The correlation function of (1,1) and (2,2) supersymmetric theories with $T\bar{T}$ deformation

Song He[a,b,1], Jia-Rui Sun[c,d,2], Yuan Sun[c,3]

[a] *Center for Theoretical Physics and College of Physics, Jilin University, Changchun 130012, People's Republic of China*

[b] *Max Planck Institute for Gravitational Physics (Albert Einstein Institute), Am Mühlenberg 1, 14476 Golm, Germany*

[c] *School of Physics and Astronomy, Sun Yat-Sen University, Guangzhou 510275, China*

[d] *Guangdong Provincial Key Laboratory of Quantum Metrology and Sensing, Sun Yat-Sen University, Guangzhou 510275, China*

Abstract

In the paper, based on recent studies on $T\bar{T}$ deformation of 2D field theory with supersymmetry, we investigate the deformed correlation functions in $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,2)$ 2D superconformal field theories. Up to the leading order in perturbation theory, we compute the correlation functions under $T\bar{T}$ deformation. The correlation functions in these undeformed theories are almost known, and together with the help of superconformal Ward identity in $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,2)$ theories respectively we can obtain the correlation functions with operator $T\bar{T}$ inserted. Finally, by employing dimensional regularization, we can work out the integrals in the first order perturbation. The study in this paper extends previous works on the correlation functions of $T\bar{T}$ deformed bosonic CFT to the supersymmetric case.

[1] hesong@jlu.edu.cn
[2] sunjiarui@sysu.edu.cn
[3] sunyuan6@mail.sysu.edu.cn

Contents

1 Introduction

Studying exactly solvable models in 2D QFT can help us get a deep understanding of general field theory. The next step is naturally taken to consider the deviation from these exactly solvable models. In the language of renormalization group flow, in general the study of deformations by turning on relevant operators is under more controllable than irrelevant deformations which may introduce infinite divergences in the UV. However, a special kind of irrelevant deformation of 2D QFT was shown to have a number of remarkable properties even in the UV [1–3]. Such deformation preserves the integrability if the undeformed theory is integrable, also the spectrum and the S-matrix can be calculated. In addition, the deformed theory can be renormalized perturbatively systematically [4].

Among these deformations there is a special one, referred to as $T\bar{T}$ deformation, have attracted much attention recently [5–28]. Here T is related to stress tensor of the theory. The deformed Lagrangian $S(\lambda)$ can be written as

$$\frac{\partial S(\lambda)}{\partial \lambda} = \int d^2 z T\bar{T}(z), \tag{1}$$

where the operator $T\bar{T}(z)$ was first introduced in [1]. For conformal field theory, it was found that the partition function of deformed theory can be computed and remains modular invariant, and one can even obtain Cardy-like formula in deformed CFT. Meanwhile, there are other different perspectives on the $T\bar{T}$ deformation [29, 30], and applications in string theory [31–42]. More interestingly, from holographic dual point of view, it is suggested that the $T\bar{T}$ deformed 2D CFT dual to AdS_3 gravity with finite cutoff in the radial direction [43]. Evidence for this non-CFT/non-AdS kind of duality including matching of the energy spectrum, holographic entanglement entropies, exact holographic renormalization and so on. For recent progress on holographic aspects of $T\bar{T}$ deformation see also [44–55].

There are many directions to generalize the $T\bar{T}$ deformation, then an interesting question to ask is that what will happen when additional symmetry is presented in the theory, for example, conformal symmetry discussed above. In [56–59] (see also [60,61]), the authors have taken into account the supersymmetry, more specific, $\mathcal{N} = (0, 1)$ and extend SUSY with $\mathcal{N} = (1, 1), (2, 0), (2, 2)$ was considered. In these studies, the supersymmetric version of $T\bar{T}$ operator appeared in eq.(1) was constructed based on the supercurrent multiplet [62], and the deformed Lagrangian is also given for free theory with or without potential. Taking $\mathcal{N} = (1,1)$ for example [56], the deformed action takes the form

$$S_\alpha = S_0 + \lambda \int d^2\sigma O(\sigma) \tag{2}$$

with

$$O(\sigma) = \int d\theta^+ d\theta^- \mathcal{O}(\zeta). \tag{3}$$

Here $\mathcal{O}(\xi) = \mathcal{J}_{+++}(\zeta)\mathcal{J}_{---}(\zeta) - \mathcal{J}_{-}(\zeta)\mathcal{J}_{+}(\zeta)$, $(\mathcal{J}_{+++}, \mathcal{J}_{-})$ and $(\mathcal{J}_{---}, \mathcal{J}_{+})$ are two pairs of superfields, which include stress energy tensor (For more details for this construct, please refer to [56]). Moreover, it was shown that the deformation constructed in this way preserves solvability and supersymmetry. Furthermore, the operator O in eq.(2) is equal to bosonic $T\bar{T}$ as appeared in eq.(1) up to total derivative terms vanished on shell

$$O = T\bar{T} + \mathrm{EOM}'s + \text{total derivatives}. \tag{4}$$

Similar relationships between bosonic $T\bar{T}$ and its supersymmetric counterparts are also hold in other extend SUSY mentioned above.

In this work, we are interested in studying the correlation functions in $T\bar{T}$ deformation of superconformal field theory perturbatively. Correlation functions are fundamental observables in QFT, thus it is of great importance to study the correlation functions in its own right. The behavior of correlation functions was studied in both $T\bar{T}$ [19, 45] and $J\bar{T}$ [17] perturbatively, and unperturbatively in deep UV region by J. Cardy [18]. Inspired by these progress, here we would like to add supersymmetry to the undeformed theory. Since we will work with Euclidean signature, we would like to focus on our attention to the superconformal field theory with $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,2)$ supersymmetry. As discussed above the operators O and $T\bar{T}$ are equal on shell up to some total derivative terms, thus we will employ the latter as the definition for $T\bar{T}$ deformation in the process of computing correlation functions. Here we have to emphasize that we only focus on the deformation region nearby the undeformed CFTs, where the CFT Ward identity still holds and it is not necessary to take account the effect of the renormalization group flow of the operator with the irrelevant deformation. Therefore, the conformal symmetry can be considered as an approximate symmetry up to the first order of the $T\bar{T}$ deformation and the correlation functions can also be obtained nearby the original theory. Moreover, both in holography and quantum field theory, these correlation functions can be also applied to obtain various interesting quantum information quantities in the deformed field theory, e.g. the Rényi entanglement entropy of local quench in various situations [63–65], entanglement negativity [66], entanglement purification [67], information metric [68, 69], etc.

The remaining parts of the paper are organized as follows. In section 2, we first briefly review the Ward identity in (1,1) superconformal and also the correlation functions in undeformed theory, then formulate the 2-,3-, and n-point (n-pt) correlation functions with $T\bar{T}$ inserted, the last step is to perform the integral in conformal perturbation theory using dimensional regularization. In section 3, we first discuss the Ward identity and undeformed correlators in (2,2) superconformal field theory. Then following the same line as section 2, we compute the 2-,3-, and n-point deformed correlation function. In section 4. We discuss the dimensional regularization methods used in section 2 and section 3. In the final section, conclusions and discussions will be given.

2 $\mathcal{N}$=(1,1) superconformal symmetry

In this section we review (1,1) superconformal symmetry and the corresponding Ward identity. The coordinates on superspace are analytic coordinates $Z = (z, \theta)$ and anti-analytic coordinates $\bar{Z} = (\bar{z}, \bar{\theta})$ where $z, \bar{z}$ are two complex coordinates and $\theta, \bar{\theta}$ are Grassmannian coordinates. The (1,1) superconformal algebra is the direct sum of (1,0) and (0,1) algebra, thus for simplicity we may subsequently only write out the analytic part. For (1,1) theory the superderivative is [70–73]

$$D = \partial_\theta + \theta\partial_z, \quad D^2 = \partial_z. \tag{5}$$

The superfield

$$J(Z) = \Theta(z) + \theta T(z) \tag{6}$$

generate analytic supercoordinates transformations of in superspace. Here $T(z)$ is stress-energy tensor of the theory and Θ is generator of supersymmetry transformations. Similar expression can be write out for $\bar{J}(Z)$

Under analytic supercoordinates transformations with parameter $E(Z)$, a local superfield $\Phi(Z,\bar{Z})$ obeys

$$\delta_E \Phi(Z,\bar{Z}) = [J_E, \Phi(Z,\bar{Z})] = \oint dZ' E(Z') J(Z') \Phi(Z,\bar{Z}) \tag{7}$$

with

$$\oint dZ \equiv \frac{1}{2\pi i} \oint dz \int d\theta. \tag{8}$$

A superfield $\Phi(Z,\bar{Z})$ is called primary superfield if it transforms as

$$\Phi(Z,\bar{Z}) \to (\partial f)^{\Delta} (\bar{\partial}\bar{f})^{\bar{\Delta}} \Phi(Z,\bar{Z}) \tag{9}$$

under conformal transformation

$$Z = (z,\theta) \to Z' = (f(z), \sqrt{\partial_z f(z)}\theta), \quad \bar{Z} = (\bar{z},\bar{\theta}) \to \bar{Z}' = (\bar{f}(\bar{z}), \sqrt{\bar{\partial}\bar{f}(\bar{z})}\bar{\theta}). \tag{10}$$

Here $\Delta, \bar{\Delta}$ are the anomalous dimensions of $\Phi(Z,\bar{Z})$. The infinitesimal version of eq.(9) is

$$\delta_E \Phi(Z,\bar{Z}) = E(Z)\partial_z \Phi(Z,\bar{Z}) + \frac{1}{2} DE(Z) D\Phi(Z,\bar{Z}) + \Delta \partial_z E(Z) \Phi(Z,\bar{Z}), \tag{11}$$

where only the analytic part of the transformation is considered. Furthermore, one can obtain the OPE between the superfield $J(Z)$ containing stress tensor $T(z)$ and primary superfield Φ with dimension Δ, which is the generalization of OPE between stress tensor and primary field $T(z)\phi(z')$ in CFT. This can be done by substituting eq.(11) back to eq.(7) and using super-Cauchy theorem [4] which implies

$$\oint dZ_1 E(Z_1) \frac{\theta_{12}}{Z_{12}} = E(Z_2) \tag{13}$$

$$\oint dZ_1 E(Z_1) \frac{1}{Z_{12}} = DE(Z_2) \tag{14}$$

$$\oint dZ_1 E(Z_1) \frac{\theta_{12}}{Z_{12}^2} = \partial_z E(Z_2) \tag{15}$$

---

[4]

$$\oint dZ' E(Z') \frac{\theta' - \theta}{Z' - Z} = E(Z) \tag{12}$$

where the SUSY invariant distance $Z_{12} = z_1 - z_2 - \theta_1\theta_2$ and $\theta_{12} = \theta_1 - \theta_2$. We then obtain the following OPE [72]

$$J(Z_1)\Phi(Z_2) = \frac{\theta_{12}}{Z_{12}}\partial_{z_2}\Phi(Z_2,\bar{Z}_2) + \frac{1}{2}\frac{1}{Z_{12}}D\Phi(Z_2,\bar{Z}_2) + \Delta\frac{\theta_{12}}{Z_{12}}\Phi(Z_2,\bar{Z}_2). \tag{16}$$

From this OPE, the $\mathcal{N} = (1,1)$ superconformal Ward identity can be written as

$$\begin{aligned}&\langle J(Z_0)\Phi_1(Z_1,\bar{Z}_1)...\Phi_n(Z_n,\bar{Z}_n)\rangle\\ =&\sum_{i=1}^{n}\Big(\frac{\theta_{0i}}{Z_{0i}}\partial_{z_i} + \frac{1}{2Z_{0i}}D_i + \Delta_i\frac{\theta_{0i}}{Z_{0i}^2}\Big)\langle\Phi_1(Z_1,\bar{Z}_1)...\Phi_n(Z_n,\bar{Z}_n)\rangle.\end{aligned} \tag{17}$$

and similar expressions for $\bar{J}(\bar{Z})$.

It is important to apply Ward identity to global superconformal transformation whose algebra osp(2|1) is a subalgebra of superconformal algebra. By employing Ward identity and the fact that correlator of primary superfields is invariant under global superconformal transformation since it is a true symmetry of the theory, these correlators will be highly constrained. And similar to the cases in bosonic CFT, it is possible to completely fix 2- and 3-point correlators up to to some constant factors. The 2-pt correlator is

$$\langle\Phi_1(Z_1,\bar{Z}_1)\Phi_2(Z_2,\bar{Z}_2)\rangle = c_{12}\frac{1}{Z_{12}^{2\Delta}\bar{Z}_{12}^{2\bar{\Delta}}}, \quad \Delta\equiv\Delta_1=\Delta_2, \quad \bar{\Delta}\equiv\bar{\Delta}_1=\bar{\Delta}_2 \tag{18}$$

with c_{12} a constant and 3-pt correlator is

$$\langle\Phi_1(Z_1,\bar{Z}_1)\Phi_2(Z_2,\bar{Z}_2)\Phi_3(Z_3,\bar{Z}_3)\rangle = \Big(\prod_{i<j=1}^{3}\frac{1}{Z_{ij}^{\Delta_{ij}}\bar{Z}_{ij}^{\bar{\Delta}_{ij}}}\Big)(c_{123} + c'_{123}\theta_{123}\bar{\theta}), \tag{19}$$

where the second factor in the right hand side can also be written as

$$c_{123} + c'_{123}\theta_{123}\bar{\theta} = c_{123}e^{c'_{123}\theta_{123}\bar{\theta}/c_{123}}. \tag{20}$$

Here c_{123}, c'_{123} are constants, $\Delta_{ij} = \Delta_i + \Delta_j - \epsilon_{ijk}\Delta_k$, and θ_{123} is defined as

$$\theta_{ijk} = \frac{1}{\sqrt{Z_{ij}Z_{jk}Z_{kl}}}(\theta_i Z_{jk} + \theta_j Z_{ki} + \theta_k Z_{ij} + \theta_i\theta_j\theta_k), \tag{21}$$

which is invariant under global conformal transformation. By definition θ_{123} is Grassmann-odd, thus $\theta_{123}^2 = 0$ and eq.(20) follows.

As for n-pt correlators with $n \geq 4$, they depend on $2n$ coordinates $z_i, \theta_i, i = 1, ..., n$, and 5 constraints corresponding to 5 generators of osp(2|1). Thus there are $2n-5$ independent variables in n-pt correlators. Actually, there exists the same number of independent osp(2|1) invariants, i.e. $2n-5$, which are [72]

$$w_j \equiv \theta_{12j},\ j = 3, ..., n, \quad U_k \equiv Z_{123k},\ k = 4, ..., n, \tag{22}$$

where θ_{12j} is defined in eq.(21) and Z_{ijkl} is an analogue of cross ratio in CFT

$$Z_{ijkl} = \frac{Z_{ij}Z_{kl}}{Z_{li}Z_{jk}}. \tag{23}$$

In terms of these variables the n-pt function can be determined as

$$\langle \Phi_1(Z_1, \bar{Z}_1)...\Phi_n(Z_n, \bar{Z}_n)\rangle = \Big(\prod_{i<j=1}^{n} \frac{1}{Z_{ij}^{\Delta_{ij}} \bar{Z}_{ij}^{\bar{\Delta}_{ij}}}\Big) f(w_i, \bar{w}_i, U_j, \bar{U}_j) \tag{24}$$

with $\sum_{i\neq j} \Delta_{ij} = 2\Delta_j, \Delta_{ij} = \Delta_{ji}$ and similar for $\bar{\Delta}_{ij}$. Here f is a function can not be fixed by global superconformal symmetry, and it depends on the theory under consideration.

With the results discussed above, we can compute the $T\bar{T}$ deformed correlators. The variation of action under $T\bar{T}$ deformation can be constructed as

$$\delta S = \lambda \int d^2 z T\bar{T}(z) = -\lambda \int d^2 z \int d\theta d\bar{\theta} J(Z)\bar{J}(\bar{Z}), \tag{25}$$

where the minus sign comes from the anti-commutation nature of θ. Thus to first order in λ the variation of n-pt correlator is

$$-\lambda \int d^2 z \int d\theta d\bar{\theta} \langle J(Z)\bar{J}(\bar{Z})\Phi(Z_1, \bar{Z}_1)...\Phi(Z_n, \bar{Z}_n)\rangle. \tag{26}$$

Note that the correlator inside the integral can be evaluated via Ward identity. In the following section we will compute eq.(26) for $n = 2, 3$ and $n \geq 4$.

2.1 2-pt correlators

In this section we will consider the 2-pt correlators with $T\bar{T}$ deformation. The undeformed correlator takes the form as eq.(18)

$$\langle \Phi_1(Z_1, \bar{Z}_1)\Phi_2(Z_2, \bar{Z}_2)\rangle = \frac{c_{12}}{Z_{12}^{2\Delta} \bar{Z}_{12}^{2\bar{\Delta}}}. \tag{27}$$

First consider only holomorphic component of stress tensor inserted in above correlator

$$\langle J(Z)\Phi_1\Phi_2\rangle = \sum_{i=1}^{2}\Big(\frac{\theta_{0i}}{Z_{0i}}\partial_{z_i} + \frac{1}{2Z_{0i}}D_i + \frac{\Delta_i\theta_{0i}}{Z_{0i}^2}\Big)\langle\Phi_1\Phi_2\rangle, \tag{28}$$

where $\theta_{0i} = \theta - \theta_i, Z_{0i} = z - z_i - \theta\theta_i$, and the derivatives on the right hand side can act on both holomorphic and antiholomorphic parts of $\langle\Phi_1\Phi_2\rangle$. For example, for holomorphic part

$$\partial_{z_1}\frac{1}{Z_{12}^{2\Delta}} = -2\Delta\frac{1}{Z_{12}^{2\Delta+1}}, \quad D_1\frac{1}{Z_{12}^{2\Delta}} = -2\Delta\frac{\theta_{12}}{Z_{12}^{2\Delta+1}}, \tag{29}$$

and for antiholomorphic part [5]

$$\partial_{z_1}\frac{1}{\bar{Z}_{12}^{2\bar{\Delta}}} = \frac{2\bar{\Delta}}{\bar{Z}_{12}^{2\bar{\Delta}-1}}\tilde{\delta}^{(2)}(z_{12})\Big(1 + \frac{2\bar{\theta}_1\bar{\theta}_2}{\bar{z}_{12}}\Big), \tag{32}$$

Therefore

$$\begin{aligned}
&\langle J(Z)\Phi_1\Phi_2\rangle\\
=&\Big(-\frac{2\Delta}{Z_{12}}\Big(\frac{\theta_{01}}{z_{01}} - \frac{\theta_{02}}{z_{02}}\Big) - \frac{\Delta\theta_{12}}{z_{12}}\Big(\frac{1}{Z_{01}} + \frac{1}{Z_{02}}\Big) + \Delta\Big(\frac{\theta_{01}}{z_{01}^2} + \frac{\theta_{02}}{z_{02}^2}\Big) - \frac{\bar{\Delta}\theta_{12}}{z_{01}}(\bar{z}_{12} + \bar{\theta}_1\bar{\theta}_2)\tilde{\delta}(z_{12})\Big)\langle\Phi_1\Phi_2\rangle\\
\equiv&P\langle\Phi_1\Phi_2\rangle.
\end{aligned} \tag{33}$$

Similarly, the correlator with antiholomorphic component of stress tensor inserted, i.e. $\langle\bar{J}(\bar{Z})\Phi_1\Phi_2\rangle$ can be obtained by making the replacement $Z \to \bar{Z}, \theta \to \bar{\theta}$ in P defined above, and we denote it as $\langle\bar{J}(\bar{Z})\Phi_1\Phi_2\rangle \equiv \bar{P}\langle\Phi_1\Phi_2\rangle$.

A simplification can be made by noting that to extract $\langle T(z)\Phi_1\Phi_2\rangle$ from eq.(33), one need to integrate eq.(33) over θ, and the δ-function term in eq.(33) contains no θ thus gives no contribution to $\langle T(z)\Phi_1\Phi_2\rangle$. In view of this point, we will neglect the δ-function terms in both P and $\bar{P}$ hereafter.

[5] Useful formulae

$$\frac{1}{\bar{Z}_{ij}} = \frac{1}{\bar{z}_{ij}} + \frac{\bar{\theta}_1\bar{\theta}_2}{\bar{z}_{ij}^2}, \quad \frac{\theta_{ij}}{Z_{ij}} = \frac{\theta_{ij}}{z_{ij}}, \quad \frac{\theta_i}{Z_{ij}} = \frac{\theta_i}{z_{ij}}. \tag{30}$$

And the differential

$$\partial_{z_1}\frac{1}{\bar{Z}_{12}} = \partial_{z_1}\Big(\frac{1}{\bar{z}_{12}} + \frac{\bar{\theta}_1\bar{\theta}_2}{\bar{z}_{12}^2}\Big) = \tilde{\delta}(z_{12})\Big(1 + \frac{2\bar{\theta}_1\bar{\theta}_2}{\bar{z}_{12}}\Big), \quad \tilde{\delta}(z_{12}) \equiv 2\pi\delta^{(2)}(z_{12}), \tag{31}$$

where $\partial_{z_1}\frac{1}{\bar{z}_{12}} = \tilde{\delta}(z_{12})$ is used.

Having obtained $\langle \bar{J}(\bar{Z})\Phi_1\Phi_2\rangle$ we are in position to consider $\langle J(Z)\bar{J}(\bar{Z})\Phi_1\Phi_2\rangle$ which follows as

$$\langle J\bar{J}\Phi_1\Phi_2\rangle = \sum_{i=1}^{2}\Big(\frac{\theta_{0i}}{Z_{0i}}\partial_{z_i} + \frac{1}{2Z_{0i}}D_i + \frac{\Delta_i\theta_{0i}}{Z_{0i}^2}\Big)\langle \bar{J}(\bar{Z})\Phi_1\Phi_2\rangle \equiv (G+F)\langle \bar{J}(\bar{Z})\Phi_1\Phi_2\rangle, \tag{34}$$

where in the second step for later convenience we name the terms involving derivatives as G, and the remaining terms as F

$$G = \sum_{i=1}^{n}\frac{\theta_{0i}}{Z_{0i}}\partial_{z_i} + \frac{1}{2Z_{0i}}D_i, \quad F = \sum_{i=1}^{n}\frac{\Delta_i\theta_{0i}}{Z_{0i}^2} \tag{35}$$

with $n=2$ in the present case. To evaluate the right hand side of eq.(34), first consider the anticommutator between P and $J = F+G$, noting P, G, F are all Grassmannian odd

$$\begin{aligned}\{J,P\}R &= J(PR) + P(JR)\\ &= FPR + G(PR) + PFR + P(GR)\\ &= FPR + (GP)R - P(GR) + PFR + P(GR)\\ &= (GP)R\end{aligned} \tag{36}$$

with $R \equiv \langle\Phi_1\Phi_2\rangle$. Hence we obtain

$$\langle J\bar{J}\Phi_1\Phi_2\rangle = (P\bar{P} + (G\bar{P}))\langle\Phi_1\Phi_2\rangle, \tag{37}$$

where the first term on the right hand side

$$\begin{aligned}P\bar{P} =& \Delta\bar{\Delta}\Big(-\frac{2}{Z_{12}}\Big(\frac{\theta_{01}}{z_{01}} - \frac{\theta_{02}}{z_{02}}\Big) - \frac{\theta_{12}}{z_{12}}\Big(\frac{1}{Z_{01}} + \frac{1}{Z_{02}}\Big) + \Big(\frac{\theta_{01}}{z_{01}^2} + \frac{\theta_{02}}{z_{02}^2}\Big)\Big)\\ &\times\Big(-\frac{2}{\bar{Z}_{12}}\Big(\frac{\bar{\theta}_{01}}{\bar{z}_{01}} - \frac{\bar{\theta}_{02}}{\bar{z}_{02}}\Big) - \frac{\bar{\theta}_{12}}{\bar{z}_{12}}\Big(\frac{1}{\bar{Z}_{01}} + \frac{1}{\bar{Z}_{02}}\Big) + \Big(\frac{\bar{\theta}_{01}}{\bar{z}_{01}^2} + \frac{\bar{\theta}_{02}}{\bar{z}_{02}^2}\Big)\Big).\end{aligned} \tag{38}$$

Here we have omitted δ-function terms in both P and $\bar{P}$ as mentioned above. The second term in eq.(37) is

$$\begin{aligned}G\bar{P} =& \bar{\Delta}\sum_i\Big(\frac{\theta_{0i}}{Z_{0i}}\partial_{z_1} + \frac{1}{2Z_{0i}}\partial_{\theta_i} + \frac{1}{2Z_{0i}}\theta_i\partial_{z_i}\Big)\\ &\times\Big(-\frac{2}{\bar{Z}_{12}}\Big(\frac{\bar{\theta}_{01}}{\bar{z}_{01}} - \frac{\bar{\theta}_{02}}{\bar{z}_{02}}\Big) - \frac{\bar{\theta}_{12}}{\bar{z}_{12}}\Big(\frac{1}{\bar{Z}_{01}} + \frac{1}{\bar{Z}_{02}}\Big) + \Big(\frac{\bar{\theta}_{01}}{\bar{z}_{01}^2} + \frac{\bar{\theta}_{02}}{\bar{z}_{02}^2}\Big)\Big),\end{aligned} \tag{39}$$

where the third term in the first bracket, i.e. $\frac{1}{2Z_{0i}}\theta_i\partial_{z_i}... = \frac{1}{2z_{0i}}\theta_i\partial_{z_i}...$, will vanish after integral over $\int d\theta$, and the second term $\partial_{\theta_i}\bar{P} = 0$ since $\bar{P}$ does not dependent on θ_i. Thus the only term needed to compute is

$$\bar{\Delta}\sum_i\Big(\frac{\theta_{0i}}{Z_{0i}}\partial_{z_i}\Big)\Big(-\frac{2}{\bar{Z}_{12}}\Big(\frac{\bar{\theta}_{01}}{\bar{z}_{01}}-\frac{\bar{\theta}_{02}}{\bar{z}_{02}}\Big)-\frac{\bar{\theta}_{12}}{\bar{z}_{12}}\Big(\frac{1}{\bar{Z}_{01}}+\frac{1}{\bar{Z}_{02}}\Big)+\Big(\frac{\bar{\theta}_{01}}{\bar{z}_{01}^2}+\frac{\bar{\theta}_{02}}{\bar{z}_{02}^2}\Big)\Big). \tag{40}$$

It turns out the contributions from the second and third terms in the second bracket are nonzero after integration $\int d\theta d\bar{\theta}$, which is

$$\int d^2z\int d\theta d\bar{\theta}G\bar{P} = 2\bar{\Delta}\int d^2z\Big(\frac{\bar{\theta}_1\bar{\theta}_2}{\bar{z}_{12}}+1\Big)\Big(\frac{\tilde{\delta}^{(2)}(z_{01})}{|z_{01}|^2}+\frac{\tilde{\delta}^{(2)}(z_{02})}{|z_{02}|^2}\Big) \tag{41}$$

where we use $\int d^2z\frac{\tilde{\delta}(z_{12})}{z_{0i}} = 0$, which can be obtained in polar coordinates. This term is divergent and it should be dropped, which can be seen as follows. By observing eq.(41), one find that it only depends on $\bar{\Delta}$ while not on Δ, in other words, this term is not symmetric under the interchange of $\bar{\Delta}$ and Δ. However $\langle\bar{J}J\Phi_1...\rangle = -\langle J\bar{J}\Phi_1...\rangle$ should holds (the minus sign appears due to $J(Z)$ is Grassmann odd), which implies the correlator $\langle J\bar{J}\Phi_1...\rangle$ should be symmetric under interchange of $\bar{\Delta}$ and Δ. From this reasoning we will drop these terms. Finally we obtain the integrals as

$$\begin{aligned}&\int d^2zd\theta d\bar{\theta}\langle J(Z)\bar{J}(\bar{Z})\Phi_1(Z_1,\bar{Z}_1)\Phi_n(Z_2,\bar{Z}_2)\rangle/\langle\Phi_1(Z_1,\bar{Z}_1)\Phi_n(Z_2,\bar{Z}_2)\rangle\\ =&\Delta\bar{\Delta}\int d^2zd\theta d\bar{\theta}\Big[\Big(-\frac{2}{Z_{12}}\Big(\frac{\theta_{01}}{z_{01}}-\frac{\theta_{02}}{z_{02}}\Big)-\frac{\theta_{12}}{Z_{12}}\Big(\frac{1}{Z_{01}}+\frac{1}{Z_{02}}\Big)+\Big(\frac{\theta_{01}}{z_{01}^2}+\frac{\theta_{02}}{z_{02}^2}\Big)\Big)\\ &\times\Big(-\frac{2}{\bar{Z}_{12}}\Big(\frac{\bar{\theta}_{01}}{\bar{z}_{01}}-\frac{\bar{\theta}_{02}}{\bar{z}_{02}}\Big)-\frac{\bar{\theta}_{12}}{\bar{Z}_{12}}\Big(\frac{1}{\bar{Z}_{01}}+\frac{1}{\bar{Z}_{02}}\Big)+\Big(\frac{\bar{\theta}_{01}}{\bar{z}_{01}^2}+\frac{\bar{\theta}_{02}}{\bar{z}_{02}^2}\Big)\Big)\Big].\end{aligned} \tag{42}$$

Expanding the integrand, there will be nine terms. We will consider the first term here and list the remaining eight terms in appendix. These integrals can be explicitly performed by employing dimensional regularization which is discussed in section 4.

More concretely, the first term is (Consider the case $\Delta=\bar{\Delta}$) [6]

$$\begin{aligned}
T_{11} \equiv & \int d^2z d\theta d\bar{\theta} \frac{4\Delta^2}{Z_{12}\bar{Z}_{12}}\Big(\frac{\theta_{01}}{z_{01}}-\frac{\theta_{02}}{z_{02}}\Big)\Big(\frac{\bar{\theta}_{01}}{\bar{z}_{01}}-\frac{\bar{\theta}_{02}}{\bar{z}_{02}}\Big) \\
=& -\frac{4\Delta^2}{Z_{12}\bar{Z}_{12}}\int d^2z\Big(\frac{1}{z_{01}}-\frac{1}{z_{02}}\Big)\Big(\frac{1}{\bar{z}_{01}}-\frac{1}{\bar{z}_{02}}\Big) \\
=& -\frac{4\Delta^2}{Z_{12}\bar{Z}_{12}}(\mathcal{I}_{11}(z_1,\bar{z}_1)+\mathcal{I}_{11}(z_2,\bar{z}_2)-\mathcal{I}_{11}(z_1,\bar{z}_2)-\mathcal{I}_{11}(z_2,\bar{z}_1)) \\
=& -\frac{4\Delta^2}{Z_{12}\bar{Z}_{12}}2\pi\Big(-\frac{2}{\epsilon}+\ln|z_{12}|^2+\gamma+\ln\pi\Big),
\end{aligned} \tag{44}$$

where in the second step we used the notation introduced in eq.(134), and γ is Euler constant and ϵ is an infinitesimal constant.

$$\mathcal{I}_{11}(z_i,\bar{z}_j) \equiv \int d^2z \frac{1}{z_{0i}\bar{z}_{0j}}. \tag{45}$$

This integral is computed in setion 4, and we only quote the results in the last line of eq.(44). ϵ is a infinitesimal constant coming from dimensional regularization. [7] Therefore putting together the results of the nine integrals leads to

$$\begin{aligned}
&\frac{1}{\langle\Phi_1(Z_1,\bar{Z}_1)\Phi_2(Z_2,\bar{Z}_2)\rangle}\int d^2z d\theta d\bar{\theta}\langle J(Z)\bar{J}(\bar{Z})\Phi_1(Z_1,\bar{Z}_1)\Phi_2(Z_2,\bar{Z}_2)\rangle \\
=& -\frac{4\pi\Delta^2}{Z_{12}\bar{Z}_{12}}\Big(-\frac{4}{\epsilon}+2\ln|z_{12}|^2+2\gamma+2\ln\pi-2\Big).
\end{aligned} \tag{47}$$

In principle by setting $\theta_{1,2}\to 0$, one can get the results for bosonic CFT, which is

$$-\frac{4\pi\Delta^2}{|z_{12}|^2}\Big(-\frac{4}{\epsilon}+2\ln|z_{12}|^2+2\gamma+2\ln\pi-2\Big). \tag{48}$$

[6] Useful relations

$$\int d\theta\frac{\theta_{01}}{Z_{01}}=\frac{1}{z_{01}}, \quad \int d\theta\frac{1}{Z_{01}}=\frac{\theta_1}{z_{01}^2}, \quad \int d\theta d\bar{\theta}\bar{\theta}\theta=1. \tag{43}$$

[7] Also T_{11} can be evaluated in an alternatively way as

$$\begin{aligned}
T_{11} =& -\frac{4\Delta^2}{Z_{12}\bar{Z}_{12}}\int d^2z\Big(\frac{1}{z_{01}}-\frac{1}{z_{02}}\Big)\Big(\frac{1}{\bar{z}_{01}}-\frac{1}{\bar{z}_{02}}\Big) \\
=& -\frac{4\Delta^2|z_{12}|^2}{Z_{12}\bar{Z}_{12}}\int d^2z\frac{1}{|z_{01}|^2|z_{02}|^2} \\
=& -\frac{4\Delta^2|z_{12}|^2}{Z_{12}\bar{Z}_{12}}\mathcal{I}_{1111}(z_1,z_2,\bar{z}_1,\bar{z}_2) \\
=& -\frac{4\Delta^2}{Z_{12}\bar{Z}_{12}}2\pi\Big(-\frac{2}{\epsilon}+\ln|z_{12}|^2+\gamma+\log\pi+\mathcal{O}(\epsilon)\Big).
\end{aligned} \tag{46}$$

which is equal to result in eq.(44). The integral in the last step was computed in [19].

Comparing this with the CFT results given in eq.(8) in [19] as

$$-\frac{4\pi\Delta^2}{|z_{12}|^2}\Big(-\frac{4}{\epsilon}+2\ln|z_{12}|^2+2\gamma+2\ln\pi-5\Big). \tag{49}$$

One can find that the last constant is different in eq.(48) and eq.(49). This difference can be understood from the way we performing the integrals. On one hand, we can use dimensional regularization to evaluate the integral directly

$$\int d^2z\frac{|z_{12}|^4}{|z_{01}|^4|z_{02}|^4}=-\frac{4\pi\Delta^2}{|z_{12}|^2}\Big(-\frac{4}{\epsilon}+2\ln|z_{12}|^2+2\gamma+2\ln\pi-5\Big), \tag{50}$$

which will result in eq.(49). On the other hand, we can compute the above integral in an indirect way as we did at the beginning, i.e., Firstly, expanding the integrand into several terms as below, then using dimensional regularization to compute each integral, finally adding up the contribution of individual term

$$\begin{aligned}\int d^2z\frac{|z_{12}|^4}{|z_{01}|^4|z_{02}|^4}&=\int d^2z\Big(\frac{1}{z_{01}^2}+\frac{1}{z_{02}^2}-\frac{1}{z_{01}z_{02}}\Big)\Big(\frac{1}{\bar{z}_{01}^2}+\frac{1}{\bar{z}_{02}^2}-\frac{1}{\bar{z}_{01}\bar{z}_{02}}\Big)\\&=-\frac{4\pi\Delta^2}{|z_{12}|^2}\Big(-\frac{4}{\epsilon}+2\ln|z_{12}|^2+2\gamma+2\ln\pi-2\Big),\end{aligned} \tag{51}$$

which leads to eq.(48). The difference between eq.(48) and eq.(49) can be eliminated by redefine ϵ.

2.2 3-pt correlators

The general form of 3pt correlators can be written as

$$\langle\Phi(Z_1,\bar{Z}_1)\Phi(Z_2,\bar{Z}_2)\Phi(Z_3,\bar{Z}_3)\rangle=cO_3\bar{O}_3e^{a\theta_{123}\bar{\theta}_{123}}, \tag{52}$$

where a,c are two undetermined constants and for later convenience we denote

$$O_3=\prod_{i<j=1}^{3}\frac{1}{Z_{ij}^{\Delta_{ij}}},\quad \bar{O}_3=\prod_{i<j=1}^{3}\frac{1}{\bar{Z}_{ij}^{\bar{\Delta}_{ij}}}. \tag{53}$$

As discussed in 2-pt correlators in the previous section, we first consider the correlator $\langle J\Phi_1\Phi_2\Phi_3\rangle$ which can be calculated by using the definition of G,F in eq.(35) as follows

$$\begin{aligned}&(G+F)O_3\bar{O}_3e^{a\theta_{123}\bar{\theta}_{123}}\\=&FO_3\bar{O}_3e^{a\theta_{123}\bar{\theta}_{123}}+[G(O_3\bar{O}_3)]e^{a\theta_{123}\bar{\theta}_{123}}+O_3\bar{O}_3[Ge^{a\theta_{123}\bar{\theta}_{123}}]\\=&(F+P)O_3\bar{O}_3e^{a\theta_{123}\bar{\theta}_{123}}+O_3(G\bar{O}_3)e^{a\theta_{123}\bar{\theta}_{123}}+O_3\bar{O}_3a[(G\theta_{123})\bar{\theta}_{123}-\theta_{123}(G\bar{\theta}_{123})]e^{a\theta_{123}\bar{\theta}_{123}}\\\to&(F+P+a(G\theta_{123})\bar{\theta}_{123})O_3\bar{O}_3e^{a\theta_{123}\bar{\theta}_{123}},\end{aligned} \tag{54}$$

where P (defined by $GO_3 \equiv PO_3$) turns out to be

$$P = \sum_{i,k,k\neq i} \frac{\Delta_{ik}}{Z_{ki}} \Big(\frac{\theta_{0i}}{z_{0i}} - \frac{\theta_{ki}}{2Z_{0i}} \Big). \tag{55}$$

In the last step of eq.(54) we have omitted the "crossing" terms such as $G\bar{O}_3, G\bar{\theta}_{123}$ (By crossing terms we mean the terms with holomorphic derivative ∂_z acting on antiholomorphic coordinates, or $\partial_{\bar{z}}$ acting on holomorphic coordinates, which will result in a δ-function as $\partial_z(1/\bar{z}) = \tilde{\delta}(z)$. Note that we have encountered crossing term as in eq.(32) in the 2pt correlator case), since these terms will vanish when integrating over θ. To be concrete, taking the term $G\bar{O}_3$ for example

$$\begin{aligned} G\bar{O}_3 = &- \Big(\frac{\theta_{12}}{z_{01}} \bar{\Delta}_{12} \bar{\theta}_1 \bar{\theta}_2 \tilde{\delta}(z_{12}) + \frac{\theta_{31}}{z_{01}} \bar{\Delta}_{13} \bar{\theta}_3 \bar{\theta}_1 \tilde{\delta}(z_{31}) + \frac{\theta_{23}}{z_{02}} \bar{\Delta}_{23} \bar{\theta}_2 \bar{\theta}_3 \tilde{\delta}(z_{23}) \Big) \bar{O}_3 \\ &- \frac{\theta_{23}}{2z_{02}} \frac{\bar{\Delta}_{23}\tilde{\delta}(z_{32})}{\bar{Z}_{12}^{\bar{\Delta}_{12}} \bar{Z}_{23}^{\bar{\Delta}_{23}-1} \bar{Z}_{31}^{\bar{\Delta}_{13}}} - \frac{\theta_{31}}{2z_{01}} \frac{\bar{\Delta}_{13}\tilde{\delta}(z_{31})}{\bar{Z}_{12}^{\bar{\Delta}_{12}} \bar{Z}_{23}^{\bar{\Delta}_{23}} \bar{Z}_{31}^{\bar{\Delta}_{13}-1}} - \frac{\theta_{12}}{2z_{01}} \frac{\bar{\Delta}_{12}\tilde{\delta}(z_{12})}{\bar{Z}_{12}^{\bar{\Delta}_{12}-1} \bar{Z}_{23}^{\bar{\Delta}_{23}} \bar{Z}_{31}^{\bar{\Delta}_{13}}}, \end{aligned} \tag{56}$$

thus $\int d\theta G\bar{O}_3 = 0$.

With $\langle J\Phi_1\Phi_2\Phi_3\rangle$ in hand, we can go on to consider $\langle J\bar{J}\Phi_1\Phi_2\Phi_3\rangle$

$$\begin{aligned} &(G+F)(\bar{G}+\bar{F})O_3\bar{O}_3 e^{a\theta_{123}\bar{\theta}_{123}} \\ =&(G+F)(\bar{P}+\bar{F} - a\theta_{123}(\bar{G}\bar{\theta}_{123}))O_3\bar{O}_3 e^{a\theta_{123}\bar{\theta}_{123}} \\ =&\Big[(F+P+a(G\theta_{123})\bar{\theta}_{123})(\bar{F}+\bar{P} - a\theta_{123}(\bar{G}\bar{\theta}_{123})) \\ &- a(G\theta_{123})(\bar{G}\bar{\theta}_{123}) + a\theta_{123}(G(\bar{G}\bar{\theta}_{123})) + G(\bar{F}+\bar{P})\Big] O_3\bar{O}_3 e^{a\theta_{123}\bar{\theta}_{123}}. \end{aligned} \tag{57}$$

Let us first focus on the last two terms which are crossing terms. After some computation the last term is

$$\int d\theta d\bar{\theta} G(\bar{P}+\bar{F}) = -2\sum_i \bar{\Delta}_i \frac{\tilde{\delta}^{(2)}(z_{0i})}{|z_{0i}|^2} + \sum_{i,k,i\neq k} \frac{\tilde{\delta}^{(2)}(z_{0i})}{|z_{0i}|^2} \frac{\bar{\theta}_k\bar{\theta}_i}{\bar{z}_{ki}} \bar{\Delta}_{ik}. \tag{58}$$

For the same reason as discussed below eq.(41), this term should be dropped out. As for the term $G(\bar{G}\bar{\theta}_{123})$, after employing the anti-commutator

$$\begin{aligned} \{G,\bar{G}\} =& \sum_i \Big(\frac{\theta_{0i}}{z_{0i}} + \frac{\theta_i}{2z_{0i}}\Big)(-\tilde{\delta}(z_{0i}))\Big(\bar{\theta}_{0i} + \frac{\bar{\theta}_i}{2}\Big)\bar{\partial}_i + \sum_i \Big(\frac{\bar{\theta}_{0i}}{\bar{z}_{0i}} + \frac{\bar{\theta}_i}{2\bar{z}_{0i}}\Big)(-\tilde{\delta}(z_{0i}))\Big(\theta_{0i} + \frac{\theta_i}{2}\Big)\partial_i \\ &+ \sum_i \Big(\frac{\theta_{0i}}{z_{0i}} + \frac{\theta_i}{2z_{0i}}\Big)(-\tilde{\delta}(z_{0i}))\Big(1 + \frac{2\bar{\theta}\bar{\theta}_i}{\bar{z}_{0i}}\Big)\frac{1}{2}\partial_{\bar{\theta}_i} + \sum_i \Big(\frac{\bar{\theta}_{0i}}{\bar{z}_{0i}} + \frac{\bar{\theta}_i}{2\bar{z}_{0i}}\Big)(-\tilde{\delta}(z_{0i}))\Big(1 + \frac{2\theta\theta_i}{z_{0i}}\Big)\frac{1}{2}\partial_{\theta_i} \end{aligned} \tag{59}$$

$G(\bar{G}\bar{\theta}_{123})$ can be written as

$$\begin{aligned}
&G(\bar{G}\bar{\theta}_{123}) \to \{G, \bar{G}\}\bar{\theta}_{123} \\
&\to \sum_i \Big(\frac{\theta_{0i}}{z_{0i}} + \frac{\theta_i}{2z_{0i}}\Big)(-\tilde{\delta}(z_{0i}))\Big(1 + \frac{2\bar{\theta}\bar{\theta}_i}{\bar{z}_{0i}}\Big)\frac{1}{2}\partial_{\bar{\theta}_i}\bar{\theta}_{123} \\
&\quad + \sum_i \Big(\frac{\theta_{0i}}{z_{0i}} + \frac{\theta_i}{2z_{0i}}\Big)(-\tilde{\delta}(z_{0i}))\Big(\bar{\theta}_{0i} + \frac{\bar{\theta}_i}{2}\Big)\bar{\partial}_i\bar{\theta}_{123}
\end{aligned} \tag{60}$$

where the term $G\bar{\theta}_{123}$ is omitted in the first step, and also for $\partial_{z_j}\bar{\theta}_{123}$ in the second step since they do not contain θ. Thus finally we get

$$\int d^2z d\theta d\bar{\theta} G(\bar{G}\bar{\theta}_{123}) = \sum_i \int d^2z \frac{-\tilde{\delta}(z_{0i})\bar{\theta}_i}{|z_{0i}|^2}\partial_{\bar{\theta}_i}\bar{\theta}_{123}, \tag{61}$$

which is also singular and should be dropped. This can be seen by noting that if we interchange the position in $\langle J(Z)\bar{J}(\bar{Z})\Phi_1...\rangle$, and to consider $\langle \bar{J}(\bar{Z})J(Z)\Phi_1...\rangle$ we will obtain a term different with eq.(61) as

$$\int d^2z d\theta d\bar{\theta} \bar{G}(G\theta_{123}) = -\sum_i \int d^2z \frac{-\tilde{\delta}(z_{0i})\theta_i}{|z_{0i}|^2}\partial_{\theta_i}\theta_{123}. \tag{62}$$

thus the appearance of eq.(61) implies the identity $\langle \bar{J}(\bar{Z})J(Z)\Phi_1...\rangle = -\langle J(Z)\bar{J}(\bar{Z})\Phi_1...\rangle$ does not hold. Thus we must drop the crossing term eq.(61). From this consideration we will omit all the crossing terms without explicitly pointing out in the following case with $n \geq 4$ point correlation functions.

Finally we obtain the 3pt correlator as

$$\begin{aligned}
&\frac{1}{\langle \Phi_1\Phi_2\Phi_3\rangle}\int d^2z d\theta d\bar{\theta}\langle J(Z)\bar{J}(\bar{Z})\Phi_1(Z_1,\bar{Z}_1)\Phi_2(Z_2,\bar{Z}_2)\Phi_3(Z_3,\bar{Z}_3)\rangle \\
&= \int d^2z\Big[\sum_{i,k,k\neq i}\frac{\Delta_{ik}}{Z_{ki}}\Big(\frac{1}{z_{0i}} + \frac{\theta_k\theta_i}{2z_{0i}^2}\Big) + \sum_i \frac{\Delta_i}{z_{0i}^2} + a\sum_i\Big(\frac{1}{z_{0i}}\partial_{z_i}\theta_{123} + \frac{\theta_i}{2z_{0i}^2}\partial_{\theta_i}\theta_{123}\Big)\bar{\theta}_{123}\Big] \\
&\quad \times(-1)\Big[\sum_{i,k,k\neq i}\frac{\bar{\Delta}_{ik}}{\bar{Z}_{ki}}\Big(\frac{1}{\bar{z}_{0i}} + \frac{\bar{\theta}_k\bar{\theta}_i}{2\bar{z}_{0i}^2}\Big) + \sum_i\frac{\bar{\Delta}_i}{\bar{z}_{0i}^2} + a\theta_{123}\sum_i\Big(\frac{1}{\bar{z}_{0i}}\partial_{\bar{z}_i}\bar{\theta}_{123} + \frac{\bar{\theta}_i}{2\bar{z}_{0i}^2}\partial_{\bar{\theta}_i}\bar{\theta}_{123}\Big)\Big] \\
&\quad - a\sum_i\Big(\frac{1}{z_{0i}}\partial_{z_i}\theta_{123} + \frac{\theta_i}{2z_{0i}^2}\partial_{\theta_i}\theta_{123}\Big)\times\sum_i\Big(\frac{1}{\bar{z}_{0i}}\partial_{\bar{z}_i}\bar{\theta}_{123} + \frac{\bar{\theta}_i}{2\bar{z}_{0i}^2}\partial_{\bar{\theta}_i}\bar{\theta}_{123}\Big).
\end{aligned} \tag{63}$$

Let us first consider the terms containing no a

$$
\begin{aligned}
&-\int d^2z \sum_{ij}\Big[\sum_{k,k\neq i}\frac{\Delta_{ik}}{Z_{ki}}\frac{1}{z_{0i}}+\frac{1}{z_{0i}^2}\Big(\sum_{k,k\neq i}\frac{\theta_k\theta_i\Delta_{ik}}{2Z_{ki}}+\Delta_i\Big)\Big]\Big[\sum_{l,l\neq j}\frac{\bar\Delta_{lj}}{\bar Z_{lj}}\frac{1}{\bar z_{0j}}+\frac{1}{\bar z_{0j}^2}\Big(\sum_{l,l\neq j}\frac{\bar\theta_l\bar\theta_j\bar\Delta_{lj}}{2\bar Z_{lj}}+\bar\Delta_j\Big)\Big]\\
&=-\sum_{ij}\Big[\mathcal{I}_{11}(z_i,\bar z_j)\sum_{k,k\neq i}\sum_{l,l\neq j}\frac{\Delta_{ik}}{Z_{ki}}\frac{\bar\Delta_{lj}}{\bar Z_{lj}}+\mathcal{I}_{22}(z_i,\bar z_j)\Big(\sum_{k,k\neq i}\frac{\theta_k\theta_i\Delta_{ik}}{2Z_{ki}}+\Delta_i\Big)\Big(\sum_{l,l\neq j}\frac{\bar\theta_l\bar\theta_j\bar\Delta_{lj}}{2\bar Z_{lj}}+\bar\Delta_j\Big)\\
&\quad+\mathcal{I}_{12}(z_i,\bar z_j)\sum_{k,k\neq i}\frac{\Delta_{ik}}{Z_{ki}}\Big(\sum_{l,l\neq j}\frac{\bar\theta_l\bar\theta_j\bar\Delta_{lj}}{2\bar Z_{lj}}+\bar\Delta_j\Big)+\mathcal{I}_{21}(z_i,\bar z_j)\Big(\sum_{k,k\neq i}\frac{\theta_k\theta_i\Delta_{ik}}{2Z_{ki}}+\Delta_i\Big)\sum_{l,l\neq j}\frac{\bar\Delta_{lj}}{\bar Z_{lj}}\Big].
\end{aligned}
\tag{64}
$$

Next evaluating the a^1-terms which contains two parts, the first part is

$$
\begin{aligned}
V_{11}\equiv&-a\sum_{ij}\int d^2z\Big[\sum_{k,k\neq i}\frac{\Delta_{ik}}{2Z_{ki}}\frac{1}{z_{0i}}+\frac{1}{z_{0i}^2}\Big(\sum_{k,k\neq i}\frac{\theta_k\theta_i\Delta_{ik}}{2Z_{ki}}+\Delta_i\Big)\Big]\theta_{123}\Big(\frac{1}{\bar z_{0j}}\partial_{\bar z_j}\bar\theta_{123}+\frac{\bar\theta_j}{2\bar z_{0j}^2}\partial_{\bar\theta_j}\bar\theta_{123}\Big)\\
&\quad-(\text{barred}\leftrightarrow\text{unbarred})\\
=&-a\sum_{ij}\Big[\mathcal{I}_{11}(z_i,\bar z_j)\sum_{k,k\neq i}\frac{\Delta_{ik}}{Z_{ki}}\theta_{123}\partial_{\bar z_j}\bar\theta_{123}+\mathcal{I}_{12}(z_i,\bar z_j)\sum_{k,k\neq i}\frac{\Delta_{ik}}{2Z_{ki}}\theta_{123}\bar\theta_j\partial_{\bar\theta_j}\bar\theta_{123}\\
&\quad+\mathcal{I}_{21}(z_i,\bar z_j)\Big(\sum_{k,k\neq i}\frac{\theta_k\theta_i\Delta_{ik}}{2Z_{ki}}+\Delta_i\Big)\theta_{123}\partial_{\bar z_j}\bar\theta_{123}+\frac{1}{2}\mathcal{I}_{22}(z_i,\bar z_j)\Big(\sum_{k,k\neq i}\frac{\theta_k\theta_i\Delta_{ik}}{2Z_{ki}}+\Delta_i\Big)\theta_{123}\bar\theta_j\partial_{\bar\theta_j}\bar\theta_{123}\Big]\\
&\quad-(\text{barred}\leftrightarrow\text{unbarred}),
\end{aligned}
\tag{65}
$$

and the second part is

$$
\begin{aligned}
V_{12}\equiv&-a\sum_{ij}\Big(\frac{1}{z_{0i}}\partial_{z_i}\theta_{123}+\frac{\theta_i}{2z_{0i}^2}\partial_{\theta_i}\theta_{123}\Big)\Big(\frac{1}{\bar z_{0j}}\partial_{\bar z_j}\bar\theta_{123}+\frac{\bar\theta_j}{2\bar z_{0j}^2}\partial_{\bar\theta_j}\bar\theta_{123}\Big)\\
=&-a\sum_{ij}\Big(\mathcal{I}_{11}(z_i,\bar z_j)\partial_{z_i}\theta_{123}\partial_{\bar z_j}\bar\theta_{123}+\frac{1}{2}\mathcal{I}_{12}(z_i,\bar z_j)(\partial_{z_i}\theta_{123})\bar\theta_j\partial_{\bar\theta_j}\bar\theta_{123}+\frac{1}{2}\mathcal{I}_{21}(z_i,\bar z_j)\theta_i\partial_{\theta_i}\theta_{123}\partial_{\bar z_j}\bar\theta_{123},
\end{aligned}
\tag{66}
$$

As for the a^2-term denoted as V_2, by observing eq.(63) we find $V_2=-aV_{12}\theta_{123}\bar\theta_{123}$, thus $V_{12}+V_2=V_{12}e^{-a\theta_{123}\bar\theta_{123}}$. In summary, the result for 3-pt correlators with $T\bar T$

perturbation to first order is

$$\begin{aligned}
&\frac{1}{\langle\Phi_1\Phi_2\Phi_3\rangle}\int d^2z d\theta d\bar\theta \langle J(Z)\bar J(\bar Z)\Phi_1(Z_1,\bar Z_1)\Phi_2(Z_2,\bar Z_2)\Phi_3(Z_3,\bar Z_3)\rangle \\
=&-\sum_{ij}\Big[-\pi(-\frac{2}{\epsilon}+\ln|z_{ij}|^2+\gamma+\ln\pi) \\
&\times\sum_{k,k\neq i}\Big(\sum_{l,l\neq j}\frac{\Delta_{ik}}{Z_{ki}}\frac{\bar\Delta_{lj}}{\bar Z_{lj}}+a\frac{\Delta_{ik}}{Z_{ki}}\theta_{123}\partial_{\bar z_j}\bar\theta_{123}+a\frac{\bar\Delta_{ik}}{\bar Z_{ki}}\partial_{z_j}\theta_{123}\bar\theta_{123}+a\partial_{z_i}\theta_{123}\partial_{\bar z_j}\bar\theta_{123}e^{-a\theta_{123}\bar\theta_{123}}\Big) \\
&+\frac{\pi}{\bar z_{ij}}\Big(\frac{a}{2}(\partial_{z_i}\theta_{123})\bar\theta_j\partial_{\bar\theta_j}\bar\theta_{123}e^{-a\theta_{123}\bar\theta_{123}}+\sum_{k,k\neq i}\frac{\Delta_{ik}}{Z_{ik}}\sum_{l,l\neq j}\frac{\bar z_{lj}\bar\Delta_{lj}}{2\bar Z_{lj}} \\
&+a\sum_{l,l\neq j}\frac{\bar z_{lj}\bar\Delta_{jl}}{2\bar Z_{lj}}\partial_{z_i}\theta_{123}\bar\theta_{123}+a\sum_{k,k\neq i}\frac{\Delta_{ik}}{2Z_{ki}}\theta_{123}\bar\theta_j\partial_{\bar\theta_j}\bar\theta_{123}\Big) \\
&-\frac{\pi}{z_{ij}}\Big(\frac{a}{2}(\theta_i\partial_{\theta_i}\theta_{123})\partial_{\bar z_j}\bar\theta_{123}e^{-a\theta_{123}\bar\theta_{123}}+a\sum_{k,k\neq i}\frac{z_{ki}\Delta_{ik}}{2Z_{ki}}\theta_{123}\partial_{\bar z_j}\bar\theta_{123} \\
&+a\theta_i\partial_{\theta_i}\theta_{123}\bar\theta_{123}\sum_{l,l\neq j}\frac{\bar\Delta_{lj}}{2\bar Z_{lj}}+\sum_{k,k\neq i}\frac{z_{ki}\Delta_{ik}}{2Z_{ki}}\sum_{l,l\neq j}\frac{\bar\Delta_{lj}}{\bar Z_{lj}}\Big],
\end{aligned} \tag{67}$$

where the identity $\sum_{k,k\neq i}\bar\Delta_{ik}=2\bar\Delta_i$. is used to simplify the final expression.

2.3 n-pt correlators

For n point with $n\geq 4$, the undeformed correlator functions take the form as

$$\langle\Phi_1(Z_1,\bar Z_1)...\Phi_n(Z_n,\bar Z_n)\rangle=O_n\bar O_n f(U_i,\bar U_i,w_k,\bar w_k) \tag{68}$$

with

$$O_n=\prod_{i<j}Z_{ij}^{-\Delta_{ij}},\quad \bar O_n=\prod_{i<j}\bar Z_{ij}^{-\bar\Delta_{ij}}. \tag{69}$$

Assuming all Φ_i have the same dimension $(\Delta,\bar\Delta)$, we have

$$A\equiv\Delta_{ij}=\frac{2\Delta}{n-1},\quad \bar\Delta\equiv\bar\Delta_{ij}=\frac{2\bar\Delta}{n-1}. \tag{70}$$

Again the crossing terms $\bar G Z_{ijkl},\bar G\theta_{ijk},\bar G O_n$ do not depend on θ and we will not consider these terms below. Now evaluate

$$\langle J\Phi_1...\Phi_n\rangle=(F+G)O_n\bar O_n f=(F+P)O_n\bar O_n f+QO_n\bar O_n, \tag{71}$$

where P takes the same form as eq.(55) with summation from 1 to n, and

$$Q \equiv (GU_i)\frac{\partial f}{\partial U_i} + (Gw_k)\frac{\partial f}{\partial w_k} = \sum_{j=1}^{n}\Big(\frac{\theta_{0j}}{Z_{0j}}\partial^R_{z_j} f + \frac{1}{2Z_{0j}}D^R_j f\Big), \tag{72}$$

where we introduced the notation $\partial^R_{z_j}, D^R_j, \partial^R_\theta$ which act on z_i, θ_i but not on $\bar{z}_i, \bar{\theta}_i$, and similarly let $\partial^L_{\bar{z}_j}, \bar{D}^L_j, \partial^L_{\bar{\theta}}$ act on $\bar{z}_i, \bar{\theta}_i$ but not on z_i, θ_i (thus $\partial^R_{z_j}(1/\bar{z}_j) = 0$). When inserting $J\bar{J}$, yields

$$\begin{aligned}&(F+G)[(\bar{F}+\bar{P})O_n\bar{O}_n f + \bar{Q}O_n\bar{O}_n]\\ =&(F+P)(\bar{F}+\bar{P})O_n\bar{O}_n f + Q(\bar{F}+\bar{P})O_n\bar{O}_n + (F+P)\bar{Q}O_n\bar{O}_n + (G\bar{Q})O_n\bar{O}_n\end{aligned} \tag{73}$$

with

$$\bar{Q} = \sum_{j=1}^{n}\Big(\Big(\frac{\bar{\theta}_{0j}}{\bar{z}_{0j}} + \frac{\bar{\theta}_j}{2\bar{z}_{0j}}\Big)\partial^L_{\bar{z}_j} f + \frac{1}{2\bar{Z}_{0j}}\partial^L_{\bar{\theta}_j} f\Big). \tag{74}$$

Naively the last term in eq.(73) looks like a crossing term, but this is not the case as can be see below

$$\begin{aligned}G\bar{Q} =& -\sum_{ij}\Big(\frac{\bar{\theta}_{0j}}{\bar{z}_{0j}} + \frac{\bar{\theta}_j}{2\bar{z}_{0j}}\Big)\Big((\partial_{\bar{z}_j}\bar{U}_i)\Big(G\frac{\partial f}{\partial \bar{U}_i}\Big) - (\partial_{\bar{z}_j}\bar{w}_i)\Big(G\frac{\partial f}{\partial \bar{w}_i}\Big)\Big)\\ &-\sum_{ij}\frac{1}{2\bar{Z}_{0j}}\Big((\partial_{\bar{\theta}_j}\bar{U}_i)\Big(G\frac{\partial f}{\partial \bar{U}_i}\Big) - (\partial_{\bar{\theta}_j}\bar{w}_i)\Big(G\frac{\partial f}{\partial \bar{w}_i}\Big)\Big),\end{aligned} \tag{75}$$

where for example one has

$$G\frac{\partial f}{\partial \bar{U}_i} = \sum_j\Big((GU_j)\frac{\partial^2 f}{\partial U_j \partial \bar{U}_i} + (Gw_j)\frac{\partial^2 f}{\partial w_j \partial \bar{U}_i}\Big) \equiv G^R\frac{\partial f}{\partial \bar{U}_i} \tag{76}$$

with G^R acting only on U_j, w_j but not on $\bar{U}_j, \bar{w}_j$. Eventually one can get

$$\int d\theta d\bar{\theta} G\bar{Q} = \sum_{ij}\Big[\frac{1}{z_{0i}}\Big(-\frac{1}{\bar{z}_{0j}}\partial^R_{z_i}\partial^L_{\bar{z}_j} f - \frac{\bar{\theta}_j}{2\bar{z}^2_{0j}}\partial^R_{z_i}\partial^L_{\bar{\theta}_j} f\Big) + \frac{\theta_i}{2z^2_{0i}}\Big(-\frac{1}{\bar{z}_{0j}}\partial^R_{\theta_n}\partial^L_{\bar{z}_j} f + \frac{\bar{\theta}_j}{2\bar{z}^2_{0j}}\partial^R_{\theta_n}\partial^L_{\bar{\theta}_j} f\Big)\Big]. \tag{77}$$

In summary the $T\bar{T}$ deformed correlator is of the form

$$\begin{aligned}
&\frac{1}{\langle\Phi_1...\Phi_n\rangle}\int d^2z d\theta d\bar{\theta}\langle J(Z)\bar{J}(\bar{Z})\Phi_1(Z_1,\bar{Z}_1)...\Phi_n(Z_n,\bar{Z}_n)\rangle\\
=&\int d^2z(-1)\sum_{ij}\Big[\sum_{k,k\neq i}\frac{\Delta_{ik}}{Z_{ki}}\frac{1}{z_{0i}}+\frac{1}{z_{0i}^2}\Big(\sum_{k,k\neq i}\frac{\theta_k\theta_i\Delta_{ik}}{2Z_{ki}}+\Delta_i\Big)\Big]\\
&\times\Big[\sum_{l,l\neq j}\frac{\bar{\Delta}_{lj}}{\bar{Z}_{lj}}\frac{1}{\bar{z}_{0j}}+\frac{1}{\bar{z}_{0j}^2}\Big(\sum_{l,l\neq j}\frac{\bar{\theta}_l\bar{\theta}_j\bar{\Delta}_{lj}}{2\bar{Z}_{lj}}+\bar{\Delta}_j\Big)\Big]\\
&-\sum_{ij}\Big[\Big(\frac{1}{z_{0i}}\partial^R_{z_i}f+\frac{\theta_i}{2z_{0i}^2}D^R_i f\Big)\frac{1}{f}\Big]\Big[\sum_{l,l\neq j}\frac{\bar{\Delta}_{lj}}{\bar{Z}_{lj}}\frac{1}{\bar{z}_{0j}}+\frac{1}{\bar{z}_{0j}^2}\Big(\sum_{l,l\neq j}\frac{\bar{\theta}_l\bar{\theta}_j\bar{\Delta}_{lj}}{2\bar{Z}_{lj}}+\bar{\Delta}_j\Big)\Big]\\
&-\sum_{ij}\Big[\sum_{k,k\neq i}\frac{\Delta_{ik}}{Z_{ki}}\frac{1}{z_{0i}}+\frac{1}{z_{0i}^2}\Big(\sum_{k,k\neq i}\frac{\theta_k\theta_i\Delta_{ik}}{2Z_{ki}}+\Delta_i\Big)\Big]\Big[\Big(\frac{1}{\bar{z}_{0j}}\partial^L_{\bar{z}_j}f+\frac{\bar{\theta}_j}{2\bar{z}_{0j}^2}\bar{D}^L_j f\Big)\frac{1}{f}\Big]\\
&+\sum_{ij}\Big[\frac{1}{z_{0i}}\Big(-\frac{1}{\bar{z}_{0j}}\partial^R_{z_i}\partial^L_{\bar{z}_j}f-\frac{\bar{\theta}_j}{2\bar{z}_{0j}^2}\partial^R_{z_i}\partial^L_{\bar{\theta}_j}f\Big)+\frac{\theta_i}{2z_{0i}^2}\Big(-\frac{1}{\bar{z}_{0j}}\partial^R_{\theta_i}\partial^L_{\bar{z}_j}f+\frac{\bar{\theta}_j}{2\bar{z}_{0j}^2}\partial^R_{\theta_i}\partial^L_{\bar{\theta}_j}f\Big)\Big].
\end{aligned}\tag{78}$$

Hence using the results for integrals in section 4, the final result is

$$\begin{aligned}
&\frac{1}{\langle\Phi_1...\Phi_n\rangle}\int d^2z d\theta d\bar{\theta}\langle J(Z)\bar{J}(\bar{Z})\Phi_1(Z_1,\bar{Z}_1)...\Phi_n(Z_n,\bar{Z}_n)\rangle\\
=&\sum_{ij}\Big[-\pi\Big(-\frac{2}{\epsilon}+\ln|z_{ij}|^2+\gamma+\ln\pi\Big)\\
&\times\Big(-\sum_{k,k\neq i}\frac{\Delta_{ik}}{Z_{ki}}\sum_{l,l\neq j}\frac{\bar{\Delta}_{lj}}{\bar{Z}_{lj}}-\partial^R_{z_i}f\sum_{l,l\neq j}\frac{\bar{\Delta}_{lj}}{f\bar{Z}_{lj}}-\sum_{k,k\neq i}\frac{\Delta_{ik}}{Z_{ki}}\partial^L_{\bar{z}_j}f\frac{1}{f}-\partial^R_{z_i}\partial^L_{\bar{z}_j}f\frac{1}{f}\Big)\\
&-\frac{\pi}{\bar{z}_{ij}}\Big(\sum_{k,k\neq i}\frac{\Delta_{ik}}{Z_{ki}}\sum_{l,l\neq j}\frac{\bar{z}_{lj}\bar{\Delta}_{lj}}{2\bar{Z}_{lj}}+\partial^R_{z_i}f\frac{1}{f}\sum_{l,l\neq j}\frac{\bar{z}_{lj}\bar{\Delta}_{lj}}{2\bar{Z}_{lj}}+\sum_{k,k\neq i}\frac{\Delta_{ik}}{2Z_{ki}}\bar{\theta}_j\partial^L_{\bar{\theta}_j}f\frac{1}{f}-\frac{\bar{\theta}_j}{2}\partial^R_{z_i}\partial^L_{\bar{\theta}_j}f\frac{1}{f}\Big)\\
&+\frac{\pi}{z_{ij}}\Big(\sum_{k,k\neq i}\frac{z_{ki}\Delta_{ik}}{2Z_{ki}}\sum_{l,l\neq j}\frac{\bar{\Delta}_{lj}}{\bar{Z}_{lj}}+\frac{\theta_i}{2f}\partial^R_{\theta_i}f\sum_{l,l\neq j}\frac{\bar{\Delta}_{lj}}{\bar{Z}_{lj}}+\sum_{k,k\neq i}\frac{z_{ki}\Delta_{ik}}{2Z_{ki}}\partial^L_{\bar{z}_j}f\frac{1}{f}-\frac{\theta_i}{2}\partial^R_{\theta_i}\partial^L_{\bar{z}_j}f\frac{1}{f}\Big)\Big].
\end{aligned}\tag{79}$$

Setting $n=4$, the above results can be used to investigate, for example, the OTOC. The superfield can be written as

$$\Phi(Z,\bar{Z})=\phi+\theta\psi_1+\bar{\theta}\psi_2+\theta\bar{\theta}f \tag{80}$$

and its conjugate

$$\Phi(Z,\bar{Z})^\dagger=\phi^\dagger-\theta\psi_2^\dagger-\bar{\theta}\psi_1^\dagger+\theta\bar{\theta}f^\dagger \tag{81}$$

To consider the OTOC involving two fields ϕ, ψ_1, from (45) in [19], at first order one of the 4pt functions needed to compute is

$$
\begin{aligned}
&\langle \phi(z_1,\bar{z}_1)\phi^\dagger(z_2,\bar{z}_2)\psi_1(z_3,\bar{z}_3)\psi_1^\dagger(z_4,\bar{z}_4)\rangle_\lambda \\
&= -\int d\theta_3 d\bar{\theta}_4 \int d^2z d\theta d\bar{\theta} \langle J(Z)\bar{J}(\bar{Z})\Phi_1(Z_1,\bar{Z}_1)\Phi^\dagger(Z_2,\bar{Z}_2)\Phi(Z_3,\bar{Z}_3)\Phi^\dagger(Z_4,\bar{Z}_4)\rangle|_{\theta_1=\bar{\theta}_1=\theta_2=\bar{\theta}_2=\theta_4=\bar{\theta}_3=0} \\
&= -\int d\theta_3\bar{\theta}_4 \Big\{ \sum_{i\neq j} \Big[-\pi\Big(-\frac{2}{\epsilon} + \ln|z_{ij}|^2 + \gamma + \ln\pi \Big) \\
&\quad \times \Big(-\sum_{k,k\neq i} \frac{\Delta_{ik}}{z_{ki}} \sum_{l,l\neq j} \frac{\bar{\Delta}_{lj}}{\bar{z}_{lj}} f - \partial^R_{z_i} f \sum_{l,l\neq j} \frac{\bar{\Delta}_{lj}}{\bar{z}_{lj}} - \sum_{k,k\neq i} \frac{\Delta_{ik}}{z_{ki}} \partial^L_{\bar{z}_j} f - \partial^L_{\bar{z}_j}\partial^R_{z_i} f \Big) \\
&\quad - \frac{\pi}{\bar{z}_{ij}} \Big(\sum_{k,k\neq i} \frac{\Delta_{ik}}{z_{ki}} \bar{\Delta}_j f + \partial^R_{z_i} f \bar{\Delta}_j + \delta_{j4} \sum_{k,k\neq i} \frac{\Delta_{ik}}{2z_{ki}} \bar{\theta}_j \partial^L_{\bar{\theta}_j} f - \delta_{j4} \frac{\bar{\theta}_j}{2} \partial^R_{z_i} \partial^L_{\bar{\theta}_j} f \Big) \\
&\quad + \frac{\pi}{z_{ij}} \Big(\Delta_i \sum_{l,l\neq j} \frac{\bar{\Delta}_{lj}}{\bar{z}_{lj}} f + \delta_{i3} \frac{\theta_i}{2} \partial^R_{\theta_i} f \sum_{l,l\neq j} \frac{\bar{\Delta}_{lj}}{\bar{z}_{lj}} + \Delta_i \partial^L_{\bar{z}_j} f - \delta_{i3} \frac{\theta_i}{2} \partial^R_{\theta_i} \partial^L_{\bar{z}_j} f \Big) \Big] \\
&\quad \times \prod_{i<j} z_{ij}^{-\Delta_{ij}} \bar{z}_{ij}^{-\bar{\Delta}_{ij}} \Big\} |_{\theta_1=\bar{\theta}_1=\theta_2=\bar{\theta}_2=\theta_4=\bar{\theta}_3=0},
\end{aligned} \tag{82}
$$

where in the integrand, we can replace $Z_{ij} \to z_{ij}, \bar{Z}_{ij} \to \bar{z}_{ij}$. In the bosonic CFT, 4-pt correlators can be expressed as conformal blocks whose universal properties are known in some cases, thus the OTOC can be computed [74], while in eq.(82) the function f is unknown in general. Thus it is more difficult to compute OTOC here.

3 $\mathcal{N}$=(2,2) superconformal symmetry

For (2,2) superconformal symmetry, the coordinates on superspace is divided into holomorphic $Z = (z, \theta, \bar{\theta})$ and antiholomorphic part $\tilde{Z} = (\bar{z}, \tilde{\theta}, \bar{\tilde{\theta}})$ respectively. In parallel with the situation in (1,1) case, (2,2) superconformal group is a direct product of (2,0) and (0,2) superconformal group which acts on Z and $\tilde{Z}$ respectively. Thus we will only write out the holomorphic coordinates explicitly hereafter. For holomorphic part the covariant derivatives are [70, 71, 75–77]

$$
D = \partial_\theta + \bar{\theta}\partial_z, \quad \bar{D} = \partial_{\bar{\theta}} + \theta\partial_z, \tag{83}
$$

which satisfy $D^2 = \bar{D}^2 = 0, \{D, \bar{D}\} = 2\partial_z$. The energy momentum superfield is

$$
J(Z) = j(z) + i\theta\bar{G}(z) + i\bar{\theta}G(z) + 2\theta\bar{\theta}T(z), \tag{84}
$$

and similar for $\bar{J}(\bar{Z})$. Here $T(z)$ is stress tensor of the theory, and $G(z), \bar{G}(z)$ are two supersymmetric generators, $j(z)$ corresponds to the U(1) symmetry of rotation of the two supersymmtries.

Super-analytic transformation can be defined via the transformation law of covariant derivatives as

$$D = (D\theta')D', \quad \bar{D} = (\bar{D}\bar{\theta}')\bar{D}'. \tag{85}$$

Superconformal primary fields are defined such that under super-analytic transformation they transform as

$$\Phi(Z) = (D\theta')^{\Delta+Q/2}(\bar{D}\bar{\theta}')^{\Delta-Q/2}\Phi'(Z') \tag{86}$$

where Δ, J are the dimension and charge of Φ respectively. The OPE between energy momentum superfield $J(Z)$ and primary superfield have been considered in [75, 77]

$$J(Z_1)\Phi(Z_2) = 2\Delta\frac{\theta_{12}\bar{\theta}_{12}}{Z_{12}^2}\Phi(Z_2)+2\frac{\theta_{12}\bar{\theta}_{12}}{Z_{12}}\partial_{z_2}\Phi(Z_2)+\frac{\theta_{12}}{Z_{12}}D\Phi(Z_2)-\frac{\bar{\theta}_{12}}{Z_{12}}\bar{D}\Phi(Z_2)+Q\frac{\Phi(Z_2)}{Z_{12}}, \tag{87}$$

where $Z_{12} = z_{12} - \theta_1\bar{\theta}_2 - \bar{\theta}_1\theta_2$ (also $\tilde{Z}_{12} = \bar{z}_{12} - \tilde{\theta}_1\bar{\tilde{\theta}}_2 - \bar{\tilde{\theta}}_1\tilde{\theta}_2$). In analogy with (1,1) case in the previous section, from this OPE, we can get the Ward identity as [8]

$$\begin{aligned}&\langle J(Z_0)\Phi_1(Z_1, \tilde{Z}_1)...\Phi_n(Z_n, \tilde{Z}_n)\rangle \\ &= \sum_{i=1}^{n}\Big(2\Delta_i\frac{\theta_{0i}\bar{\theta}_{0i}}{Z_{0i}^2} + 2\frac{\theta_{0i}\bar{\theta}_{0i}}{Z_{0i}}\partial_{z_i} + \frac{\theta_{0i}}{Z_{0i}}D_i - \frac{\bar{\theta}_{0i}}{Z_{0i}}\bar{D}_i + \frac{Q_i}{Z_{0i}}\Big)\langle\Phi_1(Z_1, \tilde{Z}_1)...\Phi_n(Z_n, \tilde{Z}_n)\rangle.\end{aligned} \tag{88}$$

In NS sector the n-pt correlators on the right hand side of eq.(88) are constrained by Ward identity corresponding to global superconformal Osp(2|2) transformation [77]. When $n = 2$, the correlator is fixed as

$$\langle\Phi(Z_1, \tilde{Z}_1)\Phi_n(Z_n, \tilde{Z}_n)\rangle = \frac{1}{Z_{12}^{2\Delta}\tilde{Z}_{12}^{2\bar{\Delta}}}e^{Q_2\frac{\theta_{12}\bar{\theta}_{12}}{Z_{12}}}e^{\bar{Q}_2\frac{\tilde{\theta}_{12}\bar{\tilde{\theta}}_{12}}{\tilde{Z}_{12}}}, \tag{89}$$

where $\Delta_1 = \Delta_2, Q_1 + Q_2 = 0$ and similar for $\bar{\Delta}, \bar{Q}$. Note here we have written the antiholomorphic part explicitly.

[8] For the $N = 2$ Super-Cauchy theorem see [75]

For $n=3$ the correlators take the form

$$\begin{aligned}\langle\Phi_1(Z_1,\tilde{Z}_1)\Phi_2(Z_2,\tilde{Z}_2\Phi_3(Z_3,\tilde{Z}_3\rangle =& \Big(\prod_{i<j}^{3} Z_{ij}^{-\Delta_{ij}}\Big)\exp\Big(\sum_{i<j} A_{ij}\frac{\theta_{ij}\bar{\theta}_{ij}}{Z_{ij}}\Big)\delta_{Q_1+Q_2+Q_3,0} \\ &\times\Big(\prod_{i<j}^{3} \tilde{Z}_{ij}^{-\bar{\Delta}_{ij}}\Big)\exp\Big(\sum_{i<j} \bar{A}_{ij}\frac{\tilde{\theta}_{ij}\bar{\tilde{\theta}}_{ij}}{\tilde{Z}_{ij}}\Big)\delta_{\bar{Q}_1+\bar{Q}_2+\bar{Q}_3,0}\end{aligned} \tag{90}$$

with $A_{ij}=-A_{ji}, \sum_{j=1,j\neq i}^{3} A_{ij}=-Q_i$, and similar for the $\bar{A}_{ij}, \bar{Q}_i$. Note that not all A_{ij} are fixed, this is because for 3-pt case there are nine coordinates $(z_i,\theta_i,\bar{\theta}_i), i=1,2,3,$ and eight generators for osp(2|2), thus there remains one degree of freedom which corresponds to the invariant quantity

$$R_{123}=\frac{\theta_{12}\bar{\theta}_{12}}{Z_{12}}+\frac{\theta_{31}\bar{\theta}_{31}}{Z_{31}}+\frac{\theta_{23}\bar{\theta}_{23}}{Z_{23}} \tag{91}$$

with $R_{123}^2=0$.

The n-pt correlators can be fixed by Ward identity up to an undetermined function

$$\begin{aligned}&\langle\Phi_1(Z_1,\tilde{Z}_1)...\Phi_2(Z_n,\tilde{Z}_n)\rangle \\ =&\Big(\prod_{i<j}^{n}\frac{1}{Z_{ij}^{\Delta_{ij}}}\frac{1}{\tilde{Z}_{ij}^{\bar{\Delta}_{ij}}}\Big)\exp\Big(\sum_{i<j}^{n}\bar{A}_{ij}\frac{\tilde{\theta}_{ij}\bar{\tilde{\theta}}_{ij}}{\tilde{Z}_{ij}}\Big)\exp\Big(\sum_{i<j}^{n}A_{ij}\frac{\theta_{ij}\bar{\theta}_{ij}}{Z_{ij}}\Big) \\ &\times f(x_1,x_2,...,x_{3n-8},\bar{x}_1,\bar{x}_2,...,\bar{x}_{3n-8})\delta_{\sum_i Q_i,0}\delta_{\sum_i \bar{Q}_i,0}, \\ &A_{ij}=-A_{ji},\ \Delta_{ij}=\Delta_{ji},\ \sum_{j,j\neq i}A_{ij}=-Q_i,\ \sum_{j,j\neq i}\Delta_{ij}=2\Delta_i,\end{aligned} \tag{92}$$

where x_i is Osp(2|2) invariant variables which may be either R_{ijk} or Z_{ijkl}

$$R_{ijk}=\frac{\theta_{ij}\bar{\theta}_{ij}}{Z_{ij}}+\frac{\theta_{jk}\bar{\theta}_{jk}}{Z_{jk}}+\frac{\theta_{ki}\bar{\theta}_{ki}}{Z_{ki}},\quad Z_{ijkl}=\frac{Z_{ij}Z_{kl}}{Z_{li}Z_{jk}}. \tag{93}$$

It should be point out that only $3n-8$ variables R_{ijk}, Z_{ijkl} are independent.

In parallel with (1,1), we can now define $T\bar{T}$ deformed correlators for (2,2) case. The variation of action under $T\bar{T}$ deformation can be constructed as

$$\delta S=\lambda\int d^2zT\bar{T}(z)=\lambda\int d^2z\int d\theta d\bar{\theta}d\tilde{\theta}\bar{\tilde{\theta}}J(Z)\bar{J}(\bar{Z}), \tag{94}$$

Also to first order the n-pt correlators is

$$-\lambda\int d^2z\int d\theta d\bar{\theta}d\tilde{\theta}d\bar{\tilde{\theta}}\langle J(Z)\bar{J}(\bar{Z})\Phi(Z_1,\bar{Z}_1)...\Phi(Z_n,\bar{Z}_n)\rangle. \tag{95}$$

In the following section we will consider eq.(95)) with $n=2,3$ and $n\geq 4$.

3.1 2-pt correlators

Up to a constant prefactor, the 2-pt correlators take the form as

$$\langle\Phi_1(Z_1,\tilde{Z}_1)\Phi_2(Z_2,\tilde{Z}_2)\rangle = \frac{1}{Z_{12}^{2\Delta}\tilde{Z}_{12}^{2\bar{\Delta}}}e^{Q_2\frac{\theta_{12}\bar{\theta}_{12}}{Z_{12}}}e^{\bar{Q}_2\frac{\tilde{\theta}_{12}\bar{\tilde{\theta}}_{12}}{\tilde{Z}_{12}}}. \tag{96}$$

To obtain $T\bar{T}$ deformed correlators, first consider only the correlators with holomorphic component of stress tensor inserted, from eq.(88), this is

$$\langle J\Phi_1\Phi_2\rangle \equiv (F+G)\langle\Phi_1\Phi_2\rangle, \tag{97}$$

where for later convenience we introduced G, F such that G contains derivatives and F does not

$$G = \sum_{i=1}^{n}\Big(2\frac{\theta_{0i}\bar{\theta}_{0i}}{Z_{0i}}\partial_{z_i} + \frac{\theta_{0i}}{Z_{0i}}D_i - \frac{\bar{\theta}_{0i}}{Z_{0i}}\bar{D}_i\Big), \quad F = \sum_{i=1}^{n}\Big(2\Delta_i\frac{\theta_{0i}\bar{\theta}_{0i}}{Z_{0i}^2} + \frac{Q_i}{Z_{0i}}\Big). \tag{98}$$

To evaluate eq.(97), firstly, let us consider the crossing terms (holomorphic derivatives ∂_z acting on antiholomorphic coordinates or vice versa) in eq.(97). In analogy with the $(1,1)$ case, it can be shown that this kind of terms vanish when integrating over $\theta, \bar{\theta}$, thus it will not contribute to the final results eq.(95). Explicitly, consider the crossing term $G\frac{1}{\tilde{Z}_{12}^{2\bar{\Delta}}}$ [9]

$$\begin{aligned} G\frac{1}{\tilde{Z}_{12}^{2\bar{\Delta}}} &= \int d\theta d\bar{\theta}\sum_i\Big(2\frac{\theta_{0i}\bar{\theta}_{0i}}{Z_{0i}}\partial_{z_i} + \frac{\theta_{0i}}{Z_{0i}}D_i - \frac{\bar{\theta}_{0i}}{Z_{0i}}\bar{D}_i\Big)\frac{1}{\tilde{Z}_{12}^{2\bar{\Delta}}} \\ &= \int d\theta d\bar{\theta}\Big(\frac{\theta_{01}}{z_{01}}\bar{\theta}_1 - \frac{\bar{\theta}_{01}}{z_{01}}\theta_1 - \frac{\theta_{02}}{z_{02}}\bar{\theta}_2 + \frac{\bar{\theta}_{02}}{z_{02}}\theta_2 + 2\Big(\frac{\theta_{01}\bar{\theta}_{01}}{z_{01}} - \frac{\theta_{02}\bar{\theta}_{02}}{z_{02}}\Big)\Big)\frac{2\bar{\Delta}}{\tilde{Z}_{12}^{2\bar{\Delta}-1}}\partial_{z_1}\frac{1}{\tilde{Z}_{12}} = 0, \end{aligned} \tag{102}$$

where in the last step we have used

$$\partial_{z_1}\frac{1}{\tilde{Z}_{12}} = -\partial_{z_2}\frac{1}{\tilde{Z}_{12}} = \tilde{\delta}(z_{12})\Big(1 + 2\frac{\tilde{\theta}_1\bar{\tilde{\theta}}_2 + \bar{\tilde{\theta}}_1\tilde{\theta}_2}{\bar{z}_{12}} + 6\frac{\tilde{\theta}_1\bar{\tilde{\theta}}_2\bar{\tilde{\theta}}_1\tilde{\theta}_2}{\bar{z}_{12}^2}\Big). \tag{103}$$

[9]Some useful expressions

$$\frac{1}{Z_{0i}} = \frac{1}{z_{0i}} + \frac{\theta_0\bar{\theta}_i + \bar{\theta}_0\theta_i}{z_{0i}^2} + 2\frac{\theta_0\bar{\theta}_i\bar{\theta}_0\theta_i}{z_{0i}^3}, \tag{99}$$

$$\frac{\theta_{0i}}{Z_{0i}} = \frac{\theta_{0i}}{z_{0i}} - \frac{\theta_0\theta_i\bar{\theta}_{0i}}{z_{0i}^2}, \quad \frac{\bar{\theta}_{0i}}{Z_{0i}} = \frac{\bar{\theta}_{0i}}{z_{0i}} - \frac{\bar{\theta}_0\bar{\theta}_i\theta_{0i}}{z_{0i}^2}, \quad \frac{\theta_{0i}\bar{\theta}_{0i}}{Z_{0i}} = \frac{\theta_{0i}\bar{\theta}_{0i}}{z_{0i}}, \tag{100}$$

$$\int d\theta d\bar{\theta}\bar{\theta}\theta = 1. \tag{101}$$

In the same manner one has $\int d\theta d\bar{\theta} G e^{\bar{Q}_2 \frac{\tilde{\theta}_{12}\bar{\tilde{\theta}}_{12}}{\tilde{Z}_{12}}} = 0$. Therefore we can derive eq.(97) in the following without considering crossing terms, which is

$$\langle J\Phi_1\Phi_2\rangle = (F+G)\langle\Phi_1\Phi_2\rangle \equiv (F+P)\langle\Phi_1\Phi_2\rangle \tag{104}$$

with

$$F = 2\Delta\Big(\frac{\theta_{01}\bar{\theta}_{01}}{z_{01}^2} + \frac{\theta_{02}\bar{\theta}_{02}}{z_{02}^2}\Big) - Q_2\Big(\frac{1}{Z_{01}} - \frac{1}{Z_{02}}\Big) \tag{105}$$

and P is defined as (similar for $\tilde{P}, \tilde{F}$)

$$\begin{aligned}
&P \equiv G\langle\Phi_1\Phi_2\rangle/\langle\Phi_1\Phi_2\rangle \\
=& -4\Delta\Big(\frac{\theta_{01}\bar{\theta}_{01}}{z_{01}} - \frac{\theta_{02}\bar{\theta}_{02}}{z_{02}}\Big)\frac{1}{Z_{12}} + \Big(\frac{\theta_{01}\bar{\theta}_{21}}{Z_{01}} + \frac{\theta_{02}\bar{\theta}_{21}}{Z_{02}}\Big)\frac{2\Delta}{Z_{12}} \\
&- \Big(\frac{\bar{\theta}_{01}\theta_{21}}{Z_{01}} + \frac{\bar{\theta}_{02}\theta_{21}}{Z_{02}}\Big)\frac{2\Delta}{Z_{12}} - 2Q_2\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2}\Big(\frac{\theta_{01}\bar{\theta}_{01}}{z_{01}} - \frac{\theta_{02}\bar{\theta}_{02}}{z_{02}}\Big) \\
&+ Q_2\Big(\frac{\theta_{01}}{Z_{01}} - \frac{\theta_{02}}{Z_{02}}\Big)\frac{\bar{\theta}_{12}}{Z_{12}} + Q_2\Big(\frac{\bar{\theta}_{01}}{Z_{01}} - \frac{\bar{\theta}_{02}}{Z_{02}}\Big)\frac{\theta_{12}}{Z_{12}}.
\end{aligned} \tag{106}$$

Having obtained eq.(97), next we can investigate $\langle J\bar{J}\Phi_1\Phi_2\rangle$

$$\begin{aligned}
&(F+G)(\tilde{F}+\tilde{G})\langle\Phi_1\Phi_2\rangle = (F+G)(\tilde{F}+\tilde{P})\langle\Phi_1\Phi_2\rangle \\
=&(F+P)(\tilde{F}+\tilde{P})O_2\tilde{O}_2 + [G(\tilde{F}+\tilde{P})]\langle\Phi_1\Phi_2\rangle.
\end{aligned} \tag{107}$$

Note that the last term is also a crossing term which can be dropped by the same reason as discussed around eq.(58). Finally, we get the first order $T\bar{T}$ deformation of 2-pt correlators

$$\begin{aligned}
&\frac{1}{\langle\Phi_1\Phi_2\rangle}\int d^2z d\theta d\bar{\theta} d\tilde{\theta} d\bar{\tilde{\theta}} \langle J\bar{J}\Phi_1\Phi_2\rangle \\
=&\int d^2z d\theta d\bar{\theta} d\tilde{\theta} d\bar{\tilde{\theta}} (F+P)(\tilde{F}+\tilde{P}) \\
=&\int d^2z\Big[-2\Delta\Big(\frac{1}{z_{01}^2} + \frac{1}{z_{02}^2}\Big) - 2Q_2\Big(\frac{\bar{\theta}_1\theta_1}{z_{01}^3} - \frac{\bar{\theta}_2\theta_2}{z_{02}^3}\Big) + 4\Delta\Big(\frac{1}{z_{01}} - \frac{1}{z_{02}}\Big)\frac{1}{Z_{12}} - \Big(\frac{\theta_1}{z_{01}^2} + \frac{\theta_2}{z_{02}^2}\Big)\frac{2\Delta\bar{\theta}_{21}}{Z_{12}} \\
&- \Big(\frac{\bar{\theta}_1}{z_{01}^2} + \frac{\bar{\theta}_2}{z_{02}^2}\Big)\frac{\theta_{21}2\Delta}{Z_{12}} + 2Q_2\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2}\Big(\frac{1}{z_{01}} - \frac{1}{z_{02}}\Big) - Q_2\Big(\frac{\theta_1}{z_{01}^2} - \frac{\theta_2}{z_{02}^2}\Big)\frac{\bar{\theta}_{12}}{Z_{12}} + Q_2\Big(\frac{\bar{\theta}_1}{z_{01}^2} - \frac{\bar{\theta}_2}{z_{02}^2}\Big)\frac{\theta_{12}}{Z_{12}}\Big] \\
&\times\Big[-2\bar{\Delta}\Big(\frac{1}{\bar{z}_{01}^2} + \frac{1}{\bar{z}_{02}^2}\Big) - 2\bar{Q}_2\Big(\frac{\bar{\tilde{\theta}}_1\tilde{\theta}_1}{\bar{z}_{01}^3} - \frac{\bar{\tilde{\theta}}_2\tilde{\theta}_2}{\bar{z}_{02}^3}\Big) + 4\bar{\Delta}\Big(\frac{1}{\bar{z}_{01}} - \frac{1}{\bar{z}_{02}}\Big)\frac{1}{\tilde{Z}_{12}} - \Big(\frac{\tilde{\theta}_1}{\bar{z}_{01}^2} + \frac{\tilde{\theta}_2}{\bar{z}_{02}^2}\Big)\frac{2\bar{\Delta}\bar{\tilde{\theta}}_{21}}{\tilde{Z}_{12}} \\
&- \Big(\frac{\bar{\tilde{\theta}}_1}{\bar{z}_{01}^2} + \frac{\bar{\tilde{\theta}}_2}{\bar{z}_{02}^2}\Big)\frac{\tilde{\theta}_{21}2\bar{\Delta}}{\tilde{Z}_{12}} + 2\bar{Q}_2\frac{\tilde{\theta}_{12}\bar{\tilde{\theta}}_{12}}{\bar{z}_{12}^2}\Big(\frac{1}{\bar{z}_{01}} - \frac{1}{\bar{z}_{02}}\Big) - \bar{Q}_2\Big(\frac{\tilde{\theta}_1}{\bar{z}_{01}^2} - \frac{\tilde{\theta}_2}{\bar{z}_{02}^2}\Big)\frac{\bar{\tilde{\theta}}_{12}}{\tilde{Z}_{12}} + \bar{Q}_2\Big(\frac{\bar{\tilde{\theta}}_1}{\bar{z}_{01}^2} - \frac{\bar{\tilde{\theta}}_2}{\bar{z}_{02}^2}\Big)\frac{\tilde{\theta}_{12}}{\tilde{Z}_{12}}\Big].
\end{aligned} \tag{108}$$

Further Performing the integral over z using dimensional regularization, yields

$$\begin{aligned}
&\frac{1}{\langle\Phi_1\Phi_2\rangle}\int d^2z d\theta d\bar{\theta} d\tilde{\theta} d\bar{\tilde{\theta}} \langle J\bar{J}\Phi_1\Phi_2\rangle \\
=&2\pi\Big(-\frac{2}{\epsilon}+\ln|z_{ij}|^2+\gamma+\ln\pi+\mathcal{O}(\epsilon)\Big)\Big(\frac{4\Delta}{Z_{12}}+2Q_2\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2}\Big)\Big(\frac{4\bar{\Delta}}{\tilde{Z}_{12}}+2\bar{Q}_2\frac{\tilde{\theta}_{12}\bar{\tilde{\theta}}_{12}}{\bar{z}_{12}^2}\Big) \\
&-\frac{\pi}{\bar{z}_{ij}}\Big(\frac{4\Delta}{Z_{12}}+2Q_2\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2}\Big)\Big(\frac{4\bar{\Delta}\bar{z}_{12}}{\tilde{Z}_{12}}+\bar{Q}_2\frac{(\tilde{\theta}_1+\tilde{\theta}_2)\bar{\tilde{\theta}}_{12}}{\bar{z}_{12}}-\bar{Q}_2\frac{(\bar{\tilde{\theta}}_1+\bar{\tilde{\theta}}_2)\tilde{\theta}_{12}}{\bar{z}_{12}}\Big) \\
&-\frac{\pi}{z_{ij}}\Big(\frac{4\Delta z_{12}}{Z_{12}}+Q_2\frac{\theta_{12}(\bar{\theta}_1+\bar{\theta}_2)}{z_{12}}-Q_2\frac{\bar{\theta}_{12}(\theta_1+\theta_2)}{z_{12}}\Big)\Big(\frac{4\bar{\Delta}}{\tilde{Z}_{12}}+2\bar{Q}_2\frac{\tilde{\theta}_{12}\bar{\tilde{\theta}}_{12}}{\bar{z}_{12}^2}\Big) \\
&+\frac{\pi}{(\bar{z}_{ij})^2}\Big(\frac{4\Delta}{Z_{12}}+2Q_2\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2}\Big)\Big(2\bar{Q}_2\bar{\tilde{\theta}}_2\tilde{\theta}_2+2\bar{Q}_2\bar{\tilde{\theta}}_1\tilde{\theta}_1\Big) \\
&+\frac{\pi}{(z_{ij})^2}\Big(2Q_2\bar{\theta}_1\theta_1+2Q_2\bar{\theta}_2\theta_2\Big)\Big(\frac{4\bar{\Delta}}{\tilde{Z}_{12}}+2\bar{Q}_2\frac{\tilde{\theta}_{12}\bar{\tilde{\theta}}_{12}}{\bar{z}_{12}^2}\Big).
\end{aligned} \tag{109}$$

3.2 3-pt correlators

Using Ward identity, the 3pt correlators take the general form as

$$\begin{aligned}
\langle\Phi_1(Z_1,\tilde{Z}_1)\Phi_2(Z_2,\tilde{Z}_2)\Phi_3(Z_3,\tilde{Z}_3)\rangle=&\Big(\prod_{i<j}^{3}Z_{ij}^{-\Delta_{ij}}\Big)\exp\Big(\sum_{i<j}A_{ij}\frac{\theta_{ij}\bar{\theta}_{ij}}{Z_{ij}}\Big)\delta_{Q_1+Q_2+Q_3,0} \\
&\times\Big(\prod_{i<j}^{3}\tilde{Z}_{ij}^{-\bar{\Delta}_{ij}}\Big)\exp\Big(\sum_{i<j}\bar{A}_{ij}\frac{\tilde{\theta}_{ij}\bar{\tilde{\theta}}_{ij}}{\tilde{Z}_{ij}}\Big)\delta_{\bar{Q}_1+\bar{Q}_2+\bar{Q}_3,0}.
\end{aligned} \tag{110}$$

Following the same line as 2-pt correlators, we first consider

$$\langle J\Phi_1\Phi_2\Phi_3\rangle=\langle G+F\rangle\langle\Phi_1\Phi_2\Phi_3\rangle. \tag{111}$$

It can be shown that the crossing terms do not contribute, i.e.

$$\int d\theta d\bar{\theta}G\Big(\prod_{i<j}^{3}\tilde{Z}_{ij}^{-\bar{\Delta}_{ij}}\Big)=0,\quad\int d\theta d\bar{\theta}G\exp\Big(\sum_{i<j}\bar{A}_{ij}\frac{\tilde{\theta}_{ij}\bar{\tilde{\theta}}_{ij}}{\tilde{Z}_{ij}}\Big)=0. \tag{112}$$

Therefore we only need to consider

$$\begin{aligned}
G\Big(\prod_{i<j}^{3}Z_{ij}^{-\Delta_{ij}}\Big)=&\sum_{i,k,i\neq k}\Big(2\frac{\theta_{0k}\bar{\theta}_{0k}}{z_{0k}}\frac{\Delta_{ik}}{Z_{ik}}+\frac{\theta_{0k}}{Z_{0k}}\frac{\bar{\theta}_{ki}\Delta_{ik}}{Z_{ik}}-\frac{\bar{\theta}_{0k}}{Z_{0k}}\frac{\theta_{ki}\Delta_{ik}}{Z_{ik}}\Big)\Big(\prod_{i<j}^{3}Z_{ij}^{-\Delta_{ij}}\Big) \\
\equiv&P_1\Big(\prod_{i<j}^{3}Z_{ij}^{-\Delta_{ij}}\Big)
\end{aligned} \tag{113}$$

and

$$
\begin{aligned}
&G\exp\Big(\sum_{i<j}A_{ij}\frac{\theta_{ij}\bar\theta_{ij}}{z_{ij}}\Big)\\
=&\sum_{j,k,j\neq k}\Big(2\frac{\theta_{0k}\bar\theta_{0k}}{z_{0k}}A_{jk}\frac{\theta_{jk}\bar\theta_{jk}}{z_{jk}^2}+\frac{\theta_{0k}}{Z_{0k}}A_{kj}\frac{\bar\theta_{kj}}{Z_{kj}}-\frac{\bar\theta_{0k}}{Z_{0k}}A_{jk}\frac{\theta_{jk}}{Z_{jk}}\Big)\exp\Big(\sum_{i<j}A_{ij}\frac{\theta_{ij}\bar\theta_{ij}}{z_{ij}}\Big) \qquad (114)\\
\equiv&P_2\exp\Big(\sum_{i<j}A_{ij}\frac{\theta_{ij}\bar\theta_{ij}}{z_{ij}}\Big).
\end{aligned}
$$

Thus we obtain $\langle J\Phi_1\Phi_2\Phi_3\rangle=(F+P_1+P_2)\langle\Phi_1\Phi_2\Phi_3\rangle$, and it follows that

$$
\begin{aligned}
\langle J\bar J\Phi_1\Phi_2\Phi_3\rangle=&(G+F)(\tilde G+\tilde F)\langle\Phi_1\Phi_2\Phi_3\rangle\\
=&(P_1+P_2+F)(\tilde P_1+\tilde P_2+\tilde F)\langle\Phi_1\Phi_2\Phi_3\rangle+[G(\tilde P_1+\tilde P_2+\tilde F)]\langle\Phi_1\Phi_2\Phi_3\rangle,
\end{aligned}
\qquad (115)
$$

where the last term should be dropped as discussed in previous sections. Substituting the expression of P_1,P_2,F into eq.(115), we have

$$
\begin{aligned}
&\frac{1}{\langle\Phi_1\Phi_2\Phi_3\rangle}\int d^2z d\theta d\bar\theta d\tilde\theta d\bar{\tilde\theta}\langle J\bar J\Phi_1\Phi_2\Phi_3\rangle\\
=&\int d^2z\Big[\sum_{i,k,i\neq k}\Big(-\frac{2}{z_{0k}}\frac{\Delta_{ik}}{Z_{ik}}+\frac{1}{z_{0k}^2}\frac{(\theta_k\bar\theta_i+\bar\theta_k\theta_i)\Delta_{ik}}{Z_{ik}}-2\frac{1}{z_{0k}}A_{ik}\frac{\theta_{ik}\bar\theta_{ik}}{z_{ik}^2}-\frac{\theta_k}{z_{0k}^2}A_{ki}\frac{\bar\theta_{ki}}{Z_{ki}}\\
&-\frac{\bar\theta_k}{z_{0k}^2}A_{ik}\frac{\theta_{ik}}{Z_{ik}}\Big)+\sum_i\Big(-2\Delta_i\frac{1}{z_{0i}^2}+2\frac{Q_i\bar\theta_i\theta_i}{z_{0i}^3}\Big)\Big]\\
&\times\Big[\sum_{j,k,j\neq k}\Big(-\frac{2}{\bar z_{0k}}\frac{\bar\Delta_{jk}}{\tilde Z_{jk}}+\frac{1}{\bar z_{0k}^2}\frac{(\tilde\theta_k\bar{\tilde\theta}_j+\bar{\tilde\theta}_k\tilde\theta_j)\bar\Delta_{jk}}{\tilde Z_{jk}}-2\frac{1}{\bar z_{0k}}\bar A_{jk}\frac{\tilde\theta_{jk}\bar{\tilde\theta}_{jk}}{\bar z_{jk}^2}-\frac{\tilde\theta_k}{\bar z_{0k}^2}\bar A_{kj}\frac{\bar{\tilde\theta}_{kj}}{\bar Z_{kj}}\\
&-\frac{\bar{\tilde\theta}_k}{\bar z_{0k}^2}\bar A_{jk}\frac{\tilde\theta_{jk}}{\bar Z_{jk}}\Big)+\sum_j\Big(-2\bar\Delta_j\frac{1}{\bar z_{0j}^2}+2\frac{\bar Q_j\bar{\tilde\theta}_j\tilde\theta_j}{\bar z_{0j}^3}\Big)\Big].
\end{aligned}
\qquad (116)
$$

Using $\sum_{i,i\neq k}\Delta_{ik}=2\Delta_k$, the first and second line of the integrand can be expressed as

$$
\begin{aligned}
&\Big[\sum_{i,k,i\neq k}\Big(-\frac{2}{z_{0i}}\frac{\Delta_{ki}}{Z_{ki}}+\frac{1}{z_{0i}^2}\frac{(\theta_i\bar\theta_k+\bar\theta_i\theta_k)\Delta_{ki}}{Z_{ki}}-2\frac{1}{z_{0i}}A_{ki}\frac{\theta_{ki}\bar\theta_{ki}}{z_{ki}^2}-\frac{\theta_i}{z_{0i}^2}A_{ik}\Big(\frac{\bar\theta_{ik}}{z_{ik}}-\frac{\theta_{ik}\bar\theta_i\bar\theta_k}{z_{ik}^2}\Big)\\
&-\frac{\bar\theta_i}{z_{0i}^2}A_{ki}\Big(\frac{\theta_{ki}}{z_{ki}}-\frac{\theta_k\theta_i\bar\theta_{ki}}{z_{ik}^2}\Big)\Big)+\sum_i\Big(-2\Delta_i\frac{1}{z_{0i}^2}+2\frac{Q_i\bar\theta_i\theta_i}{z_{0i}^3}\Big)\Big]\\
=&\sum_{i,k,i\neq k}\Big(-\frac{1}{z_{0i}}\Big(\frac{2\Delta_{ki}}{Z_{ki}}+\frac{2\theta_{ki}\bar\theta_{ki}A_{ki}}{z_{ki}^2}\Big)-\frac{1}{z_{0i}^2}\Big(\frac{z_{ki}\Delta_{ki}}{Z_{ki}}+A_{ik}\frac{\theta_i\bar\theta_{ik}-\bar\theta_i\theta_{ik}}{z_{ik}}\Big)\Big)+\sum_i\frac{2Q_i\bar\theta_i\theta_i}{z_{0i}^3}.
\end{aligned}
\qquad (117)
$$

Consequently, the final result for 3-pt correlators are

$$\frac{1}{\langle\Phi_1\Phi_2\Phi_3\rangle}\int d^2z d\theta d\bar{\theta} d\tilde{\theta} d\bar{\tilde{\theta}} \langle J\bar{J}\Phi_1\Phi_2\Phi_3\rangle$$

$$\begin{aligned}
=&\sum_{ij}\Big[\Big(-\pi(-\frac{2}{\epsilon}+\ln|z_{ij}|^2+\gamma+\ln\pi+\mathcal{O}(\epsilon))\Big)\sum_{k,i\neq k}\Big(\frac{2\Delta_{ki}}{Z_{ki}}+\frac{2\theta_{ki}\bar{\theta}_{ki}A_{ki}}{z_{ki}^2}\Big)\sum_{l,j\neq l}\Big(\frac{2\bar{\Delta}_{lj}}{\tilde{Z}_{lj}}+\frac{2\tilde{\theta}_{lj}\bar{\tilde{\theta}}_{lj}\bar{A}_{lj}}{\bar{z}_{lj}^2}\Big)\\
&+\frac{\pi}{\bar{z}_{ij}}\sum_{k,i\neq k}\Big(\frac{2\Delta_{ki}}{Z_{ki}}+\frac{2\theta_{ki}\bar{\theta}_{ki}A_{ki}}{z_{ki}^2}\Big)\sum_{l,j\neq l}\Big(\frac{\bar{z}_{lj}\bar{\Delta}_{lj}}{\tilde{Z}_{lj}}+\bar{A}_{jl}\frac{\tilde{\theta}_j\bar{\tilde{\theta}}_{jl}-\bar{\tilde{\theta}}_j\tilde{\theta}_{jl}}{\bar{z}_{jl}}\Big)\\
&-\frac{\pi}{z_{ij}}\sum_{k,i\neq k}\Big(\frac{z_{ki}\Delta_{ki}}{Z_{ki}}+A_{ik}\frac{\theta_i\bar{\theta}_{ik}-\bar{\theta}_i\theta_{ik}}{z_{ik}}\Big)\sum_{l,j\neq l}\Big(\frac{2\bar{\Delta}_{lj}}{\tilde{Z}_{lj}}+\frac{2\tilde{\theta}_{lj}\bar{\tilde{\theta}}_{lj}\bar{A}_{lj}}{\bar{z}_{lj}^2}\Big)\\
&-\frac{\pi}{\bar{z}_{ij}^2}\sum_{k,i\neq k}\Big(\frac{2\Delta_{ki}}{Z_{ki}}+\frac{2\theta_{ki}\bar{\theta}_{ki}A_{ki}}{z_{ki}^2}\Big)2\bar{Q}_j\bar{\tilde{\theta}}_j\theta_j-\frac{\pi}{z_{ij}^2}2Q_i\bar{\theta}_i\theta_i\sum_{l,j\neq l}\Big(\frac{2\bar{\Delta}_{lj}}{\tilde{Z}_{lj}}+\frac{2\tilde{\theta}_{lj}\bar{\tilde{\theta}}_{lj}\bar{A}_{lj}}{\bar{z}_{lj}^2}\Big)\Big].
\end{aligned} \tag{118}$$

3.3 n-pt correlators

The n-pt function can be fixed by Ward identity up to an undetermined function

$$\begin{aligned}
&\langle\Phi_1(Z_1,\tilde{Z}_1)...\Phi_n(Z_n,\tilde{Z}_n)\rangle\\
=&\Big(\prod_{i<j}^{n}\frac{1}{Z_{ij}^{\Delta_{ij}}}\frac{1}{\tilde{Z}_{ij}^{\bar{\Delta}_{ij}}}\Big)\exp\Big(\sum_{i<j}^{n}\bar{A}_{ij}\frac{\tilde{\theta}_{ij}\bar{\tilde{\theta}}_{ij}}{\tilde{Z}_{ij}}\Big)\exp\Big(\sum_{i<j}^{n}A_{ij}\frac{\theta_{ij}\bar{\theta}_{ij}}{Z_{ij}}\Big)\\
&\times f(x_1,x_2,...,x_{3n-8},\bar{x}_1,\bar{x}_2,...,\bar{x}_{3n-8})\delta_{\sum_i Q_i,0}\delta_{\sum_i \bar{Q}_i,0},
\end{aligned} \tag{119}$$

where x_i can be either of the following invariant variables

$$R_{ijk}=\frac{\theta_{ij}\bar{\theta}_{ij}}{Z_{ij}}+\frac{\theta_{jk}\bar{\theta}_{jk}}{Z_{jk}}+\frac{\theta_{ki}\bar{\theta}_{ki}}{Z_{ki}},\quad Z_{ijkl}=\frac{Z_{ij}Z_{kl}}{Z_{li}Z_{jk}}. \tag{120}$$

Note that only $3n-8$ of R_{ijk} and Z_{ijkl} are independent.

Let us first consider only holomorphic component $J(Z)$ inserted

$$\langle J\Phi_1...\Phi_n\rangle=(G+F)\langle\Phi_1...\Phi_n\rangle, \tag{121}$$

where we will encounter new crossing terms $G\tilde{R}_{ijk}$, $G\tilde{Z}_{ijkl}$ in addition to these appeared in eq.(112). By using eq.(103) it can be checked that they will vanish, i.e.

$$\int d\theta d\bar{\theta} G\tilde{R}_{ijk}=0,\quad \int d\theta d\bar{\theta} G\tilde{Z}_{ijkl}=0. \tag{122}$$

Thus we will not consider crossing terms in eq.(121), then

$$\begin{aligned}(G+F)\langle\Phi_1...\Phi_n\rangle &= (P_1+P_2+F)\langle\Phi_1...\Phi_n\rangle\\ =&(P_1+P_2+F)\langle\Phi_1...\Phi_n\rangle + Q\langle\Phi_1...\Phi_n\rangle\frac{1}{f},\end{aligned} \tag{123}$$

where P_1, P_2 be of the same form as defined in eq.(113) and eq.(114). Here Q equals Gf, which is

$$\begin{aligned} Q =&\Big(\sum_{R_{ijk}}(GR_{ijk})\frac{\partial f}{\partial R_{ijk}} + \sum_{Z_{ijkl}}(GZ_{ijkl})\frac{\partial f}{\partial Z_{ijkl}}\Big)\\ =&\sum_n\Big(\frac{2\theta_{0n}\bar{\theta}_{0n}}{z_{0n}}\Big((\partial_{z_n}R)\frac{\partial f}{\partial R} + (\partial_{z_n}Z)\frac{\partial f}{\partial Z}\Big) + \frac{\theta_{0n}}{Z_{0n}}\Big((\partial_{\theta_n}R)\frac{\partial f}{\partial R} + (\partial_{\theta_n}Z)\frac{\partial f}{\partial Z}\Big)\\ &-\frac{\bar{\theta}_{0n}}{Z_{0n}}\Big((\partial_{\bar{\theta}_n}R)\frac{\partial f}{\partial R} + (\partial_{\bar{\theta}_n}Z)\frac{\partial f}{\partial Z}\Big)\Big)\\ \equiv&\sum_n\Big(\frac{2\theta_{0n}\bar{\theta}_{0n}}{z_{0n}}\partial^R_{z_n}f + \frac{\theta_{0n}}{Z_{0n}}\partial^R_{\theta_n}f - \frac{\bar{\theta}_{0n}}{Z_{0n}}\partial^R_{\bar{\theta}_n}f\Big),\end{aligned} \tag{124}$$

where for simplicity we have abbreviated R_{ijk} as R, Z_{ijkl} as Z and suppressed the summation $\sum_{R_{ijk}}, \sum_{Z_{ijkl}}$. Note in the first step in eq.(124) we omit the terms vanishing after integration over $\theta, \bar{\theta}$. Following the same way we introduce $\tilde{Q}$ as

$$\tilde{Q} \equiv \sum_n\Big(\frac{2\tilde{\theta}_{0n}\bar{\tilde{\theta}}_{0n}}{\bar{z}_{0n}}\partial^L_{\bar{z}_n}f + \frac{\tilde{\theta}_{0n}}{\tilde{Z}_{0n}}\partial^L_{\tilde{\theta}_n}f - \frac{\bar{\tilde{\theta}}_{0n}}{\tilde{Z}_{0n}}\partial^L_{\bar{\tilde{\theta}}_n}f\Big). \tag{125}$$

Next consider $\langle J\bar{J}\Phi_1...\Phi_n\rangle$, which is

$$\begin{aligned} &(G+F)(\tilde{G}+\tilde{F})\langle\Phi_1...\Phi_n\rangle\\ =&(F+P)(\tilde{F}+\tilde{F})\langle\Phi_1...\Phi_n\rangle + Q(\tilde{F}+\tilde{P})\langle\Phi_1...\Phi_n\rangle/f + (F+P)\tilde{Q}\langle\Phi_1...\Phi_n\rangle/f\\ &+(G\tilde{Q})\langle\Phi_1...\Phi_n\rangle/f + [G(\tilde{P}+\tilde{F})]\langle\Phi_1...\Phi_n\rangle,\end{aligned} \tag{126}$$

where the last term should be dropped as discussed in previous sections. And the term $(G\tilde{Q})\langle\Phi_1...\Phi_n\rangle/f$ is very similar to the (1,1) case as discussed in eq.(75), which is not a crossing term. Actually,

$$G\tilde{Q} = \sum_{i,n}\Big(\frac{2\theta_{0i}\bar{\theta}_{0i}}{z_{0i}}\partial^R_{z_i} + \frac{\theta_{0i}}{Z_{0i}}\partial^R_{\theta_i} - \frac{\bar{\theta}_{0i}}{Z_{0i}}\partial^R_{\bar{\theta}_i}\Big)\Big(\frac{2\tilde{\theta}_{0n}\bar{\tilde{\theta}}_{0n}}{\bar{z}_{0n}}\partial^L_{\bar{z}_n} + \frac{\tilde{\theta}_{0n}}{\tilde{Z}_{0n}}\partial^L_{\tilde{\theta}_n} - \frac{\bar{\tilde{\theta}}_{0n}}{\tilde{Z}_{0n}}\partial^L_{\bar{\tilde{\theta}}_n}\Big)f, \tag{127}$$

thus

$$\int d\theta d\bar{\theta}d\tilde{\theta}d\bar{\tilde{\theta}}G\tilde{Q} = \sum_{i,n}\Big(\frac{2}{z_{0i}}\partial^R_{z_i} + \frac{\theta_i}{z^2_{0i}}\partial^R_{\theta_i} + \frac{\bar{\theta}_i}{z^2_{0i}}\partial^R_{\bar{\theta}_i}\Big)\Big(\frac{2}{\bar{z}_{0n}}\partial^L_{\bar{z}_n} + \frac{\tilde{\theta}_n}{\bar{z}^2_{0n}}\partial^L_{\tilde{\theta}_n} + \frac{\bar{\tilde{\theta}}_n}{\bar{z}^2_{0n}}\partial^L_{\bar{\tilde{\theta}}_n}\Big)f. \tag{128}$$

Gathering all the results together, we then have

$$
\begin{aligned}
&\frac{1}{\langle\Phi_1...\Phi_n\rangle}\int d^2z d\theta d\bar{\theta} d\tilde{\theta} d\bar{\tilde{\theta}}\langle J\bar{J}\Phi_1...\Phi_n\rangle \\
=&\int d^2z\Big[\sum_{i,k,i\neq k}\Big(-\frac{2}{z_{0k}}\frac{\Delta_{ik}}{Z_{ik}}+\frac{1}{z_{0k}^2}\frac{(\theta_k\bar{\theta}_i+\bar{\theta}_k\theta_i)\Delta_{ik}}{Z_{ik}}-2\frac{1}{z_{0k}}A_{jk}\frac{\theta_{jk}\bar{\theta}_{jk}}{z_{jk}^2}-\frac{\theta_k}{z_{0k}^2}A_{kj}\frac{\bar{\theta}_{kj}}{Z_{kj}} \\
&-\frac{\bar{\theta}_k}{z_{0k}^2}A_{jk}\frac{\theta_{jk}}{Z_{jk}}\Big)+\sum_i\Big(-2\Delta_i\frac{1}{z_{0i}^2}+2\frac{Q_i\bar{\theta}_i\theta_i}{z_{0i}^3}\Big)\Big] \\
&\times\Big[\sum_{i,k,i\neq k}\Big(-\frac{2}{\bar{z}_{0k}}\frac{\bar{\Delta}_{ik}}{\tilde{Z}_{ik}}+\frac{1}{\bar{z}_{0k}^2}\frac{(\tilde{\theta}_k\bar{\tilde{\theta}}_i+\bar{\tilde{\theta}}_k\tilde{\theta}_i)\bar{\Delta}_{ik}}{\tilde{Z}_{ik}}-2\frac{1}{\bar{z}_{0k}}\bar{A}_{jk}\frac{\tilde{\theta}_{jk}\bar{\tilde{\theta}}_{jk}}{\bar{z}_{jk}^2}-\frac{\tilde{\theta}_k}{\bar{z}_{0k}^2}\bar{A}_{kj}\frac{\bar{\tilde{\theta}}_{kj}}{\bar{Z}_{kj}} \\
&-\frac{\bar{\tilde{\theta}}_k}{\bar{z}_{0k}^2}\bar{A}_{jk}\frac{\tilde{\theta}_{jk}}{\bar{Z}_{jk}}\Big)+\sum_i\Big(-2\bar{\Delta}_i\frac{1}{\bar{z}_{0i}^2}+2\frac{\bar{Q}_i\bar{\tilde{\theta}}_i\tilde{\theta}_i}{\bar{z}_{0i}^3}\Big)\Big]+\sum_n\Big(\frac{-2}{z_{0n}}\partial^R_{z_n}f-\frac{\theta_n}{z_{0n}^2}\partial^R_{\theta_n}f-\frac{\bar{\theta}_n}{z_{0n}^2}\partial^R_{\bar{\theta}_n}f\Big)\frac{1}{f} \\
&\times\Big[\sum_{i,k,i\neq k}\Big(-\frac{2}{\bar{z}_{0k}}\frac{\bar{\Delta}_{ik}}{\tilde{Z}_{ik}}+\frac{1}{\bar{z}_{0k}^2}\frac{(\tilde{\theta}_k\bar{\tilde{\theta}}_i+\bar{\tilde{\theta}}_k\tilde{\theta}_i)\bar{\Delta}_{ik}}{\tilde{Z}_{ik}}-2\frac{1}{\bar{z}_{0k}}\bar{A}_{jk}\frac{\tilde{\theta}_{jk}\bar{\tilde{\theta}}_{jk}}{\bar{z}_{jk}^2}-\frac{\tilde{\theta}_k}{\bar{z}_{0k}^2}\bar{A}_{kj}\frac{\bar{\tilde{\theta}}_{kj}}{\bar{Z}_{kj}} \\
&-\frac{\bar{\tilde{\theta}}_k}{\bar{z}_{0k}^2}\bar{A}_{jk}\frac{\tilde{\theta}_{jk}}{\bar{Z}_{jk}}\Big)+\sum_i\Big(-2\bar{\Delta}_i\frac{1}{\bar{z}_{0i}^2}+2\frac{\bar{Q}_i\bar{\tilde{\theta}}_i\tilde{\theta}_i}{\bar{z}_{0i}^3}\Big)\Big] \\
&+\Big[\sum_{i,k,i\neq k}\Big(-\frac{2}{z_{0k}}\frac{\Delta_{ik}}{Z_{ik}}+\frac{1}{z_{0k}^2}\frac{(\theta_k\bar{\theta}_i+\bar{\theta}_k\theta_i)\Delta_{ik}}{Z_{ik}}-2\frac{1}{z_{0k}}A_{jk}\frac{\theta_{jk}\bar{\theta}_{jk}}{z_{jk}^2}-\frac{\theta_k}{z_{0k}^2}A_{kj}\frac{\bar{\theta}_{kj}}{Z_{kj}} \\
&-\frac{\bar{\theta}_k}{z_{0k}^2}A_{jk}\frac{\theta_{jk}}{Z_{jk}}\Big)+\sum_i\Big(-2\Delta_i\frac{1}{z_{0i}^2}+2\frac{Q_i\bar{\theta}_i\theta_i}{z_{0i}^3}\Big)\Big]\times\sum_n\Big(\frac{-2}{\bar{z}_{0n}}\partial^L_{\bar{z}_n}f-\frac{\tilde{\theta}_n}{\bar{z}_{0n}^2}\partial^L_{\tilde{\theta}_n}f-\frac{\bar{\tilde{\theta}}_n}{\bar{z}_{0n}^2}\partial^L_{\bar{\tilde{\theta}}_n}f\Big)\frac{1}{f} \\
&+\int d^2z\Big(\sum_{i,n}\Big(\frac{2}{z_{0i}}\partial^R_{z_i}+\frac{\theta_i}{z_{0i}^2}\partial^R_{\theta_i}+\frac{\bar{\theta}_i}{z_{0i}^2}\partial^R_{\bar{\theta}_i}\Big)\Big(\frac{2}{\bar{z}_{0n}}\partial^L_{\bar{z}_n}+\frac{\tilde{\theta}_n}{\tilde{z}_{0n}^2}\partial^L_{\tilde{\theta}_n}+\frac{\bar{\tilde{\theta}}_n}{\tilde{z}_{0n}^2}\partial^L_{\bar{\tilde{\theta}}_n}\Big)f\Big)\frac{1}{f}
\end{aligned}
\tag{129}
$$

Note that the first term of the integrand has the same form as 3-pt correlators in eq.(116) except for the summation here runs from 1 to n instead of 3 in eq.(116). After integration the final result is

$$\frac{1}{\langle\Phi_1...\Phi_n\rangle}\int d^2zd\theta d\bar{\theta}d\tilde{\theta}d\bar{\tilde{\theta}}\langle J\bar{J}\Phi_1...\Phi_n\rangle$$
$$=\sum_{ij}\Big(-\pi(-\frac{2}{\epsilon}+\ln|z_{ij}|^2+\gamma+\ln\pi+\mathcal{O}(\epsilon))\Big)\Big(\sum_{k,i\neq k}\Big(\frac{2\Delta_{ki}}{Z_{ki}}+\frac{2\theta_{ki}\bar{\theta}_{ki}A_{ki}}{z_{ki}^2}\Big)\sum_{l,j\neq l}\Big(\frac{2\bar{\Delta}_{lj}}{\tilde{Z}_{lj}}+\frac{2\tilde{\theta}_{lj}\bar{\tilde{\theta}}_{lj}\bar{A}_{lj}}{\bar{z}_{lj}^2}\Big)$$
$$+2\partial^R_{z_i}f\frac{1}{f}\sum_{k,k\neq j}\Big(\frac{2\bar{\Delta}_{kj}}{\tilde{Z}_{kj}}+\frac{2\tilde{\theta}_{kj}\bar{\tilde{\theta}}_{kj}\bar{A}_{kj}}{\bar{z}_{kj}^2}\Big)+\sum_{k,k\neq j}\Big(\frac{2\Delta_{ki}}{Z_{ki}}+\frac{2\theta_{ki}\bar{\theta}_{ki}A_{ki}}{z_{ki}^2}\Big)2\partial^L_{\bar{z}_j}f\frac{1}{f}+4\partial^L_{\bar{z}_j}\partial^R_{z_i}f\frac{1}{f}\Big)$$
$$+\sum_{ij}\frac{\pi}{\bar{z}_{ij}}\Big(\sum_{k,i\neq k}\Big(\frac{2\Delta_{ki}}{Z_{ki}}+\frac{2\theta_{ki}\bar{\theta}_{ki}A_{ki}}{z_{ki}^2}\Big)\sum_{l,j\neq l}\Big(\frac{\bar{z}_{lj}\bar{\Delta}_{lj}}{\tilde{Z}_{lj}}+\bar{A}_{jl}\frac{\tilde{\theta}_j\bar{\tilde{\theta}}_{jl}-\bar{\tilde{\theta}}_j\tilde{\theta}_{jl}}{\bar{z}_{jl}}\Big)$$
$$+2\partial^R_{z_i}f\sum_{k,k\neq j}\Big(\frac{\bar{z}_{kj}\bar{\Delta}_{kj}}{\tilde{Z}_{kj}}+\bar{A}_{jk}\frac{\tilde{\theta}_j\bar{\tilde{\theta}}_{jk}-\bar{\tilde{\theta}}_j\tilde{\theta}_{jk}}{\bar{z}_{jk}}\Big)\frac{1}{f}+\sum_{k,k\neq i}\Big(\frac{2\Delta_{ki}}{Z_{ki}}+\frac{2\theta_{ki}\bar{\theta}_{ki}A_{ki}}{z_{ki}^2}\Big)\Big(\tilde{\theta}_j\partial^L_{\tilde{\theta}_j}f+\bar{\tilde{\theta}}_j\partial^L_{\bar{\tilde{\theta}}_j}f\Big)\frac{1}{f}$$
$$+(2\bar{\tilde{\theta}}_j\partial^L_{\bar{\tilde{\theta}}_j}\partial^R_{z_i}f+2\tilde{\theta}_j\partial^L_{\tilde{\theta}_j}\partial^R_{z_i}f)\frac{1}{f}\Big)$$
$$+\sum_{ij}\frac{-\pi}{z_{ij}}\Big(\sum_{k,i\neq k}\Big(\frac{z_{ki}\Delta_{ki}}{Z_{ki}}+A_{ik}\frac{\theta_i\bar{\theta}_{ik}-\bar{\theta}_i\theta_{ik}}{z_{ik}}\Big)\sum_{l,j\neq l}\Big(\frac{2\bar{\Delta}_{lj}}{\tilde{Z}_{lj}}+\frac{2\tilde{\theta}_{lj}\bar{\tilde{\theta}}_{lj}\bar{A}_{lj}}{\bar{z}_{lj}^2}\Big)$$
$$+\Big(\theta_i\partial^R_{\theta_i}f+\bar{\theta}_i\partial^R_{\bar{\theta}_j}f\Big)\sum_{k,k\neq j}\Big(\frac{2\bar{\Delta}_{kj}}{\tilde{Z}_{kj}}+\frac{2\tilde{\theta}_{kj}\bar{\tilde{\theta}}_{kj}\bar{A}_{kj}}{\bar{z}_{kj}^2}\Big)\frac{1}{f}+\sum_{k,k\neq i}\Big(\frac{z_{ki}\Delta_{ki}}{Z_{ki}}+A_{ik}\frac{\theta_i\bar{\theta}_{ik}-\bar{\theta}_i\theta_{ik}}{z_{ik}}\Big)2\partial^L_{\bar{z}_j}f\frac{1}{f}$$
$$+(2\theta_i\partial^R_{\theta_i}\partial^L_{\bar{z}_j}f+2\bar{\theta}_i\partial^R_{\bar{\theta}_i}\partial^L_{\bar{z}_j}f)\frac{1}{f}\Big)$$
$$-\sum_{ij}\frac{\pi}{(\bar{z}_{ij})^2}\Big(4\partial^R_{z_i}f\frac{1}{f}\bar{Q}_j\bar{\tilde{\theta}}_j\tilde{\theta}_j+\sum_{k,i\neq k}\Big(\frac{2\Delta_{ki}}{Z_{ki}}+\frac{2\theta_{ki}\bar{\theta}_{ki}A_{ki}}{z_{ki}^2}\Big)2\bar{Q}_j\bar{\tilde{\theta}}_j\theta_j\Big)$$
$$-\sum_{ij}\frac{\pi}{(z_{ij})^2}\Big(4Q_i\bar{\theta}_i\theta_i\partial^L_{\bar{z}_j}f\frac{1}{f}+2Q_i\bar{\theta}_i\theta_i\sum_{l,j\neq l}\Big(\frac{2\bar{\Delta}_{lj}}{\tilde{Z}_{lj}}+\frac{2\tilde{\theta}_{lj}\bar{\tilde{\theta}}_{lj}\bar{A}_{lj}}{\bar{z}_{lj}^2}\Big)\Big)\tag{130}$$

As an application, we briefly discuss the 4-pt functions that might be useful in the study of the deformed OTOC. The superfield in (2,2) superspace takes the form

$$\Phi(Z,\tilde{Z})=\phi+\theta\psi_1+...,\tag{131}$$

where there are total 16 terms at the right hand side, and we only explicitly write out the first two components since we are only interested in correlators involving ϕ,ψ_1 as we did in (1,1) case. The conjugated superfield then is

$$\Phi(Z,\tilde{Z})^\dagger=\phi^\dagger-\bar{\theta}\psi_1^\dagger+...\tag{132}$$

Thus the following operator appeared in first order perturbation of OTOC

$$\begin{aligned}&\langle\phi(z_1,\bar z_1)\phi^\dagger(z_2,\bar z_2)\psi_1(z_3,\bar z_3)\psi_1^\dagger(z_4,\bar z_4)\rangle_\lambda\\ =&-\int d\theta_3 d\bar\theta_4\int d^2z d\theta d\bar\theta\langle J(Z)\bar J(\tilde Z)\Phi(Z_1,\tilde Z_1)\\ &\times\Phi^\dagger(Z_2,\tilde Z_2)\Phi(Z_3,\tilde Z_3)\Phi^\dagger(Z_4,\tilde Z_4)\rangle|_{\theta_1=\bar\theta_1=\theta_2=\bar\theta_2=\theta_4=\bar\theta_3=0,\tilde\theta_i=\bar{\tilde\theta}_i=0}\end{aligned}\tag{133}$$

can be computed by utilizing eq.(130).

4 Dimensional regularization

Using Feynman parametrization and dimensional regularization one can obtain the following basic integral [17] (Let $z_1\neq z_2$) [10]

$$\mathcal{I}_{11}(z_1,\bar z_2)=\int d^2z\frac{1}{z_{01}\bar z_{02}}=-\pi\Big(-\frac{2}{\epsilon}+\ln|z_{12}|^2+\gamma+\ln\pi\Big)+O(\epsilon)\tag{135}$$

with ϵ being a infinitesimal constant. Next consider $\mathcal{I}_{12}(z_1,\bar z_2)$ with $z_1\neq z_2$

$$\begin{aligned}\int d^2z\frac{1}{z_{01}\bar z_{02}^2}&=\int d^2z\frac{\bar z_{01}z_{02}^2}{|z_{01}|^2|z_{02}|^4}\\ &=2\int_0^1 du(1-u)\int d^2z\frac{\bar z_{01}z_{02}^2}{(u|z_{01}|^2+(1-u)|z_{02}|^2)^3}\\ &=2\int_0^1 du(1-u)\int d^2y\frac{2uz_{12}|y|^2-(1-u)u^2z_{12}^2\bar z_{12}}{(|y|^2+(1-u)u|z_{12}|^2)^3}\\ &=2z_{12}\int_0^1 duu(1-u)\int d^2y\frac{2|y|^2-A^2}{(|y|^2+A^2)^3}\\ &=2z_{12}\int_0^1 duu(1-u)V_d\int d\rho\rho^{d-1}\frac{2\rho^2-A^2}{(\rho^2+A^2)^3}\\ &=2z_{12}\int_0^1 duu(1-u)V_dA^{-2}\frac{1}{4}=\frac{\pi}{\bar z_{12}},\end{aligned}\tag{136}$$

where in the last step $d=2$ is set directly since there is no divergence in the integral, and analytical continuation of the dimension is not required. Here $V_d=2\pi^{d/2}/\Gamma(d/2)$

[10] The notation of integrals is taken the same form as [19]

$$\mathcal{I}_{a_1,\cdots,a_m,b_1,\cdots,b_n}(z_{i_1},\cdots,z_{i_m},\bar z_{j_1},\cdots,\bar z_{j_n})\equiv\int\frac{d^2z}{(z-z_{i_1})^{a_1}\cdots(z-z_{i_m})^{a_m}(\bar z-\bar z_{j_1})^{b_1}\cdots(\bar z-\bar z_{j_n})^{b_n}}.\tag{134}$$

is the area of $(d-1)$-sphere with unit radius, also we denote $A^2 = (1-u)u|z_{12}|^2$ and use the coordinates transformation

$$z = y + uz_1 + (1-u)z_2, \quad z_{01} = y - (1-u)z_{12}, \quad z_{02} = y + uz_{12} \tag{137}$$

Let us mention that the result in eq.(136) is consistent with eq.(135), i.e. they satisfy $\partial_{\bar{z}_2}\mathcal{I}_{11}(z_i, \bar{z}_j) = \mathcal{I}_{12}(z_i, \bar{z}_j)$.

For $\mathcal{I}_{22}(z_1, \bar{z}_2)$ with $z_1 \neq z_2$, similarly we can obtain

$$\begin{aligned}
\int d^2z \frac{1}{z_{01}^2 \bar{z}_{02}^2} &= \int d^2z \frac{\bar{z}_{01}^2 z_{02}^2}{|z_{01}|^4 |z_{02}|^4} \\
&= 6\int_0^1 duu(1-u) \int d^2y \frac{(\bar{y} - (1-u)\bar{z}_{12})^2 (y + uz_{12})^2}{(|y|^2 + (1-u)u|z_{12}|^2)^4} \\
&= 6\int_0^1 duu(1-u) \int d^2y \frac{|y|^4 - 4|y|^2 u(1-u)|z_{12}|^2 + (1-u)^2 u^2 |z_{12}|^4}{(|y|^2 + (1-u)u|z_{12}|^2)^4} \\
&= 6\int_0^1 duu(1-u) \int d^2y \frac{|y|^4 - 4|y|^2 A^2 + A^4}{(|y|^2 + A^2)^4} = 0.
\end{aligned} \tag{138}$$

In summary, by using dimensional regularization we can obtain the following basic integrals which appear in $\mathcal{N}$ =(1,1) case

$$\begin{aligned}
&\mathcal{I}_{11}(z_i, \bar{z}_j) = -\pi(-\frac{2}{\epsilon} + \ln|z_{ij}|^2 + \gamma + \ln\pi + \mathcal{O}(\epsilon)), \\
&\mathcal{I}_{12}(z_i, \bar{z}_j) = \frac{\pi}{\bar{z}_{ij}}, \quad \mathcal{I}_{21}(z_i, \bar{z}_j) = -\frac{\pi}{z_{ij}}, \quad \mathcal{I}_{22}(z_i, \bar{z}_j) = 0, \\
&\mathcal{I}_{11}(z_i, \bar{z}_i) = 0, \quad \mathcal{I}_{12}(z_i, \bar{z}_i) = 0, \quad \mathcal{I}_{22}(z_i, \bar{z}_i) = 0,
\end{aligned} \tag{139}$$

where in the last line the integrals with two points coincide are listed. For these integrals by translation symmetry, we can set $z_i = 0$, thus there is no scale in the integrals and we can set these integrals equal zero in dimensional regularization. Note that the integral $\mathcal{I}_{22}(z_i, \bar{z}_j)$ is proportional to a delta function $\delta^{(2)}(z_{ij})$ in (B.7) of [17]. However, we will omit this delta function here due to the fact that once we let $z_i = z_j$ in $\mathcal{I}_{22}(z_i, \bar{z}_j)$, as mentioned above, by translation symmetry there is no scale in the integral. Thus the term $\delta^{(2)}(z_{ij})$ in (B.7) of [17] is simply replaced by zero in eq.(139).

By using Feynman parametrization, following the same line as above, we can also obtain the integrals needed in the $\mathcal{N} = (2,2)$ case, which are

$$\begin{aligned}
&\mathcal{I}_{13}(z_i, \bar{z}_j) = \frac{\pi}{(\bar{z}_{ij})^2}, \quad \mathcal{I}_{31}(z_i, \bar{z}_j) = \frac{\pi}{(z_{ij})^2}, \\
&\mathcal{I}_{23}(z_i, \bar{z}_j) = \mathcal{I}_{32}(z_i, \bar{z}_j) = \mathcal{I}_{33}(z_i, \bar{z}_j) = 0,
\end{aligned} \tag{140}$$

where we also let the integrals with two points coinciding with each other vanish.

5 Conclusions

In the paper we investigated the correlation functions with $T\bar{T}$ deformation for $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,2)$ superconformal field theory perturbatively to the first order in the coupling constant. This extends previous work on the correlation function from bosonic CFT to supersymmetry case. Much like the bosonic CFT, the undeformed 2- and 3-point function almost fixed by global superconformal symmetry, while the n-pt ($n \geq 4$) functions dependent on a undetermined function f. By using superconformal Ward identities, we work out the correlation functions with $T\bar{T}$ operator inserted. It is shown that all the integral in first order perturbation can be decomposed into several basic integral as listed in the last section. As a consequence, we only need to evaluate these integrals, which have been done with dimensional regularization. As a possible application, we briefly mentioned the OTOC under deformation. Unlike the bosonic CFT, where the conformal blocks in 4-pt functions can be used to evaluate the OTOC, for superconformal field theory, there is an unknown function f in 4-pt functions. Thus more information about the function f is required to study the OTOC of superconformal field theory.

In the present paper we only considered the effect of $T\bar{T}$ deformation on correlation functions perturbatively near the IR conformal fixed point. Since $T\bar{T}$ deformation is believed to have good behavior in the UV, it is interesting to study the correlation functions of superconformal theory in the UV as what has been done for the bosonic CFT in [18]. Another interesting problem is to study the correlation functions in $\mathcal{N} = (1,0)$ and $\mathcal{N} = (2,0)$ theories, which exist for Lorentz signature. Possibly, one can also consider the $J\bar{T}$ deformation in supersymmetry theory recently studied in [61].

Acknowledgements

The authors are grateful to Bin Chen, Hongfei Shu, Rong-Xin Miao, Jiahui Bao, Chao Yu for useful discussions. SH thanks the Yukawa Institute for Theoretical Physics at Kyoto University. Discussions during the workshop YITP-T-19-03 "Quantum Information and String Theory 2019" were useful to complete this work. SH also would like to appreciate the financial support from Jilin University and Max Planck Partner group. JRS is supported by the National Natural Science Foundation of China under Grant No. 11675272. The Project is also funded by China Postdoctoral Science Foundation (No. 2019M653137).

A Integrals in 2pt-correlators

There are nine terms in eq.(42), the first one have been considered in eq.(44). Below by using the integrals in section 4 we list the remaining eight terms in the integral eq.(42). The second term

$$\begin{aligned}T_{22} \equiv &\int d^2z d\theta d\bar{\theta}\frac{\Delta\theta_{12}}{Z_{12}}\Big(\frac{1}{Z_{01}}+\frac{1}{Z_{02}}\Big)\frac{\Delta\bar{\theta}_{12}}{\bar{Z}_{12}}\Big(\frac{1}{\bar{Z}_{01}}+\frac{1}{\bar{Z}_{02}}\Big)\\ =&\frac{\Delta^2\theta_{12}\bar{\theta}_{12}}{Z_{12}\bar{Z}_{12}}\int d^2z\int d\theta\Big(\frac{1}{Z_{01}}+\frac{1}{Z_{02}}\Big)\int d\bar{\theta}\Big(\frac{1}{\bar{Z}_{01}}+\frac{1}{\bar{Z}_{02}}\Big)\\ =&\frac{\Delta^2\theta_{12}\bar{\theta}_{12}}{Z_{12}\bar{Z}_{12}}\int d^2z\Big(\frac{\theta_1}{z_{01}^2}+\frac{\theta_2}{z_{02}^2}\Big)\Big(\frac{\bar{\theta}_1}{\bar{z}_{01}^2}+\frac{\bar{\theta}_2}{\bar{z}_{02}^2}\Big)\\ =&-\frac{\Delta^2\theta_1\theta_2\bar{\theta}_1\bar{\theta}_2}{Z_{12}\bar{Z}_{12}}(2\mathcal{I}_{22}(z_1,\bar{z}_1)+\mathcal{I}_{22}(z_1,\bar{z}_2)+\mathcal{I}_{22}(z_2,\bar{z}_1))=0.\end{aligned} \tag{141}$$

The third term

$$\begin{aligned}T_{33} \equiv&\Delta^2\int d^2z d\theta d\bar{\theta}\Big(\frac{\theta_{01}}{Z_{01}^2}+\frac{\theta_{02}}{Z_{02}^2}\Big)\Big(\frac{\bar{\theta}_{01}}{\bar{Z}_{01}^2}+\frac{\bar{\theta}_{02}}{\bar{Z}_{02}^2}\Big)\\ =&-\Delta^2\int d^2z\Big(\frac{1}{z_{01}^2}+\frac{1}{z_{02}^2}\Big)\Big(\frac{1}{\bar{z}_{01}^2}+\frac{1}{\bar{z}_{02}^2}\Big)\\ =&-\Delta^2(2\mathcal{I}_{22}(z_1,\bar{z}_1)+\mathcal{I}_{22}(z_1,\bar{z}_2)+\mathcal{I}_{22}(z_2,\bar{z}_1))=0.\end{aligned} \tag{142}$$

The fourth term

$$\begin{aligned}T_{12} \equiv&\int d^2z d\theta d\bar{\theta}\frac{2\Delta^2}{Z_{12}}\Big(\frac{\theta_{01}}{Z_{01}}-\frac{\theta_{02}}{Z_{02}}\Big)\frac{\bar{\theta}_{12}}{\bar{Z}_{12}}\Big(\frac{1}{\bar{Z}_{01}}+\frac{1}{\bar{Z}_{02}}\Big)\\ =&\frac{2\Delta^2\bar{\theta}_1\bar{\theta}_2}{Z_{12}\bar{Z}_{12}}\int d^2z\Big(\frac{1}{z_{01}}-\frac{1}{z_{02}}\Big)\Big(\frac{1}{\bar{z}_{01}^2}+\frac{1}{\bar{z}_{02}^2}\Big)\\ =&\frac{2\Delta^2\bar{\theta}_1\bar{\theta}_2}{Z_{12}\bar{Z}_{12}}(\mathcal{I}_{12}(z_1,\bar{z}_1)+\mathcal{I}_{12}(z_1,\bar{z}_2)-\mathcal{I}_{12}(z_2,\bar{z}_1)-\mathcal{I}_{12}(z_2,\bar{z}_2))\\ =&\frac{2\Delta^2\bar{\theta}_1\bar{\theta}_2}{Z_{12}\bar{Z}_{12}}(\mathcal{I}_{12}(z_1,\bar{z}_2)-\mathcal{I}_{12}(z_2,\bar{z}_1))=\frac{2\Delta^2\bar{\theta}_1\bar{\theta}_2}{Z_{12}\bar{Z}_{12}}\frac{2\pi}{\bar{z}_{12}}.\end{aligned} \tag{143}$$

The fifth term

$$T_{21} \equiv \frac{2\Delta^2\theta_1\theta_2}{Z_{12}\bar{Z}_{12}}(\mathcal{I}_{12}(\bar{z}_1, z_2) - \mathcal{I}_{12}(\bar{z}_2, z_1)) = \frac{2\Delta^2\theta_1\theta_2}{Z_{12}\bar{Z}_{12}}\frac{2\pi}{z_{12}}. \tag{144}$$

The sixth term

$$\begin{aligned}
T_{13} \equiv &-\int d^2z d\theta d\bar{\theta}\frac{2\Delta^2}{Z_{12}}\Big(\frac{\theta_{01}}{Z_{01}} - \frac{\theta_{02}}{Z_{02}}\Big)\Big(\frac{\bar{\theta}_{01}}{\bar{Z}_{01}^2} + \frac{\bar{\theta}_{02}}{\bar{Z}_{02}^2}\Big)\\
=&\frac{2\Delta^2}{Z_{12}}\int d^2z\Big(\frac{1}{z_{01}} - \frac{1}{z_{02}}\Big)\Big(\frac{1}{\bar{z}_{01}^2} + \frac{1}{\bar{z}_{02}^2}\Big)\\
=&\frac{2\Delta^2}{Z_{12}}(\mathcal{I}_{12}(z_1, \bar{z}_2) - \mathcal{I}_{12}(z_2, \bar{z}_1)) = \frac{2\Delta^2}{Z_{12}}\frac{2\pi}{\bar{z}_{12}}.
\end{aligned} \tag{145}$$

The seventh term

$$T_{31} \equiv \frac{2\Delta^2}{\bar{Z}_{12}}(\mathcal{I}_{12}(\bar{z}_1, z_2) - \mathcal{I}_{12}(\bar{z}_2, z_1)) = \frac{2\Delta^2}{\bar{Z}_{12}}\frac{2\pi}{z_{12}}. \tag{146}$$

The eighth term

$$\begin{aligned}
T_{23} \equiv &-\int d^2z d\theta d\bar{\theta}\frac{\Delta^2\theta_{12}}{Z_{12}}\Big(\frac{1}{Z_{01}} + \frac{1}{Z_{02}}\Big)\Big(\frac{\bar{\theta}_{01}}{\bar{Z}_{01}^2} + \frac{\bar{\theta}_{02}}{\bar{Z}_{02}^2}\Big)\\
= &-\frac{\Delta^2\theta_1\theta_2}{Z_{12}}\int d^2z\Big(\frac{1}{z_{01}^2} + \frac{1}{z_{02}^2}\Big)\Big(\frac{1}{\bar{z}_{01}^2} + \frac{1}{\bar{z}_{02}^2}\Big)\\
= &-\frac{\Delta^2\theta_1\theta_2}{Z_{12}}(2\mathcal{I}_{22}(z_1, \bar{z}_1) + \mathcal{I}_{22}(z_1, \bar{z}_2) + \mathcal{I}_{22}(z_2, \bar{z}_1)) = 0.
\end{aligned} \tag{147}$$

The ninth term

$$T_{32} \equiv -\frac{\Delta^2\bar{\theta}_1\bar{\theta}_2}{\bar{Z}_{12}}(2\mathcal{I}_{22}(z_1, \bar{z}_1) + \mathcal{I}_{22}(z_1, \bar{z}_2) + \mathcal{I}_{22}(z_2, \bar{z}_1)) = 0. \tag{148}$$

Finally, the total contribution from the eight terms is

$$T_{12} + T_{21} + T_{13} + T_{23} = \frac{8\pi\Delta^2}{Z_{12}\bar{Z}_{12}}. \tag{149}$$

References

[1] A. B. Zamolodchikov, "Expectation value of composite field T anti-T in two-dimensional quantum field theory," hep-th/0401146.

[2] F. A. Smirnov and A. B. Zamolodchikov, "On space of integrable quantum field theories," Nucl. Phys. B **915**, 363 (2017) [arXiv:1608.05499 [hep-th]].

[3] A. Cavaglia, S. Negro, I. M. Szecsenyi and R. Tateo, "$T\bar{T}$-deformed 2D Quantum Field Theories," JHEP **1610**, 112 (2016) [arXiv:1608.05534 [hep-th]].

[4] V. Rosenhaus and M. Smolkin, "Integrability and Renormalization under $T\bar{T}$," arXiv:1909.02640 [hep-th].

[5] O. Aharony and T. Vaknin, "The TT* deformation at large central charge," JHEP **1805**, 166 (2018) [arXiv:1803.00100 [hep-th]].

[6] S. Datta and Y. Jiang, "$T\bar{T}$-deformed partition functions," JHEP **1808**, 106 (2018) [arXiv:1806.07426 [hep-th]].

[7] O. Aharony, S. Datta, A. Giveon, Y. Jiang and D. Kutasov, "Modular invariance and uniqueness of $T\bar{T}$-deformed CFT," JHEP **1901**, 086 (2019) [arXiv:1808.02492 [hep-th]].

[8] R. Conti, S. Negro and R. Tateo, "The $\mathrm{T}\overline{\mathrm{T}}$ perturbation and its geometric interpretation," JHEP **1902**, 085 (2019) [arXiv:1809.09593 [hep-th]].

[9] G. Bonelli, N. Doroud and M. Zhu, "$T\bar{T}$-deformations in closed form," JHEP **1806**, 149 (2018) [arXiv:1804.10967 [hep-th]].

[10] S. Dubovsky, V. Gorbenko and G. Hernandez-Chifflet, "$T\overline{T}$ partition function from topological gravity," JHEP **1809**, 158 (2018) [arXiv:1805.07386 [hep-th]].

[11] R. Conti, L. Iannella, S. Negro and R. Tateo, "Generalised Born-Infeld models, Lax operators and the $\mathrm{T}\overline{\mathrm{T}}$ perturbation," JHEP **1811**, 007 (2018) [arXiv:1806.11515 [hep-th]].

[12] L. Santilli and M. Tierz, "Large N phase transition in $T\overline{T}$ -deformed 2d Yang-Mills theory on the sphere," JHEP **1901**, 054 (2019) [arXiv:1810.05404 [hep-th]].

[13] Y. Jiang, "Expectation value of $\mathrm{T}\overline{\mathrm{T}}$ operator in curved spacetimes," arXiv:1903.07561 [hep-th].

[14] R. Conti, S. Negro and R. Tateo, "Conserved currents and $\mathrm{T}\bar{\mathrm{T}}_s$ irrelevant deformations of 2D integrable field theories," arXiv:1904.09141 [hep-th].

[15] J. Cardy, "$T\overline{T}$ deformations of non-Lorentz invariant field theories," arXiv:1809.07849 [hep-th].

[16] M. Guica, "An integrable Lorentz-breaking deformation of two-dimensional CFTs," SciPost Phys. **5**, no. 5, 048 (2018) [arXiv:1710.08415 [hep-th]].

[17] M. Guica, "On correlation functions in $J\bar{T}$-deformed CFTs," J. Phys. A **52**, no. 18, 184003 (2019) [arXiv:1902.01434 [hep-th]].

[18] J. Cardy, "$T\overline{T}$ deformation of correlation functions," arXiv:1907.03394 [hep-th].

[19] S. He and H. Shu, "Correlation functions, entanglement and chaos in the $T\bar{T}/J\bar{T}$-deformed CFTs," arXiv:1907.12603 [hep-th].

[20] W. Donnelly and V. Shyam, "Entanglement entropy and $T\overline{T}$ deformation," Phys. Rev. Lett. **121**, no. 13, 131602 (2018) doi:10.1103/PhysRevLett.121.131602 [arXiv:1806.07444 [hep-th]].

[21] P. Wang, H. Wu and H. Yang, "The dual geometries of $T\bar{T}$ deformed CFT_2 and highly excited states of CFT_2," arXiv:1811.07758 [hep-th].

[22] Y. Sun and J. R. Sun, Phys. Rev. D **99**, no. 10, 106008 (2019) doi:10.1103/PhysRevD.99.106008 [arXiv:1901.08796 [hep-th]].

[23] Y. Jiang, "Lectures on solvable irrelevant deformations of 2d quantum field theory," arXiv:1904.13376 [hep-th].

[24] H. S. Jeong, K. Y. Kim and M. Nishida, "Entanglement and Rényi entropy of multiple intervals in $T\overline{T}$-deformed CFT and holography," Phys. Rev. D **100**, no. 10, 106015 (2019) [arXiv:1906.03894 [hep-th]].

[25] S. Grieninger, "Entanglement entropy and $T\overline{T}$ deformations beyond antipodal points from holography," JHEP **1911**, 171 (2019) doi:10.1007/JHEP11(2019)171 [arXiv:1908.10372 [hep-th]].

[26] A. Banerjee, A. Bhattacharyya and S. Chakraborty, "Entanglement Entropy for TT deformed CFT in general dimensions," Nucl. Phys. B **948**, 114775 (2019) [arXiv:1904.00716 [hep-th]].

[27] D. J. Gross, J. Kruthoff, A. Rolph and E. Shaghoulian, "$T\overline{T}$ in AdS_2 and Quantum Mechanics," arXiv:1907.04873 [hep-th].

[28] D. J. Gross, J. Kruthoff, A. Rolph and E. Shaghoulian, "Hamiltonian deformations in quantum mechanics, $T\bar{T}$, and SYK," arXiv:1912.06132 [hep-th].

[29] S. Dubovsky, V. Gorbenko and M. Mirbabayi, "Asymptotic fragility, near AdS_2 holography and $T\overline{T}$," JHEP **1709**, 136 (2017) [arXiv:1706.06604 [hep-th]].

[30] J. Cardy, "The $T\overline{T}$ deformation of quantum field theory as random geometry," JHEP **1810**, 186 (2018) [arXiv:1801.06895 [hep-th]].

[31] A. Giveon, N. Itzhaki and D. Kutasov, "$\mathrm{T}\overline{\mathrm{T}}$ and LST," JHEP **1707**, 122 (2017) [arXiv:1701.05576 [hep-th]].

[32] A. Giveon, N. Itzhaki and D. Kutasov, "A solvable irrelevant deformation of AdS_3/CFT_2," JHEP **1712**, 155 (2017) [arXiv:1707.05800 [hep-th]].

[33] M. Asrat, A. Giveon, N. Itzhaki and D. Kutasov, "Holography Beyond AdS," Nucl. Phys. B **932**, 241 (2018) [arXiv:1711.02690 [hep-th]].

[34] G. Giribet, "$T\bar{T}$-deformations, AdS/CFT and correlation functions," JHEP **1802**, 114 (2018) [arXiv:1711.02716 [hep-th]].

[35] M. Baggio and A. Sfondrini, "Strings on NS-NS Backgrounds as Integrable Deformations," Phys. Rev. D **98**, no. 2, 021902 (2018) [arXiv:1804.01998 [hep-th]].

[36] L. Apolo and W. Song, "Strings on warped AdS_3 via $\mathrm{T}\bar{\mathrm{J}}$ deformations," JHEP **1810**, 165 (2018) [arXiv:1806.10127 [hep-th]].

[37] J. P. Babaro, V. F. Foit, G. Giribet and M. Leoni, "$T\overline{T}$ type deformation in the presence of a boundary," JHEP **1808**, 096 (2018) [arXiv:1806.10713 [hep-th]].

[38] S. Chakraborty, "Wilson loop in a $T\bar{T}$ like deformed CFT_2," Nucl. Phys. B **938**, 605 (2019) [arXiv:1809.01915 [hep-th]].

[39] L. Apolo, S. Detournay and W. Song, "TsT, $T\bar{T}$ and black strings," arXiv:1911.12359 [hep-th].

[40] T. Araujo, E. O. Colgain, Y. Sakatani, M. M. Sheikh-Jabbari and H. Yavartanoo, "Holographic integration of $T\bar{T}$ & $J\bar{T}$ via $O(d,d)$," JHEP **1903**, 168 (2019) [arXiv:1811.03050 [hep-th]].

[41] A. Giveon, "Comments on $T\bar{T}$, $J\bar{T}$ and String Theory," arXiv:1903.06883 [hep-th].

[42] S. Chakraborty, A. Giveon and D. Kutasov, "$T\bar{T}$, $J\bar{T}$, $T\bar{J}$ and String Theory," arXiv:1905.00051 [hep-th].

[43] L. McGough, M. Mezei and H. Verlinde, "Moving the CFT into the bulk with $T\overline{T}$," JHEP **1804**, 010 (2018) [arXiv:1611.03470 [hep-th]].

[44] V. Shyam, "Background independent holographic dual to $T\bar{T}$-deformed CFT with large central charge in 2 dimensions," JHEP **1710**, 108 (2017) [arXiv:1707.08118 [hep-th]].

[45] P. Kraus, J. Liu and D. Marolf, "Cutoff AdS_3 versus the $T\overline{T}$ deformation," JHEP **1807**, 027 (2018) [arXiv:1801.02714 [hep-th]].

[46] W. Cottrell and A. Hashimoto, "Comments on $T\bar{T}$ double trace deformations and boundary conditions," Phys. Lett. B **789**, 251 (2019) [arXiv:1801.09708 [hep-th]].

[47] A. Bzowski and M. Guica, "The holographic interpretation of $J\bar{T}$-deformed CFTs," JHEP **1901**, 198 (2019) [arXiv:1803.09753 [hep-th]].

[48] M. Taylor, "TT deformations in general dimensions," arXiv:1805.10287 [hep-th].

[49] T. Hartman, J. Kruthoff, E. Shaghoulian and A. Tajdini, "Holography at finite cutoff with a T^2 deformation," JHEP **1903**, 004 (2019) [arXiv:1807.11401 [hep-th]].

[50] V. Shyam, "Finite Cutoff AdS_5 Holography and the Generalized Gradient Flow," JHEP **1812**, 086 (2018) [arXiv:1808.07760 [hep-th]].

[51] P. Caputa, S. Datta and V. Shyam, "Sphere partition functions & cut-off AdS," JHEP **1905**, 112 (2019) [arXiv:1902.10893 [hep-th]].

[52] L. Apolo and W. Song, "Heating up holography for single-trace $J\bar{T}$-deformations," arXiv:1907.03745 [hep-th].

[53] B. Chen, L. Chen and P. X. Hao, "Entanglement entropy in $T\overline{T}$-deformed CFT," Phys. Rev. D **98**, no. 8, 086025 (2018) [arXiv:1807.08293 [hep-th]].

[54] B. Chen, L. Chen and C. Y. Zhang, "Surface/State correspondence and $T\overline{T}$ deformation," arXiv:1907.12110 [hep-th].

[55] M. He and Y. h. Gao, "On the symmetry of $T\bar{T}$ deformed CFT," arXiv:1910.09390 [hep-th].

[56] M. Baggio, A. Sfondrini, G. Tartaglino-Mazzucchelli and H. Walsh, "On $T\overline{T}$ deformations and supersymmetry," JHEP **1906**, 063 (2019) [arXiv:1811.00533 [hep-th]].

[57] C. K. Chang, C. Ferko and S. Sethi, "Supersymmetry and $T\overline{T}$ deformations," JHEP **1904**, 131 (2019) [arXiv:1811.01895 [hep-th]].

[58] H. Jiang, A. Sfondrini and G. Tartaglino-Mazzucchelli, "$T\bar{T}$ deformations with $\mathcal{N} = (0, 2)$ supersymmetry," arXiv:1904.04760 [hep-th].

[59] C. K. Chang, C. Ferko, S. Sethi, A. Sfondrini and G. Tartaglino-Mazzucchelli, "$T\bar{T}$ Flows and (2,2) Supersymmetry," arXiv:1906.00467 [hep-th].

[60] E. A. Coleman, J. Aguilera-Damia, D. Z. Freedman and R. M. Soni, "$T\bar{T}$-Deformed Actions and (1,1) Supersymmetry," arXiv:1906.05439 [hep-th].

[61] H. Jiang and G. Tartaglino-Mazzucchelli, "Supersymmetric $J\bar{T}$ and $T\bar{J}$ deformations," arXiv:1911.05631 [hep-th].

[62] T. T. Dumitrescu and N. Seiberg, "Supercurrents and Brane Currents in Diverse Dimensions," JHEP **1107**, 095 (2011) [arXiv:1106.0031 [hep-th]].

[63] M. Nozaki, T. Numasawa and T. Takayanagi, "Quantum Entanglement of Local Operators in Conformal Field Theories," Phys. Rev. Lett. **112**, 111602 (2014) [arXiv:1401.0539 [hep-th]].

[64] S. He, T. Numasawa, T. Takayanagi and K. Watanabe, "Quantum dimension as entanglement entropy in two dimensional conformal field theories," Phys. Rev. D **90**, no. 4, 041701 (2014) [arXiv:1403.0702 [hep-th]].

[65] S. He, "Conformal bootstrap to Rényi entropy in 2D Liouville and super-Liouville CFTs," Phys. Rev. D **99**, no. 2, 026005 (2019) [arXiv:1711.00624 [hep-th]].

[66] P. Calabrese, J. Cardy and E. Tonni, Phys. Rev. Lett. **109**, 130502 (2012) [arXiv:1206.3092 [cond-mat.stat-mech]].

[67] T. Takayanagi and K. Umemoto, "Entanglement of purification through holographic duality," Nature Phys. **14**, no. 6, 573 (2018) [arXiv:1708.09393 [hep-th]].

[68] M. Miyaji, T. Numasawa, N. Shiba, T. Takayanagi and K. Watanabe, Phys. Rev. Lett. **115**, no. 26, 261602 (2015) [arXiv:1507.07555 [hep-th]].

[69] Y. Suzuki, T. Takayanagi and K. Umemoto, "Entanglement Wedges from Information Metric in Conformal Field Theories," Phys. Rev. Lett. **123**, no. 22, 221601 (2019) [arXiv:1908.09939 [hep-th]].

[70] P. C. West, "Introduction to supersymmetry and supergravity," Singapore, Singapore: World Scientific (1990) 425 p

[71] R. Blumenhagen and E. Plauschinn, "Introduction to conformal field theory : with applications to String theory," Lect. Notes Phys. **779**, 1 (2009).

[72] J. Fuchs, "Superconformal Ward Identities and the WZW Model," Nucl. Phys. B **286**, 455 (1987).

[73] J. L. Petersen, "Conformal And Superconformal Two-dimensional Field Theories," In *Ahrenshoop 1985, Proceedings, Special Topics In Gauge Field Theories*, 1-36 and Copenhagen Univ. - NBI-HE-85-31 (85,REC.JAN.86) 35p

[74] D. A. Roberts and D. Stanford, "Two-dimensional conformal field theory and the butterfly effect," Phys. Rev. Lett. **115**, no. 13, 131603 (2015) [arXiv:1412.5123 [hep-th]].

[75] P. Di Vecchia, J. L. Petersen and H. B. Zheng, "N=2 Extended Superconformal Theories in Two-Dimensions," Phys. Lett. **162B**, 327 (1985).

[76] P. Di Vecchia, V. G. Knizhnik, J. L. Petersen and P. Rossi, “A Supersymmetric Wess-Zumino Lagrangian in Two-Dimensions,” Nucl. Phys. B **253**, 701 (1985).

[77] E. B. Kiritsis, “The Structure of $N = 2$ Superconformally Invariant 'Minimal' Theories: Operator Algebra and Correlation Functions,” Phys. Rev. D **36**, 3048 (1987).