Performance optimization of low-dissipation thermal machines revisited

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We revisit the optimization of performance of finite-time Carnot machines satisfying the low-dissipation assumption. The standard procedure seeks to optimize an objective function, such as power output of the engine, over the durations of contacts between the working medium and the heat reservoirs. This procedure may lead to unwieldy equations at the optimum of some objective functions. We propose an alternate scheme in which the output or input work is first optimized for a given cycle time, followed by an optimization of another objective function over the cycle time. This optimization problem is solved in a much simplified manner, with closed-form expressions for figures of merit. The approach is demonstrated for various objective functions, both for engines as well as refrigerators.

I. INTRODUCTION

Optimization of performance of finite-time thermal machines has been intensely studied for many years now [1–5]. In recent years, the low-dissipation model has been proposed and applied to heat engines and refrigerators with presumably large cycle times and so, close to the reversible limit. The low-dissipation regime is characterized by the following dependence: the entropy generated in a heat-exchange process is inversely proportional to the duration of the process. It was initially derived for a mesoscopic, brownian heat engine treated within stochastic thermodynamic framework [6], and was later adapted for finite-time macroscopic engines [7]. It is observed at the optimal performance of quantum dot Carnot engine based on the master equation approach [8], and within a perturbative approach for slowly driven open quantum systems [9].

Because of its simplicity, the low-dissipation model has attracted a lot of attention [10–18]. Furthermore, there is no explicit requirement on the form of heat-transfer law, or the temperature difference between the heat reservoirs to be small, unlike in endoreversible models [7]. Still, the optimization problem may become cumbersome, or even intractable, with some objective functions. In this paper, we propose an alternate two-step optimization scheme which yields the optimal solution in a quite simplified manner, while predicting the essential characteristics of the model, such as closed-form expressions for figures of merit as well as the bounds satisfied by them within the domain of applicability of the model. The utility of the approach is demonstrated on various objective functions.

The plan of the paper is as follows. In Section II, we briefly describe the basic features low-dissipation Carnot engine. In Section III, we first optimize the work output for a given cycle time and discuss its main features. In Section IV, we optimize other objective functions for the

engine, giving explicit expressions for the efficiency. In Section V, we treat the model of a refrigerator and derive expressions for the coefficient of performance at optimum of different objective functions. Sections VI is devoted to a discussion of some novel features of the work-optimized model and Section VII contains the conclusions.

II. LOW-DISSIPATION MODEL

Consider a two heat-reservoirs set up, with hot (h) and cold (c) temperatures, T_h and T_c. A heat engine runs through a four-step cycle by coupling to these reservoirs alternately. The cycle consists of two thermal contacts lasting for time intervals τ_h and τ_c, and two adiabatic steps whose time intervals are considered negligible in comparison to the other time scales. Now, the change in entropy of the working medium during heat transfer at the hot/cold contact, can be split as: $\Delta S_j = \Delta_{\rm rev} S_j + \Delta_{\rm ir} S_j$, with $j = h, c$. Here, the first term accounts for a reversible heat transfer, whereas the second term denotes an irreversible entropy generation during the process. Now, the low-dissipation behavior is quantified as: $T_j \Delta_{\rm ir} S_j = \sigma_j/t_j + O(1/t_j^2)$, where σ_j is the dissipation constant [7, 13], and the higher order terms are considered neglegible due to the large durations. Thus at the hot and the cold contact, we respectively have

$$\Delta S_h = \frac{Q_h}{T_h} + \frac{\sigma_h}{T_h t_h}, \tag{1}$$

$$\Delta S_c = -\frac{Q_c}{T_c} + \frac{\sigma_c}{T_c t_c}, \tag{2}$$

where $Q_j > 0$. Given that the other two steps in the heat cycle are adiabatic—with no entropy changes—the cyclic process within the working medium implies $\Delta S_h + \Delta S_c = 0$. In other words, $\Delta S_h = -\Delta S_c = \Delta S > 0$, where the value ΔS is preassigned. Then the amount of heat exchanged with each reservoir can be written as:

$$Q_h = T_h \Delta S - \frac{\sigma_h}{t_h}, \tag{3}$$

$$Q_c = T_c \Delta S + \frac{\sigma_c}{t_c}. \tag{4}$$

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The work extracted in a cycle with the time period $t \approx t_h + t_c$ is, $W = Q_h - Q_c$, given by

$$W(t_h, t_c) = \Delta T \Delta S - \frac{\sigma_h}{t_h} - \frac{\sigma_c}{t_c}, \qquad (5)$$

where $\Delta T = T_h - T_c$. Now, in the standard optimzations of the LD model, the parameter ΔS is held fixed during variations of the time intervals. Clearly, for a given ΔS, as each $t_j \to \infty$, the work approaches its maximum value of $\Delta T \Delta S \equiv W_{\rm rev}$, which is referred to as the reversible work, under conditions of a fixed ΔS. Therefore, the difference $W_{\rm rev} - W$ represents the lost work due to entropy production. Recently, we showed that each loss term above can be obtained from a linear-irreversible engine running between an infinite heat reservoir and a finite heat sink or source for a given time (t_h or t_c) [19].

III. OPTIMAL WORK FOR A GIVEN CYCLE TIME

Now, instead of choosing t_h and t_c as the control parameters that may be tuned in order to optimize the overall performance of the engine [6, 7, 13], let us define t_h and t_c in terms of the fraction of the total cycle time as: $t_h = \gamma t$, and $t_c = (1 - \gamma)t$, obtaining

$$W(\gamma, t) = \Delta T \Delta S - \left(\frac{\sigma_h}{\gamma} + \frac{\sigma_c}{1 - \gamma} \right) \frac{1}{t}. \qquad (6)$$

As a first step towards optimization of the engine's performance, we maximize the irreversible work W for a fixed value of the time interval t. This amounts to tuning the parameter γ. Thus, setting:

$$\left. \frac{\partial W}{\partial \gamma} \right|_{t, \Delta S} = 0, \qquad (7)$$

we obtain the optimum value of γ as

$$\hat{\gamma} = \frac{\sqrt{\sigma_h}}{\sqrt{\sigma_h} + \sqrt{\sigma_c}}, \qquad (8)$$

which is function only of the ratio of the dissipation constants. In the following, we seek to optimize the sub-class of low-dissipation models which operate at optimal work in a given time. Thus, in our model, the (maximum) work output for a cycle of time t is:

$$\hat{W}(t) = \Delta T \Delta S - \frac{(\sqrt{\sigma_h} + \sqrt{\sigma_c})^2}{t}, \qquad (9)$$

which is equivalent to finite-time availability [20, 21] for the low-dissipation engine.

Then, the heat absorbed from the hot reservoir is:

$$\hat{Q}_h(t) = T_h \Delta S - \frac{\sqrt{\sigma_h}(\sqrt{\sigma_h} + \sqrt{\sigma_c})}{t}. \qquad (10)$$

Thus, the efficiency under these conditions is: $\hat{\eta}(t) = \hat{W}/\hat{Q}_h$.

Similarly, the heat rejected to the cold reservoir is:

$$\hat{Q}_c(t) = T_c \Delta S + \frac{\sqrt{\sigma_c}(\sqrt{\sigma_h} + \sqrt{\sigma_c})}{t}. \qquad (11)$$

We notice two limiting cases here. For a finite value of σ_h, if we have $\sigma_c \ll \sigma_h$, which is equivalent to the condition $\hat{\gamma} \approx 1$, Eqs. (9)-(11) simplify to: $\hat{W}(t) \approx \Delta T \Delta S - \sigma_h/t$, $\hat{Q}_h(t) \approx T_h \Delta S - \sigma_h/t$, and $\hat{Q}_c \approx T_c \Delta S$. In other words, the heat exchange at the cold end approaches its reversible value, in this limit. On the other hand, for a given finite value of σ_c, if we have $\sigma_h \ll \sigma_c$, which is equivalent to the condition $\hat{\gamma} \approx 0$, Eqs. (9)-(11) simplify to: $\hat{W}(t) \approx \Delta T \Delta S - \sigma_c/t$, $\hat{Q}_h \approx T_h \Delta S$, and $\hat{Q}_c(t) \approx T_c \Delta S + \sigma_c/t$. Thus, when the strength of dissipation at the hot end is negligible as compared to the cold end, then, at optimal work, the heat exchange at the hot end can be approximated to be reversible.

Further, upon eliminating time t from Eqs. (10) and (11), we obtain the following interesting equality:

$$\frac{\hat{\gamma} \hat{Q}_c + (1 - \hat{\gamma}) \hat{Q}_h}{\hat{\gamma} T_c + (1 - \hat{\gamma}) T_h} = \Delta S. \qquad (12)$$

This also makes it clear that as $\hat{\gamma} \to 1$, $\hat{Q}_c \to T_c \Delta S$. Further, this limit implies $\hat{t}_h/t \to 1$, while $\hat{t}_c/t \to 0$. Thus, for a smaller dissipation at a thermal contact, one has to spend a smaller fraction of the total given time for that process. Similarly, as $\hat{\gamma} \to 0$, $\hat{Q}_h \to T_h \Delta S$, and analogous conclusions can be drawn.

Equivalently, we may consider the ratio of dissipations at the hot to cold contacts, as follows:

$$\frac{T_h \Delta_{\rm ir} \hat{S}_h}{T_c \Delta_{\rm ir} \hat{S}_c} = \frac{\hat{\gamma}}{1 - \hat{\gamma}} = \sqrt{\frac{\sigma_h}{\sigma_c}}. \qquad (13)$$

In this sense, $\hat{\gamma} \to 0$ can be regarded as the limit in which the dissipation at the hot end becomes negligible *in comparison* to the dissipation at the cold end, and so on. A related fact is that the average rate of dissipation at the hot and the cold end become equal at optimum work:

$$\frac{T_h \Delta_{\rm ir} \hat{S}_h}{\hat{t}_h} = \frac{T_c \Delta_{\rm ir} \hat{S}_c}{\hat{t}_c} = \frac{(\sqrt{\sigma_h} + \sqrt{\sigma_c})^2}{t^2}. \qquad (14)$$

IV. OPTIMIZATION: SECOND STEP

The optimal work derived above, Eq. (9), is still a function of the chosen cycle time t. An appropriate value of this cycle time may be selected as the one which optimizes another chosen objective function. In the following, we show that, for a variety of objective functions—popular in the study of finite-time thermodynamics—it is relatively easy to perform this optimization and thus to find an optimal cycle time. Interestingly, this procedure also yields a closed-form expression for the corresponding figure of merit, along with its lower and upper bounds

set by the allowed parameter range. We show the utility of this approach for low-dissipation engines as well as refrigerators.

A. Power output

After knowing the optimal work as a function of the cycle time, we may like to extract this work at the fastest rate. An appropriate objective function to optimize is then the average power output, defined as

$$P \equiv \frac{\hat{W}(t)}{t} = \frac{\Delta T \Delta S}{t} - \frac{(\sqrt{\sigma_h} + \sqrt{\sigma_c})^2}{t^2}. \qquad (15)$$

Note that the power output is defined relative to the optimal work in time t. Then, t^*, corresponding to the maximum of this power, is obtained by setting $\partial P/\partial t = 0$, which yields

$$t^* = \frac{2(\sqrt{\sigma_h} + \sqrt{\sigma_c})^2}{\Delta T \Delta S}, \qquad (16)$$

with the optimal allocation of times for the thermal contacts: $t_h = \hat{\gamma} t^*$, and $t_c = (1 - \hat{\gamma}) t^*$. The optimal amounts of heat and work are:

$$Q_h^* = \left[T_h - \frac{\hat{\gamma}}{2} \Delta T \right] \Delta S, \qquad (17)$$

$$W^* = \frac{\Delta T \Delta S}{2}, \qquad (18)$$

from which the efficiency at maximum power, $\eta^* = W^*/Q_h^*$, follows in the well-known form [6, 13, 22]:

$$\eta^* = \frac{\eta_C}{2 - \hat{\gamma} \eta_C}. \qquad (19)$$

Note that the same optimum for power may also be obtained by performing optimization simultaneously over the pair of variables t_h and t_c [13], which is the standard approach in literature. However, this approach often becomes involved and an analytic solution becomes hard to obtain with other objective functions, in general. In the following, we highlight the utility of the present two-step optimization approach, for the case of engines as well as refrigerators.

B. Per-unit-time efficiency

First proposed by Ma [23], this objective function was optimized for the endoreversible model in Ref. [24]. Our first step is to optimize the work output for a given time t, as described above, and calculate the efficiency at this optimal work, denoted by $\hat{\eta}(t)$. As the second step, we optimize the function:

$$\dot{\hat{\eta}} \equiv \frac{\hat{\eta}(t)}{t}, \qquad (20)$$

w.r.t. time t. The solution can be easily worked out and the efficiency at optimal $\dot{\hat{\eta}}$ is given by:

$$\hat{\eta}^* = \frac{1}{\hat{\gamma}} \left(1 - \sqrt{1 - \hat{\gamma} \eta_C} \right), \qquad (21)$$

which is bounded as: $\eta_C/2 \leqslant \hat{\eta}^* \leqslant 1 - \sqrt{1 - \eta_C}$. Thus the results from the endoreversible model [24] are derived within the low-dissipation model too, in a simple manner.

C. Efficient power

An objective function, defined as the product of efficiency of the engine and its power output [25], was optimized for the low-dissipation model with the standard optimization [26], but the solution turns out to be highly involved. In the present approach, at optimal work for the given cycle time t, the efficient power is defined as:

$$\hat{P}_\eta(t) = \hat{\eta}(t) \frac{\hat{W}(t)}{t}. \qquad (22)$$

The optimum of the above function ($\partial \hat{P}_\eta/\partial t = 0$) is easily evaluated by just solving a quadratic equation in t. Finally, the efficiency at optimal efficient power is obtained in a simple closed form:

$$\eta^* = \frac{1}{2\hat{\gamma}} \left[3 - \sqrt{9 - 8\hat{\gamma} \eta_C} \right], \qquad (23)$$

which is bounded as follows:

$$\frac{2}{3} \eta_C \leq \eta^* \leq \frac{1}{2} \left[3 - \sqrt{9 - 8\eta_C} \right], \qquad (24)$$

as $\hat{\gamma}$ interpolates in the interval $[0, 1]$. These bounds were also obtained in Ref. [14, 26].

V. REFRIGERATOR

Analogous to the heat engine, one may consider the operation of a refrigerator by inverting the thermal and work flows. So, in this case, the entropy generated at the hot and the cold contact is respectively given by:

$$\Delta S_{\mathrm{ir},c} = \Delta S - \frac{Q_c}{T_c}, \qquad (25)$$

and

$$\Delta S_{\mathrm{ir},h} = \frac{Q_h}{T_h} - \Delta S. \qquad (26)$$

Here, $\Delta S > 0$ is the entropy change of the working medium at the cold contact. Q_c is the heat extracted from cold reservoir, while Q_h is the heat dumped into the hot reservoir. Within the low-dissipation assumption, the input work to drive the refrigerator, $W = Q_h - Q_c$, is given by

$$W(\gamma, t) = \Delta T \Delta S + \left(\frac{\sigma_h}{\gamma} + \frac{\sigma_c}{1 - \gamma} \right) \frac{1}{t}. \qquad (27)$$

As expected, the input work is more than the reversible work, in case of an irreversible refrigerator. Then, minimizing the irreversible work w.r.t to γ, for a given time t, we obtain—as in case of the engine—the optimal value, $\hat{\gamma} = \sqrt{\sigma_h}/(\sqrt{\sigma_h} + \sqrt{\sigma_c})$. So, the optimal input work is given by:

$$\hat{W}(t) = \Delta T \Delta S + \frac{(\sqrt{\sigma_h} + \sqrt{\sigma_c})^2}{t}, \qquad (28)$$

and the optimal heat extracted from the cold reservoir is

$$\hat{Q}_c(t) = T_c \Delta S - \frac{\sqrt{\sigma_c}(\sqrt{\sigma_h} + \sqrt{\sigma_c})}{t}. \qquad (29)$$

The next step would be to obtain an optimal cycle time corresponding to a chosen objective function, as discussed below.

A. Cooling power

We consider the cooling power of the refrigerator, operating with optimal work input for a given cycle time, given by : $\hat{Q}_c(t)/t$. The optimal cycle time that maximizes this cooling power is found to be:

$$t^* = \frac{2\sqrt{\sigma_c}(\sqrt{\sigma_h} + \sqrt{\sigma_c})}{T_c \Delta S}. \qquad (30)$$

The corresponding optimal amounts of heat exchanged with reservoirs are:

$$\hat{Q}_c^* = \frac{T_c \Delta S}{2}, \qquad (31)$$

$$\hat{Q}_h^* = T_h \Delta S + \sqrt{\frac{\sigma_h}{\sigma_c}} \frac{T_c \Delta S}{2}. \qquad (32)$$

Finally, the coefficient of performance (COP) of the refrigerator is defined as $\xi = Q_c/(Q_h - Q_c)$, and, at optimum cooling power, COP is evaluated to be

$$\xi^* = \xi_C \left(2 + \frac{\xi_C}{1 - \hat{\gamma}}\right)^{-1}, \qquad (33)$$

where $\xi_C = T_c/(T_h - T_c)$ is the Carnot coefficient. The above expression is bounded as $0 \leq \xi^* \leq \xi_C/(2 + \xi_C)$, and is also obtained in other studies, such as exoreversible refrigerators with only the internal irreversibilities [27], and within a global linear-irreversible framework for total entropy production [28], with a parameter equivalent to $\hat{\gamma}$ and defined in the range $[0, 1]$.

It may be noted that a two-parameter, direct optimization problem cannot be set up with the standard definition of cooling power which, from Eq. (25), can be written as:

$$\frac{Q_c}{t} = \left(T_c \Delta S - \frac{\sigma_c}{t_c}\right) \frac{1}{t}. \qquad (34)$$

Clearly, with $t = t_h + t_c$, the optimum of cooling power does not exist under the variations of both t_h and t_c.

B. Per-unit-time COP

This objective function was investigated in Ref. [29] for the endoreversible model. It is a criterion for refrigerators, analogous to the function used in Section IIIB on engines. Again, we use the optimal work condition for a given cycle time, and evaluate $\hat{\xi}(t) \equiv \hat{Q}_c/\hat{W}$, using Eqs. (28) and (29). Then, we optimize the function:

$$\dot{\hat{\xi}} \equiv \frac{\hat{\xi}(t)}{t}, \qquad (35)$$

w.r.t time t. The COP at the optimal $\dot{\hat{\xi}}$ is evaluated to be:

$$\hat{\xi}^* = (1 - \hat{\gamma}) \left[\sqrt{1 + \frac{\xi_C}{1 - \hat{\gamma}}} - 1\right]. \qquad (36)$$

which is bounded as: $0 \leq \hat{\xi}^* \leq \sqrt{1 + \xi_C} - 1$. The bounds match with the findings of Ref. [29].

C. χ-criterion

In the literature on the optimal performance of refrigerators, χ-criterion is defined as: $\chi = \xi Q_c/t$. This has been studied within endoreversible [30] as well as low-dissipation models [10, 11]. As pointed out above, the calculations may become involved for such objective functions, thus making the analytic solution intractable [10]. However, the present approach of two-step optimization leads directly to an exact expression for the COP as:

$$\xi^* = \frac{1 - \hat{\gamma}}{2} \left[\sqrt{9 + \frac{8\xi_C}{1 - \hat{\gamma}}} - 3\right]. \qquad (37)$$

Interestingly, a similar formula as above is obtained for the so-called minimally nonlinear irreversible refrigerators [31], with an equivalent parameter defined in the range [0,1]. Moreover, the above formula satisfies the following bounds:

$$0 \leq \xi^* \leq \frac{1}{2} \left[\sqrt{9 + 8\xi_C} - 3\right], \qquad (38)$$

which are obtainable under the standard optimization [10].

VI. DISCUSSION

In this section, we further highlight some features of the low-dissipation model, in particular, those related to the first step of work optimization for a given cycle time. We start by studying the notion of lost work in this model.

The lost work is equivalent to the energy which is made unavailable for work, due to the irreversible process [20].

As a way to characterize irreversibility in nonequilibrium thermodynamics, it can also be related to the concept of thermodynamic length in finite-time thermodynamics [32]. It is defined by the difference: $W_{\text{lost}} = W_{\text{rev}} - W$. Therefore, optimizing the work in the irreversibile process as above, implies minimizing the lost work. Since, $W_{\text{rev}} = \Delta T \Delta S$, from Eq. (9), we have

$$\hat{W}_{\text{lost}} = \frac{(\sqrt{\sigma_h} + \sqrt{\sigma_c})^2}{t}, \tag{39}$$

as the minimum lost work in the low-dissipation engine with a given cycle time t. Further, according to Gouy-Stodola theorem [20, 33, 34], the lost work is directly proportional to the total entropy produced in the irreversibile process, where the constant of proportionality is given by a reference temperature T_0. Then, the above theorem as applied at optimal work, implies:

$$\hat{W}_{\text{lost}} = T_0 \Delta_{\text{tot}} \hat{S}. \tag{40}$$

Now, the total entropy produced per cycle (of a given time t) is the sum of entropies produced at the hot and cold steps. Under optimal work:

$$\Delta_{\text{tot}} \hat{S} = \Delta_{\text{ir}} \hat{S}_h + \Delta_{\text{ir}} \hat{S}_c, \tag{41}$$
$$= \frac{\sigma_h}{\hat{\gamma} T_h t} + \frac{\sigma_c}{(1 - \hat{\gamma}) T_c t},$$
$$= \left[\frac{\sqrt{\sigma_h}}{T_h} + \frac{\sqrt{\sigma_c}}{T_c} \right] \frac{\sqrt{\sigma_h} + \sqrt{\sigma_c}}{t},$$
$$= \left[\frac{\hat{\gamma}}{T_h} + \frac{1 - \hat{\gamma}}{T_c} \right] \frac{(\sqrt{\sigma_h} + \sqrt{\sigma_c})^2}{t}. \tag{42}$$

Comparing Eq. (39) with (42), we obtain:

$$T_0 = \left[\frac{\hat{\gamma}}{T_h} + \frac{1 - \hat{\gamma}}{T_c} \right]^{-1}, \tag{43}$$

a weighted harmonic mean of the reservoir temperatures. Clearly, $T_c \leq T_0 \leq T_h$. It may be noted that the apparent form of T_0 as a harmonic mean, holds for the case when the dissipation constants σ_j, and hence the weights $\{\hat{\gamma}, 1 - \hat{\gamma}\}$, are temperature-independent. The specific form of T_0 is different if $\hat{\gamma}$ depends on temperatures. For instance, based on a general, microscopic approach [9], the dissipation constants are found to obey: $\sigma_c / \sigma_h = (T_c/T_h)^{2\alpha}$, where $\alpha \in R$. Thereupon, T_0 comes out in the form:

$$T_0 = \frac{T_h^\alpha + T_c^\alpha}{T_h^{\alpha-1} + T_c^{\alpha-1}}, \tag{44}$$

which is known by the name of Lehmer mean [35]. Clearly, some standard means are subsumed in this general case. In particular, for $\alpha = 0$, we obtain the symmetric harmonic mean. For $\alpha = 1/2$, we obtain the geometric mean, $T_0 = \sqrt{T_h T_c}$, and the arithmetic mean, for $\alpha = 1$.

Finally, it is interesting to rewrite Eq. (42), as follows:

$$T_0 \Delta_{\text{tot}} \hat{S} = \frac{(\sqrt{\sigma_h} + \sqrt{\sigma_c})^2}{t}. \tag{45}$$

Thus, we note that if the lost work is minimized for a fixed cycle time, it yields a concrete expression for the reference temperature relating the lost work to the total entropy produced (Gouy-Stodola theorem). Secondly, the total entropy produced is rendered inversely proportional to the cycle time, which is reminiscent of the low-dissipation behavior, but now applied to the cycle as a whole. Consequently, one may as well identify an effective dissipation constant for the total cycle, in Eq. (45), as $(\sqrt{\sigma_h} + \sqrt{\sigma_c})^2$. In this sense, the first step of optimizing work may be seen as a natural step, since it extends the low-dissipation behavior from the individual processes to the overall cycle.

VII. CONCLUSIONS

We have proposed and demonstrated the utility of an alternate, two-step optimization procedure for low-dissipation thermal machines, whereby the irreversible work is optimized first for a given cycle time, and then a second objective function is optimized, yielding the optimal cycle time for the operation of the device. It is important to emphasize that, in general, the optimum with the alternate procedure will not coincide with the global optimum of the chosen objective function in the standard approach (simultaneous variation of the two contact times), except in case of power output. Actually, the standard approach may not yield a tractable solution for the global optimization problem, such as χ-criterion for low-dissipation refrigerators [10]. Further, a two-parameter optimization problem may not be well-defined for some objective function, e.g. cooling power of low-dissipation refrigerator. On the other hand, the present approach yields, rather easily, closed-form expressions as well as bounds of figures of merit for such functions. Interestingly, these bounds match with the corresponding bounds (whenever these can be derived) from the standard approach. We have applied this approach to a few objective functions, such as efficient power, per-unit-time efficiency, and χ-criterion, which are not easy to treat within the standard approach.

Further, it is observed for various objective functions that the optimal results come out equivalent to those obtained from other models, such as the endoreversible, minimally nonlinear irreversible approaches and global linear-irreversibile framework. This analogy between the low-dissipation and the endoreversible models [18] as well as other approaches [10, 12, 28] needs to be further explored. One reason is that the constraints for optimization, such as keeping ΔS fixed [36], are the same in some procedures. It is hoped that the proposed scheme will provide insights into the connections between optimal behavior of different irreversible models. Finally, it is straightforward to extend the present analysis to a multi-reservoirs scenario [13].

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ACKNOWLEDGMENTS

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