincide. Since the original formulation for spin-field perturbations of vacuum, asymptotically flat, non-extremal black holes [6, 7] (compiled in a review in [16]), the MST method has been extended in a variety of ways to encompass electrically-charged black holes and/or a nonzero cosmological constant [17–19]. However, to the best of our knowledge, there has been no prior MST work on extremal black holes. The extension is not straightforward since the coincidence of the inner and outer horizons converts a pair of regular singular points of the radial equation into a single irregular singular point, changing the character of the series solutions. We side-step the difficulty by starting with a functional expansion adapted to extremal Kerr and develop a Leaver-MST method accordingly. The MST-type series that we derive in extremal Kerr provides a practical and efficient formulation for analytic evaluation of integer-spin perturbations of this spacetime.

As an application, we compute the late-time behavior (“tail”) of the perturbing field. In the non-extremal case, the tail arises from a branch point that the radial solutions possess at the origin of the complex frequency plane (i.e., at zero frequency) [15, 20, 21]. In the extremal case there is an additional branch point at the so-called superradiant bound frequency that must also be considered [22]. In previous work [23, 24], we have computed the extremal Kerr tail from the branch point at the superradiant bound frequency, showing that the asymptotic decay of the perturbing field, whether it be metric, vector potential, or scalar, is $1/v$ off the horizon and $1/\sqrt{v}$ on the horizon (v being advanced time). The difference in the rates on and off the horizon accounts for the divergent growth of transverse derivatives at the horizon [25, 26], a phenomenon named after its discoverer, Aretakis [27, 28]. In these calculations we used the method of matched asymptotic expansions (MAE) to compute the leading late-time behavior, which comes from the behaviour of the modes near the superradiant bound frequency. In this work we show that the transfer function (i.e., the fixed frequency modes of the retarded Green function) used in MAE calculations is recovered exactly by the leading-order term in the MST series that we derive. This unites previously distinct techniques, provides a more rigorous justification for the MAE, and shows how it can be systematically corrected to arbitrary order in frequency. Moreover, in the aforementioned MAE calculations the late-time rates due to the superradiant bound frequency were reported under the assumption that the tail due to the branch point at the origin is subleading. Here we justify this assumption, showing that the tail from the origin is in indeed subleading at the horizon, going as $v^{-3-2\ell}$ along the future event horizon (see Eq.(133)). We also derive asymptotic decay rates at future null and timelike infinity. We find that these rates are: $u^{-2+s-\ell}$ along future null infinity (see Eq.(130)) and $t^{-3-2\ell}$ at future timelike infinity (see Eq.(122)), where t is Boyer-Lindquist time, u is retarded time and ℓ is the multipole number in the decomposition in angular functions (i.e., spin-weighted spheroidal harmonics [29, 30]). Our results are for nonaxisymmetric, integer-spin field perturbations¹.

The rest of this paper is organized as follows. In Sec. II we introduce the Teukolsky equation and its retarded Green function. In Sec. III we develop the MST formalism for extremal Kerr. In Sec. V we apply the MST formalism to obtain the formal contribution to the Green function from the branch cut down from the origin and derive the corresponding leading-order late-time tail. In Sec. VI we obtain the formal contribution to the Green function from the branch cut down from the superradiant bound frequency. In Sec. VII we show that, in the limit to the superradiant bound frequency, the MST method recovers the MAE results. In App.A we give Leaver’s [5] original expressions and relate them to ours.

We follow the notation and units of Sec.VIII.B of Ref. [5]. In particular, we choose $c = G = 1$ and the unusual choice $M = 1/2$ for the mass of the black hole.

¹ The axisymmetric case is already considered in [23, 24] with the exception of the tail at future null infinity. Also, it should be straightforward to extend our results to half-integer values for the spin of the field.

II. PERTURBATIONS OF EXTREMAL KERR

A. Retarded Green function

Scalar (spin $s = 0$), electromagnetic ($s = \pm 1$), and gravitational ($s = \pm 2$) perturbations² Ψ of an extremal Kerr black hole are governed by a single “master” equation first derived by Teukolsky [4]. This is a $(3 + 1)$ -dimensional, second-order wave equation. Our main study concerns the retarded Green function of this equation, where x^μ and $x^{\mu'}$ are spacetime points³. The retarded Green function $G(x^\mu, x^{\mu'})$ is defined to vanish when x^μ is outside the causal future of $x^{\mu'}$. We employ Boyer-Lindquist coordinates $\{t \in \mathbb{R}, r \in (M, \infty), \theta \in [0, \pi], \phi \in [0, 2\pi)\}$ outside the event horizon of the black hole and install the Kinnersley tetrad [31]. We denote the mass of the black hole by M and let $r_H := M$ denote the radius of the event horizon at extremality. Henceforth we choose units such that $M = 1/2$, and so $r_H = 1/2$. Instead of the Boyer-Lindquist radial coordinate r we shall use the shifted radial coordinate

$$x := r - r_H = r - 1/2 \in (0, \infty). \quad (1)$$

In the shifted Boyer-Lindquist coordinates, the retarded Green function satisfies the fundamental equation [4]

$$\mathcal{O}[G(x^\mu, x^{\mu'})] = \delta(t - t')\delta(x - x')\delta(\cos\theta - \cos\theta')\delta(\phi - \phi'), \quad (2)$$

where \mathcal{O} is the Teukolsky operator. In the metric signature $(-+++)$ that we use, \mathcal{O} corresponds to minus the operator in the left-hand-side of Eq.(4.7) of [4].

In order to calculate the retarded Green function, we mode-decompose into spin-weighted spheroidal harmonics ${}_sS_{\ell m \omega}$ [29, 30] and make use of the axisymmetry and stationarity of the spacetime. Explicitly, we decompose G as

$$G(x^\mu, x^{\mu'}) = -\frac{x'^{2s}}{2\pi} \sum_{\ell=|s|}^{\infty} \sum_{m=-\ell}^{\ell} \int_{-\infty+ic}^{\infty+ic} e^{-i\omega t + im\phi} {}_s\mathcal{Z}_{\ell m \omega}(\theta, \theta') \tilde{g}_{\ell m \omega}(x, x') d\omega, \quad (3)$$

where

$${}_s\mathcal{Z}_{\ell m \omega}(\theta, \theta') := {}_sS_{\ell m \omega}(\theta) {}_sS_{\ell m \omega}^*(\theta'), \quad (4)$$

and $c > 0$ ensures the integration contour of the inverse Laplace transform is in the analytic region of the transfer function $\tilde{g}_{\ell m \omega}$. By the symmetries of the Kerr spacetime, we have set $t' = 0$ and $\phi' = 0$ without loss of generality.

The spin-weighted spheroidal harmonics ${}_sS_{\ell m \omega}$ are understood to be evaluated at extremality, i.e., for black hole angular momentum a per unit mass equal to the mass, i.e., $a = M = 1/2$. These angular functions satisfy the following ordinary differential equation:

$$\left(\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d}{d\theta} \right) + \frac{\omega^2 \cos^2\theta}{4} - \frac{m^2}{\sin^2\theta} - \omega s \cos\theta - \frac{2ms \cos\theta}{\sin^2\theta} - s^2 \cot^2\theta + s + {}_sA_{\ell m \omega} \right) {}_sS_{\ell m \omega}(\theta) = 0, \quad (5)$$

where ${}_sA_{\ell m \omega}$ is a separation constant. Together with the boundary conditions of regularity at $\theta = 0$ and π , this equation poses an eigenvalue problem with eigenvalue ${}_sA_{\ell m \omega}$. For real frequencies, the spin-weighted spheroidal harmonics form a strongly complete set of eigenfunctions, whereas for complex frequencies they only form a weakly complete set [32]. Following [4], it is also convenient to define the quantity ${}_s\lambda_{\ell m \omega} :=$

² Fermion ($s = \pm 1/2$) field perturbations also obey the Teukolsky equation. However, in this paper we assume integer s , which simplifies some of the formulas.

³ In a common abuse of language, we use the same symbol to denote spacetime points and their coordinates.

${}_sA_{\ell m \omega} + M^2 \omega^2 - 2Mm\omega = {}_sA_{\ell m \omega} + \omega^2/4 - m\omega$. Our convention is to normalize the spin-weighted spheroidal harmonics such that

$$\int_0^{2\pi} \int_0^\pi e^{i(m-m')\phi} {}_sS_{\ell m \omega}(\theta) {}_sS_{\ell' m' \omega}(\theta) \sin \theta d\theta d\phi = 2\pi \delta_{\ell \ell'} \delta_{mm'}. \quad (6)$$

We now give some useful properties of the angular eigenfunctions and their eigenvalues. From the angular equation, (5) the following symmetries are manifest:

$${}_sA_{\ell m \omega} + s = -{}_sA_{\ell m \omega} - s, \quad {}_sA_{\ell m \omega}^* = {}_sA_{\ell m \omega}^*, \quad {}_sA_{\ell m \omega} = {}_sA_{\ell, -m, -\omega}, \quad (7)$$

for the angular eigenvalue and

$${}_s\mathcal{X}_{\ell m \omega}(\theta, \theta') = -{}_s\mathcal{X}_{\ell, -m, -\omega}(\theta, \theta') = {}_s\mathcal{X}_{\ell, -m, -\omega}(\pi - \theta, \pi - \theta'), \quad {}_s\mathcal{X}_{\ell m \omega}^*(\theta, \theta') = {}_s\mathcal{X}_{\ell m \omega}^*(\theta, \theta'), \quad (8)$$

for the angular eigenfunction product.

In its turn, routine separation of variables reveals that the transfer function obeys the ordinary differential equation

$$\mathcal{L}[\tilde{g}_{\ell m \omega}(x, x')] = -\delta(x - x'), \quad (9)$$

where

$$\mathcal{L} := x^{-2s} \frac{d}{dx} \left(x^{2s+2} \frac{d}{dx} \right) + V(x), \quad (10)$$

with V given by

$$V(x) := (k + m) \left(k + (k + m)(x + 1)^2 + 2isx \right) + \frac{k^2}{4x^2} + \frac{k^2}{x} + \frac{k(m - is)}{x} - {}_s\lambda_{\ell m \omega}. \quad (11)$$

Here, we have introduced a shifted frequency,

$$k := \omega - m\Omega_H = \omega - m, \quad (12)$$

where $\Omega_H := 1/(2r_H) = 1$ is the horizon frequency. The value $k = 0$ (i.e., $\omega = m$) corresponds to the so-called superradiant bound frequency (in the literature, this frequency is also called horizon frequency or critical frequency; we use these terms interchangeably to denote $k = 0$).

In this paper we will carry out an in-depth analysis of the transfer function $\tilde{g}_{\ell m \omega}(x, x')$. For that purpose, we first define some homogeneous solutions of the radial equation (9).

B. Radial Teukolsky equation

The homogeneous version of the radial equation (9),

$$\mathcal{L}[R_{\ell m \omega}] = 0, \quad (13)$$

where $R_{\ell m \omega} = R_{\ell m \omega}(x)$ is a radial function, is the key equation for frequency-domain perturbations of extremal Kerr. This second-order, linear ordinary differential equation has rank-1 irregular singular points at infinity ($r = \infty$, i.e., $x = \infty$) and at the horizon ($r = 1/2$, i.e., $x = 0$). This classifies it as a doubly-confluent Heun equation [33]. This is in contrast with the radial equation in subextremal Kerr, which instead possesses two regular singular points (at the Cauchy and event horizons) and only one irregular singular point (at $x = \infty$), thus classifying it as a confluent Heun equation. The main purpose of this paper is to, first, develop analytic

techniques for solving Eq. (13) and, second, use these techniques to obtain late-time tails of perturbations of extreme black holes.

In applications, it is natural to consider four different solutions to (13), which, following Leaver's notation [5], we denote by $R_{\pm}^{(0)}$ and $R_{\pm}^{(\infty)}$. These are defined according to boundary conditions imposed at the event horizon $x = 0$ and infinity⁴. In our notation, “0/∞” refers to the horizon/infinity, while “+/-” means a purely outgoing/incoming wave. For example, $R_+^{(\infty)}$ corresponds to the solution with purely outgoing radiation at infinity, while $R_+^{(0)}$ corresponds to radiation entering the black hole. Based on these properties, and following a more standard notation in the literature, we shall also denote $R_+^{(\infty)}$ by the upgoing radial solution $R_{\ell m \omega}^{\text{up}}$ and $R_+^{(0)}$ by the ingoing radial solution $R_{\ell m \omega}^{\text{in}}$. The two notations are interchangeable. Mathematically, the “in” and “up” functions satisfy the following boundary conditions:

$$R_{\ell m \omega}^{\text{in}} := R_+^{(0)} \sim \begin{cases} \mathcal{T}_{\text{in}} e^{ik/(2x)} x^{-2s} e^{-i\omega \ln x}, & x \rightarrow 0^+, \\ \mathcal{J}_{\text{in}} \frac{e^{-i\omega(x+\ln x)}}{x} + \mathcal{R}_{\text{in}} \frac{e^{i\omega(x+\ln x)}}{x^{1+2s}}, & x \rightarrow \infty, \end{cases} \quad (14)$$

and

$$R_{\ell m \omega}^{\text{up}} := R_+^{(\infty)} \sim \begin{cases} \mathcal{R}_{\text{up}} e^{ik/(2x)} x^{-2s} e^{-i\omega \ln x} + \mathcal{J}_{\text{up}} e^{-ik/(2x)} e^{i\omega \ln x}, & x \rightarrow 0^+, \\ \mathcal{J}_{\text{up}} \frac{e^{i\omega(x+\ln x)}}{x^{1+2s}}, & x \rightarrow \infty, \end{cases} \quad (15)$$

where $\mathcal{J}_{\text{in/up}}$, $\mathcal{R}_{\text{in/up}}$ and $\mathcal{T}_{\text{in/up}}$ are, respectively, complex-valued incidence, reflection and transmission coefficients of the ingoing/upgoing solutions.

The other homogeneous radial solutions $R_-^{(0)}$ and $R_-^{(\infty)}$ obey boundary conditions such that they are purely outgoing from the horizon and purely ingoing from infinity, respectively (see Eqs.(A2) and (A3)). Also following standard notation in the literature, we shall denote $R_-^{(0)}$ by the outgoing radial solution $R_{\ell m \omega}^{\text{out}}$. It satisfies

$$R_{\ell m \omega}^{\text{out}} := R_-^{(0)} \sim \mathcal{T}_{\text{out}} e^{-ik/(2x)} e^{i\omega \ln x}, \quad x \rightarrow 0^+, \quad (16)$$

where \mathcal{T}_{out} is its transmission coefficient.

Clearly, any pair of the above solutions, such as the pair $\{R_-^{(0)}, R_-^{(\infty)}\}$ or $\{R_+^{(0)}, R_+^{(\infty)}\}$, forms a complete set of linearly independent solutions of the homogeneous radial equation.

It shall prove useful to also normalize radial solutions and coefficients in a different way. We adopt the notation of placing a hat over a radial function or coefficient to indicate that quantity normalized via the corresponding transmission coefficient:

$$\hat{R}_{\ell m \omega}^{\text{in/up/out}} := \frac{R_{\ell m \omega}^{\text{in/up/out}}}{\mathcal{T}_{\text{in/up/out}}}, \quad \hat{\mathcal{J}}_{\text{in/up}} := \frac{\mathcal{J}_{\text{in/up}}}{\mathcal{T}_{\text{in/up}}}, \quad \hat{\mathcal{R}}_{\text{in/up}} := \frac{\mathcal{R}_{\text{in/up}}}{\mathcal{T}_{\text{in/up}}}. \quad (17)$$

In particular, it follows from Eq.(14) that

$$\hat{R}_{\ell m \omega}^{\text{in}} \sim e^{ik/(2x)} x^{-2s} e^{-i\omega \ln x}, \quad x \rightarrow 0^+, \quad (18)$$

$$\hat{R}_{\ell m \omega}^{\text{up}} \sim \frac{e^{i\omega(x+\ln x)}}{x^{1+2s}}, \quad x \rightarrow \infty. \quad (19)$$

In both of the asymptotic expressions in (18) and (19), applying the transformation $\{m \rightarrow -m, \omega \rightarrow -\omega^*\}$ is equivalent to complex-conjugating them. Similarly, using the symmetries in Eq.(7), it is easy to see that

⁴ It is worth noting that the asymptotics for $x \rightarrow 0^+$ and $x \rightarrow \infty$ in, respectively, Eqs.(14) and (15), are only true boundary conditions (i.e., specify the solutions uniquely when giving specific values to the transmission coefficients) when $\text{Im}(\omega) \geq 0$; the values of the solutions for $\text{Im}(\omega) < 0$ are determined by analytic continuation from $\text{Im}(\omega) \geq 0$. The reason is that, for $\text{Im}(\omega) < 0$, the transmitted waves become exponentially dominant solutions in those asymptotic regions: $e^{-i\omega \ln x} \gg e^{i\omega \ln x}$ for $x \rightarrow 0^+$ for in, and $e^{i\omega(x+\ln x)} \gg e^{-i\omega(x+\ln x)}$ for $x \rightarrow \infty$ for up; any exponentially subdominant solutions that may be part of the “boundary conditions” in those regions are not unambiguously determined by the asymptotic expressions.

applying $\{m \rightarrow -m, \omega \rightarrow -\omega^*\}$ on the radial operator \mathcal{L} in Eq.(13) is also equivalent to complex-conjugating it. It then follows that

$$\hat{R}_{\ell m \omega}^{\text{in}}(x) = \hat{R}_{\ell, -m, -\omega^*}^{\text{in}*}(x), \quad \hat{R}_{\ell m \omega}^{\text{up}}(x) = \hat{R}_{\ell, -m, -\omega^*}^{\text{up}*}(x). \quad (20)$$

Similarly, $\hat{\mathcal{J}}_{\text{in/up}}$ and $\hat{\mathcal{R}}_{\text{in/up}}$ are all complex-conjugated under $\{m \rightarrow -m, \omega \rightarrow -\omega^*\}$. This is the main reason for choosing the normalization as in the hatted radial quantities.

C. Transfer function

The method we adopt for constructing the transfer function involves a set of linearly independent homogeneous solutions of the radial differential equation (13). The radial solutions yielding the *retarded* Green function of the Teukolsky equation are the above in and up solutions, which correspond to solutions Ψ of the Teukolsky equation having no radiation coming out of the white hole or from past null infinity, respectively. The transfer function $\tilde{g}_{\ell m \omega}(x, x')$ corresponding to the retarded Green function is thus given by

$$\tilde{g}_{\ell m \omega}(x, x') = -\frac{R_+^{(0)}(x_<)R_+^{(\infty)}(x_>)}{\mathcal{W}}, \quad (21)$$

where $x_< := \min(x, x')$, $x_> := \max(x, x')$, \mathcal{W} is the constant scaled Wronskian,

$$\mathcal{W} := \Delta^{s+1} W[R_+^{(0)}, R_+^{(\infty)}] = 2i\omega \mathcal{J}_{\text{in}} \mathcal{T}_{\text{up}}, \quad (22)$$

and

$$\Delta := (r - r_H)^2 = x^2. \quad (23)$$

We use the notation

$$W[R_1, R_2] := R_1 \frac{dR_2}{dx} - R_2 \frac{dR_1}{dx} \quad (24)$$

for the actual Wronskian, where R_1 and R_2 are any two solutions of the homogeneous radial equation. We may equivalently express the transfer function $\tilde{g}_{\ell m \omega}(x, x')$ in terms of the hatted quantities as

$$\tilde{g}_{\ell m \omega}(x, x') = -\frac{\hat{R}_{\ell m \omega}^{\text{in}}(x_<)\hat{R}_{\ell m \omega}^{\text{up}}(x_>)}{\hat{\mathcal{W}}}, \quad (25)$$

where $\hat{\mathcal{W}}$ is the scaled Wronskian,

$$\hat{\mathcal{W}} := \Delta^{s+1} W[\hat{R}_{\ell m \omega}^{\text{in}}, \hat{R}_{\ell m \omega}^{\text{up}}] = 2i\omega \hat{\mathcal{J}}_{\text{in}} = ik \hat{\mathcal{T}}_{\text{up}}. \quad (26)$$

In the next-to-last equality in Eq.(26) we have evaluated the radial solutions for $x \rightarrow \infty$ and in its last equality we have evaluated them for $x \rightarrow 0^+$. It readily follows from Eq.(20) that $\hat{\mathcal{W}}$ is complex-conjugated under $\{m \rightarrow -m, \omega \rightarrow -\omega^*\}$ ⁵.

We now give other scaled Wronskian identities which will be useful for our later calculations. From the asymptotics in Eqs.(14), (15) and (16), it is straightforward to find

$$\Delta^{s+1} W[\hat{R}_{\ell m \omega}^{\text{out}}, \hat{R}_{\ell m \omega}^{\text{up}}] = -ik \hat{\mathcal{R}}_{\text{up}} \quad (27)$$

⁵ We note that there is a typographical error in Eq.3.17 [34]: the minus sign should not be present in its right hand side, with no consequences at all for any of the results in [34].

and

$$\Delta^{s+1} W \left[\hat{R}_{\ell m \omega}^{\text{out}}, \hat{R}_{\ell m \omega}^{\text{in}} \right] = -ik. \quad (28)$$

The last Wronskian identity that we give is

$$\Delta^{s+1} W \left[\hat{R}_{\ell m \omega}^{\text{in}}, \Delta^{-s} \hat{R}_{\ell m \omega}^{\text{up}*} \Big|_{-s} \right] = -2i\omega \hat{\mathcal{R}}_{\text{in}} = ik \hat{\mathcal{R}}_{\text{up}}^* \Big|_{-s}, \quad (29)$$

which is only valid for $\omega \in \mathbb{R}$. We note that, for $\omega \in \mathbb{R}$, if R is a solution of the radial Teukolsky equation for spin $s \neq 0$, then, because the corresponding radial operator \mathcal{L} is not self-adjoint, R^* is not a solution of the equation (for any spin), but $\Delta^{-s} R^*|_{-s}$ is a solution of the equation for spin s .

From Eqs.(25) and (26), it follows that the symmetry of Eq.(20) carries over to the transfer function modes: $\tilde{g}_{\ell m \omega}(x, x') = \tilde{g}_{\ell, -m, -\omega}^*(x, x')$. Applying $\{m \rightarrow -m, \omega \rightarrow -\omega^*\}$ to the exponential factors in the integrand in Eq.(3) is also equivalent to complex-conjugating them. If the whole integrand in Eq.(3) transformed in this way, then the ℓ -modes of the retarded Green function $G(x^\mu, x^{\mu'})$, as well as $G(x^\mu, x^{\mu'})$ itself, would be real-valued. However, under the transformation $\{m \rightarrow -m, \omega \rightarrow -\omega^*\}$, Eq.(8) shows that the angular factor ${}_s\mathcal{Z}_{\ell m \omega}(\theta, \theta')$ not only becomes complex-conjugated but also undergoes $\{\theta \rightarrow \pi - \theta, \theta' \rightarrow \pi - \theta'\}$ (or, equivalently, it undergoes $s \rightarrow -s$). This means that $G(x^\mu, x^{\mu'})$ is generally not real-valued (although it is real-valued on the equator $\forall s$ and everywhere for $s = 0$): complex-conjugating it is equivalent to taking $\{\theta \rightarrow \pi - \theta, \theta' \rightarrow \pi - \theta'\}$. Mathematically, the fact that $G(x^\mu, x^{\mu'})$ is not real-valued for $s \neq 0$ can be traced back to the fact that the Teukolsky operator is not self-adjoint for $s \neq 0$.

III. MST METHOD

In this section we bring forth the MST machinery and use it to derive practical analytic expressions for the various physically relevant solutions to the homogeneous radial Teukolsky equation and associated scattering amplitudes. We end the section with an exploration of the asymptotic behaviour of the series coefficients and renormalized angular momentum as $\omega \rightarrow \{0, m\}$.

We note that most of the results in this section have been numerically validated in [34, 35]. The numerical validation performed in these references consists of checking: (i) that values from different expressions in this paper for the same quantity (such as $\hat{\mathcal{W}}$ in Eq.(26)) agree with each other; (ii) that such values are consistent with limiting values from the MST formalism in subextremal Kerr evaluated for near-extremal values of a ; (iii) that values of quasinormal modes (which are poles in the complex frequency plane of the transfer function) obtained using expressions here agree with tabulated values in [36].

A. Series representations for the radial solutions

1. Radial series

Inspired by the work of Leaver [5] and MST [6, 7, 16], we make an ansatz for the homogeneous radial solutions as a sum over irregular confluent hypergeometric functions U . Our choices, which are based on Eqs.(191) and (192) of Ref. [5] (see Appendix A for further justification), are

$$\begin{aligned} R_{\pm}^{(\infty)} &= \zeta_{\pm}^{(\infty)} x^{-s+\nu} e^{ik/(2x)} e^{\pm i\omega x} (2\omega)^{\nu+1} e^{-i\pi\chi_s/2} e^{\mp i\pi(\nu+1/2)} \\ &\times \sum_{n=-\infty}^{\infty} \left(\frac{\Gamma(q_n^\nu + \chi_s)}{\Gamma(q_n^\nu - \chi_s)} \right)^{1/2} \left(\frac{\Gamma(q_n^\nu \pm \chi_s)}{\Gamma(q_n^\nu \mp \chi_s)} \right)^{1/2} (-2i\omega x)^n a_n^\nu U(q_n^\nu \pm \chi_s, 2q_n^\nu, \mp 2i\omega x) \end{aligned} \quad (30)$$

and

$$R_{\pm}^{(0)} = \zeta_{\pm}^{(0)} x^{-s-\nu-1} e^{i\omega x} e^{\pm ik/(2x)} k^{\nu+1} e^{-i\pi\chi_{-s}/2} e^{\mp i\pi(\nu+1/2)} \\ \times \sum_{n=-\infty}^{\infty} \left(\frac{\Gamma(q_n^{\nu} - \chi_{-s})}{\Gamma(q_n^{\nu} + \chi_{-s})} \right)^{1/2} \left(\frac{\Gamma(q_n^{\nu} \pm \chi_{-s})}{\Gamma(q_n^{\nu} \mp \chi_{-s})} \right)^{1/2} \frac{\Gamma(q_n^{\nu} + \chi_s)}{\Gamma(q_n^{\nu} - \chi_s)} \left(\frac{-ik}{x} \right)^n a_n^{\nu} U \left(q_n^{\nu} \pm \chi_{-s}, 2q_n^{\nu}, \mp \frac{ik}{x} \right), \quad (31)$$

where a_n^{ν} are series coefficients. Here we have defined

$$\chi_s := s - i\omega, \quad \chi_{-s} = -s - i\omega, \quad (32)$$

as well as

$$q_n^{\nu} := n + \nu + 1. \quad (33)$$

We have also introduced an auxiliary parameter ν , the so-called renormalized angular momentum, which plays an important role in all MST analyses. The normalization constants $\zeta_{\pm}^{(0)}$ and $\zeta_{\pm}^{(\infty)}$ will be chosen in Sec. III B such that the radial solutions are symmetric under $\nu \rightarrow -1 - \nu$. We have directly checked that Eqs.(30) and (31) satisfy the homogeneous radial equation (13) as long as the coefficients a_n^{ν} satisfy a certain recurrence relation (see Eq.(47) below). We shall deal with the series coefficients a_n^{ν} and with ν in the next subsection.

It follows from Eqs.(30) and (31), together with the analytical properties of the irregular confluent hypergeometric function U (as well as the prefactors $\omega^{\nu+1}$ and $k^{\nu+1}$, respectively), that, in principle, $\omega = 0$ is a branch point of the solutions $R_{\pm}^{(\infty)}$ and $\omega = m$ (i.e., $k = 0$) is a branch point of the solutions $R_{\pm}^{(0)}$. As we shall see in Secs.V and VI, these are indeed branch points of the radial solutions and they carry over to the transfer function.

We note that, in subextremal Kerr, while the corresponding $R_{\pm}^{(\infty)}$ solutions are similarly expressed in terms of the irregular confluent hypergeometric U -functions, the corresponding $R_{\pm}^{(0)}$ solutions are instead expressed in terms of the regular hypergeometric ${}_2F_1$ -functions. This is due to the aforementioned fact that the event horizon is a regular singular point of the radial equation in subextremal Kerr whereas it is an irregular singular point in extremal Kerr. As a consequence, the transfer function in subextremal Kerr only possesses a branch point at the origin, $\omega = 0$, which is responsible for the late-time decay of the linear perturbations [15, 20, 21]. The branch point at $\omega = m$ is a new feature of the extremal configuration and gives rise to the Aretakis phenomenon of field perturbations on the horizon hole [23, 24].

To simplify future calculations, we now give slightly more compact expressions for the radial solutions separately and obtain their transmission coefficients. For the ingoing and upgoing solutions, which are the ones of main interest here, the above expressions simplify a little further:

$$R_{\ell m \omega}^{\text{up}} = f_{\text{up}}(x, \omega) \sum_{n=-\infty}^{\infty} A_n^{\text{up}}(x, \omega) U(q_n^{\nu} + \chi_s, 2q_n^{\nu}, -2i\omega x) \quad (34)$$

and

$$R_{\ell m \omega}^{\text{in}} = f_{\text{in}}(x, k) \sum_{n=-\infty}^{\infty} A_n^{\text{in}}(x, k) U(q_n^{\nu} + \chi_{-s}, 2q_n^{\nu}, -\frac{ik}{x}), \quad (35)$$

where

$$f_{\text{up}}(x, \omega) := \zeta_+^{(\infty)} x^{-s+\nu} e^{ik/(2x)} e^{i\omega x} (2\omega)^{\nu+1} e^{-i\chi_s \pi/2} e^{-i\pi(\nu+\frac{1}{2})}, \quad (36a)$$

$$f_{\text{in}}(x, k) := \zeta_+^{(0)} x^{-s-\nu-1} e^{i\omega x} e^{ik/(2x)} k^{\nu+1} e^{-i\pi\chi_{-s}/2} e^{-i\pi(\nu+\frac{1}{2})}, \quad (36b)$$

and

$$A_n^{\text{up}}(x, \omega) := \frac{\Gamma(q_n^{\nu} + \chi_s)}{\Gamma(q_n^{\nu} - \chi_s)} (-2i\omega x)^n a_n^{\nu}, \quad (37a)$$

$$A_n^{\text{in}}(x, k) := \frac{\Gamma(q_n^{\nu} + \chi_s)}{\Gamma(q_n^{\nu} - \chi_s)} \left(\frac{-ik}{x} \right)^n a_n^{\nu}. \quad (37b)$$

The reason for writing ω as the argument of f_{up} but k as that of f_{in} is to make manifest their branch points at $\omega = 0$ and $k = 0$ respectively (we write the arguments of A_n^{in} and A_n^{up} merely out of notational consistency, not to denote any branch points in these functions). The transmission coefficients, defined via Eqs.(14) and (15), readily follow from Eqs.(34) and (35). Using Eq.13.2.6 [37], we obtain

$$\mathcal{T}_{\text{in}} = \zeta_+^{(0)} k^{\nu+1} (-ik)^{-\nu-1+s+i\omega} e^{-i\pi\chi_{-s}/2} e^{-i\pi(\nu+\frac{1}{2})} \sum_{n=-\infty}^{\infty} \frac{\Gamma(q_n^\nu + \chi_s)}{\Gamma(q_n^\nu - \chi_s)} a_n^\nu, \quad (38)$$

and

$$\mathcal{T}_{\text{up}} = \zeta_+^{(\infty)} e^{-i\chi_s\pi/2} e^{-i\pi(\nu+\frac{1}{2})} (-2i\omega)^{-\nu-1-s+i\omega} (2\omega)^{\nu+1} \sum_{n=-\infty}^{\infty} \frac{\Gamma(q_n^\nu + \chi_s)}{\Gamma(q_n^\nu - \chi_s)} a_n^\nu. \quad (39)$$

It will also be useful to give explicit expressions for $R_-^{(0)} \equiv R_{\ell m \omega}^{\text{out}}$ and its transmission coefficient near the horizon. From Eq.(31) we find that

$$R_{\ell m \omega}^{\text{out}} = f_{\text{out}}(x, k) \sum_{n=-\infty}^{\infty} A_n^{\text{out}}(x, k) U\left(q_n^\nu - \chi_{-s}, 2q_n^\nu, \frac{ik}{x}\right), \quad (40)$$

where

$$f_{\text{out}}(x, k) := \zeta_-^{(0)} x^{-s-\nu-1} e^{i\omega x} e^{-ik/(2x)} k^{\nu+1} e^{-i\pi\chi_{-s}/2} e^{i\pi(\nu+\frac{1}{2})} \quad (41)$$

and

$$A_n^{\text{out}}(x, k) := \left(\frac{-ik}{x}\right)^n a_n^\nu. \quad (42)$$

The coefficient of the outgoing solution near the horizon is readily obtained from Eq.(40) and Eq.13.2.6 [37]:

$$\mathcal{T}_{\text{out}} = \zeta_-^{(0)} k^{\nu+1} (ik)^{-\nu-1-s-i\omega} e^{-i\pi\chi_{-s}/2} e^{i\pi(\nu+\frac{1}{2})} \sum_{n=-\infty}^{\infty} (-1)^n a_n^\nu. \quad (43)$$

2. Series coefficients and renormalized angular momentum

Here we give recurrence relations for the MST coefficients a_n^ν and discuss the properties of the solutions of these relations.

Using⁶

$$\frac{1}{z} \hat{H}_L^+(-\eta, z) = -i \frac{(L+1-i\eta)}{(L+1)(2L+1)} \hat{H}_{L+1}^+(-\eta, z) + \frac{\eta}{L(L+1)} \hat{H}_L^+(-\eta, z) + i \frac{(L+i\eta)}{L(2L+1)} \hat{H}_{L-1}^+(-\eta, z), \quad (44)$$

$$\frac{d}{dz} \hat{H}_L^+(-\eta, z) = i \frac{L(L+1-i\eta)}{(L+1)(2L+1)} \hat{H}_{L+1}^+(-\eta, z) + \frac{\eta}{L(L+1)} \hat{H}_L^+(-\eta, z) + \frac{(L+1)(L+i\eta)}{L(2L+1)} \hat{H}_{L-1}^+(-\eta, z), \quad (45)$$

where

$$\hat{H}_L^+(\eta, z) := e^{iz} (-2iz)^{L+1} U(L+1+i\eta, 2L+2, -2iz), \quad (46)$$

⁶ Eqs.(44) and (45) are given below Eq.3.15 in [14] but here we correct a typographical error there of an extra factor of i in front of $\hat{H}_{L+1}^+(-\eta, z)$ and of $\hat{H}_{L-1}^+(-\eta, z)$.

we find that Eqs.(30) and (31) satisfy Eq.(13) as long as the coefficients a_n^ν satisfy the following bilateral recurrence relation:

$$\alpha_n a_{n+1}^\nu + \beta_n a_n^\nu + \gamma_n a_{n-1}^\nu = 0, \quad n \in \mathbb{Z}, \quad (47)$$

where

$$\begin{aligned} \alpha_n &:= \frac{\epsilon(q_n^\nu + \chi_s)(q_n^\nu - \chi_{-s})}{q_n^\nu(2q_n^\nu + 1)}, \\ \beta_n &:= (q_n^\nu - 1)q_n^\nu - {}_s\bar{A}_{\ell m \omega} - \epsilon \frac{\chi_s \chi_{-s}}{(q_n^\nu - 1)q_n^\nu}, \\ \gamma_n &:= \frac{\epsilon(q_n^\nu - 1 - \chi_s)(q_n^\nu - 1 + \chi_{-s})}{(q_n^\nu - 1)(2q_n^\nu - 3)}, \end{aligned} \quad (48)$$

and ${}_s\bar{A}_{\ell m \omega} := -\frac{7}{4}\omega^2 + s(s+1) + {}_sA_{\ell m \omega}$. Here we have defined

$$\epsilon := \omega k.$$

We note that, under $\nu \rightarrow -\nu - 1$, α_n transforms to γ_{-n} and β_n transforms to β_{-n} . We choose the normalization $a_0^{-\nu-1} = a_0^\nu = 1$ and, therefore, $a_{-n}^{-\nu-1} = a_{-n}^\nu$, $\forall n \in \mathbb{Z}$, directly follows.

The bilateral recurrence relations Eq.(47) may be solved in the following way. First, define the ratios

$$R_n := \frac{a_n^\nu}{a_{n-1}^\nu}, \quad L_n := \frac{a_n^\nu}{a_{n+1}^\nu}. \quad (49)$$

Then, using the recurrence relations, express these ratios as the following continued fractions:

$$R_n = -\frac{\gamma_n}{\beta_n + \alpha_n R_{n+1}} = -\frac{\gamma_n}{\beta_n} \cdot \frac{\alpha_n \gamma_{n+1}}{\beta_{n+1}} \cdot \frac{\alpha_{n+1} \gamma_{n+2}}{\beta_{n+2}} \dots \quad (50)$$

$$L_n = -\frac{\alpha_n}{\beta_n + \gamma_n L_{n-1}} = -\frac{\alpha_n}{\beta_n} \cdot \frac{\alpha_{n-1} \gamma_n}{\beta_{n-1}} \cdot \frac{\alpha_{n-2} \gamma_{n-1}}{\beta_{n-2}} \dots \quad (51)$$

Now, once R_n and L_n have been obtained (either numerically to within a certain prescribed precision or analytically up to a certain order in an expansion parameter), respectively, $\forall n > 0$ and $\forall n < 0$, then one can obtain $a_n^\nu = R_n a_{n-1}^\nu$, $\forall n > 0$, by starting from $n = 1$ (given a certain choice for a_0^ν as a normalization choice, such as ours, $a_0^\nu = 1$). Similarly, one can obtain $a_n^\nu = L_n a_{n+1}^\nu$, $\forall n < 0$.

To investigate the convergence of the continued fractions we carry out the large- n asymptotics of the series coefficients a_n^ν . In order to find the large- n behaviour of the solutions of the recurrence relations Eq.(47), we apply Theorem 2.3 of Ref. [38]. We find that there exists a pair of solutions, say $a_n^{(1)}$ and $a_n^{(2)}$, which satisfy

$$\frac{a_{n+1}^{(1)}}{a_n^{(1)}} \sim -\frac{\epsilon}{2} \frac{1}{n^2}, \quad \frac{a_{n+1}^{(2)}}{a_n^{(2)}} \sim -\frac{2}{\epsilon} n^2, \quad n \rightarrow +\infty, \quad (52)$$

and another pair, say $b_n^{(1)}$ and $b_n^{(2)}$, which satisfy

$$\left| \frac{b_n^{(1)}}{b_{n+1}^{(1)}} \right| \sim \frac{|\epsilon|}{2} \frac{1}{n^2}, \quad \left| \frac{b_n^{(2)}}{b_{n+1}^{(2)}} \right| \sim \frac{2}{|\epsilon|} n^2, \quad n \rightarrow -\infty. \quad (53)$$

Since $\lim_{n \rightarrow +\infty} a_n^{(1)}/a_n^{(2)} = 0$, $a_n^{(1)}$ is said to be a *minimal* solution (which is unique up to a normalization) and $a_n^{(2)}$ a *dominant* solution as $n \rightarrow +\infty$; similarly, $b_n^{(1)}$ is said to be a minimal solution and $b_n^{(2)}$ a dominant solution as $n \rightarrow -\infty$. The continued fraction in Eq.(50) [resp. Eq.(51)] converges when applied to a minimal solution as $n \rightarrow +\infty$ [resp. $n \rightarrow -\infty$] (see theorem 1.1 of Ref. [38]). The renormalized angular momentum parameter ν is determined by the consistency requirement that the minimal solution as $n \rightarrow \infty$ coincides with the minimal solution as $n \rightarrow -\infty$, i.e., that $a_n^{(1)} = b_n^{(1)}$. The series coefficients a_n^ν shall henceforth denote the corresponding unique minimal solution of the recurrence relations Eq.(47) as $n \rightarrow \pm\infty$ with the normalization choice $a_0^\nu = 1$.

The choice of a_n^ν as a minimal solution both as $n \rightarrow \infty$ and $n \rightarrow -\infty$ guarantees [5] that the series in Eq.(30) converges everywhere except on $r = M$ and that the series in Eq.(31) converges everywhere except on $r = \infty$. This means that the value of ν is fixed via the following condition:

$$R_n L_{n-1} = 1. \quad (54)$$

Equivalently, one may impose the condition

$$\beta_n + \alpha_n R_{n+1} + \gamma_n L_{n-1} = 0. \quad (55)$$

This is a transcendental equation for ν where R_n and L_n may be obtained from the continued fractions in Eqs.(50) and (51). One is free to choose the value of $n \in \mathbb{Z}$ in Eqs.(54) and (55).

The recurrence relations Eq.(47) in extremal Kerr are the same as in subextremal Kerr (see, e.g., Eqs.123 and 124 in [16]) when taking the extremal limit $a \rightarrow M$, except for a change in the signs of α_n and of γ_n . Such sign changes simply amount to $a_n^\nu \rightarrow (-1)^n a_n^\nu$ and they do not affect Eq.(54), which determines ν . Therefore, general properties of ν in subextremal Kerr which have been derived in the literature from its defining equation are also satisfied by ν in extremal Kerr. We next note some properties and symmetries exhibited by the MST construction and which relate to the renormalized angular momentum parameter ν^7 :

- The MST formalism is fundamentally invariant under $\nu \rightarrow -\nu - 1$. The reason is that ν was introduced as a parameter in the radial ordinary differential equation through the combination ' $\nu(\nu + 1)$ ' only (see Eq.119 [16] in the subextremal case). This leads, in particular, to the symmetry

$$a_n^{-\nu-1} = a_{-n}^\nu \quad (56)$$

observed previously. As long as we require the boundary conditions of the radial solutions to also be invariant under $\nu \rightarrow -\nu - 1$, the radial solutions will also satisfy this symmetry (see Eq.(57)).

- If ν is a solution of Eq.(55), then so is $\nu + k$, for any $k \in \mathbb{Z}$. The reason is that ν only appears in Eqs.(54) and (55) in the combination $\nu + n$, where $n \in \mathbb{Z}$.
- Applying $\nu \rightarrow \nu^*$ and $\omega \rightarrow \omega^*$ to all coefficients α_n , β_n and γ_n is equivalent, from their definitions, to complex-conjugating them⁸. This implies, from Eq.(55), that applying $\omega \rightarrow \omega^*$ to ν is equivalent to complex-conjugating ν . We note that these transformation properties are, however, not necessarily true if ω lies on a branch cut of ν ; in that case, ν is not necessarily real when ω is real.
- It has been shown (analytically, but with an assumption which is supported numerically) in subextremal Kerr in [16, 40] and in extremal Kerr in [34, 35] that, for ω real, ν is either real-valued or else complex-valued with a real part that is equal to a half-integer number.
- It follows from the above property that, for ω real, complex-conjugation of ν can be achieved by applying the MST symmetries of $\nu \rightarrow -\nu - 1$ and addition of an integer to ν .
- The series coefficients α_n , β_n and γ_n are all invariant⁹ under $m \rightarrow -m$ and $\omega \rightarrow -\omega$, and, therefore, so is ν .

We note that throughout the paper we make use of the symmetry (56) in the n -sums of the MST series.

⁷ We note that the parameter ν appears in other analyses of black hole perturbations which do not use the MST formalism. See, for example, [39] in subextremal Kerr, where ν is related to the monodromy of the upgoing radial solution at the irregular singular point $r = \infty$. See also Sec.III E here.

⁸ For β_n , we use the first two properties in Eq.(7).

⁹ For β_n , we use the last property in Eq.(7).

B. Symmetric decomposition of the radial solutions

We now write the radial solutions in a form which is manifestly invariant under $\nu \rightarrow -\nu - 1$. For that purpose, we use Eq. 13.2.42 [37] to write the solutions in terms of the *regular* confluent hypergeometric function M as¹⁰:

$$R_{\pm}^{(\infty)} = R_{\infty, \pm}^{\nu} + R_{\infty, \pm}^{-\nu-1}, \quad (57a)$$

$$R_{\pm}^{(0)} = R_{0, \pm}^{\nu} + R_{0, \pm}^{-\nu-1}, \quad (57b)$$

where

$$R_{\infty, \pm}^{\nu} := \zeta_{\pm}^{(\infty)} x^{-s+\nu} e^{ik/(2x)} e^{\pm i\omega x} (2\omega)^{\nu+1} e^{-\pi i \chi_s/2} e^{\mp i\pi(\nu+1/2)} \times \sum_{n=-\infty}^{\infty} \left(\frac{\Gamma(q_n^{\nu} + \chi_s)}{\Gamma(q_n^{\nu} - \chi_s)} \right)^{\frac{1}{2}} \left(\frac{\Gamma(q_n^{\nu} \pm \chi_s)}{\Gamma(q_n^{\nu} \mp \chi_s)} \right)^{\frac{1}{2}} \frac{\Gamma(1-2q_n^{\nu})}{\Gamma(1-q_n^{\nu} \pm \chi_s)} (-2i\omega x)^n a_n^{\nu} M(q_n^{\nu} \pm \chi_s, 2q_n^{\nu}, \mp 2i\omega x) \quad (58)$$

and

$$R_{0, \pm}^{-\nu-1} := \zeta_{\pm}^{(0)} |_{\nu} x^{-s-\nu-1} e^{i\omega x} e^{\pm ik/(2x)} k^{\nu+1} e^{-\pi i \chi_{-s}/2} e^{\mp i\pi(\nu+1/2)} \times \sum_{n=-\infty}^{\infty} \left(\frac{\Gamma(q_n^{\nu} - \chi_{-s})}{\Gamma(q_n^{\nu} + \chi_{-s})} \right)^{\frac{1}{2}} \left(\frac{\Gamma(q_n^{\nu} \pm \chi_{-s})}{\Gamma(q_n^{\nu} \mp \chi_{-s})} \right)^{\frac{1}{2}} \frac{\Gamma(q_n^{\nu} + \chi_s)}{\Gamma(q_n^{\nu} - \chi_s)} \frac{\Gamma(1-2q_n^{\nu})}{\Gamma(1-q_n^{\nu} \pm \chi_{-s})} \left(-\frac{ik}{x} \right)^n a_n^{\nu} M\left(q_n^{\nu} \pm \chi_{-s}, 2q_n^{\nu}, \mp \frac{ik}{x}\right). \quad (59)$$

The decompositions (57a) and (57b) simplify for the in and up solutions. For these, we find

$$R_{\infty, +}^{\nu} = \zeta_{+}^{(\infty)} x^{-s+\nu} e^{ik/(2x)} e^{i\omega x} (2\omega)^{\nu+1} e^{-\pi i \chi_s/2} e^{-i\pi(\nu+\frac{1}{2})} \times \sum_{n=-\infty}^{\infty} \frac{\Gamma(q_n^{\nu} + \chi_s) \Gamma(1-2q_n^{\nu})}{\Gamma(q_n^{\nu} - \chi_s) \Gamma(1-q_n^{\nu} + \chi_s)} (-2i\omega x)^n a_n^{\nu} M(q_n^{\nu} + \chi_s, 2q_n^{\nu}, -2i\omega x), \quad (60a)$$

$$R_{\infty, +}^{-\nu-1} = \zeta_{+}^{(\infty)} x^{-\nu-1-s} e^{ik/(2x)} e^{i\omega x} (2\omega)^{\nu+1} (-2i\omega)^{-2\nu-1} e^{-\pi i \chi_s/2} e^{-i\pi(\nu+\frac{1}{2})} \times \sum_{n=-\infty}^{\infty} \frac{\Gamma(2q_n^{\nu}-1)}{\Gamma(q_n^{\nu} - \chi_s)} (-2i\omega x)^{-n} a_n^{\nu} M(1-q_n^{\nu} + \chi_s, 2(1-q_n^{\nu}), -2i\omega x), \quad (60b)$$

and

$$R_{0, +}^{\nu} = \zeta_{+}^{(0)} x^{-s+\nu} e^{ik/(2x)} e^{i\omega x} k^{\nu+1} (-ik)^{-2\nu-1} e^{-\pi i \chi_{-s}/2} e^{-i\pi(\nu+\frac{1}{2})} \times \sum_{n=-\infty}^{\infty} \frac{\Gamma(q_n^{\nu} + \chi_s) \Gamma(2q_n^{\nu}-1)}{\Gamma(q_n^{\nu} - \chi_s) \Gamma(q_n^{\nu} + \chi_{-s})} a_n^{\nu} \left(\frac{ix}{k} \right)^n M\left(1-q_n^{\nu} + \chi_{-s}, 2(1-q_n^{\nu}), -\frac{ik}{x}\right), \quad (61a)$$

$$R_{0, +}^{-\nu-1} = \zeta_{+}^{(0)} x^{-s-\nu-1} e^{i\omega x} e^{ik/(2x)} k^{\nu+1} e^{-\pi i \chi_{-s}/2} e^{-i\pi(\nu+1/2)} \times \sum_{n=-\infty}^{\infty} \frac{\Gamma(q_n^{\nu} + \chi_s) \Gamma(1-2q_n^{\nu})}{\Gamma(q_n^{\nu} - \chi_s) \Gamma(1-q_n^{\nu} + \chi_{-s})} \left(\frac{k}{ix} \right)^n a_n^{\nu} M\left(q_n^{\nu} + \chi_{-s}, 2q_n^{\nu}, -\frac{ik}{x}\right). \quad (61b)$$

A transformation property we have used here is

$$\frac{\Gamma(q_n^{\nu} + \chi_s)}{\Gamma(q_n^{\nu} - \chi_s)} \rightarrow \frac{\sin(\pi(\nu - i\omega))}{\sin(\pi(\nu + i\omega))} \frac{\Gamma(q_n^{\nu} + \chi_s)}{\Gamma(q_n^{\nu} - \chi_s)}, \quad \text{under } \nu \rightarrow -\nu - 1, \quad (62)$$

for $s \in \mathbb{Z}$.

¹⁰ We note that here we choose to use the subindex “ ∞ ” for the solutions which, in the subextremal case, correspond to those in [16] with subindex “ C ”, referring to Coulomb wavefunctions. The reason is that in the extremal case both the ingoing and upgoing solutions have representations in terms of Coulomb wavefunctions.

Equations (57a) and (57b) put into manifest the symmetry of the radial solutions under $\nu \rightarrow -\nu - 1$ provided the ν -dependent normalizations are chosen appropriately. In particular, for Eqs.(60a)–(61b) to respect the symmetry under $\nu \rightarrow -\nu - 1$ in Eqs. (57) we require

$$\zeta_+^{(0)} \rightarrow e^{-2i\pi(\nu+\frac{1}{2})} (-ik)^{-2\nu-1} k^{2\nu+1} \frac{\sin(\pi(\nu+i\omega))}{\sin(\pi(\nu-i\omega))} \zeta_+^{(0)}, \quad \text{under } \nu \rightarrow -\nu - 1, \quad (63a)$$

$$\zeta_+^{(\infty)} \rightarrow e^{-2i\pi(\nu+\frac{1}{2})} (-i\omega)^{-2\nu-1} \omega^{2\nu+1} \frac{\sin(\pi(\nu+i\omega))}{\sin(\pi(\nu-i\omega))} \zeta_+^{(\infty)}, \quad \text{under } \nu \rightarrow -\nu - 1. \quad (63b)$$

The relations (63) are satisfied by

$$\zeta_+^{(0)} = k^{-\nu} (-ik)^\nu e^{i\pi\nu} \left(\frac{\sin(\pi(\nu-i\omega))}{\sin(\pi(\nu+i\omega))} \right)^{1/2}, \quad (64a)$$

$$\zeta_+^{(\infty)} = \omega^{-\nu} (-i\omega)^\nu e^{i\pi\nu} \left(\frac{\sin(\pi(\nu-i\omega))}{\sin(\pi(\nu+i\omega))} \right)^{1/2}, \quad (64b)$$

thereby fixing our normalization of the radial solutions.

C. Matching radial solutions

We now match the radial solutions. As both the in and up series solutions appearing in the right hand sides of Eq.(57) converge in $M < r < \infty$, we proceed similarly to Sec.4.4 [16] and match within the large overlap region of convergence. The $\nu \rightarrow -\nu - 1$ symmetry of the radial solutions halves the amount of work necessary to match the solutions. Consequently, we choose to explicitly match $R_{\infty,+}^\nu$ to $R_{0,+}^\nu$ and obtain the $R_{\infty,+}^{-\nu-1}$ to $R_{0,+}^{-\nu-1}$ by symmetry.

Now, we write the hypergeometric functions appearing in $R_{\infty,+}^\nu$ and $R_{0,+}^\nu$ as power series in x and match the coefficients. From Eq.(60a) and Eq.13.2.2 of [37] we readily obtain

$$R_{\infty,+}^\nu = \zeta_+^{(\infty)} x^{-s+\nu} e^{ik/(2x)} e^{i\omega x} (2\omega)^{\nu+1} e^{-i\pi\chi_s/2} e^{-i\pi(\nu+1/2)} \sum_{p=-\infty}^{\infty} \left(\sum_{n=-\infty}^p D_{n,p-n} \right) x^p, \quad (65)$$

where

$$D_{n,j} := \frac{\Gamma(q_n^\nu + \chi_s) \Gamma(1 - 2q_n^\nu)}{\Gamma(q_n^\nu - \chi_s) \Gamma(1 - q_n^\nu + \chi_s)} \frac{(q_n^\nu + \chi_s)_j}{(2q_n^\nu)_j j!} a_n^\nu (-2i\omega)^{n+j}, \quad (66)$$

and where $(z)_n := \Gamma(z+n)/\Gamma(z)$ denotes the Pochhammer symbol. Similarly, we obtain

$$R_{0,+}^\nu = \zeta_+^{(0)} x^{-s+\nu} e^{ik/(2x)} e^{i\omega x} k^{\nu+1} (-ik)^{-2\nu-1} e^{-i\pi/2} e^{-i\pi\chi_s/2} e^{-i\pi\nu} \sum_{p=-\infty}^{\infty} \left(\sum_{n=p}^{\infty} C_{n,n-p} \right) x^p, \quad (67)$$

where

$$C_{n,j} := \frac{\Gamma(q_n^\nu + \chi_s)}{\Gamma(q_n^\nu - \chi_s)} \frac{\Gamma(2q_n^\nu - 1)}{\Gamma(q_n^\nu + \chi_s)} \frac{(1 - q_n^\nu + \chi_s)_j}{(2 - 2q_n^\nu)_j j!} a_n^\nu (-ik)^{j-n}. \quad (68)$$

By comparing Eqs.(65) and (67) we see that they are proportional:

$$R_{0,+}^\nu = K_\nu R_{\infty,+}^\nu, \quad (69)$$

with

$$K_\nu := \frac{\zeta_+^{(0)}}{\zeta_+^{(\infty)}} k^{\nu+1} (-ik)^{-2\nu-1} (2\omega)^{-\nu-1} e^{i\pi s} \frac{\sum_{n=p}^{\infty} C_{n,n-p}}{\sum_{n=-\infty}^p D_{n,p-n}}, \quad (70)$$

and p is an arbitrary integer. From Eqs. (69) and (57b) it follows that

$$R_{\ell m \omega}^{\text{in}} = K_{\nu} R_{\infty, +}^{\nu} + K_{-\nu-1} R_{\infty, +}^{-\nu-1}. \quad (71)$$

This series representation of the in solution is valid at $r = \infty$ and will be used in the next section to find the ingoing radial coefficients at infinity.

Finally, a representation for the up solution which is valid in the horizon limit follows trivially from Eqs.(57a) and (69):

$$R_{\ell m \omega}^{\text{up}} = (K_{\nu})^{-1} R_{0, +}^{\nu} + (K_{-\nu-1})^{-1} R_{0, +}^{-\nu-1}. \quad (72)$$

This series representation will be used in a later section to find the upgoing radial coefficients at the horizon.

D. Radial coefficients (scattering amplitudes)

In Sec.III B we obtained series representations for the transmission coefficients of the in, up and out radial solutions. In this subsection we derive series representations for the remaining radial coefficients: the incidence and reflection amplitudes.

1. Scattering amplitudes at infinity

In order to obtain the radial coefficients at radial infinity, we split $R_{\infty, +}^{\nu}$ into two pieces: one which is purely ingoing and the other one purely outgoing at infinity. We note that these ingoing and outgoing pieces are, of course, proportional to, respectively, $R_{-}^{(\infty)}$ and $R_{+}^{(\infty)}$. For notation compactness, we label these new ingoing and outgoing solutions with a new variable: R_{+}^{ν} and R_{-}^{ν} , respectively. Specifically, using Eq. 6.7(7) Vol.1 [41], we split $R_{\infty, +}^{\nu}$ in Eq.(60a) as

$$R_{\infty, +}^{\nu} = R_{+}^{\nu} + R_{-}^{\nu}, \quad (73)$$

where

$$\begin{aligned} R_{+}^{\nu} := & \zeta_{+}^{(\infty)} x^{-s+\nu} e^{ik/(2x)} e^{-i\omega x} (2\omega)^{\nu+1} e^{-\pi\omega(1\pm 2)/2} e^{-i\pi\nu(1\mp 1) - \pi i(s+1)/2} \\ & \times \frac{\sin(\pi(\nu + i\omega))}{\sin(2\pi\nu)} \sum_{n=-\infty}^{\infty} (-2i\omega x)^n a_n^{\nu} U(q_n^{\nu} - \chi_s, 2q_n^{\nu}, 2i\omega x), \end{aligned} \quad (74)$$

and

$$\begin{aligned} R_{-}^{\nu} := & \zeta_{+}^{(\infty)} x^{-s+\nu} e^{ik/(2x)} e^{i\omega x} (2\omega)^{\nu+1} e^{-\pi\omega(1\pm 2)/2} e^{-i\pi\nu(1\pm 1) - \pi i(s+1)/2} \\ & \times \frac{\sin(\pi(\nu + i\omega))}{\sin(2\pi\nu)} \sum_{n=-\infty}^{\infty} \frac{\Gamma(q_n^{\nu} + \chi_s)}{\Gamma(q_n^{\nu} - \chi_s)} (-2i\omega x)^n a_n^{\nu} U(q_n^{\nu} + \chi_s, 2q_n^{\nu}, -2i\omega x), \end{aligned} \quad (75)$$

where the upper/lower signs respectively correspond to $\text{Re}(\omega x) > 0/\text{Re}(\omega x) < 0$. In deriving these relations we have assumed $s \in \mathbb{Z}$. By comparing Eq.(75) with Eq.(34) we see that R_{-}^{ν} is proportional to $R_{\text{up}} := R_{+}^{(\infty)}$ ¹¹. Similarly, comparing Eq.(74) with Eq.(30), it follows that R_{+}^{ν} is proportional to $R_{-}^{(\infty)}$. Therefore, R_{\pm}^{ν} are

¹¹ Mathematically, this comes from the fact that in Eq.(57) we split the irregular U -function appearing in the series for $R_{\ell m \omega}^{\text{up}}$ into two regular M -functions; in Eq.(73), in some sense we ‘undo’ this transformation by splitting each of these M -functions back into U -functions.

homogeneous solutions of the Teukolsky equation. It then follows from Eq.(73) that $R_{\infty,+}^\nu$ is also a homogeneous solution of the Teukolsky equation and from Eq.(69) that so is $R_{0,+}^\nu$.

In order to find the large- x asymptotics of these solutions we use Eq.13.7.3 [37] to obtain:

$$R_+^\nu \sim \mathcal{T}_+ \frac{e^{-i\omega(x+\ln x)}}{x}, \quad x \rightarrow \infty, \quad (76)$$

and

$$R_-^\nu \sim \mathcal{T}_- \frac{e^{i\omega(x+\ln x)}}{x^{1+2s}}, \quad x \rightarrow \infty, \quad (77)$$

for which

$$\mathcal{T}_+ := \zeta_+^{(\infty)} 2^{s-i\omega} \omega^{\nu+s} (i\omega)^{-\nu-i\omega} e^{-\pi\omega(1\pm 2)/2} e^{-i\pi\nu(1\mp 1)} e^{-\pi i} \frac{\sin(\pi(\nu+i\omega))}{\sin(2\pi\nu)} \sum_{n=-\infty}^{\infty} (-1)^n a_n^\nu, \quad (78a)$$

$$\mathcal{T}_- := \zeta_+^{(\infty)} 2^{-s+i\omega} \omega^{\nu-s} (-i\omega)^{-\nu+i\omega} e^{-\pi\omega(1\pm 2)/2} e^{-i\pi\nu(1\pm 1)} \frac{\sin(\pi(\nu+i\omega))}{\sin(2\pi\nu)} \sum_{n=-\infty}^{\infty} \frac{\Gamma(q_n^\nu + \chi_s)}{\Gamma(q_n^\nu - \chi_s)} a_n^\nu, \quad (78b)$$

where the upper/lower signs respectively correspond to $\text{Re}(\omega) > 0/\text{Re}(\omega) < 0$, and where we have again used that $s \in \mathbb{Z}$. This shows again, more explicitly, that R_+^ν and R_-^ν are proportional to, respectively, $R_+^{(\infty)}$ and $R_-^{(\infty)}$. Following the hatted notation in Eq.(17), we define $\hat{R}_+^\nu := R_+^\nu/\mathcal{T}_+$.

From now on and for the rest of this subsection, we choose the upper signs, i.e., we restrict ourselves to $\text{Re}(\omega) > 0$ – one can get the results for $\text{Re}(\omega) < 0$ from the symmetries in (20). It is then easy to check that:

$$R_+^{-\nu-1} = \mathcal{E}_+ R_+^\nu, \quad R_-^{-\nu-1} = \mathcal{E}_- R_-^\nu, \quad (79)$$

where

$$\mathcal{E}_+ := -(-i\omega)^{-2\nu} (i\omega)^{2\nu} e^{-2\pi i\nu}, \quad \mathcal{E}_- := e^{2\pi i\nu} \frac{\sin(\pi(\nu-i\omega))}{\sin(\pi(\nu+i\omega))}. \quad (80)$$

The first expression can be explicitly written as

$$\mathcal{E}_+ = \begin{cases} -1, & \arg \omega \in (0, \pi/2], \\ -e^{-4\pi i\nu}, & \arg \omega \in (-\pi/2, 0]. \end{cases} \quad (81)$$

As seen above, R_\pm^ν are solutions of the homogeneous Teukolsky equation and so, from Eq. (79), so are $R_\pm^{-\nu-1}$, as well as $R_{\infty,+}^{-\nu-1}$ and $R_{0,+}^{-\nu-1}$.

From Eqs. (71), (73), and (79) we have

$$R_{\ell m \omega}^{\text{in}} = (K_\nu + \mathcal{E}_+ K_{-\nu-1}) R_+^\nu + (K_\nu + \mathcal{E}_- K_{-\nu-1}) R_-^\nu. \quad (82)$$

Finally, using the large- x asymptotics of Eqs. (76) and (77) in Eq.(71) we obtain the incidence and reflection coefficients at infinity, as defined via Eq. (14):

$$\mathcal{I}_{\text{in}} = (K_\nu + \mathcal{E}_+ K_{-\nu-1}) \zeta_+^{(\infty)} 2^{s-i\omega} \omega^{\nu+s} (i\omega)^{-\nu-i\omega} e^{-3\pi\omega/2} e^{-\pi i} \frac{\sin(\pi(\nu+i\omega))}{\sin(2\pi\nu)} \sum_{n=-\infty}^{\infty} (-1)^n a_n^\nu, \quad (83)$$

for the incidence coefficient and

$$\begin{aligned} \mathcal{R}_{\text{in}} &= (K_\nu + \mathcal{E}_- K_{-\nu-1}) \zeta_+^{(\infty)} 2^{-s+i\omega} \omega^{\nu-s} (-i\omega)^{-\nu+i\omega} e^{-3\pi\omega/2} e^{-2\pi i\nu} \\ &\quad \times \frac{\sin(\pi(\nu+i\omega))}{\sin(2\pi\nu)} \sum_{n=-\infty}^{\infty} \frac{\Gamma(q_n^\nu + \chi_s)}{\Gamma(q_n^\nu - \chi_s)} a_n^\nu \end{aligned} \quad (84)$$

for the reflection coefficient.

2. Scattering amplitudes at the horizon

The reader may wish to also make use of the upgoing scattering amplitudes at the horizon, \mathcal{R}_{up} and \mathcal{J}_{up} . These could be obtained from the ingoing coefficients at infinity of the previous subsection combined with the Wronskian relations in Eq. (26) and, if $\omega \in \mathbb{R}$, Eq. (29). For completeness, in this subsection we derive these coefficients at the horizon directly - this will yield alternative expressions for these coefficients which are not readily obtainable from the coefficients at infinity combined with the Wronskian relations.

We find the upgoing coefficients at the horizon by asymptotic expansion of the matching expression (72), taking $x \rightarrow 0$ while keeping $|k|$ finite. The small- x asymptotics of $R_{0,+}^\nu$ directly follow from Eqs. (61a) and the large-argument behavior of the Kummer M functions (which crucially depends on the angle of approach, which here corresponds to the argument of k) given in p.278 of [41]. The result is

$$R_{0,+}^\nu \sim \zeta_+^{(0)} k^\nu (-ik)^{-2\nu} e^{-i\pi(2\nu+\chi-s)/2} \left(\mathcal{A}_\nu(-ik)^s \left(\frac{e^{i\pi\varsigma}}{-ik} \right)^{-i\omega-\nu} e^{ik/(2x)} x^{-i\omega-2s} + \mathcal{B}_\nu(-ik)^{-s-i\omega+\nu} e^{-ik/(2x)} x^{i\omega} \right), \quad (85)$$

with $\varsigma := -\text{sgn}(\text{Re } k)$ and

$$\mathcal{A}_\nu := \frac{\sin(\pi(\nu - i\omega))}{\sin(2\pi\nu)} \sum_{n=-\infty}^{\infty} \frac{\Gamma(q_n^\nu + \chi_s)}{\Gamma(q_n^\nu - \chi_s)} a_n^\nu, \quad (86)$$

$$\mathcal{B}_\nu := -\pi \csc(2\pi\nu) \sum_{n=-\infty}^{\infty} \frac{\Gamma(q_n^\nu + \chi_s)}{\Gamma(q_n^\nu - \chi_s) \Gamma(q_n^\nu + \chi_{-s}) \Gamma(1 - q_n^\nu + \chi_{-s})} a_n^\nu. \quad (87)$$

The complete small- x asymptotics of the upgoing solution are found by invoking the $\nu \leftrightarrow -\nu - 1$ exchange symmetry of (72). Making the appropriate identifications with Eq.(15) leads to the radial coefficients

$$\mathcal{R}_{\text{up}} = \mathcal{A}_\nu (K_\nu)^{-1} \zeta_+^{(0)} e^{-\pi(2i\nu+\omega)/2} k^{\nu+s} (-ik)^{-2\nu} \left(\frac{e^{i\pi\varsigma}}{-ik} \right)^{-\nu-i\omega} + (\nu \rightarrow -\nu - 1) \quad (88)$$

and

$$\mathcal{J}_{\text{up}} = \mathcal{B}_\nu (K_\nu)^{-1} \zeta_+^{(0)} (-1)^s e^{-\pi(2i\nu+\omega)/2} k^{\nu-s} (-ik)^{-\nu-i\omega} + (\nu \rightarrow -\nu - 1), \quad (89)$$

where “ $(\nu \rightarrow -\nu - 1)$ ” denotes the term preceeding it evaluated under this transformation.

E. Series coefficients and renormalized angular momentum for $\omega \rightarrow 0, m$

Later in the paper we shall be interested in obtaining the contribution to the Green function coming from the branch points at $\omega = 0$ and $\omega = m$. For that purpose, in this subsection we analyze the series coefficients a_n^ν and the renormalized angular momentum ν as $\omega \rightarrow 0, m$.

Let us first consider an expansion for ϵ small of the recurrence relations satisfied by the series coefficients a_n^ν . It is clear from Eq.(48) that $\alpha_n = O(\epsilon)$, $\beta_n = O(1)$ and $\gamma_n = O(\epsilon)$, except for possible special values of ν such that β_n or the denominators of α_n or γ_n vanish for $\epsilon \rightarrow 0$. Furthermore, from Eq.(52), we have that $R_n = O(\epsilon)$ and $L_{-n} = O(\epsilon)$ for sufficiently large n . Therefore, except for the mentioned possible special cases, the order of a_n^ν for small ϵ increases as $|n|$ increases. We will not consider these possible special cases, since in principle they should not affect the behaviour of the radial solutions to *leading* order for small ϵ (see Eq.172 [16]). Therefore, barring these special cases, we have that

$$a_n^\nu = O\left(\epsilon^{|n|}\right), \quad \epsilon \rightarrow 0. \quad (90)$$

Obtaining the leading-order behaviour of ν for small ϵ is trivial. One just imposes that Eq.(54) is satisfied to $O(1)$ for small ϵ with the choice of, e.g., $n = 1$, and uses the property, shown above, that $R_2 = O(\epsilon)$ and $L_{-1} = O(\epsilon)$. This requires that both the $O(1)$ and $O(\epsilon)$ terms in β_0 are zero. Requiring first the $O(1)$ term in β_0 to be zero readily yields

$$\lim_{\omega \rightarrow 0, m} \nu(\nu + 1) = \lim_{\omega \rightarrow 0, m} {}_s\bar{A}_{\ell m \omega}. \quad (91)$$

This result agrees with Leaver [5]. Note that if we had used Eq.(54) with $n = k$, for some $k \in \mathbb{Z}$, we would have found the same equation as Eq.(91) but with ν shifted by k . Therefore, fixing the value of n in the condition Eq.(54) eliminates the symmetry of the MST formalism under $\nu \rightarrow \nu + k$, $k \in \mathbb{Z}$ and in this paper we choose $n = 1$ in (54) to determine ν .

In the limit $\omega \rightarrow 0$, we have ${}_s\bar{A}_{\ell m \omega} \rightarrow \bar{A}_{\ell m 0} = \ell(\ell + 1)$, and so Eq.(91) implies that $\nu = \ell$ or $\nu = -\ell - 1$ for $\omega = 0$. In its turn, in the superradiant bound limit, Eq.(91) implies that

$$\lim_{\omega \rightarrow m} \nu = \nu_{c, \pm} := -\frac{1}{2} \pm \sqrt{\frac{1}{4} + {}_sK_{\ell m} - 2m^2}, \quad (92)$$

where ${}_sK_{\ell m} := ({}_sA_{\ell m \omega} + M^2\omega^2 + s(s + 1))|_{\omega=m} = {}_sA_{\ell, m, m} + m^2/4 + s(s + 1)$ is the separation constant of [23, 24, 42] commonly used in the Kerr/CFT literature, for example in [43]. As expected, the relationship $\nu_{c, \pm} = -\nu_{c, \mp} - 1$ is satisfied. The parameter h in [23, 24], which determines the rate of the horizon instability of a field perturbation to extremal Kerr, is equal to $-\nu_{c, -}$. We also note that $-(\nu_{c, \pm} + 1/2)^2$ is equal to δ^2 , in Eq.A6 [44] (which is the same as δ^2 in the appendix of [45]) for $a = M = 1/2$ at $\omega = m$. As mentioned in Sec.III A 2, it has been observed that, for ω real, ν is either real or else it is complex with a real part equal to a half-integer number. Similar properties were observed by us in Eq.(23) of [23] and Eq.(67) of [24] for h . Finally, a property which we shall utilize later is that $\nu_{c, \pm}$ is invariant under $s \rightarrow -s$, since ${}_sK_{\ell m} = -{}_sK_{\ell m}$ follows from Eq.(7).

Requiring the $O(\epsilon)$ term in β_0 to be zero would yield the next-to-leading order term for ν . Orders higher than leading order are easy to find, then, by introducing in Eq.(54) an expansion for $\bar{A}_{\ell m}$ for either ω small or k small, which is assumed to be known (this is certainly true for ω small – see [46]) and a corresponding expansion for ν with undetermined coefficients. These coefficients can be found by iteratively solving the equation to higher orders. For example, for small $|\omega|$ (and any $\arg \omega$), we have

$$\nu = \ell + \nu_2 \omega^2 + \nu_3 \omega^3 + O(\omega^4), \quad (93)$$

where

$$\nu_2 := \frac{(-15\ell^4 - 30\ell^3 - 6\ell^2 s^2 - 4\ell^2 - 6\ell s^2 + 11\ell - 3s^4 + 6s^2)}{2\ell(\ell + 1)(2\ell + 1)(4\ell^2 + 4\ell - 3)}, \quad (94a)$$

$$\nu_3 := \frac{m}{(\ell - 1)\ell^2(\ell + 1)^2(\ell + 2)(2\ell - 1)(2\ell + 1)(2\ell + 3)} \left(5\ell^6 + 15\ell^5 + \ell^4(3s^2 + 2) + 3\ell^3(2s^2 - 7) + \ell^2(3s^4 - 6s^2 - 7) + 3\ell(s^4 - 3s^2 + 2) + s^2(5s^4 - 16s^2 + 11) \right). \quad (94b)$$

IV. ANALYTICAL PROPERTIES OF THE TRANSFER FUNCTION

In pioneering work [20], Leaver deformed the Laplace integral contour corresponding to Eq.(3) in Schwarzschild spacetime into the complex frequency plane. In doing so, he revealed how various types of singularities in the transfer function contribute to the full Green function in their own ways. Subtleties aside, the qualitative picture drawn by Leaver goes as follows. At “early” times, direct propagation on the future light cone derives from a large- $|\omega|$ arc (the only contribution which does not come from a singularity in the transfer function). At very late times, the field exhibits a power-law time dependence deriving from frequencies near the branch point(s)

on the real axis. At “intermediate” times, the field takes the form of a decaying sinusoid coming from the quasinormal modes¹².

Indeed, Leaver’s picture is supported by the asymptotic theory of Laplace transforms [48], in which the late-time behaviour of a function of time, say $f(t)$ as $t \rightarrow \infty$, is related to the asymptotics of its Laplace transform, $\tilde{f}(\omega)$, near its uppermost singular point ω_0 , or points $\{\omega_i\}_{i=0,1,2,\dots}$ if more than one happen to lie on the same abscissa, in the complex- ω plane.

As mentioned in Sec. III A, in extremal Kerr, the radial solutions $R_{\pm}^{(\infty)}$ in principle possess a branch point at zero frequency ($\omega = 0$) and $R_{\pm}^{(0)}$ possess a branch point at the superradiant bound frequency ($\omega = m$). As we shall explicitly show in the following sections, the radial solutions indeed possess these branch points and they carry over to the transfer function. Whereas the branch point at $\omega = 0$ already exists in subextremal Kerr, with associated late-time decay having been analyzed in [15, 20, 21], the branch point at $\omega = m$ is new to extremal Kerr. The late-time decay of the master field on the horizon due to the emergent branch point at $\omega = m$ has been analyzed in Refs. [23, 24].

Apart from the “physical” branch points at $\omega = 0$ and m that give rise to the late-time decay in Leaver’s picture, the various mode quantities may also possess other branch points which may be deemed “unphysical” in a certain sense. These “unphysical” branch points are of two types. The first type of “unphysical” branch points comes from the angular eigenvalue ${}_sA_{\ell m \omega}$ and eigenfunction ${}_sS_{\ell m \omega}$ [49–51], and in principle carry over to the renormalized angular momentum ν and the MST series coefficients a_n^ν . It has been shown that the angular branch points do not lie on the real axis when $s = 0$ [50]. Numerical evidence suggests that this is also the case for $s \neq 0$ (for this, one can make use of the MATHEMATICA toolkit in [52]). Furthermore, it has been shown that the angular branch points vanish upon summation over ℓ and m [15], contributing nothing to the full Green function. The second type of “unphysical” branch points is observed directly in ν and a_n^ν . Numerical evidence presented in [34, 35] suggests that these two quantities possess discontinuities which may be removed by using the above “MST symmetries” of addition of an integer to ν and/or transformation $\nu \rightarrow -\nu - 1$. Other than singularities of the removable type and possible angular branch points (inherited from the eigenvalue), ν and a_n^ν do not appear to display any further discontinuities. In conclusion, we are confident that potential discontinuities in ${}_sA_{\ell m \omega}$, ${}_sS_{\ell m \omega}$, ν and a_n^ν will not influence any physical quantities such as the scalar/electromagnetic/gravitational wave tail. They will thus be ignored in subsequent calculations.

Apart from the above mentioned physical branch points at $\omega = 0$ and m , the transfer function also possesses poles in the complex-frequency plane corresponding to the quasinormal modes. In Fig. 1 we schematically represent the various physical singularities of the transfer function in the complex-frequency plane, as well as the integration contours for the Green function.

There is strong numerical evidence [34, 36, 53] that the uppermost singular points of the transfer function in extremal Kerr are the branch points at the origin ($\omega = 0$) and the superradiant bound ($k = 0$). A complimentary analytical argument may be found in Sec. VII B of [34]. Furthermore, there is a rigorous result for mode stability in the very recent [54] and rigorous linear stability results exist for the specific case of *axisymmetric* scalar field perturbations in [55].

Using our MST expressions, in the next three sections we provide a formalism for obtaining the discontinuity in the transfer function across the branch cuts (BCs) extending from $\omega = 0$ and from $k = 0$ to arbitrary order in the frequency. We explicitly calculate the leading-order tail due to the $\omega = 0$ branch point and the leading-order transfer function near $k = 0$ (which yields the known leading order tail due to the $k = 0$ BC). The former is a new result, whereas we find that the latter agrees with the existing results in [23, 24] obtained using MAE.

¹² Although perhaps less known, there is also a contribution at “intermediate” times from frequencies along the branch cut which are not “near” the branch point [10, 47]

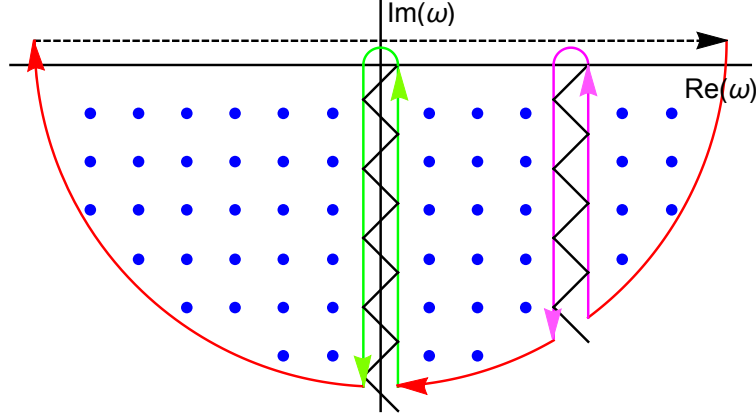


FIG. 1. Schematic representation in the complex frequency plane of the singularities of the transfer function and integration contours for extremal Kerr. The physical singularities are: (i) blue dots correspond to simple poles (quasi-normal modes; for simplicity, we plot them symmetrically with respect to the negative imaginary axis, although that is generally not the case in Kerr spacetime); (ii) two criss-crossed black lines corresponding to branch cuts down from the origin $\omega = 0$ and the superradiant-bound frequency $\omega = m$ (here represented for $m > 0$). The dashed (black) straight line corresponds to the original integration contour in Eq.(3). This contour may be deformed so as to yield an integration over a high-frequency arc (red semicircle) together with integrals around the two branch cuts (green and pink contours wrapped around the criss-crossed lines). N.B.: here we omit any “unphysical” (see text) branch points in the transfer function.

V. BRANCH CUT FROM THE ORIGIN AND TAIL

As advanced in Sec.III A, in extremal Kerr, the upgoing radial solution possesses a branch point at $\omega = 0$ in the complex frequency plane, which carries over to the transfer function, similarly to what happens in subextremal Kerr. We choose the corresponding BC to point down the negative imaginary axis.

We shall use the notation that, if $A = A(\omega)$ is a function of the frequency possessing a branch point at $\omega = 0$, then

$$\delta A := A(\omega) - A(\omega e^{2\pi i}) \quad (\text{change in function } A \text{ across BC extending from } \omega = 0). \quad (95)$$

In particular, the discontinuity in the transfer function across the BC is

$$\delta \tilde{g}_{\ell m \omega}(x, x') := \tilde{g}_{\ell m \omega}(x, x')|_{\omega=-i\sigma} - \tilde{g}_{\ell m \omega}(x, x')|_{\omega=-i\sigma e^{2\pi i}}, \quad (96)$$

where $\sigma > 0$. The contribution to an (ℓ, m) -mode of the Green function due to the branch point at the origin is then obtained by, essentially, integrating the discontinuity in the transfer function, $\delta \tilde{g}_{\ell m \omega}(x, x')$, along the corresponding BC:

$$\delta G_{\ell m}(x^\mu, x'^\mu) := -i e^{im\phi} \int_0^\infty d\sigma \, e^{-\sigma t} \delta \tilde{g}_{\ell m \omega}(x, x')_s \mathcal{E}_{\ell m \omega}(\theta, \theta')|_{\omega=-i\sigma}. \quad (97)$$

As explained earlier, the late-time behaviour of $\delta G_{\ell m}$ will be given by the small- σ behaviour of the integrand in Eq.(97).

In this section, we lay out the MST formalism for calculating the discontinuity of the transfer function across the BC down from the origin. We also calculate, in separate subsections, the contribution of this BC to leading-order for late-times for a field point at: (i) timelike infinity ($t \rightarrow \infty$, r_* finite), (ii) the future event horizon ($u \rightarrow \infty$, v finite) and (iii) future null infinity ($v \rightarrow \infty$, u finite), where $u := t - r_*$ is retarded “time”, $v := t + r_*$ is advanced “time” and $r_* := x - 1/(2x) + \ln x$.

A. Discontinuity in the up modes

Here, we give an analytic expression for the discontinuity in the up solution across the BC originating from $\omega = 0$. For this purpose, we will adapt to extremal Kerr a technique used in [14] in Schwarzschild spacetime and in [15] in subextremal Kerr spacetime.

Our starting point is Eq.(34) for the up radial solution. In it, we identify the source of the discontinuity across the BC down the negative imaginary axis. There are two factors (leaving aside $\zeta_+^{(\infty)}$) which are discontinuous across this cut: ω^ν and the confluent hypergeometric function U . The analytic continuation of the first factor is trivial:

$$\omega^\nu \rightarrow e^{2\pi i\nu} \omega^\nu \quad \text{as} \quad \omega \rightarrow e^{2\pi i} \omega. \quad (98)$$

The analytic continuation of the second factor is given in Eq.(13.2.12) [37]. Combining the analytic continuation of the two factors, we can write

$$\left. \frac{R_{\ell m \omega}^{\text{up}}}{\zeta_+^{(\infty)}} \right|_{\omega \rightarrow e^{2\pi i} \omega} = \frac{f_{\text{up}}(x, \omega)}{\zeta_+^{(\infty)}(\omega)} e^{2\pi i\nu} \sum_{n=-\infty}^{\infty} A_n^{\text{up}}(x, \omega) \left(\left(1 - e^{-2\pi i b}\right) \frac{\Gamma(1-b)}{\Gamma(1+d-b)} M(d, b, -2i\omega x) + e^{-2\pi i b} U(d, b, -2i\omega x) \right), \quad (99)$$

where

$$d := q_n^\nu + \chi_s, \quad b := 2q_n^\nu. \quad (100)$$

We find it convenient to normalize the up solution by its transmission coefficient, i.e., to use $\hat{R}_{\ell m \omega}^{\text{up}}$ instead of $R_{\ell m \omega}^{\text{up}}$. The analytic continuation of this coefficient (“normalized” via $\zeta_+^{(\infty)}$) follows trivially from (39):

$$\left. \frac{\mathcal{T}_{\text{up}}}{\zeta_+^{(\infty)}} \right|_{\omega \rightarrow e^{2\pi i} \omega} = e^{-2\pi\omega} \frac{\mathcal{T}_{\text{up}}}{\zeta_+^{(\infty)}}. \quad (101)$$

We next use Eq.(6.7.7) of Vol.1 [41] to express the M function in Eq.(99) in terms of U functions, as well as the following straightforward identity,

$$e^{2\pi i\nu} (1 - e^{-2\pi i b}) \frac{\Gamma(1-b)\Gamma(b)e^{-\pi i a}}{\Gamma(1+d-b)\Gamma(b-d)} = e^{-2\pi\omega} - e^{-2\pi i\nu}, \quad (102)$$

in order to obtain

$$\delta \hat{R}_{\ell m \omega}^{\text{up}} = \frac{f_{\text{up}}}{\mathcal{T}_{\text{up}}} e^{-2i\omega x} e^{2\pi(\omega + i\nu)} (e^{-2\pi i\nu} - e^{-2\pi\omega}) \sum_{n=-\infty}^{\infty} (-2i\omega x)^n a_n^\nu U(b-d, b, 2i\omega x). \quad (103)$$

With all the quantities on the right hand side of Eq.(103) evaluated just on the right side of the BC, i.e., for $\omega = \lim_{c \rightarrow 0+} (-i\sigma + c)$, assuming $\sigma > 0$ throughout, we have an analytic expression for the discontinuity across the negative imaginary axis of the up solution normalized to have unit transmission coefficient. By comparison with Eq.(74), we observe that this discontinuity is proportional to the solution R_+^ν :

$$\delta \hat{R}_{\ell m \omega}^{\text{up}} = i q(\sigma) \left. \hat{R}_+^\nu \right|_{\omega = -i\sigma}, \quad (104)$$

where the BC “strength” $q(\sigma)$ is given by

$$q(\sigma) := -2 \sin(2\pi\nu) e^{2\pi\omega} \frac{\mathcal{T}_+}{\mathcal{T}_{\text{up}}}, \quad (105)$$

with all quantities on the right hand side evaluated at $\omega = \lim_{c \rightarrow 0+} (-i\sigma + c)$, with $c > 0$. We note that \hat{R}_+^ν has a branch point at $\omega = 0$. But, because of the boundary conditions Eq.(76) that it satisfies at infinity (with a

wave term $e^{-i\omega(x+\ln x)}$ being the complex-conjugate of the wave term $e^{+i\omega(x+\ln x)}$ in the boundary conditions for $\hat{R}_{\ell m \omega}^{\text{up}}$, its BC lies on the *positive* imaginary axis. Since \hat{R}_+^ν in Eq.(104) is evaluated on the *negative* imaginary axis, there is no ambiguity as to its value there.

The proportionality relation in Eq.(104) was to be expected for the following reason. The function $\hat{R}_{\ell m \omega}^{\text{up}}$ is a solution of the homogeneous radial equation having the same (purely-outgoing, $e^{+i\omega(x+\ln x)}$) asymptotic behaviour at radial infinity whether it is evaluated at ω or at $\omega e^{2\pi i}$. As mentioned in Sec.II, however, this behaviour corresponds to an exponentially dominant solution when $\text{Im}(\omega) < 0$, as is the case for ω on the negative imaginary axis. While the solutions $\hat{R}_{\ell m \omega}^{\text{up}}$ at ω and at $\omega e^{2\pi i}$ have the same dominant asymptotic behaviour, they differ in the amount of the subdominant solution (which is purely-ingoing, $e^{-i\omega(x+\ln x)}$, at radial infinity) that they contain. This means that $\delta \hat{R}_{\ell m \omega}^{\text{up}}$ is a solution of the homogeneous radial equation and satisfies a purely-ingoing boundary condition at radial infinity (and for ω on the negative imaginary axis). Since R_+^ν is also a solution of the homogeneous radial equation and, from (76), it is purely-ingoing at radial infinity, it follows that it must be proportional to $\delta \hat{R}_{\ell m \omega}^{\text{up}}$. Eq.(104) is this proportionality relation. This expression for the discontinuity of the upgoing radial solution across the BC allows us to find the corresponding discontinuity of the transfer function that we give in the next subsection.

B. Discontinuity in the transfer function

In [15] it was shown that the discontinuity in the transfer function across the BC is given by

$$\delta \tilde{g}_{\ell m \omega}(x, x') = -2i\sigma \frac{q(\sigma)}{\mathcal{W}^+ + \mathcal{W}^-} \left[\hat{R}_{\ell m \omega}^{\text{in}}(x) \hat{R}_{\ell m \omega}^{\text{in}}(x') \right]_{\omega = -i\sigma}. \quad (106)$$

Here, $\mathcal{W}^{+/-}$ is equal to \mathcal{W} evaluated, respectively, on the right/left of the BC and, as always, $\sigma > 0$. Eq.(106) was proven in [15] in subextremal Kerr and with the subextremal counterparts of the Wronskian \mathcal{W} and the BC strength q . However, it is trivial to see that Eq.(106) is also valid in extremal Kerr with the Wronskian \mathcal{W} and BC strength q as defined in Eq.(104), which is used in order to derive Eq.(106).

We note the following pertinent point for the small- ω asymptotics of the transfer function. Infinite series of confluent hypergeometric U -functions whose last argument goes to zero as $\omega \rightarrow 0$ and whose second argument grows with n at least like $2n$, such as the series for $R_{\ell m \omega}^{\text{up}}$ in Eq.(34) and for R_+^ν in Eq.(74), are not amenable to asymptotics for $\omega \rightarrow 0$ (see, e.g., Eq.(13.2.16) [37] together with Eq.(90)). On the other hand, Eq.(106) offers an expression for $\delta \tilde{g}_{\ell m \omega}$ as proportional to the ingoing solution: $R_{\ell m \omega}^{\text{in}}(x) R_{\ell m \omega}^{\text{in}}(x')$. The ingoing solution has the representation Eq.(35) containing U -functions whose last argument does not go to zero as $\omega \rightarrow 0$ and is thus amenable to $\omega \rightarrow 0$ asymptotics. This allows us to find the small- ω asymptotics of $\delta \tilde{g}_{\ell m \omega}$.

An expression for the denominator in Eq.(106) in terms of the ingoing radial coefficients is also given in [15]¹³ (it is valid in extremal Kerr as well as subextremal Kerr):

$$\mathcal{W}^+ \mathcal{W}^- = \left(2\sigma \frac{\mathcal{J}_{\text{in}}}{\mathcal{I}_{\text{in}}} \right)^2 + 4i\sigma^2 q(\sigma) \frac{\mathcal{J}_{\text{in}} \mathcal{R}_{\text{in}}}{(\mathcal{I}_{\text{in}})^2}, \quad (107)$$

where all the quantities are meant to be evaluated in the limit to the negative imaginary axis from the fourth quadrant.

Here we have laid the foundation for obtaining the discontinuity of the transfer function $\tilde{g}_{\ell m \omega}$ across the negative imaginary axis for any $\sigma := i\omega > 0$. From Eqs.(106), (107), (105) and the expressions for the radial coefficients and for $\hat{R}_{\ell m \omega}^{\text{in}}$ derived in the previous sections, one can obtain the discontinuity of the transfer function across the BC down from $\omega = 0$ and so, via Eq.(97), the late-time tail of the (ℓ, m) -modes of the

¹³ We note that there is a typographical error in Eq.5.10 [15]: the sign of the second term should be ‘+’ instead of ‘-’.

retarded Green function. This could be done either exactly, e.g., via a semianalytic/numeric evaluation of the infinite sums or analytically up to arbitrary order at late times by systematically expanding the expressions for small frequency, in a manner similar to [15] in subextremal Kerr or to [14] in Schwarzschild¹⁴. In the next subsections we shall do the latter to leading order: we provide the leading small- ω behaviour of various quantities and use them to obtain the leading late-time behaviour of an (ℓ, m) -mode of the Green function. We note that the results in these subsections are not valid in the axisymmetric case ($m = 0$), where the branch points at the origin and at the critical frequency coincide at $\omega = 0$. Ref. [24] gives the late-time asymptotics on the horizon in this axisymmetric case.

C. Small- ω asymptotics of radius independent quantities appearing in $\delta\tilde{g}_{\ell m \omega}$

In order to compute the late-time behaviour of the contribution to the Green function due to the branch point at $\omega = 0$, we require the small- ω asymptotics of the radius-independent constituents of $\delta\tilde{g}_{\ell m \omega}$ in Eq.(106), namely, $q(\sigma)$ and $\mathcal{W}^+\mathcal{W}^-$. Here we provide the necessary asymptotic expressions for these terms.

First of all, from Eq.(70) we obtain the needed asymptotics for the matching coefficients

$$K_\nu \sim \frac{\zeta_+^{(0)}}{\zeta_+^{(\infty)}} (-1)^{\ell+1} m^{-\ell} 2^{-\ell} \nu_2 \frac{\Gamma(2\ell+1)\Gamma(2\ell+2)}{\Gamma^2(\ell+1-s)} \omega^{-\ell}, \quad K_{-\nu-1} \ll K_\nu, \quad \omega \rightarrow 0, \quad (108)$$

where we have used

$$\Gamma(1 - q_0^\nu + \chi_s) \sim \frac{(-1)^{\ell+s} i}{\Gamma(1 + \ell - s) \omega}, \quad \Gamma(1 - 2q_0^\nu) \sim \frac{1}{2\Gamma(2\ell+2) \nu_2 \omega^2}, \quad \omega \rightarrow 0, \quad (109)$$

and ν_2 is given in Eq.(94).

With K_ν as $\omega \rightarrow 0$ at hand, the small- ω asymptotics of the scattering amplitudes follow from Eqs.(39), (83), (84), (78a) and (38). From these equations we obtain, respectively,

$$\mathcal{T}_{\text{up}} \sim \zeta_+^{(\infty)} e^{-i\pi\ell/2} 2^{-s} \frac{\Gamma(\ell+1+s)}{\Gamma(\ell+1-s)} \omega^{-s+i\omega}, \quad \omega \rightarrow 0, \quad (110)$$

$$\mathcal{J}_{\text{in}} \sim \zeta_+^{(0)} m^{-\ell} 2^{-\ell+s-1} e^{i\pi(1-\ell)/2} \frac{\Gamma(2\ell+1)\Gamma(2\ell+2)}{\Gamma^2(\ell+1-s)} \omega^{-\ell-1+s-i\omega}, \quad \omega \rightarrow 0, \quad (111)$$

$$\mathcal{R}_{\text{in}} \sim -\zeta_+^{(0)} e^{i\pi(1+\ell)/2} m^{-\ell} 2^{-\ell-s-1} \frac{\Gamma(2\ell+1)\Gamma(2\ell+2)\Gamma(\ell+1+s)}{\Gamma^3(\ell+1-s)} \omega^{-\ell-s-1+i\omega}, \quad \omega \rightarrow 0, \quad (112)$$

$$\mathcal{T}_+ \sim \zeta_+^{(\infty)} e^{i\pi(\ell-1)/2} \frac{2^{s-1}}{\nu_2} \omega^{s-i-1\omega}, \quad \omega \rightarrow 0, \quad (113)$$

and

$$\mathcal{T}_{\text{in}} \sim \zeta_+^{(0)} (-1)^s e^{-i\pi\ell/2} m^s \frac{\Gamma(\ell+1+s)}{\Gamma(\ell+1-s)}, \quad \omega \rightarrow 0, \quad (114)$$

where we have used

$$\frac{\sin(\pi(\nu + i\omega))}{\sin(2\pi\nu)} \sim \frac{(-1)^\ell i}{2\nu_2 \omega}, \quad \omega \rightarrow 0, \quad (115)$$

¹⁴ See also [47, 56], where use is made of other high-order techniques, which could be adapted to Kerr.

as well as Eq.(108) for \mathcal{J}_{in} and \mathcal{R}_{in} .

Equation (22) gives the scaled Wronskian in terms of the up transmission coefficient and the in incidence coefficient, defined via Eqs.(15) and (14), with small- ω asymptotics given in Eqs.(110) and (111). It follows that the asymptotics for the scaled Wronskian are ¹⁵

$$\mathcal{W} \sim (-1)^{\ell+1} \zeta_+^{(\infty)} \zeta_+^{(0)} (2m)^{-\ell} \frac{\Gamma(2\ell+1)\Gamma(2\ell+2)\Gamma(\ell+1+s)}{\Gamma^3(\ell+1-s)} \omega^{-\ell}, \quad \omega \rightarrow 0. \quad (116)$$

Lastly, from the various asymptotics above we obtain the asymptotics for the BC strength q in Eq.(105) and the Wronskian factor $\mathcal{W}^+ \mathcal{W}^-$ in Eq.(107):

$$q(\sigma) \sim e^{-\pi i(\ell-1/2)} 2^{1+2s} \pi \frac{\Gamma(\ell+1-s)}{\Gamma(\ell+1+s)} \omega^{2s-2i\omega+1}, \quad \omega \rightarrow 0, \quad (117)$$

and

$$\mathcal{W}^+ \mathcal{W}^- \sim (-1)^s 2^{2(s-\ell)} m^{-2(\ell+s)} \frac{\Gamma^2(2\ell+1)\Gamma^2(2\ell+2)}{\Gamma^2(\ell+1-s)\Gamma^2(\ell+1+s)} \omega^{2(s-\ell-i\omega)}, \quad \omega \rightarrow 0. \quad (118)$$

D. Late-time $\omega = 0$ tail at finite radii

The last quantity in Eq.(106) for which we need the asymptotics is the ingoing radial function. Given the small- ω behaviour of the a_n^ν in Eq.(90), it follows that the leading-order coefficient in the n -sum in Eq.(35) is given by the $n = 0$ term as long as x is finite. Thus, from Eqs.(35), (90) and (93), we straightforwardly obtain, for x finite,

$$R_{\ell m \omega}^{\text{in}} \sim \zeta_+^{(0)} e^{i\pi(s+1)/2} x^{-s-\ell-1} e^{-\frac{im}{2x}} m^{\ell+1} \frac{\Gamma(\ell+s+1)}{\Gamma(\ell-s+1)} U\left(\ell+1-s, 2\ell+2, \frac{im}{x}\right), \quad \omega \rightarrow 0. \quad (119)$$

We are now poised to carry out the integral along the BC of the small-frequency asymptotics of the transfer function. From Eq.(106), together with the asymptotics in Eqs.(114), (117), (118) and (119), we find

$$\delta \tilde{g}_{\ell m \omega} \sim i g_{\ell m}^{(f)}(x, x') \sigma^{2\ell+2}, \quad \omega \rightarrow 0, \quad (120)$$

where

$$g_{\ell m}^{(f)}(x, x') := (-1)^{\ell+s} e^{\pi i(s+1)/2} \pi 2^{2(\ell+1)} m^{2(2\ell+1)} \frac{\Gamma^3(\ell-s+1)\Gamma(\ell+s+1)}{\Gamma^2(2\ell+1)\Gamma^2(2\ell+2)} (x \cdot x')^{-\ell-s-1} e^{-im(1/x+1/x')/2} \\ \times U\left(\ell+1-s, 2\ell+2, \frac{im}{x}\right) U\left(\ell+1-s, 2\ell+2, \frac{im}{x'}\right) \quad (121)$$

is independent of σ and symmetric under interchange of x and x' .

Finally, performing the integration in Eq.(97) using (120), we find

$$\delta \mathcal{G}_{\ell m} \sim \Gamma(2\ell+3) e^{im\phi} {}_s\mathcal{F}_{\ell m 0}(\theta, \theta') g_{\ell m}^{(f)}(x, x') t^{-3-2\ell}, \quad t \rightarrow \infty, \quad (x \text{ and } x' \text{ finite}) \quad (122)$$

for the leading late-time behaviour of the Green function (ℓ, m) -modes due to the branch point at $\omega = 0$. The angular factor evaluated at the origin, ${}_s\mathcal{F}_{\ell m 0}$, so reduces to a product of the well-known spin-weighted *spherical* harmonics [57, 58]. We note that the result in Eq.(122) is valid for a field point approaching timelike infinity and a source point at an arbitrary finite radius away from the horizon (since we have kept r and r' fixed and finite while taking $t \rightarrow \infty$).

¹⁵ Although the Wronskian possesses a BC down from $\omega = 0$, its discontinuity only appears at higher order in ω , not at the leading order of Eq.(116) – for the explicit details (such as the order in which it appears) in the case of subextremal Kerr, see [15].

E. Late-time $\omega = 0$ tail at radial infinity

The results of the previous subsection are not valid as the field point x approaches radial infinity. In this section we modify the asymptotic analysis in order to obtain the late-time behavior when $x \rightarrow \infty$ and x' is finite. As opposed to the finite radii case above, here we do not use directly the formalism of Sec. V B. Instead, we use the original expression for the transfer function, i.e., Eq.(25), and find the asymptotics as $\omega \rightarrow 0$, while $x\omega \rightarrow \infty$ for each quantity in the expression. This can be accomplished by taking a parameter λ to zero while fixing $\bar{x} := \lambda^{3/2}x$ and $\bar{\omega} := \omega/\lambda$, say.

For the in solution, Eq.(119) shows that it has a finite limit as $\omega \rightarrow 0$ and contributes only to the overall amplitude of the late-time behaviour due to the corresponding BC. That is, at arbitrary finite radius we have

$$R_{\ell m \omega}^{\text{in}} = O(1), \quad \omega \rightarrow 0. \quad (123)$$

Given the small- ω behaviour of the a_n'' in Eq.(90), of ν in Eq.(93) and the small- ω but large- $x\omega$ behaviour of the confluent hypergeometric function U in Eq.(13.7.3) [37], it follows that the leading-order term in the n -sum in Eq.(34) for $R_{\ell m \omega}^{\text{up}}$ is given by the $n = 0$ term. From Eqs.(34), (90) and (93), we straightforwardly obtain, as $\omega \rightarrow 0$ with x large and $x\omega$ not necessarily small,

$$R_{\ell m \omega}^{\text{up}} \sim R_0^{\text{up}} \omega^{\ell+1} U(\ell+1-i\omega+s, 2\ell+2, -2i\omega x), \quad (124)$$

where

$$R_0^{\text{up}} := \zeta_+^{(\infty)} 2^{\ell+1} \frac{\Gamma(\ell+s+1)}{\Gamma(\ell-s+1)} x^{-s+\ell} e^{-i\pi s/2} e^{-\pi i(\ell+1/2)} e^{i\omega x}. \quad (125)$$

We note that although we could have applied Eq.13.7.3 [37] already at this stage, we find it better to compute the discontinuity in $R_{\ell m \omega}^{\text{up}}$ before applying the large- ωx asymptotics.

The discontinuity across the BC down from $\omega = 0$, to leading-order for small frequency, is then due to the small-frequency asymptotics of the up solution, Eq.(124). From Eq.(124) and the analytic continuation of Eq.13.2.12 [37], together with Eq.13.7.2 [37], it follows that

$$\begin{aligned} \delta R_{\ell m \omega}^{\text{up}} &\sim R_0^{\text{up}} \omega^{\ell+1} \left(U(\ell+1-i\omega+s, 2\ell+2, -2i\omega x) - U(\ell+1-i\omega+s, 2\ell+2, -2i\omega e^{2\pi i} x) \right), \quad \omega \rightarrow 0, \\ &\sim R_0^{\text{up}} \omega^{\ell+1} \frac{2\pi(-1)^{s-\ell+1} \Gamma(\ell-s+1)}{\Gamma(2\ell+2)} M(\ell+1+s, 2\ell+2, -2i\omega x), \quad \omega \rightarrow 0, \end{aligned} \quad (126)$$

$$\sim 2\pi R_0^{\text{up}} (-2ix)^{-\ell-1-s} \omega^{1-s+i\omega}, \quad \omega \rightarrow 0, \quad x\omega \rightarrow \infty, \quad (127)$$

where we have discarded the subdominant term in the asymptotics proportional to $e^{-2i\omega x}$, which is exponentially suppressed when evaluated on the negative imaginary axis where we compute the BC integral.

We can now carry out the integral along the BC of the small-frequency asymptotics for the transfer function. From Eq.(25), and putting together the asymptotics of Eqs.(127), (116) and (119), we obtain

$$\delta \tilde{g}_{\ell m \omega}(x, x') \sim i g_{\ell m}^{(\infty)}(x, x') \sigma^{1+\ell-s} e^{\sigma x}, \quad \omega \rightarrow 0, \quad x\omega \rightarrow \infty, \quad x' \text{ finite}, \quad (128)$$

where

$$\begin{aligned} g_{\ell m}^{(\infty)}(x, x') &:= 2\pi i m^{2\ell+1} 2^{\ell-s} e^{\pi i s} \frac{\Gamma(1+\ell+s) \Gamma(1+\ell-s)}{\Gamma(2\ell+1) \Gamma(2\ell+2)} x^{-2s-1} (x')^{-s-\ell-1} e^{-im/(2x')} \\ &\times U\left(1+\ell-s, 2\ell+2, \frac{im}{x'}\right). \end{aligned} \quad (129)$$

Finally, for the leading late-time behaviour of the Green function (ℓ, m) -modes due to the branch point at $\omega = 0$, we obtain, using Eqs.(97) and (128),

$$\delta \mathcal{G}_{\ell m} \sim \Gamma(2+\ell-s) e^{im\phi} {}_s\mathcal{Z}_{\ell m 0}(\theta, \theta') g_{\ell m}^{(\infty)}(x, x') (t-x)^{s-\ell-2}, \quad t, x \rightarrow \infty, \quad (x' \text{ finite}). \quad (130)$$

The appearance of the retarded time $t - x$ in Minkowski spacetime in Eq.(130) is expected, as our result holds at late time along future null infinity.

F. Late-time $\omega = 0$ tail on the horizon

In the limit $x_{<} \rightarrow 0$ keeping $x_{>}$ finite and nonzero, the asymptotics are carried out in a way similar to those in Sec.V D. From Eqs.(35), (90) and (93) (or directly from Eq.(119)), together with Eq.13.2.6 [37], we have

$$R_{\ell m \omega}^{\text{in}} \sim \zeta_+^{(0)} (-1)^s e^{i\pi\ell/2} m^s \frac{\Gamma(\ell + s + 1)}{\Gamma(\ell - s + 1)} x^{-2s} e^{-\frac{im}{2x}}, \quad x, \omega \rightarrow 0. \quad (131)$$

Therefore, in principle, the late-time asymptotics for the Green function modes would follow from those in Eq.(122) after accounting for the different factor in $R_{\ell m \omega}^{\text{in}}(x_{<})$ coming from Eq.(131) in the present $x_{<} \rightarrow 0$ case instead of that from Eq.(119) in the $x_{<}$ finite and nonzero case. Additionally, we must also take care of the factors $e^{-\frac{im}{2x_{<}}}$ and $x_{<}^{-2s}$ for $x_{<} \rightarrow 0$ coming from Eq.(131). Following [23], one may decompose the Green function with respect to “ingoing azimuthal angle” $\psi := \phi - 1/(2x)$ and advanced time v coordinates instead of, respectively, ϕ and t . This effectively amounts to replacing ϕ and t in Eq.(3) by, respectively, ψ and v , and then multiplying the transfer function $\tilde{g}_{\ell m \omega}$ by the corresponding correcting factor. In the limit $\omega \rightarrow 0$, this correcting factor is merely $e^{im/(2x)}$. To banish the singular x^{-2s} factor, we rescale the real-valued Kinnersley tetrad vectors by $\ell^\mu \rightarrow \Delta \ell^\mu$ and $n^\mu \rightarrow \Delta^{-1} n^\mu$, which rescales the Teukolsky master function by $\Psi \rightarrow \Delta^s \Psi$.¹⁶

We denote the Green function with respect to ingoing coordinates and the regular (rescaled Kinnersley) tetrad by \mathbf{G} , the corresponding integral of its modes around the BC down from $\omega = 0$ (i.e., the equivalent of Eq.(97)) by $\delta \mathbf{G}_{\ell m}$ and its associated transfer function by $\mathbf{g}_{\ell m \omega}$. The resulting large- v asymptotics of the Green function modes with $x_{<} \rightarrow 0$ thus can be readily obtained from Eq.(122) by replacing ϕ and t by, respectively, ψ and v , and $g_{\ell m}^{(f)}(x, x')$ by

$$\begin{aligned} \mathbf{g}_{\ell m}^{(0)}(0, x') &:= e^{\pi i(\ell-s)/2} 2^{2(\ell+1)} \pi m^{3\ell+1+s} \frac{\Gamma^3(\ell-s+1) \Gamma(\ell+s+1)}{\Gamma^2(2\ell+1) \Gamma^2(2\ell+2)} (x')^{-\ell-s-1} e^{-im/(2x')} \\ &\times U\left(\ell+1-s, 2\ell+2, \frac{im}{x'}\right). \end{aligned} \quad (132)$$

The expression for this function $\mathbf{g}_{\ell m}^{(0)}$ is obtained by multiplying $g_{\ell m}^{(f)}(x, x')$ in Eq.(121) by $e^{im/(2x_{<})}$ times the ratio of the right hand side of Eq.(131) to the right hand side of Eq.(119), both evaluated at $x = x_{<} \rightarrow 0$. Therefore, from Eq.(122) with the appropriate transformations, we have

$$\delta \mathbf{G}_{\ell m} \sim \Gamma(2\ell+3) e^{im\psi} {}_s \mathcal{Z}_{\ell m 0}(\theta, \theta') \mathbf{g}_{\ell m}^{(0)}(0, x') v^{-3-2\ell}, \quad v \rightarrow \infty, \quad (x=0 \text{ and } x' \text{ finite}), \quad (133)$$

for the decay of the field along the future horizon of extremal Kerr. This decay appears to be faster than the corresponding one coming from the $k = 0$ branch point as found in [23]¹⁷. In [23] it was assumed that the dominant behaviour of the Green function at late-times on the horizon comes from the $k = 0$ branch point; here we have shown that this is indeed the case. Finally, we note that this decay is the same as that on the future horizon of subextremal Kerr [61].

G. Summary of $\omega = 0$ tail results

Lastly, we summarize our results for the tails from the origin as measured at various radii (x zero, finite, and infinite) and connect with previous results for subextremal Kerr. We find that, for source points fixed at finite

¹⁶ Here, we assume that the source of the field is compactly supported away from the event horizon so that the transformation of the Green function under the tetrad boost is unambiguously evaluated at the field point $x_{<}$.

¹⁷ While it is not obvious that the $\omega = 0$ tail is subleading for modes with $m/\ell \lesssim .74$, where the decay from the branch point at $k = 0$ is given in Eq.(72) of [24] as v^{-h} with $h = -\nu_{c,-}$ (see Sec.III E), the large- ℓ asymptotics of h [59, 60] suggest this is so.

x	$\omega = 0$ tail	$k = 0$ tail (real part of decay exponent)	
		$m/L \lesssim .74$	$m/L \gtrsim .74$
finite	$t^{-3-2\ell}$	t^{-2h}	t^{-1}
horizon	$v^{-3-2\ell}$	v^{-s-h}	$v^{-s-1/2}$
infinity	$u^{-2+s-\ell}$	u^{-2h}	u^{-1}

TABLE I. Late-time tails of nonaxisymmetric mode perturbations due to the branch points in the complex frequency plane at $\omega = 0$ and $k = 0$. The rates are reported in advanced (ingoing) time v and rescaled Kinnersley tetrad on the horizon, retarded (outgoing) time u and Kinnersley tetrad at infinity, and Boyer-Lindquist time t and Kinnersley tetrad for points in-between. The rates for the $\omega = 0$ tail are computed in Secs. **V D**, **V E** and **V F**, whereas the rates for the $k = 0$ have been computed elsewhere [23, 24, 53]. Here L is given by $L := \ell + 1/2$. We note that the critical value of 0.74 is only approximate and obtained in the large- L limit. N.B.: $h = -\nu_{c,-}$ (see Sec. **III E**), with its values given in Eq.(67) [24].

radii $0 < x' < \infty$, the late-time contribution to the Green function from the $\omega = 0$ branch point at various field points x is given by:

- on the horizon: $\delta G_{\ell m} \simeq v^{-3-2\ell}$, $v \rightarrow \infty$;
- at finite radii: $\delta G_{\ell m} \simeq t^{-3-2\ell}$, $t \rightarrow \infty$;
- at null infinity: $\delta G_{\ell m} \simeq u^{s-\ell-2}$, $u \rightarrow \infty$,

where $A \simeq B$ means A is asymptotic to B up to multiplication by a “time”-independent factor. Evidently, when including all the modes, the dominant contribution is for the lowest multipole $\ell = |s|$. We also remark that the “true decay” of the master field at late-time results from the $k = 0$ branch point, which dominates the $\omega = 0$ tail in all cases. For quick reference we have tabulated our main tail results, including the previously-obtained $k = 0$ tails, in Table **I**.

Interestingly, the rates for the extremal Kerr tails coming from the branch point at $\omega = 0$ are identical to the subextremal ones [61, 62]. Mathematically, the agreement arises from the fact that both the subextremal and extremal radial differential equations carry a rank-1 irregular singular point at infinity, yielding confluent hypergeometric series solutions convergent at infinity in both cases. In fact, the up radial series solution for subextremal Kerr (convergent and outgoing at infinity), as given in Eq.(3.16) of [6], has a smooth limit as $a \rightarrow M$ to the extremal radial series. Therefore, in hindsight, it is no surprise that the two cases have identical tails from the origin.

VI. BRANCH CUT FROM THE CRITICAL FREQUENCY

The branch point at $\omega = 0$ in the up modes, which we investigated in Sec. **V A**, is due to the fact that these modes are defined by imposing boundary conditions ($\sim e^{i\omega(x+\ln x)}$) as $x \rightarrow \infty$, with $x = \infty$ being an *irregular* singular point of the radial ordinary differential equation. Similarly, we expect that the in modes, which are defined by imposing boundary conditions containing the term $e^{-ik \ln x}$ as $x \rightarrow 0^+$, with $x = 0$ being also an irregular singular point, possess a branch point at $k = 0$. As advanced in Sec. **III A**, this is indeed the case. Just like the branch point at the origin $\omega = 0$ of the up modes, the branch point at the critical frequency $k = 0$ of the in modes carries over to the transfer function $\tilde{g}_{\ell m \omega}$. We choose the corresponding BC to run down parallel to the negative imaginary axis. In this section we develop the MST formalism for the calculation of the discontinuity of the ingoing radial solution and of the transfer modes across the BC down from $k = 0$. This provides all the necessary expressions for calculating the *full* contribution to the Green function from the BC down from $k = 0$.

We shall use the notation that, if $B = B(k)$ is a function of k possessing a branch point at $k = 0$,

$$\bar{\delta}B := B(k) - B(ke^{2\pi i}) \quad (\text{change in function } B \text{ across BC extending from } k = 0).$$

In particular,

$$\bar{\delta}\tilde{g}_{\ell m \omega}(x, x') := \tilde{g}_{\ell m \omega}(x, x')|_{\omega=m-i\bar{\sigma}} - \tilde{g}_{\ell m \omega}(x, x')|_{\omega=(m-i\bar{\sigma})e^{2\pi i}}, \quad (134)$$

where $\bar{\sigma} > 0$. Similarly to Eq.(97) for the BC from $\omega = 0$, the contribution to an (ℓ, m) -mode of the Green function due to the BC from $k = 0$ is then given by:

$$\bar{\delta}G_{\ell m}(x^\mu, x'^\mu) := -i e^{im(\phi-t)} \int_0^\infty d\bar{\sigma} e^{-\bar{\sigma}t} \bar{\delta}\tilde{g}_{\ell m \omega}(x, x') {}_s\mathcal{Z}_{\ell m \omega}(\theta, \theta')|_{\omega=m-i\bar{\sigma}}. \quad (135)$$

The late-time behaviour of $\bar{\delta}G_{\ell m}$ will be given by the small- $\bar{\sigma}$ behaviour of the integrand in Eq.(135).

A. Discontinuity in the ingoing modes

As is apparent by comparison of Eqs.(34) and (35), the rôle played by the in modes in the discontinuity of the transfer function across the BC from $k = 0$ is, in many ways, similar to that played by the up modes in the corresponding discontinuity from $\omega = 0$. Therefore, in order to calculate the discontinuity in the in modes across the BC from $k = 0$, we proceed similarly to Sec. V A for the up modes in the $\omega = 0$ case. Our starting point is Eq.(35) and we use Eq.13.2.12 [37] to obtain

$$\left. \frac{R_{\ell m \omega}^{\text{in}}}{\zeta_+^{(0)}} \right|_{k \rightarrow ke^{2\pi i}} = \frac{f_{\text{in}}(k)}{\zeta_+^{(0)}(k)} e^{2\pi i \nu} \sum_{n=-\infty}^{\infty} A_n^{\text{in}}(k) \left(\frac{(1 - e^{-2\pi i b}) \Gamma(1 - b)}{\Gamma(1 + \bar{d} - b)} M\left(\bar{d}, b, -\frac{ik}{x}\right) + e^{-2\pi i b} U\left(\bar{d}, b, -\frac{ik}{x}\right) \right), \quad (136)$$

where $\bar{d} := q_n^\nu + \chi_{-s}$, and, as in Eq. (100), $b = 2q_n^\nu$. Eq. (136) is, trivially, the equivalent for the in modes of Eq. (99) for the up modes. The right hand side of Eq.(136) can be obtained from that of Eq. (99) under $\zeta_+^{(\infty)} \rightarrow \zeta_+^{(0)}$, $f_{\text{up}} \rightarrow f_{\text{in}}$, $d = q_n^\nu + \chi_s \rightarrow \bar{d} = q_n^\nu + \chi_{-s}$ (note that \bar{d} is equal to d under $s \rightarrow -s$) and $-2i\omega x \rightarrow -ik/x$ (note that, under this latter transformation, one obtains $A_n^{\text{up}} \rightarrow A_n^{\text{in}}$). This means, for example, that here we will need the combination on the left hand side of Eq.(102) with $d \rightarrow \bar{d}$ but, since the result on its right hand side is independent of s , Eq.(102) is equally valid with $d \rightarrow \bar{d}$.

In similarity with Eq. (104) and the argument below it, while $\hat{R}_{\ell m \omega}^{\text{in}}$ is a purely-ingoing solution into the horizon ($\sim e^{-i\omega \ln x}$, which is an exponentially-dominant solution when $\text{Im}(k) < 0$), $\bar{\delta}\hat{R}_{\ell m \omega}^{\text{in}}$ must be a purely-outgoing solution from the horizon ($\sim e^{+i\omega \ln x}$, which is exponentially-subdominant when $\text{Im}(k) < 0$). Therefore, $\bar{\delta}\hat{R}_{\ell m \omega}^{\text{in}}$ must be proportional to $\hat{R}_{\ell m \omega}^{\text{out}}$. We confirm this in the following development.

From Eq.(38), the discontinuity in the ('normalized') transmission coefficient of the ingoing radial solution is

$$\left. \frac{\mathcal{T}_{\text{in}}}{\zeta_+^{(0)}} \right|_{k \rightarrow ke^{2\pi i}} = e^{-2\pi \omega} \frac{\mathcal{T}_{\text{in}}(k)}{\zeta_+^{(0)}(k)}. \quad (137)$$

We note that the discontinuity factor $e^{-2\pi \omega}$ is exactly the same as that for the corresponding upgoing coefficient in Eq.(101). The various symmetries that we have noted are required in order to obtain the ingoing results from the upgoing results imply that Eq.(103) holds with $\delta\hat{R}_{\ell m \omega}^{\text{up}} \rightarrow \bar{\delta}\hat{R}_{\ell m \omega}^{\text{in}}$, $f_{\text{up}} \rightarrow f_{\text{in}}$, $\mathcal{T}_{\text{up}} \rightarrow \mathcal{T}_{\text{in}}$ and $-2i\omega x \rightarrow -ik/x$. That is, we have

$$\bar{\delta}\hat{R}_{\ell m \omega}^{\text{in}} = \frac{f_{\text{in}}}{\mathcal{T}_{\text{in}}} e^{-ik/x} e^{2\pi(\omega + i\nu)} (e^{-2\pi i \nu} - e^{-2\pi \omega}) \sum_{n=-\infty}^{\infty} \left(\frac{-ik}{x} \right)^n a_n^\nu U\left(b - \bar{d}, b, \frac{ik}{x}\right). \quad (138)$$

By comparison with Eq.(40), and using Eqs.(38) and (43), we confirm that this discontinuity is proportional to the solution $\hat{R}_{\ell m \omega}^{\text{out}}$:

$$\bar{\delta} \hat{R}_{\ell m \omega}^{\text{in}} = i \bar{q}(\bar{\sigma}) \left. \hat{R}_{\ell m \omega}^{\text{out}} \right|_{\omega=m-i\bar{\sigma}}, \quad (139)$$

where

$$\bar{q}(\bar{\sigma}) := i(-ik)^{\nu+1-s-i\omega} (ik)^{-\nu-1-s-i\omega} (e^{2\pi i\nu} - e^{2\pi\omega}) \frac{\sum_{n=-\infty}^{\infty} (-1)^n a_n^{\nu}}{\sum_{n=-\infty}^{\infty} \frac{\Gamma(q_n^{\nu} + \chi_s)}{\Gamma(q_n^{\nu} - \chi_s)} a_n^{\nu}}, \quad (140)$$

with all quantities on the right hand side evaluated at $\omega = \lim_{c \rightarrow 0+} (m - i\bar{\sigma} + c)$, assuming $\bar{\sigma} > 0$ throughout. Eq.(139) is the ingoing-solution equivalent down from $k = 0$ of the upgoing solution discontinuity down from $\omega = 0$ given in Eq.(104). Similarly to $\hat{R}_{\ell m \omega}^{\nu}$ there, $\hat{R}_{\ell m \omega}^{\text{out}}$ here has a branch point at $k = 0$ but its BC lies *upwards* from $k = 0$. Since, in Eq.(139), $\hat{R}_{\ell m \omega}^{\text{out}}$ is evaluated *down* from $k = 0$, there is no ambiguity as to its value in this equation.

B. Discontinuity in the transfer function across the critical frequency branch cut

Since $R_{\ell m \omega}^{\text{up}}/\mathcal{T}_{\text{up}}$ does not possess a branch point at $k = 0$, from Eqs.(25), (22) and the first expression for $\hat{\mathcal{W}}$ given in Eq.(26), it readily follows that the discontinuity in the transfer function across the BC from the critical frequency is given by:

$$\bar{\delta} \tilde{g}_{\ell m \omega} = \frac{R_{\ell m \omega}^{\text{up}}}{2i\omega \mathcal{T}_{\text{up}}} \bar{\delta} \left(\frac{R_{\ell m \omega}^{\text{in}}}{\mathcal{J}_{\text{in}}} \right). \quad (141)$$

We now use the second expression for $\hat{\mathcal{W}}$ in Eq.(26) in order to mirror down from $k = 0$ the calculation down from $\omega = 0$ with the ingoing solution now playing the rôle of the upgoing solution. Similarly to Eq.(106), we use Eqs.(139) and (28) to obtain

$$\bar{\delta} \tilde{g}_{\ell m \omega}(x, x') = -\bar{\sigma} \frac{\bar{q}(\bar{\sigma})}{\hat{\mathcal{W}}^+ \hat{\mathcal{W}}^-} \left[\hat{R}_{\ell m \omega}^{\text{up}}(x) \hat{R}_{\ell m \omega}^{\text{up}}(x') \right]_{\omega=m-i\bar{\sigma}}, \quad (142)$$

where $\hat{\mathcal{W}}^{+/-}$ is defined to be equal to $\hat{\mathcal{W}}$ evaluated, respectively, on the right/left of the BC down from $k = 0$ and, as always, $\bar{\sigma} > 0$.

In order to obtain an expression for the Wronskian factor in the denominator in Eq.(142) we proceed similarly to the corresponding Wronskian factor in Eq.(107), which was obtained in [15]. Namely, from Eq.(139),

$$\hat{\mathcal{W}}^- = \Delta^{s+1} W[\hat{R}_{\ell m \omega}^{\text{in}}(ke^{2\pi i}), \hat{R}_{\ell m \omega}^{\text{up}}] = \Delta^{s+1} \left(W[\hat{R}_{\ell m \omega}^{\text{in}}(k), \hat{R}_{\ell m \omega}^{\text{up}}] - i \bar{q}(\bar{\sigma}) W[\hat{R}_{\ell m \omega}^{\text{out}}, \hat{R}_{\ell m \omega}^{\text{up}}] \right). \quad (143)$$

Combining Eqs.(26), (143) and (27), we obtain

$$\hat{\mathcal{W}}^+ \hat{\mathcal{W}}^- = -k^2 \hat{\mathcal{J}}_{\text{up}} (\hat{\mathcal{J}}_{\text{up}} + \bar{q}(\bar{\sigma}) \hat{\mathcal{R}}_{\text{up}}). \quad (144)$$

It is understood that all radial coefficients in Eq.(144) which possess a BC lying *down* from $k = 0$ are to be evaluated to the right of the BC i.e., at $\omega = \lim_{c \rightarrow 0+} (m - i\bar{\sigma} + c)$.

Eqs.(142), (135), (140) and (144), together with the appropriate expressions for the radial coefficients and for $\hat{R}_{\ell m \omega}^{\text{up}}$ given in the previous sections, provide all the expressions that would be needed for explicitly calculating the *full* contribution of the BC from $k = 0$ to the Green function. The leading order contribution has in fact already been calculated in [23, 24] using MAE and in the next section we show that the leading order in the MST series for the radial solutions yield the corresponding MAE expressions. As mentioned, we have set up the formalism that allows one to obtain the contribution from the BC from $k = 0$ up to arbitrary order (or exactly if calculating the expresions semianalytically/numerically) but we shall not undertake this endeavour in this paper.

VII. SMALL k ASYMPTOTICS AND LINK WITH MATCHED ASYMPTOTIC EXPANSIONS

In investigations of the Aretakis phenomenon, the generic late-time decay of extremal Kerr excitations was derived using the method of matched asymptotic expansions (MAE) [23, 24]. In the MAE, the transfer function is obtained by finding expressions for the radial solutions valid in a “near zone” $x \ll 1$ and a “far zone” $x \gg k$ which are matched in an overlap region, $k \ll x \ll 1$. These MAE expressions for the radial solutions are obtained by approximating the radial potential (11) accordingly in these limits. In this section, we show that the transfer function obtained with the MAE in fact corresponds to the $n = 0$ terms, appropriately approximated for $k \rightarrow 0$, in the MST series representations for the radial solutions that we derived in Sec. III. This result puts the MAE result on a firmer footing as, in some sense, the “leading order” term in the global MST construction, where truncating a MST series to a higher $|n|$ essentially corresponds to truncation to a higher order in k . A similar viewpoint applies to small ω , with the terms in the MST series appropriately approximated for $\omega \rightarrow 0$ instead of $k \rightarrow 0$, as seen in the previous section.

A. Radial solutions near the superradiant bound

The MST series solutions given in Sec. III converge at all frequencies, generalizing previously obtained asymptotic solutions valid only as the frequency tends to zero [63, 64] or to the superradiant bound [44]. We now demonstrate that our MST series solutions, when restricted to frequencies in the neighborhood of $k = 0$, recover the known MAE expressions.

To start, recall from Secs. III A 2 and III E that the order of a_n^ν for small k increases as the summation-index $|n|$ increases. This property, together with the small- k asymptotics of the U functions and the other factors appearing in the appropriate summands, implies that the leading-order behaviour of both $R_\pm^{(\infty)}$ and $R_\pm^{(0)}$ as $k \rightarrow 0$ is contained in the $n = 0$ terms in Eqs. (30) and (31). Taking the limit $k \rightarrow 0$ in Eq. (30) while keeping x fixed and finite defines the so-called “far-zone limit”. Explicitly, in the far-zone limit we have

$$R_\pm^{(\infty)} \sim \zeta_\pm^{(c,\infty)} x^{-s+\nu_c} e^{-i\pi\chi_{s|c}/2} e^{\mp i\pi(\nu_c+1/2)} e^{\pm imx} (2m)^{\nu_c+1} \times \left(\frac{\Gamma(q_0^{\nu_c} + \chi_{s|c})}{\Gamma(q_0^{\nu_c} - \chi_{s|c})} \right)^{\frac{1}{2}} \left(\frac{\Gamma(q_0^{\nu_c} \pm \chi_{s|c})}{\Gamma(q_0^{\nu_c} \mp \chi_{s|c})} \right)^{\frac{1}{2}} U(q_0^{\nu_c} \pm \chi_{s|c}, 2q_0^{\nu_c}, \mp 2imx), \quad k \rightarrow 0, \quad (145)$$

where

$$\zeta_\pm^{(c,\infty)} := \zeta_\pm^{(\infty)}|_{\omega=m}.$$

The parameter ν_c can be chosen to be either $\nu_{c,-}$ or $\nu_{c,+}$ in Eq. (92), and we have defined $q_0^{\nu_c} := \nu_c + 1$ and $\chi_{s|c} := \chi_s|_{\omega=m} = s - im$. Using Eqs. (13.2.40) and (13.2.42) in [37], we recast the up solution as given in Eq. (145) in the more familiar form [43]

$$R_+^{(\infty)} \sim e^{-i\pi\chi_{s|c}} e^{-imx} \left(P x^{-\nu_c-1-s} M(-\nu_c + im - s, -2\nu_c, 2imx) + Q x^{\nu_c-s} M(1 + \nu_c + im - s, 2(1 + \nu_c), 2imx) \right), \quad k \rightarrow 0, \quad (146)$$

where

$$P := \zeta_\pm^{(c,\infty)} (2m)^{-\nu_c} \frac{\Gamma(2\nu_c + 1)}{\Gamma(1 + \nu_c - \chi_{s|c})}, \quad Q := P|_{\nu_c \rightarrow -\nu_c-1}. \quad (147)$$

Turning now to the convergent series solutions at the horizon (31), we obtain the so-called “near-zone limit”

by taking $k \rightarrow 0$ while fixing k/x

$$R_{\pm}^{(0)} \sim \zeta_{\pm}^{(c,0)} x^{-s-\nu_c-1} k^{\nu_c+1} e^{\pm ik/(2x)} e^{-i\pi\chi_{-s|c}/2} e^{\mp i\pi(\nu_c+1/2)} \left(\frac{\Gamma(q_0^{\nu_c} - \chi_{-s|c})}{\Gamma(q_0^{\nu_c} + \chi_{-s|c})} \right)^{1/2} \left(\frac{\Gamma(q_0^{\nu_c} \pm \chi_{-s|c})}{\Gamma(q_0^{\nu_c} \mp \chi_{-s|c})} \right)^{\frac{1}{2}} \quad (148)$$

$$\times \frac{\Gamma(q_0^{\nu_c} + \chi_{s|c})}{\Gamma(q_0^{\nu_c} - \chi_{s|c})} U\left(q_0^{\nu_c} \pm \chi_{-s|c}, 2q_0^{\nu_c}, \mp \frac{ik}{x}\right), \quad k \rightarrow 0, \quad k/x \text{ fixed},$$

where

$$\zeta_{\pm}^{(c,0)} := \zeta_{\pm}^{(0)}|_{\omega=m}.$$

Using Eq.(13.14.3) of [37], we find that the near-zone ingoing solution given in Eq.(148) simplifies to

$$R_+^{(0)} \sim A x^{-s} W_{im,1/2+\nu_c} \left(-\frac{ik}{x} \right), \quad k \rightarrow 0, \quad k/x \text{ fixed}, \quad (149)$$

where $W_{\alpha,\beta}(z)$ is the irregular Whittaker function and

$$A := \zeta_+^{(c,0)} k^{\nu_c+1} (-ik)^{-\nu_c-1} e^{-i\chi_{-s|c}\pi/2} e^{-\pi i(\nu_c+\frac{1}{2})} \frac{\Gamma(q_0^{\nu_c} + \chi_{s|c})}{\Gamma(q_0^{\nu_c} - \chi_{s|c})}. \quad (150)$$

We remind the reader that the critical parameter ν_c is related to the “weight” h of Refs. [23, 24] by $\nu_{c,-} = -h$. After replacing ν_c for $-h$, it easily seen that the known MAE expressions for the near and far radial functions [24, 43, 65] are recovered as limits of our MST solutions (146) and (149).

B. Transfer function near the superradiant bound

Lastly, we derive an expression for the “near-far” transfer function corresponding to the asymptotic solutions (149), (146) and compare with [24]. This provides a nontrivial check of our expressions for the scattering coefficients (39) and (83) which form the Wronskian (22). Taking the small- k asymptotics of these quantities, we find

$$\mathcal{W} \sim \zeta_+^{(c,0)} \zeta_+^{(c,\infty)} e^{-\pi i(\nu_c+1/2+\chi_{-s|c})} \frac{\Gamma(1+\nu_c+\chi_{s|c})}{\Gamma(1+\nu_c-\chi_{s|c})} \frac{\sin(\pi(\nu_c+im))}{\sin(2\pi\nu_c)} \times k^{\nu_c+1} \left(S_{\nu_c}(-ik)^{-2\nu_c-1} - e^{-i\pi(\nu_c+1/2)} S_{-\nu_c-1} \right), \quad k \rightarrow 0, \quad (151)$$

where

$$S_{\nu_c} := (2m)^{-\nu_c} \frac{\Gamma(2\nu_c+1)\Gamma(s-\nu_c-im)}{\Gamma(-2\nu_c-1)\Gamma(\nu_c+1-im-s)}. \quad (152)$$

Introducing the quantities

$$\hat{A} := \frac{\Gamma(-2\nu_c)}{\Gamma(-\nu_c-s-im)}, \quad \hat{B} := \frac{\Gamma(2+2\nu_c)}{\Gamma(1+\nu_c-s-im)}, \quad \mathcal{R} := -\frac{\Gamma(2+2\nu_c)\Gamma(-\nu_c-im+s)}{\Gamma(1+\nu_c-im+s)\Gamma(-2\nu_c)} (-2im)^{-1-2\nu_c}, \quad (153)$$

as used in [24], we find from (22), (149), (146), and (151) that the “near-far” ($k \rightarrow 0$, k/x finite, and x' finite) transfer function may be written as

$$\tilde{g}_{\ell m \omega}(x, x') \sim -\frac{(-ik)^{-\nu_c-1}}{\mathcal{R}\hat{B}(-ik)^{-2\nu_c-1}-\hat{A}} x^{-s} W_{im+s,1/2+\nu_c} \left(\frac{-ik}{x} \right) e^{-imx'} \times \left(\mathcal{R} x'^{-\nu_c-1-s} M(-\nu_c+im-s, -2\nu_c, 2imx') + x'^{\nu_c-s} M(1+\nu_c+im-s, 2(1+\nu_c), 2imx') \right), \quad (154)$$

which is in agreement with [24].

Appendix A: Radial Solutions à la Leaver

The MST method that we have developed in this paper builds on series representations to the extremal Teukolsky equation originally obtained by Leaver [5]. In this appendix we provide a brief recapitulation of Leaver's solutions, emphasizing important ingredients for our MST analysis. In particular, we give the radial asymptotics of Leaver's Coulomb function representations (correcting a typo in [5]) and relate his series coefficients to our MST coefficients a_n^ν .

In Eqs.(191) and (192) of [5] Leaver provides radial Teukolsky solutions in terms of Coulomb wave functions G and F (see, e.g., Sec.33.2 [37] for definitions). These solutions read

$$R_\pm^{(\infty)} = x^{-s-1} e^{ik/(2x)} \sum_{L=-\infty}^{\infty} a_L \left(G_{L+\nu}(-i\chi_s, \omega x) \pm i F_{L+\nu}(-i\chi_s, \omega x) \right), \quad \text{convergent for } x > 0, \quad (\text{A1a})$$

$$R_\pm^{(0)} = x^{-s} e^{i\omega x} \sum_{L=-\infty}^{\infty} b_L \left(G_{L+\nu}(-i\chi_{-s}, k/(2x)) \pm i F_{L+\nu}(-i\chi_{-s}, k/(2x)) \right), \quad \text{convergent for } x < \infty, \quad (\text{A1b})$$

where $\chi_{\pm s}$ is defined in Eq.(32). Leaver's series coefficients a_L and b_L satisfy distinct three-term recurrence relations (Eqs.(186) and Eqs.(188) of [5]). The auxiliary parameter ν is again the renormalized angular momentum parameter, which we describe in Sec.III A 2.

Using Eq.(109) [5], we find the following asymptotic behaviours near the event horizon¹⁸

$$\begin{aligned} R_+^{(0)} &\sim \left(\sum_{L=-\infty}^{\infty} b_L e^{i(-(L+\nu)\pi/2 + \tilde{\sigma}_L)} \right) e^{ik/(2x)} x^{-2s} e^{-i\omega \ln x} k^{i\omega+s}, & x \rightarrow 0^+, \\ R_-^{(0)} &\sim \left(\sum_{L=-\infty}^{\infty} b_L e^{-i(-(L+\nu)\pi/2 + \tilde{\sigma}_L)} \right) e^{-ik/(2x)} e^{i\omega \ln x} k^{-i\omega-s}, & x \rightarrow 0^+, \end{aligned} \quad (\text{A2})$$

where $\tilde{\sigma}_L$ is given by the right hand side of Eq.110 [5] with $\eta := -\omega - is$ replaced by $\tilde{\eta} := -\omega + is$ (Leaver's η and $\tilde{\eta}$ are equivalent to our $-i\chi_s$ and $-i\chi_{-s}$, respectively). Similarly, near radial infinity, Eq.(196) [5] yields, after applying Eq.(13.7.3) [37],

$$\begin{aligned} R_+^{(\infty)} &\sim \left(\sum_{L=-\infty}^{\infty} a_L e^{-\frac{i\pi}{2}(L+\nu+s-i\omega)+i\sigma_L} \right) (-2i\omega)^{-s+i\omega} x^{-1-2s} e^{i\omega(x+\ln x)}, & x \rightarrow \infty, \\ R_-^{(\infty)} &\sim \left(\sum_{L=-\infty}^{\infty} a_L e^{\frac{i\pi}{2}(L+\nu-s+i\omega)-i\sigma_L} \right) (2i\omega)^{s-i\omega} x^{-1} e^{-i\omega(x+\ln x)}, & x \rightarrow \infty. \end{aligned} \quad (\text{A3})$$

We reiterate that the coefficients a_L for $R_\pm^{(\infty)}$ are different from the coefficients b_L for $R_\pm^{(0)}$.

Lastly, we relate Leaver's radial solutions to the solutions (30) and (31) used in our analysis. To do so, we first use Eq.(125) of Ref. [5] to rewrite the Coulomb wavefunctions that appear in Eqs. (191) and (192) [5] in terms of the irregular confluent hypergeometric function U . Leaver's series coefficients are then related to the ones in our ansatz by (mapping Leaver's index L to our index n):

$$a_L = \left(\frac{\Gamma(q_n^\nu + \chi_s)}{\Gamma(q_n^\nu - \chi_s)} \right)^{1/2} \zeta_\pm^{(\infty)} i^n a_n^\nu, \quad (\text{A4})$$

$$b_L = \left(\frac{\Gamma(q_n^\nu - \chi_{-s})}{\Gamma(q_n^\nu + \chi_{-s})} \right)^{1/2} \frac{\Gamma(q_n^\nu + \chi_s)}{\Gamma(q_n^\nu - \chi_s)} \zeta_\pm^{(0)} i^n a_n^\nu. \quad (\text{A5})$$

A key point in our building of the MST formalism has been providing series representations for all radial solutions in terms of *the same* series coefficients (namely, a_n^ν).

¹⁸ Eq.(A2) differs from Eqs.(193) and (194) in [5] in that x^{-2s} appears in $R_+^{(0)}$ instead of in $R_-^{(0)}$ in our expressions. As our expressions are consistent with the "peeling off property" of zero rest mass fields [4, 66] and Eq.5.6 [4], we are confident in their correctness.

ACKNOWLEDGMENTS

We thank Sam Gralla for contributions in the early stages of this work. M.C. is thankful to Luís Felipe Longo for useful discussions. M.C. acknowledges partial financial support by CNPq (Brazil), process numbers 308556/2014-3 and 310200/2017-2. During the course of this work, PZ was supported by NSF grant 1506027 to the University of Arizona.

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