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# Spectral and Hodge theory of "Witt" incomplete cusp edge spaces

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**Abstract.** Incomplete cusp edges model the behavior of the Weil–Petersson metric on the compactified Riemann moduli space near the interior of a divisor. Assuming such a space is Witt, we construct a fundamental solution to the heat equation, and using a precise description of its asymptotic behavior at the singular set, we prove that the Hodge–Laplacian on differential forms is essentially self-adjoint, with discrete spectrum satisfying Weyl asymptotics. We go on to prove bounds on the growth of $L^2$-harmonic forms at the singular set and to prove a Hodge theorem, namely that the space of $L^2$-harmonic forms is naturally isomorphic to the middle-perversity intersection cohomology. Moreover, we develop an asymptotic expansion for the heat trace near $t = 0$.

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## Contents

## 1. Introduction

On a compact manifold $M$ with boundary $\partial M$ which is the total space of a fiber bundle

$$Z \hookrightarrow \partial M \xrightarrow{\pi} Y, \tag{1.1}$$

with $Z, Y$ closed manifolds, an *incomplete cusp edge metric* $g_{\text{ice}}$ is, roughly speaking, a smooth Riemannian metric on the interior of $M$ which near the boundary takes the form

$$g_{\text{ice}} = dx^2 + x^{2k} g_Z + \pi^* g_Y + \widetilde{g}, \qquad k > 1, \tag{1.2}$$

where $g_Y$ is a metric on the base $Y$, $g_Z$ is positive definite restricted to the fibers, $x$ is the distance to the boundary (to first order), and $\widetilde{g}$ is a higher order term. Thus near the boundary $(M, g_{\text{ice}})$ is a bundle of geometric horns over a smooth Riemannian manifold $Y$. When $k = 3$, such metrics model the singular behavior of the Weil–Petersson metric on the moduli space of Riemann surfaces, as we discuss below.

In this paper, we study the Hodge–Laplacian

$$\Delta := \Delta^{g_{\text{ice}}} = d\delta + \delta d \tag{1.3}$$

acting on differential forms. Our first result shows that under conditions which contain the main examples of interest, one need not impose "ideal boundary conditions" at $\partial M$ in order to obtain a self-adjoint operator.

**Theorem 1.1.** *Let $(M, g_{\text{ice}})$ be an incomplete cusp edge manifold that is "Witt," meaning that either* $\dim Z = f$ *is odd or*

$$H^{f/2}(Z) = \{0\}. \tag{1.4}$$

*Assume furthermore that $g = g_{\text{ice}}$ satisfies (2.7)–(2.8) below and that the parameter $k$ in (4.1) satisfies*

$$k \geq 3. \tag{1.5}$$

*Then the Hodge–Laplacian $\Delta^{g_{\text{ice}}}$ acting on differential forms is essentially self-adjoint and has discrete spectrum.*

Thus, by the spectral theorem [36], there exists an orthonormal basis of $L^2(\Omega^p(M))$ of eigenforms $\Delta^{g_{\text{ice}}} \alpha_{j,p} = \lambda_{j,p}^2 \alpha_{j,p}$. We also prove that the distribution of eigenvalues satisfies "Weyl asymptotics," concretely, for fixed degree $p$

$$\#\{j \mid \lambda_{j,p}^2 < \lambda^2\} = c_n \operatorname{Vol}(M, g_{\text{ice}}) \lambda^n + o(\lambda^n) \text{ as } \lambda \to \infty. \tag{1.6}$$

See §4.2 for the proofs of Theorem 1.1 and of the asymptotic formula in (1.6).

Having established these fundamental properties of the Hodge–Laplacian on such spaces, we turn to the next natural topic: Hodge Theory. Here the object of study is "Hodge cohomology," or the space of $L^2$ harmonic forms,

$$\mathcal{H}^p_{L^2}(M, g_{\text{ice}}) = \{\alpha \in L^2(\Omega^p(M), g_{\text{ice}}) \mid d\alpha = 0 = \delta\alpha\}, \tag{1.7}$$

and one phrasing of the Hodge theory problem is to find a parametrization for $\mathcal{H}^*_{L^2}(M, g_{\mathrm{ice}})$ in terms of a topological invariant. As described in [18], in analogous settings the relevant topological space for Hodge theoretic statements is not the manifold $M$, but the stratified space $X$ obtained by collapsing the fibration at the boundary over the base,

$$X := M/\{p \sim q \mid p, q \in \partial M \text{ and } \pi(p) = \pi(q)\}. \tag{1.8}$$

In §4.3 we will prove the following.

**Theorem 1.2.** *For a cusp edge space* $(M, g_{\mathrm{ice}})$ *whose link* $Z$ *satisfies the Witt condition* (1.4)*, there is a natural isomorphism*

$$\mathcal{H}^*_{L^2}(M, g_{\mathrm{ice}}) \simeq IH_{\overline{\mathfrak{m}}}(X), \tag{1.9}$$

*where* $IH_{\overline{\mathfrak{m}}}$ *is the middle perversity intersection cohomology of* $X$*. Furthermore, differential forms* $\gamma \in \mathcal{H}^*_{L^2}(M, g_{\mathrm{ice}})$ *admit asymptotic expansions at the boundary of* $M$*.*

*Moreover, if* $Z \simeq \mathbb{S}^f$*, the sphere of dimension* $f$*, then* $X$ *is homeomorphic to a differentiable manifold and the isomorphism* (1.9) *becomes*

$$\mathcal{H}^*_{L^2}(M, g_{\mathrm{ice}}) \simeq H^*_{dR}(X), \tag{1.10}$$

*where the latter is the de Rham cohomology of* $X$*.*

We recall the relevant facts about intersection cohomology, originally defined by Goresky and MacPherson in [13, 14], in §4.3 below. The equivalence in (1.9) will follow using the arguments from Hunsicker and Rochon’s recent work [20] on iterated fibered cusp edge metrics (which are *complete*, non-compact Riemannian manifolds). To elaborate on the asymptotic expansion for $L^2$-harmonic forms $\gamma$, we will show in Lemma 4.5 below that in fact

$$\mathcal{H}^*_{L^2}(M, g_{\mathrm{ice}}) = \{\alpha \in L^2(\Omega^p(M), g_{\mathrm{ice}}) \mid \Delta^{g_{\mathrm{ice}}}\alpha = 0\},$$

(that the former is included into the latter is obvious), and we show that elements in the $L^2$ kernel of $\Delta^{g_{\mathrm{ice}}}$ have expansions at $\partial M$ analogous to Taylor expansions but with non-integer powers, a statement which can be be interpreted as a sort of elliptic regularity at the boundary of $M$.

One application of these results, and to putative further work we describe below, is to the analysis on the Riemann moduli spaces $\mathcal{M}_{\gamma,\ell}$ of Riemann surfaces of genus $\gamma \geq 0$ with $\ell \geq 0$ marked points. These spaces carry a natural $L^2$ metric, the Weil–Petersson metric $g_{WP}$, which near the interior of a divisor is an incomplete cusp edge metric with $k = 3$. In general divisors may intersect with normal crossings, but in at least two cases only one divisor is present.

**Theorem 1.3.** *Let $\mathcal{M}_{1,1}$ (also known as the moduli space of elliptic curves) and $\mathcal{M}_{0,4}$ be the spaces of, respectively, once punctured Riemann surfaces of genus* 1 *and* 4 *times punctured Riemann surfaces of genus zero, modulo conformal diffeomorphism. Then the Hodge–Laplacian $\Delta^{g_{WP}}$ on differential forms is essentially self-adjoint on $L^2$ with core domain $C^\infty_{c,\mathrm{orb}}$ (see Theorem 3.7) with discrete spectrum and Weyl asymptotics, and if $\overline{\mathcal{M}}_{1,1}$ and $\overline{\mathcal{M}}_{0,4}$ denote the Deligne–Mumford compactifications (see e.g. [17, 38]). Then the de Rham cohomology spaces $H_{dR}(\overline{\mathcal{M}}_{1,1})$ are naturally isomorphic to $\mathcal{H}^*_{L^2}(\overline{\mathcal{M}}_{1,1}, g_{WP})$, and the same holds for $\overline{\mathcal{M}}_{0,4}$.*

We discuss the proof at the end of §4.3, though this is really a direct application of our results together with the recent work on the structure of the Weil–Petersson metric near a divisor in [27] and [32].

This article is partly motivated by Ji, Mazzeo, Müller, and Vasy's work [21] on the spectral theory of the (scalar) Laplace–Beltrami operator on the Riemann moduli spaces $\mathcal{M}_g$, for which it was shown by methods different from ours that it is essentially self-adjoint and its eigenvalues satisfy a Weyl asymptotic formula. Here they analyze incomplete cusp edge spaces with normal crossings, and find in particular that the value $k = 3$ in (1.5) is critical; indeed *for values $k < 3$ one does not expect self-adjointness*. It would be interesting (though more complex) to find a parametrization of the space of closed extensions of incomplete cusp edge Laplacians with $k < 3$, which is expected to be infinite dimensional, e.g. by [3].

In contrast with [21], since our eventual goal is Hodge and index theory on moduli space, our main technical contribution is the construction and detailed description of the heat kernel $H = \exp(-t\Delta^{g_{\mathrm{ice}}})$. Indeed, our approach to establishing Theorem 1.1 (which justifies the use of the word "the" in the previous sentence) and Theorem 1.2, is to develop in Theorem 3.7 below a precise understanding of the behavior of a fundamental solution to the heat equation, which we only conclude is *the* heat kernel after using it to prove Theorem 1.1; we establish asymptotic expressions for it at the singular set, uniformly down to time $t = 0$, obtaining in particular in Corollary 4.4, an asymptotic formula for its trace (which has potential applications to index theory, since our method for analyzing $\Delta^{g_{\mathrm{ice}}}$ may be used for other natural elliptic differential operators on these spaces as well) and fine mapping properties of $\Delta^{g_{\mathrm{ice}}}$ which allow us to analyze its kernel, i.e. harmonic forms. This is all described in detail in §4.

Essential self-adjointness of a differential operator $P$ is typically a statement about the decay of $L^2$ sections $u$ for which $Pu \in L^2$. (Here the derivative is taken in the distributional sense.) The set of such sections is denoted

$$\mathcal{D}_{\max} := \mathcal{D}_{\max}(\Delta^{g_{\mathrm{ice}}}) = \{u \in L^2 \mid Pu \in L^2\}. \tag{1.11}$$

This is the largest subset of $L^2$ which is a closed subspace in the graph norm $\|u\|_\Gamma = \|u\|_{L^2} + \|Pu\|_{L^2}$. On the other hand, the smallest such closed extension from the domain $C^\infty_c(M)$ is the closure, i.e. the minimal domain

$$\mathcal{D}_{\min} := \mathcal{D}_{\min}(\Delta^{g_{\mathrm{ice}}}) = \{u \in L^2 \mid \exists u_k \in C^\infty_c(M) \text{ with } \lim_{k\to\infty} \|u_k - u\|_\Gamma = 0\}. \tag{1.12}$$

The essential self-adjointness statement in Theorem 1.1 says that the smallest closed extension is equal to the largest, i.e. that

$$\mathcal{D}_{\max} = \mathcal{D}_{\min}, \tag{1.13}$$

and therefore there is exactly one closed extension. On the other hand, $\mathcal{D}_{\max}$ is dual to $\mathcal{D}_{\min}$ with respect to $L^2$ and thus if (1.13) holds then $P$ with core domain $C_c^\infty(M)$ has exactly one closed extension, which we denote by $\mathcal{D} = \mathcal{D}_{\min} = \mathcal{D}_{\max}$ and $(P, \mathcal{D})$ is a self-adjoint, unbounded operator on $L^2$. Equation (1.13) is a statement about decay in the sense that to prove it we will show that a differential form $\alpha \in \mathcal{D}_{\max}$ decays fast enough near $\partial M$ that it can be approximated in the graph norm by compactly supported smooth forms. This we do using the heat kernel.

Recall that the heat kernel $H$ is a section of the form bundle $\Pi\colon \mathrm{End}(\Lambda) \longrightarrow M^\circ \times M^\circ \times [0,\infty)$, where $M^\circ$ is the interior of $M$ and $\mathrm{End}(\Lambda)$ is the vector bundle whose fiber over $(p,q,t)$ is $\mathrm{End}(\Lambda_q^*(M); \Lambda_p^*(M))$, smooth on the interior $M^\circ \times M^\circ \times [0,\infty)_t$, which solves

$$(\partial_t + \Delta^{g_{\mathrm{ice}}})H = 0 \quad \text{and} \quad H_t \longrightarrow \mathrm{Id}, \text{strongly as } t \downarrow 0. \tag{1.14}$$

For a compactly supported smooth differential form $\alpha$, the differential form

$$\beta(\omega, t) := \int_M H(w, \widetilde{w}, t)\alpha(\widetilde{w})\, d\mathrm{Vol}_{g_{\mathrm{ice}}}(\widetilde{w})$$

solves the heat equation $(\partial_t + \Delta^{g_{\mathrm{ice}}})\beta = 0$ with initial data $\beta|_{t=0} = \alpha$. One consequence of our precise description of $H$ in Theorem 3.7 below will be the following.

**Theorem 1.4.** *On a Witt incomplete cusp edge space $(M, g_{\mathrm{ice}})$ with metric satisfying the assumptions in (2.7)–(2.8) below together with (1.5), there exists a fundamental solution to the heat equation $H_t = H(w, \widetilde{w}, t)$ in the sense of (1.14) such that for $t > 0$*

$$H_t\colon L^2(M; \Omega^*(M)) \longrightarrow \mathcal{D}_{\min}, \tag{1.15}$$

*and such that $H_t$ and $\partial_t H_t$ are bounded, self-adjoint operators on $L^2$.*

Theorem 1.4 implies the essential self-adjointness statement; indeed the fundamental solution $H_t$ directly gives a sequence (indeed a path) of sections on $\mathcal{D}_{\min}$ which approaches a given form in $\mathcal{D}_{\max}$. Namely,

$$\alpha \in \mathcal{D}_{\max} \implies H_t\alpha \to \alpha \text{ in } \mathcal{D}_{\min} \text{ as } t \downarrow 0. \tag{1.16}$$

As we see now, the proof of this is straightforward functional analysis given the conclusions of Theorem 1.4.

*Proof of essential self-adjointness using Theorem 1.4.* The proof has nothing to do with the fine structure of incomplete cusp edge spaces, it depends only on the soft properties of the fundamental solution $H$ in Theorem 1.4. To emphasize this, let $(M, g)$ be any Riemannian manifold and $P$ a differential operator of order 2 acting on sections of a vector bundle $E$ with hermitian metric $G$, such that $P$ is symmetric on $L^2(M; E)$. For $t > 0$, let $H_t$ be a smooth section of $\mathrm{End}(E) \to M \times M$, which depends smoothly on $t$ and satisfies

$$(\partial_t + P)H_t = 0, \quad \lim_{t\to 0} H_t = \mathrm{Id}, \quad \text{and} \quad H_t\colon L^2(M; E) \longrightarrow \mathcal{D}_{\min}, \tag{1.17}$$

where the above limit holds in the strong topology on $L^2$, and furthermore such that $H_t$ and $\partial_t H_t$ are self-adjoint on $L^2$.

Let $u \in \mathcal{D}_{\max}(P)$, i.e. $u \in L^2$, $Pu \in L^2$. We will show that $u \in \mathcal{D}_{\min}(P)$ as well, and thus $\mathcal{D}_{\min} = \mathcal{D}_{\max}$. Indeed, we will show that

$$H_t u \to u \text{ in } \mathcal{D}_{\max}, \quad \text{i.e. that } H_t u \to u \text{ and } PH_t u \to Pu \text{ in } L^2. \tag{1.18}$$

This suffices to prove that $u \in \mathcal{D}_{\min}$ since $H_t u \in \mathcal{D}_{\min}$ by assumption and $\mathcal{D}_{\min}$ is a closed subspace of $\mathcal{D}_{\max}$ in the graph norm. To prove (1.18), we note first that $H_t u \to u$ in $L^2$ trivially since $H_t \to \mathrm{Id}$ in the strong topology on $L^2$. Also note that since $u \in \mathcal{D}_{\max}$, $Pu \in L^2$, so $H_t Pu \to Pu$ in $L^2$ also. Of course, this is not what we want; we want $PH_t u \to Pu$, but in fact we claim that

$$u \in \mathcal{D}_{\max} \implies PH_t u = H_t Pu, \tag{1.19}$$

which will establish (1.13).

It remains to prove (1.19). Note that for $u \in \mathcal{D}_{\max}$ and $v \in L^2$, then $\langle H_t Pu, v\rangle_{L^2} = \langle Pu, H_t v\rangle_{L^2}$ by self-adjointness of $H_t$ on $L^2$, while $\langle Pu, H_t v\rangle_{L^2} = \langle u, PH_t v\rangle_{L^2}$. Indeed, the adjoint domain of $\mathcal{D}_{\min}$ is $\mathcal{D}_{\max}$, so for any $f \in \mathcal{D}_{\min}$, $g \in \mathcal{D}_{\max}$, $\langle Pf, g\rangle_{L^2} = \langle f, Pg\rangle_{L^2}$. But, then since $PH_t = -\partial_t H_t$ we see that

$$\langle H_t Pu, v\rangle_{L^2} = -\langle u, \partial_t H_t v\rangle_{L^2}.$$

But $\partial_t H_t$ is self-adjoint on $L^2$ so

$$\langle u, \partial_t H_t v\rangle_{L^2} = \langle \partial_t H_t u, v\rangle_{L^2} = -\langle PH_t u, v\rangle_{L^2},$$

and thus $\langle H_t Pu, v\rangle_{L^2} = \langle PH_t u, v\rangle_{L^2}$ for all $u \in \mathcal{D}_{\max}$, $v \in L^2$, i.e. (1.19) holds. □

The central vehicle for the construction of the heat kernel is the construction of a manifold with corners $M^2_{\mathrm{heat}}$ via iterated radial blowup of the natural domain of the heat kernel, namely the space $M \times M \times [0, \infty)_t$; thus the interiors of these two spaces are diffeomorphic, and the blowup process furnishes a "blowdown" map

$$\beta\colon M^2_{\mathrm{heat}} \longrightarrow M \times M \times [0, \infty)_t, \tag{1.20}$$

which encodes deeper information about the relationship between the various *boundary hypersurfaces* (codimension one boundary faces) of $M^2_{\text{heat}}$ and those of $M \times M \times [0,\infty)_t$. The upshot is that the heat kernel $H$, which lives a priori on the latter space, pulls back via $\beta$ to be "nice" (precisely to be polyhomogeneous, see Appendix A) on $M^2_{\text{heat}}$. In fact, in §3 we will construct a parametrix $K$ for the heat equation directly on $M^2_{\text{heat}}$. To obtain the actual heat kernel $H$ we use a Neumann series argument to iterate away the error.

The latter process builds on what is now a substantial body of work on analysis (in particular the structure of heat kernels) on singular and non-compact Riemannian spaces, going back at least to the work of Cheeger on manifolds with conical singularities [8–10]. Our approach here is more closely related to Melrose's geometric microlocal analysis on asymptotically cylindrical manifolds [30] (a non-compact example) and Mooers' paper [33] on manifolds with conical singularities (an incomplete, singular example). The general procedure, which one sees in both the parabolic and elliptic settings, is to express the relevant differential operator as an element in the universal enveloping algebra of a Lie algebra of vector fields, and to "resolve" this Lie algebra via radial blowup of the underlying space.

It is useful to compare our work with Mazzeo–Vertman [28], in which the authors study analytic torsion on incomplete edge spaces, which are the $k = 1$ case of incomplete cusp edges, as their work also involves a heat kernel construction using blowup analysis, which is slightly simpler in their context as the resolved double space has one less blowup (and thus the triple space is simpler). Still, the basic outline of the proof is analogous in both cases; a parametrix for the heat kernel is constructed and this parametrix is modified by a Neumann series argument to construct a fundamental solution to the heat equation.

One phenomenon revealed by our results is that the space of self-adjoint extensions of the Hodge–Laplacian can be much smaller for incomplete cusp edge spaces than it is for related incomplete edge spaces. For example, a Witt space (this is a topological condition and has nothing to do with the value of $k$) that is incomplete edge may have infinitely many self-adjoint extensions if the family of induced operators on the fibers have small non-zero eigenvalues [3]. One expects that the zero mode in the fiber (the space of fiber harmonic forms) makes a similar contribution in both the cusp and cone cases, in particular that an incomplete cusp edge space which is not Witt will have an infinite dimensional space of closed extensions on which "Cheeger ideal boundary conditions" must be imposed to make the operator self-adjoint, as is the case in [3].

A second closely related work is Grieser–Hunsicker [16], which uses also quasihomogeneous radial blowups, in this case to construct a Green's function for elliptic operators on a certain class of complete Riemannian manifolds (called "$\phi$-manifolds") which require similar analysis. There are many other related works in a similar vein including, just to name a few, Albin–Rochon [4], Brüning–Seeley [7], Gil–Krainer–Mendoza [12], Lesch [23], Schultze [35], and Grieser's notes on parametrix constructions for heat kernels [15]. For analysis of moduli space, to give just a sample recent work, we refer the reader to the papers of Liu–Sun–Yao, for example [24, 25].

## 2. Incomplete cusp edge differential geometry

We begin be recalling the differential topology of the underlying singular space $X$, which we take to be a smoothly stratified space in the sense of [2, Sec. 2.1, Def. 1] with only a single singular stratum $Y$. This means in particular, as described in *loc. cit.*, that $X \setminus Y$ is dense in $X$, that there is a tubular neighborhood $Y \subset T$ and a retraction $\pi_Y : T \longrightarrow Y$ which is a locally trivial fibration with fibre the cone $C(Z) := [0,1) \times Z/\{0\} \times Z$ with $Z$ a closed manifold, and that we are given a "radial function" $\rho : T \longrightarrow [0,\infty)$ which is proper and such that $\rho^{-1}(0) = Y$. Moreover, $Y$ is given a fixed atlas of charts $\mathcal{U}_Y = \{(\phi, \mathcal{U})\}$ where $\phi$ is a trivialization $\pi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U} \times C(Z)$, the transition functions of which preserve the rays of each conic fibre as well as the radial variable $\rho$. As explained in [2, Sect. 2], there is a resolution $\varphi : M \longrightarrow X$, obtained essentially by opening up the tips of the cone fibers, such that $\varphi^{-1}(Y) = \partial M$ and such that the radial function $\rho$ lifts to a smooth *boundary defining function* of $M$ which we call, henceforth, $x$. The boundary $\partial M$ then becomes the total space of a smooth fibration with base $Y$ and typical fibre $Z$. A choice of boundary defining function $x$, meaning a function $x \in C^\infty(M)$ with $\{x = 0\} = \partial M$ and $dx$ non-vanishing on $\partial M$, fixes (after possibly scaling $x$ by a constant) a tubular neighborhood of $\partial M$

$$\mathcal{U} \simeq \partial M \times [0,1)_x, \tag{2.1}$$

and $\mathcal{U}$ forms a locally trivial fibration over $\phi(\partial M) = Y$ with typical fiber $C_1(Z)$. A local trivialization near a point $p \in Y$ then takes the form

$$V \times C(Z), \tag{2.2}$$

with $V$ a neighborhood of $p$ in $Y$, for local coordinates $y$ on the base and $z$ on $Z$, then

$$(x, y, z) \text{ form a coordinate chart on } M \text{ in a neighborhood of } \phi^{-1}(p). \tag{2.3}$$

Let

$$f := \dim Z, \quad b := \dim Y. \tag{2.4}$$

We will consider differential forms and vector fields which are of approximately unit size with respect to Riemannian metrics of the type in (4.1). These are the incomplete cusp edge forms, which are sections of the incomplete cusp edge form bundle, ${}^{\mathrm{ice}}\Lambda^*(M)$, whose smooth sections are generated locally over the base by the forms

$$dx, \quad dy_i \quad (i = 1, \dots, b = \dim Y), \quad x^k dz_\alpha \quad (\alpha = 1, \dots, f = \dim Z). \tag{2.5}$$

Correspondingly, we will use the space of vector fields which are locally $C^\infty(M)$ linear combinations of the vector fields

$$\partial_x, \quad \partial_{y_i}, \quad x^{-k}\partial_{z_\alpha}. \tag{2.6}$$

These vector fields are local sections of a bundle ${}^{\mathrm{ice}}TM$ which is dual to ${}^{\mathrm{ice}}T^*M = {}^{\mathrm{ice}}\Lambda^1(M)$. We denote sections of ${}^{\mathrm{ice}}TM$ by $\mathcal{V}_{\mathrm{ice}}$.

We consider metrics $g$ on $M$ which are positive-definite sections of $\mathrm{Sym}^{0,2}({}^{\mathrm{ice}}T^*M)$. This means that they are smooth linear combinations of the symmetric products of $dx, dy_i$ and $x^k dz_\alpha$ which are positive-definite up to and including over the boundary $x = 0$. We will assume slightly more structure at $x = 0$ than merely assuming $g$ is positive definite; to discuss this structure we first build some examples. Specifically, we consider those metrics arising from submersion metrics on $\partial M$. Concretely, consider a metric $\pi^*h + \mathbf{k}$, where $h$ is a Riemannian metric on $Y$ and $\mathbf{k} \in \mathrm{Sym}^{0,2}(\partial M)$, has the property that its restriction to any fiber is positive definite. Then the metrics $\pi^*h + x^{2k}\mathbf{k}$ form a family of metrics on $\partial M$ and thus we obtain a metric $g_0 = dx^2 + \pi^*h + x^{2k}\mathbf{k}$ on $\mathcal{U}$. The metric $g_0$ is an *exact incomplete cusp edge metric*. Note that in coordinates $(x, y, z)$ such a metric takes the form

$$g_0 = \begin{pmatrix} dx & dy^i & x^k dz^\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (h_{ij}) & x^k(\mathbf{k}_{i\alpha}) \\ 0 & x^k(\mathbf{k}_{\alpha i}) & \mathbf{k}_{\alpha\beta} \end{pmatrix} \begin{pmatrix} dx \\ dy^j \\ x^k dz^\beta \end{pmatrix}. \tag{2.7}$$

In general we consider a metric $g$ of the form

$$g - g_0 = O(x^k, g_0), \tag{2.8}$$

where $g_0$ is an exact incomplete cusp edge metric and $O(x^k, g_0)$ refers to a $O(x^k)$ norm bound with respect to the exact incomplete cusp edge metric $g_0$ as in (2.7), and furthermore we assume that the $O(x^k, g_0)$ term is *polyhomogeneous conormal*, a regularity assumption defined precisely in Appendix A, which roughly speaking means that the coefficients have an asymptotic expansion at $x = 0$ analogous to a Taylor expansion but with non-integer powers and with precise derivative bounds on the error terms. Metrics satisfying these assumptions are what we refer to henceforth as *incomplete cusp edge metrics*. (Note that the assumptions on $g$ are stronger than merely assuming that $g \in \mathrm{Sym}^{0,2}({}^{\mathrm{ice}}T^*M)$, as the latter space contains e.g. $x(x^k dz \otimes_{\mathrm{sym}} dx)$, which does not obey the error bound.)

**Remark 2.1.** As is shown in [27] (see the introduction for further discussion) with previous results for example in [38, 39], the Weil–Petersson metric on moduli space takes the form (2.8) near the interior of a divisor and satisfies the polyhomogeneity assumption.

To understand the form the Hodge–de Rham operator takes on $\mathcal{U}$, we use the decomposition for the exterior derivative from [5, Prop. 10.1], elaborated in [6, Prop. 3.4] to show that there is a flat connection on the bundle of fiber harmonic forms. Note that the choice of a submersion metric $\pi^*h + \mathbf{k}$ on $\partial M$ induces a connection on the bundle $T\partial M$, i.e. a choice of horizontal space $T_H\partial M$ on which the map $\pi_*$ restricts to an isomorphicm of the fibres to $TY$. (Indeed this is just the space perpendicular to the vertical tangent bundle $T(\partial M/Y)$.) Correspondingly there is a decomposition of the form bundle

$$\Lambda^d(\partial M) = \sum_{p+q=d} \Lambda^{p,q}(\partial M),$$

where $\Lambda^{p,q}(\partial M) = \Lambda^p T*_H M \otimes \Lambda^q T^*(\partial M/Y)$, and where $T*_H M = \pi^* T^* Y$ and $T^*(\partial M/Y)$ is its orthocomplement. Thus differential forms on $\partial M$ can be written as linear combinations

$$\pi^*\alpha \wedge \beta, \quad \alpha \in \Omega^p(Y),\ \beta \in \Omega^q T^*(\partial M/Y), \tag{2.9}$$

and, for $y \in Y$, identifying $\Lambda T^*(\partial M/Y)$ over $\pi^{-1}(y)$ with $\Lambda(\pi^{-1}(Y))$ via the inclusion $\iota\colon \pi^{-1}(y) \longrightarrow \partial M$, we can define a fiber exterior derivative

$$d_{\partial M/Y}(\pi^*\alpha \wedge \beta) = \pi^*\alpha \wedge d_{\partial M/Y}\beta \tag{2.10}$$

(where on the right-hand side $d_{\partial M/Y}$ is the differential on the fibre).

There is a decomposition of the exterior derivative, which we denote using the convenient notation from [3, Sect. 1]

$$d_{\partial M} = d^{0,1}_{\partial M} + d^{1,0}_{\partial M} + d^{2,-1}_{\partial M},$$

where $d^{0,1}_{\partial M} = d_{\partial M/Y}$ while $d^{1,0}_{\partial M}$ is the operator (denoted by $\delta_Y$ in [5, Prop. 10.1]) defined using a connection ${}^{\partial M/Y}\nabla$ on the vertical tangent space $T(\partial M/Y)$ — in particular we can fix a submersion metric $g^{\partial M}$ and define our vertical connection using its vertical projections and Levi-Civita connection. Here $d^{2,-1}_{\partial M} = R$ is defined in terms of the curvature of the fibration. Their crucial properties in this context are that $d^{1,0}_{\partial M} d^{0,1}_{\partial M} = -d^{0,1}_{\partial M} d^{1,0}_{\partial M}$ and that (having chosen a connection on the fibration) they behave nicely with respect to the decomposition of differential forms

$$\Omega^r(\partial M) = \bigoplus_{p+q=r} \Omega^{p,q}(\partial M),$$

where $\Omega^{p,q}$ is the $C^\infty(\partial M)$ linear span of homogeneous forms $\alpha \otimes \beta$ where $\alpha$ is a horizontal form of degree $p$ and $\beta$ is a vertical form of degree $q$; specifically

$$d^{j,k}_{\partial M}\colon \Omega^{p,q}(\partial M) \longrightarrow \Omega^{p+j,q+k}(\partial M).$$

We now discuss vertical harmonic forms. Let $(d_{\partial M/Y})^*$ denote the adjoint of $d_{\partial M/Y}$ with respect to our fixed submersion metric, and write

$$\eth_{\partial M/Y} = d_{\partial M/Y} + (d_{\partial M/Y})^*.$$

Over the base $Y$ we have the bundle of vertical harmonic forms $\mathcal{H}^*(\partial M/Y) \longrightarrow Y$ whose fibers are $\ker \eth_{\partial M/Y}$. A fiber harmonic form can be thought of as a linear combination of forms as in (2.9) where $\beta$ satisfies $\eth_{\partial M/Y}\beta \equiv 0$, in particular the smooth sections of $\mathcal{H}^*(\partial M/Y)$ are naturally isomorphic to a subspace of the sections of $\Omega^*(\partial M)$, and we denote by $\Pi_0$ the $L^2$-orthogonal projection onto the closure of the subspace generated by these forms. Thus, incidentally, $\mathcal{H}^*(\partial M/Y)$ inherits a flat connection from the operator $\Pi_0 d^{1,0}_{\partial M}\Pi_0$.

Shifting the focus back to our collar neighborhood $\mathcal{U}$ of $\partial M$, we can, by thinking of the $[0,1)_x$ factor in $[0,1)_x \times \partial M$ as lying in the base of the induced fibration with typical fibre $Z$ (and base $[0,1)_x \times Y$), repeat the above argument and obtain a bundle of vertical harmonic forms over $[0,1)_x \times Y$, sections of which, again, my be thought of as linear combinations of forms as in (2.9), but now with $\alpha \in \Omega^*([0,1)_x \times Y)$. For us it is most convenient to work with fiber harmonic forms living over our tubular neighborhood $\mathcal{U}$ *which are also of bounded length* with respect to our ice-metric $g$. Thus we take $\mathcal{H}$ to be the direct sum of the spaces

$$\mathcal{H} = \bigoplus_{q=0}^{f} \mathcal{H}^q, \quad \text{where } \mathcal{H}^q := \pi^*\Omega([0,1)_x \times Y) \wedge x^{kq}\mathcal{H}^q_{\partial M/Y}. \tag{2.11}$$

Denote the projection onto the space of fiber harmonic forms by

$$\Pi_{\mathcal{H}} \colon x^{s_0}L^2({}^{\text{ice}}\Lambda^*) \longrightarrow x^{s_0}\mathcal{H}, \tag{2.12}$$

where $\Pi_{\mathcal{H}}$ is the $L^2$-orthogonal projection onto the closure of the subspace of $\Omega^*_{\text{ice}}(\mathcal{U}) := C^\infty(\mathcal{U}; {}^{\text{ice}}\Lambda^*)$ given by viewing sections of $\mathcal{H}$ as lying over $\mathcal{U}$. Then a form $\mu \in \Omega^*_{\text{ice}}(\mathcal{U})$ can be written locally as a linear combination of forms

$$\mu = a\,dx \wedge \pi^*\alpha \otimes x^{kp}\beta + b \wedge \pi^*\alpha' \otimes x^{kp}\beta',$$
$$a, b \in C^\infty([0,1)_x \times Y),\ \alpha, \alpha' \in \Omega^*(Y),\ \beta, \beta' \in \Lambda^p(\partial M/Y),$$

and

$$\Pi_{\mathcal{H}}\mu = a\,dx \wedge \pi^*\alpha \otimes x^{kp}\Pi_0\beta + b \wedge \pi^*\alpha' \otimes x^{kp}\Pi_0\beta',$$

with $\Pi_0$ as above. Since

$$\Pi_0\beta = 0 \iff \exists\gamma,\ \Delta_{\partial M/Y}\gamma = \beta,$$

solving term by term for a form $\mu$ expanded in $x$ near $\partial M$ shows that for $\mu \in x^{s_0}\Omega^*_{\text{ice}}(\mathcal{U})$, $p \in \mathbb{N}$,

$$\Pi_{\mathcal{H}}\mu = O_{{}^{\text{ice}}\Lambda^*}(x^{s_0+p}) \iff \exists\gamma \in x^{s_0+p}\Omega^*_{\text{ice}}(\mathcal{U}),\ \text{such that } \Delta_{\partial M/Y}\gamma = \mu. \tag{2.13}$$

We now compute the Hodge–de Rham operator for an exact ice-metric $g_0$. There are decompositions of $d_M$ and the dual $\delta_M$ on $\mathcal{U}$ corresponding to that of $d_{\partial M}$, obtained by the orthogonal decomposing of the ice-tangent bundle

$$ {}^{\mathrm{ice}}T = T_H \partial M \times x^{-k} T(\partial M/Y) \times T[0,1]_x. $$

Writing differential forms $\alpha = \iota(dx)dx \wedge \alpha + dx \wedge \iota(dx)\alpha$, we then have

$$ d|_{\mathcal{U}} = \begin{pmatrix} x^{-k} d_{\partial M/Z} + \delta_Y + x^k R & 0 \\ \partial_x + kx^{-1}\mathbf{N} & -(x^{-k} d_{\partial M/Z} + \delta_Y + x^k R) \end{pmatrix}, \tag{2.14} $$

and taking adjoints with respect to $g_0$ and writing $\delta_{\partial M/Z} := (d_{\partial M/Z})^*$,

$$ \delta|_{\mathcal{U}} = \begin{pmatrix} x^{-k}\delta_{\partial M/Y} + (d^{1,0}_{\partial M})^* + x^k R^* & -\partial_x - kx^{-1}(f - \mathbf{N}) \\ 0 & -(x^{-k}\delta_{\partial M/Y} + (d^{1,0}_{\partial M})^* + x^k R^*) \end{pmatrix}. \tag{2.15} $$

To state the main result we will need regarding the structure of the Hodge–de Rham operator, we first point out that the operators $d$ and $\delta$ are both elements in the algebra of differential operators $\mathrm{Diff}^*_{\mathrm{ice}}(M; {}^{\mathrm{ice}}\Lambda^* M)$ generated by the ice-vector fields $\mathcal{V}_{\mathrm{ice}}$ and the smooth (or more generally polyhomogeneous) endomorphisms of ${}^{\mathrm{ice}}\Lambda^* M$. In fact, for any $X \in \mathcal{V}_{\mathrm{ice}}$, the operator $\nabla_X \in \mathrm{Diff}^1_{\mathrm{ice}}(M; {}^{\mathrm{ice}}\Lambda^* M)$; indeed, one can check that $\nabla_X \in \mathrm{Diff}^1_{\mathrm{ice}}(M; {}^{\mathrm{ice}}TM)$ using the the Koszul formula, from which the claim follows.

**Proposition 2.2.** *Let $g$ be an incomplete cusp edge metric as above, in particular satisfying (2.8) for some exact ice-metric $g_0$. The Hodge–de Rham operator $\eth = d + \delta$ decomposes as*

$$ \eth = \eth_0 + P + E, \quad P \in x^k \mathrm{Diff}^1_{\mathrm{ice}}, \quad E \in x^{k-1}\,\mathrm{End}({}^{\mathrm{ice}}\Lambda^* M), \tag{2.16} $$

*where $\eth_0 = d + \delta_{g_0}$ is the Hodge–de Rham operator for $g_0$, so*

$$ \eth_0 = \begin{pmatrix} x^{-k}\eth_{\partial M/Y} + \eth^{1,0}_{\partial M} + x^k S & -\partial_x - kx^{-1}(f - \mathbf{N}) \\ \partial_x + kx^{-1}\mathbf{N} & -(x^{-k}\eth_{\partial M/Y} + \eth^{1,0}_{\partial M} + x^k S) \end{pmatrix}, $$

*where $\eth^{1,0}_{\partial M} = d^{1,0}_{\partial M} + (d^{1,0}_{\partial M})^*$ and $S = R + R^*$. Here $\eth_{\partial M/Y}$ depends on the base $Y$ parametrically, and acting on vertical differential forms is equal to the Hodge–de Rham operator for the Riemannian manifold $\boldsymbol{k}|_y$.*

We remark further on the space $x^k \mathrm{Diff}_{\mathrm{ice}}(M; {}^{\mathrm{ice}}\Lambda^* M)$ of operators among which the error $P$ in the proposition lies. Such operators are in particular b-differential operators on ice-forms with polyhomogeneous coefficients

$$ P \in \mathrm{Diff}^1_{\mathrm{b,phg}}(M; {}^{\mathrm{ice}}\Lambda^* M). \tag{2.17} $$

This is the space of differential operators generated by $\mathcal{V}_{\mathrm{b,phg}}$, the polyhomogeneous vector fields tangent to the boundary $\partial M$. Concretely, it satisfies

$$P = ax\partial_x + b^i\partial_{y^i} + c^\alpha\partial_{z^\alpha} + d \tag{2.18}$$

for polyhomogeneous, bounded endomorphisms $a, b^i, c^\alpha, d$, and where repeated indices are summed over. This follows from $x^k\mathcal{V}_{\mathrm{ice}} \subset \mathcal{V}_b$. In general, an element $Q \in \mathrm{Diff}^m_{\mathrm{b,phg}}(M; {}^{\mathrm{ice}}\Lambda^* M)$ also satisfies

$$Q(x^k\gamma) = O(x^k) \tag{2.19}$$

for $\gamma \in C^\infty(M; {}^{\mathrm{ice}}\Lambda^* M)$, and is given locally by polyhomogeneous linear combinations of $x\partial_x, \partial_y, \partial_z$, i.e.

$$Q = \sum_{i+|\alpha|+|\beta|\le m} a_{i,\alpha,\beta}(x\partial_x)^i\partial_y^\alpha\partial_z^\beta,$$

where $a_{i,\alpha,\beta}$ is a polyhomogeneous bounded endomorphism of ${}^{\mathrm{ice}}\Lambda^* M$.

*Proof of Proposition 2.2.* We will write the Hodge-de Rham operators $d+\delta$ in terms of the Levi-Cevita connection and exterior multiplication $\epsilon$ (defined as the operator which takes a differential form $\omega$ to the endomorphism $\mu \mapsto \omega\wedge\mu$. By [5, Prop. 3.53] we can write $d+\delta = \mathrm{Tr}\, cl_g \circ {}^g\nabla$ where $cl = \epsilon - \iota$ for $\epsilon$ exterior multiplication on $\Lambda^*$, $\iota$ its dual with respect to $g$, and ${}^g\nabla$ is the Levi-Cevita connection on differential forms. We choose an orthonormal frame for the exact metric $g_0$ in the standard way, i.e. let $g_0$ be induced by a submersion metric $g^{\partial M}$ on $\partial M$ and let $\{f^\alpha\}\cup\{e^i\}$ be an orthonormal frame of $T^*(\partial M)$ where the $f^\alpha$ are horizontal and the $e^i$ vertical differential forms. Then $\{dx, f^\alpha, x^k e^i\}$ is an orthonormal basis for $g_0$ and by Gram-Schmidt there is an orthonormal basis of the form

$$\{\omega^0 = dx + O_{\Omega^1_{\mathrm{ice}}}(x^k),\ \omega^\alpha = f^\alpha + O_{\Omega^1_{\mathrm{ice}}}(x^k),\ \eta^i = x^k e^i + O_{\Omega^1_{\mathrm{ice}}}(x^k)\},$$

where $\Omega^1_{\mathrm{ice}}(x^k)$ a polyhomogeneous differential 1-form $\beta$ with $\|\beta\|_g = O(x^k)$. Correspondingly the dual vector fields satisfy $\omega_0 - \partial_x, \omega_\alpha - f_\alpha, \eta_i - e_i \in x^k\mathcal{V}_{\mathrm{ice}}$. Moreover, for $X \in \mathcal{V}_{\mathrm{ice}}$, the tensor ${}^g\nabla_X - {}^{g_0}\nabla_X$ is $O(x^{k-1})$ as an endomorphism of ${}^{\mathrm{ice}}\Lambda^* M$, while $\mathrm{cl}_g - \mathrm{cl}_{g_0} = O(x^k)$, so

$$\begin{aligned} d+\delta &= cl_g(\xi)^g\nabla_\xi + \sum_{\alpha=1}^{b} cl_g(\omega^\alpha)^g\nabla_{\omega_\alpha} + x^{-k}\sum_{i=1}^{f} cl_g(\eta^i)^g\nabla_{\eta_i} \\ &= cl_{g_0}(dx)^g\nabla_{\partial_x} + \sum_{\alpha=1}^{b} cl_{g_0}(f^\alpha)^g\nabla_{f_\alpha} + x^{-k}\sum_{i=1}^{f} cl_{g_0}(e^i)^g\nabla_{e_i} + x^k\mathrm{Diff}^1_{\mathrm{ice}} \end{aligned}$$

$$= cl_{g_0}(dx)^{g_0}\nabla_{\partial_x} + \sum_{\alpha=1}^{b} cl_{g_0}(f^\alpha)^{g_0}\nabla_{f_\alpha} + x^{-k}\sum_{i=1}^{f} cl_{g_0}(e^i)^{g_0}\nabla_{e_i} + O_{\mathrm{End}(^{\mathrm{ice}}\Lambda^*)}(x^{k-1}) + x^k \mathrm{Diff}^1_{\mathrm{ice}}, \tag{2.20}$$

which is what we wanted. □

The Hodge–Laplacian $\Delta = \eth^2 = d\delta + \delta d$ can now be decomposed along the same lines. Proposition 2.2 together with the anti-commutation of $\eth_{\partial M/Y}$ and $\eth_H$ gives:

**Proposition 2.3.** *Locally over the base, $\Delta$ can be decomposed as follows*

$$\Delta = \Delta_0 + x^{-k}\widetilde{P} + x^{-1}\widetilde{E}, \tag{2.21}$$

*where $\Delta_0 = \eth_0^2$, i.e.*

$$\Delta_0 = \mathrm{Id}_{2\times 2}\Big(-\partial_x^2 - \frac{kf}{x}\partial_x + \frac{1}{x^{2k}}\Delta_{\partial M/Y} + \Delta_H\Big) + \begin{pmatrix} k\,\mathbf{N}(1-k(f-\mathbf{N}))x^{-2} & -2kx^{-k-1}d_{\partial M/Z} \\ -2kx^{-k-1}\delta_{\partial M/Y} & k(f-\mathbf{N})(1-k\,\mathbf{N})x^{-2} \end{pmatrix}, \tag{2.22}$$

*where $\Delta_H = (\eth^{1,0}_{\partial M})^2$, $\Delta_{\partial M/Y} = \eth^2_{\partial M/Y}$, and*

$$\widetilde{P} = \eth_{\partial M/Y} P + P\eth_{\partial M/Y}$$

*with $P$ as in Proposition 2.2 and $\widetilde{E} \in \mathrm{Diff}^2_{\mathrm{b,phg}}(M)$.*

## 3. The heat kernel

In this section we construct a manifold with corners $M^2_{\mathrm{heat}}$ as in (1.20) together with a fundamental solution to the heat equation which is a polyhomogeneous conormal distribution on $M^2_{\mathrm{heat}}$ with prescribed leading order terms in its asymptotic expansions at the various faces (see Theorem 3.7). To do so, after the construction of $M^2_{\mathrm{heat}}$, we perform a parametrix construction and then use this parametrix to obtain the fundamental solution itself via a Neumann series.

**3.1. Heat double space.** The space $M^2_{\mathrm{heat}}$ is obtained by performing three consecutive quasihomogeneous radial blowups of $M \times M \times [0,\infty)_t$. Here $M^2_{\mathrm{heat}}$ is a manifold with corners which is a resolution of $M \times M \times [0,\infty)_t$ in the sense that there is a map $\beta$ as in (1.20) with the property that $\beta^* C^\infty(M \times M \times [0,\infty)_t) \subset C^\infty(M^2_{\mathrm{heat}})$ is a proper subset — exactly which "additional" smooth functions appear on $M^2_{\mathrm{heat}}$

is the main content of the construction, as we discuss now in detail. To describe the construction we follow the development in [16] closely.

A quasihomogeneous blowup of a manifold with corners (mwc) $X$ is a mwc $[X;Y]_{\text{q-hom}}$ constructed from: (1) a boundary p-submanifold $Y \subset X$, and (2) an extension of $Y$ of order $a$ in $X$. We define these objects in detail now.

Recall that as $X$ is a mwc, near every point $p \in X$ there is a neighborhood $\mathcal{V}$ which is diffeomorphic to a an open subset of $\mathbb{R}^k_+ \times \mathbb{R}^{n-k}$, and thus there exist coordinates functions $(x_1, \dots, x_k, y_1, \dots, y_{n-k})$ with $x_i \geq 0$ for all $i = 1, \dots, k$ with independent differentials on $\mathcal{V}$. A *p-submanifold* $Y$ is an embedded mwc $Y \subset X$ such that for each $p \in Y$ there exist such local coordinates on an open set $\mathcal{V} \ni p$ such that

$$Y \cap \mathcal{V} = \{x' = (x_1, \dots, x_r) = 0, y' = (y_1, \dots, y_m) = 0\}, \quad \text{where } r < k, m \leq n-k, \tag{3.1}$$

so $y'' = (x_{r+1}, \dots, x_k, y_{m+1}, \dots, y_{n-k})$ are local coordinates on $Y$. Given a boundary p-submanifold $Y$ (i.e. a p-submanifold $Y$ which is a subset of a boundary hypersurface (bhs) of $X$), we need in addition an extension of $Y$ to a an *interior* p-submanifold $\widetilde{Y}$ with $\widetilde{Y} \cap \partial X = Y$. Given such $\widetilde{Y}$, locally we can take a coordinate neighborhood $\mathcal{V}$ with coordinates $z = (x', y', y'')$ as above such that,

$$Y \cap \mathcal{V} = \{x' = 0 = y'\} \quad \text{and} \quad \widetilde{Y} \cap \mathcal{V} = \{y' = 0\}.$$

To add flexibility to the choice of the extension, we define an *extension of $Y$ or order $a \in \mathbb{N}$* to be an equivalence class of p-submanifolds $\widetilde{Y}$ with $\partial X \cap \widetilde{Y} = Y$ which agree to order $a$ at $Y$, in the sense that for $\widetilde{Y}, \widetilde{Y}'$ two such extension and coordinates chosen as above for $\widetilde{Y}$, then $\widetilde{Y}' \cap \mathcal{V} = \{y' = G(x', y'')\}$ satisfies $D_{x',y''}G = O(|x'|^a)$.

Given such data, i.e. an mwc $X$, a boundary p-submanifold $Y \subset X$ and $\widetilde{Y}$ an interior extension to order $a$ of $Y$, one can define the quasihomogeneous blowup

$$\beta\colon [X, Y]_{\text{q-hom}} \longrightarrow X \tag{3.2}$$

as follows. On each coordinate chart in $\mathcal{V}$ in the previous paragraph, with coordinates $z = (x', y', y'')$ we define the quasihomogeneous cylindrical decomposition (see [16, eq. 12],

$$\begin{aligned} R &:= (x_1^{2a} + \dots + x_r^{2a} + y_1^2 + \dots + y_m^2)^{1/2a}, \\ S_a^+ &:= \{(\omega, \nu) \in \mathbb{R}^r_+ \times \mathbb{R}^m : R(\omega, \nu)\} = 1, \end{aligned} \tag{3.3}$$

so that, in an open rectangle $\mathcal{V}' \times \mathcal{V}'' \subset \mathcal{V}$ where $\mathcal{V}' = \{(x', y') : |x'|, |y'| < c\}$, $\mathcal{V}'' = \{y'' : |y''| < c\}$, we have the map

$$\begin{gathered} \beta|_{loc}\colon [\mathcal{V}; Y \cap \mathcal{V}]_{\text{q-hom}} = S_a^+ \times [0, \epsilon)_R \times \mathcal{V}'' \longrightarrow \mathcal{V} \\ ((\omega, \nu), R, y'') \longmapsto (R\omega, R^a \nu, y''). \end{gathered}$$

The open mwc's can be patches together to invariantly define the total space of a the resolution in (3.2).

From this construction it is clear that the function $R$ in (3.3) is smooth on $[\mathcal{V}; Y \cap \mathcal{V}]_{\text{q-hom}}$. The locus $\{R = 0\}$ is a boundary hypersurface of the (open) mwc $[\mathcal{V}; Y \cap \mathcal{V}]_{\text{q-hom}}$. Picking a covering of $Y$ by a finite collection of such coordinate charts, $\mathcal{V}_i, i \in I$, each with its corresponding function $R_i$, and choosing a partition of unity subordinate to $\mathcal{V}_i$, the function $\rho = \sum_{i \in I} \xi_i R_i$ is then a *boundary defining function* for an introduced boundary hypersurface. More precisely, define, for a mwc $X$,

$$\mathcal{M}^1(X) = \mathcal{M}(X) = \{H \subset M : H \text{ a bhs of } X\}.$$

Then

$$\mathcal{M}([X, Y]_{\text{q-hom}}) = \mathcal{M}(X) \cup \text{ff}_Y, \quad \text{ff}_Y := \{\rho = 0\}, \tag{3.4}$$

where each $H \in \mathcal{M}(X)$ lifts to a bhs by taking the closure of the pullback of lift, $\text{cl}(\beta^{-1}(H \setminus \partial H))$. (Alternatively one can take $\mathcal{M}(X)$ to be the set of *open* bhs, and then write $\{\rho = 0\}^\circ$ instead.) Here $\rho$ is a boundary defining function for $\text{ff}_Y$, in particular $\rho$ is smooth on $[X, Y]_{\text{q-hom}}$ whereas it is not smooth as a function on $X$. Moreover the ratios of functions vanishing at $Y$ are now smooth on certain open subsets of $[\mathcal{V}, \mathcal{V} \cap Y]_{\text{q-hom}}$, for example, notation as in the previous paragraph, $y_i / x_j^a$ (defined is smooth away from the closure of the lift of $x_j = 0$ via $\beta_{\mathcal{V}}$. When $a = 1$ this is just a homogeneous radial blow up.

For a detailed definition of such spaces we refer to Melrose's work [31, Chapter 5] which contains a more general construction which does not assume that one has in particular a fixed extension for the manifold $N$ away from the boundary, (whereas here we fix once and for all a boundary defining function $x$ as in (2.1), which will give all the desired extensions below). See also [16, 22].

We will need a slight extension of the concept of quasihomogeneous blowup, which are sufficient for the elliptic equations studied in [16], to include the presence of the time variable $t$. There will be an additional defining function $s$ for the boundary p-submanifolds $Y$ will blow up at a different homogeneity than that of the other defining functions; that is, with $x', y', y''$ coordinates as in (3.1), we will have $x' = (s, x_1, \dots, x_r)$, and we will want to blow up so that $s / x_i^{2a}$ is smooth for $i = 1, \dots, r$. Luckily, in all cases below, the function $s$ can be defined on a full tubular neighborhood of $Y \subset \mathcal{V}$ in such a way that for some (open) mwc $\mathcal{V}'$ we have $\mathcal{V} = \mathbb{R}^+_s \times \mathcal{V}'$. This gives a special bhs $H_s := \{s = 0\}$ in the open mwc $\mathbb{R}^+_s \times \mathcal{V}'$. We then blow up quasihomogeneously but with $s$ being "parabolic" with respect to the $y$ variables, namely we will have a boundary defining function, first defined locally on coordinate charts in $\mathcal{O} \subset \mathcal{V}'$, by

$$\widetilde{R} = (s + x_1^{2a} + \dots + x_r^{2a} + y_1^2 + \dots + y_m^2)^{1/2a}, \tag{3.5}$$

and, parallel to the simpler quasihomogeneous case above, defining

$$\widetilde{S}^+_a := \{(\sigma, \omega, \nu) \in \mathbb{R}^{r+1}_+ \times \mathbb{R}^m : \widetilde{R}(\sigma, \omega, \nu)\} = 1,$$

the (locally defined) resolutions

$$[0,\epsilon)_{\widetilde{R}} \times \widetilde{S}_a^+ \longrightarrow \mathbb{R}_s^+ \times \mathcal{O}, \quad (R,(\sigma,\omega,\nu)) \mapsto (\sigma R^{2a}, R\omega, R^a\nu, y'')$$

patch together to form a global resolution which we continue to call $[X,Y]_{\text{q-hom}}$. We continue to refer to these as quasihomogeneous blow ups.

We now construct the heat double space $M^2_{\text{heat}}$ via three blow ups. We first define the blow ups iteratively, so that $M^2_{\text{heat}}$ is at least defined, and then circle back to discuss each blow up in detail, defining explicit coordinate charts near each introduced boundary hypersurface which will be used in subsequent computations.

We begin by considering $M \times M \times [0,\infty)_t$. Consider the subset

$$\mathcal{B}_0 := \partial M \times_{\text{fib}} \partial M \times \{0\} \subset M \times M \times [0,\infty)_t \tag{3.6}$$

where the fiber diagonal $\partial M \times_{\text{fib}} \times \partial M$ is the inverse image of diag $Y$ via $\pi\times\pi\colon \partial M \times \partial M \longrightarrow Y \times Y$. Blowing up homogeneously to form $[M \times M \times [0,\infty)_t; \mathcal{B}_0]$ gives a manifold with corners with new bhs $\mathrm{ff}_1$. We let $\rho_0$ be a bdf of $\mathrm{ff}_1$ and write $\rho_t$ for bdf of the lift of $\{t=0\}$ to the blow up. We may also define the fiber diagonal of the tubular neighborhood of the boundary $\mathcal{U}\times\mathcal{U}\times\{0\} \subset M \times M \times [0,\infty)_t$ using the fibration $\mathcal{U} = \partial M \times [0,1)_x \longrightarrow Y \times [0,1)_x$, so that $\mathcal{U}\times_{\text{fib}}\mathcal{U} = \partial M \times_{\text{fib}} \times \partial M \times [0,1)_x$ and consider the proper tranforms of this set, and intersect it with $\mathrm{ff}_1$, i.e. define

$$\mathcal{B}_1 := \mathrm{ff}_1 \cap \mathrm{cl}(\partial M \times_{\text{fib}} \times \partial M \times (0,1)_x \times \{0\}_t).$$

This we blow up quasihomogeneously so that $\rho_t$ plays the role of the slow bdf $t$ in (3.5) to form $[[M \times M \times [0,\infty)_t; \mathcal{B}_0]; \mathcal{B}_1]_{\text{q-hom}}$ Finally we blow up, homogeneously, the lift of the diagonal at $t=0$, that is the proper transform of $\mathrm{diag}(M)\times\{t=0\}$; setting $\mathcal{B}_2 := \mathrm{cl}(\mathrm{diag}(M^\circ)\times\{0\}_t)$ with the closure in $[[M \times M \times [0,\infty)_t; \mathcal{B}_0]; \mathcal{B}_1]_{\text{q-hom}}$ we have

$$M^2_{\text{heat}} = [[[M \times M \times [0,\infty)_t; \mathcal{B}_0]; \mathcal{B}_1]_{\text{q-hom}}; \mathcal{B}_2]. \tag{3.7}$$

We now discuss this space in more detail at each step, including explicit coordinate functions.

*1. The blow up of $\mathcal{B}_0$, the fiber diagonal in the corner.* This is the subset of $\partial M \times \partial M \times \{0\} \subset M \times M \times [0,\infty)_t$ consisting of points $(p,q,0)$ with $\pi(p) = \pi(q)$ where $\pi$ is the projection of the fibration $\partial M$ onto its base. If local coordinates $(x,y,z)$ are chosen as in (2.3) and identical local coordinates $(\tilde{x},\tilde{y},\tilde{z})$ are fixed on the second copy of $M$ in the product $M \times M \times [0,\infty)_t$, then in this local coordinate chart $\mathcal{B}_0$ is given by $\{x = \widetilde{x} = t = y - \widetilde{y} = 0\}$. We can define the intermediate blow up space

$$M^2_{\text{heat},1} := [M \times M \times [0,\infty)_t; \mathcal{B}_0], \tag{3.8}$$

with $t \sim x^2 \sim \widetilde{x}^2 \sim |y-\widetilde{y}|^2$ at the introduced bhs. To be precise, $M^2_{\text{heat},1}$ is the parabolic blowup in time of the set $\mathcal{B}_0$ as defined in [30, Chapter 7]. In particular there is a blowdown map $\beta_1: M^2_{\mathrm{heat},1} \longrightarrow M^2 \times [0,\infty)_t$, and polar coordinates on $M^2_{\mathrm{heat},1}$ near $\beta_1^{-1}(\mathcal{B}_0)$ (once coordinates $y, z$ are chosen on $\partial M$) are given by

$$\begin{aligned} \rho &= \left(t + x^2 + \widetilde{x}^2 + |y - \widetilde{y}|^2\right)^{1/2}, \\ \phi &= \left(\frac{t}{\rho^2}, \frac{x}{\rho}, \frac{\widetilde{x}}{\rho}, \frac{y-\widetilde{y}}{\rho}\right) \\ &= (\phi_t, \phi_x, \phi_{\widetilde{x}}, \phi_y), \text{ along with } \widetilde{y}, z, \widetilde{z}. \end{aligned} \tag{3.9}$$

The set $\{\rho = 0\}$ is a boundary hypersurface on $M^2_{\mathrm{heat},1}$ introduced by the blowup; we call it $\mathrm{ff}_1$; we will see that only the projection of the heat kernel onto the zero mode in $Z$ is relevant at the face $\mathrm{ff}_1$. Letting $s = x/\widetilde{x}$, the interior of $\mathrm{ff}_1$ is the total space of a fiber bundle over $Y \times (0,\infty)_s$, which is the fiber product $\partial M \times_{\mathrm{fib}} \partial M \times_{\mathrm{fib}} TY \times \mathbb{R}_{t'}$ where $t'$ is a rescaled time variable (see (3.11) below). Indeed, the map from $\mathrm{ff}_1$ to the base $Y$ is simply $\beta_1|_{\mathrm{ff}_1}$

2. *Blow up of* $\mathcal{B}_1$. The preceding blow up does not resolve the term $\frac{t}{x^{2k}}\Delta_{\partial M/Y}$ in $t(\partial_t + \Delta)$ (see (2.22)). To accomplish this, we blowup the subset of $\mathrm{ff}_1$ defined in polar coordinates by

$$\mathcal{B}_1 := \{\rho = 0, \phi_t = \phi_y = 0, \phi_x = \phi_{\widetilde{x}}\}, \tag{3.10}$$

i.e. by $\rho = 0, \phi = (0, 1/\sqrt{2}, 1/\sqrt{2}, 0)$, quasihomogeneously so that near the new face, ff, the function $t/x^{2k}$ is smooth, and furthermore so that $t\partial_x^2$ is non-degenerate, the latter condition being satisfied if $(x-\widetilde{x})/\sqrt{t}$ is smooth up to the interior of ff. Near $\mathcal{B}_1$ we can use projective coordinates

$$\widetilde{x}, \quad s = x/\widetilde{x}, \quad \eta = \frac{y-\widetilde{y}}{\widetilde{x}}, \quad t' = t/\widetilde{x}^2, \tag{3.11}$$

along with $\widetilde{y}, z, \widetilde{z}$. The quasihomogeneous blow up of $\mathcal{B}_1$ creates another intermediate space $M^2_{\mathrm{heat},2}$. This space has $t' \sim |\eta|^2 \sim (s-1)^2 \sim \widetilde{x}^{2(k-1)}$, and we have polar coordinates near ff given by

$$\begin{aligned} \overline{\rho} &= \left((t/\widetilde{x}^2) + \widetilde{x}^{2(k-1)} + (s-1)^2 + (|y-\widetilde{y}|/\widetilde{x})^2\right)^{1/2(k-1)}, \\ \overline{\phi} &:= (\overline{\phi}_t, \overline{\phi}_{\widetilde{x}}, \overline{\psi}_x, \overline{\psi}_y) = \left(\frac{t}{\widetilde{x}^2\overline{\rho}^{2(k-1)}}, \frac{\widetilde{x}}{\overline{\rho}}, \frac{x-\widetilde{x}}{\widetilde{x}\overline{\rho}^{(k-1)}}, \frac{y-\widetilde{y}}{\widetilde{x}\overline{\rho}^{(k-1)}}\right) \text{ along with } \widetilde{y}, z, \widetilde{z}. \end{aligned} \tag{3.12}$$

Let

$$\beta_2: M^2_{\mathrm{heat},2} \longrightarrow M \times M \times [0,\infty)_t \tag{3.13}$$

denote the blowdown map. Then, similar to the setup at $\mathrm{ff}_1$, if we define $\sigma = (x-\widetilde{x})/\widetilde{x}$, the interior of ff is a bundle over $Y \times \mathbb{R}_\sigma$ whose fiber over $p \in Y$ is isomorphic to $T_pY \times Z^2 \times \mathbb{R}_{\widetilde{T}}$ for $\widetilde{T}$ the rescaled time variable below.

See Remark 3.4 below for further discussion of the need for the two distinct blown up faces ff and $\mathrm{ff}_1$.

*3. Blow up of the time equals zero diagonal,* $\mathcal{B}_2 := \mathrm{cl}(\beta_2(\mathrm{diag}(M^\circ) \times \{t = 0\}))$. Note that $\mathcal{B}_2$ intersects the face ff at $\overline{\phi} = (1, 0, 0, 0)$, so near the intersection, defining the functions

$$\widetilde{x}, \quad \sigma = \frac{s-1}{\widetilde{x}^{k-1}} = \frac{x - \widetilde{x}}{\widetilde{x}^k}, \quad \widetilde{\eta} = \frac{y - \widetilde{y}}{\widetilde{x}^k}, \quad \widetilde{T} = \frac{t'}{\widetilde{x}^{2(k-1)}} = \frac{t}{\widetilde{x}^{2k}}, \tag{3.14}$$

we have the projective coordinates

$$\widetilde{x}, \widetilde{y}, \sigma, \widetilde{\eta}, \widetilde{T}, z, \widetilde{z}. \tag{3.15}$$

The full heat space is $M^2_{\mathrm{heat}}$ is the parabolic blow up of $\mathcal{B}_2$ in $M^2_{\mathrm{heat},2}$, and has has $\widetilde{T} \sim \sigma^2 \sim (z - \widetilde{z})^2$ at the introduced bhs. The face tf introduced by the final blowup satisfies

$$\mathrm{tf}^\circ \simeq {}^{\mathrm{ice}}TM, \tag{3.16}$$

where ${}^{\mathrm{ice}}TM$ is the incomplete cusp edge tangent bundle defined in (2.6). Concretely, in coordinates $(x, y, z)$ if we set

$$\xi = \frac{x - \widetilde{x}}{\sqrt{t}}, \quad \eta_i = \frac{y_i - \widetilde{y}_i}{\sqrt{t}}, \quad \zeta_\alpha = \frac{z_\alpha - \widetilde{z}_\alpha}{\sqrt{t}}\, \widetilde{x}^k, \quad \tau = \frac{\sqrt{t}}{\widetilde{x}^k}, \tag{3.17}$$

then $(x, y, z, \xi, \eta, \zeta, \tau)$ (or $(\widetilde{x}, \widetilde{y}, \widetilde{z}, \xi, \eta, \zeta, \tau)$) form local coordinates near the intersection of tf with ff and away from $t = 0$, and the association $\xi \mapsto \partial_x$, $\eta_i \mapsto \partial_{y_i}$, $\zeta_\alpha \mapsto x^{-k}\partial_{z_\alpha}$ induces the map.

In summary, we have constructed a manifold with corners $M^2_{\mathrm{heat}}$, depicted in Figure 1, with a blowdown map $\beta$ as in (1.20), such that $M^2_{\mathrm{heat}}$ has six total faces, three of them being the lifts of the standard faces

$$\begin{gathered} \mathrm{lf} := \mathrm{cl}(\beta^{-1}(\{x = 0\}^\circ)), \quad \mathrm{rf} := \mathrm{cl}(\beta^{-1}(\{\widetilde{x} = 0\}^\circ)), \\ \mathrm{tb} := \mathrm{cl}(\beta^{-1}(\{t = 0\}^\circ)), \end{gathered} \tag{3.18}$$

and then the three faces $\mathrm{ff}_1$, ff, and tf constructed (in that order) by radial blowup as described above. Denoting the set of the six boundary hypersurfaces by $\mathcal{M}(M^2_{\mathrm{heat}}) = \{\mathrm{lf}, \mathrm{rf}, \mathrm{tb}, \mathrm{ff}_1, \mathrm{ff}, \mathrm{tf}\}$, and given $\bullet \in \mathcal{M}(M^2_{\mathrm{heat}})$, below we will let $\rho_\bullet$ denote a boundary defining function for $\bullet$, so $\rho_\bullet \in C^\infty(M^2_{\mathrm{heat}})$ satisfies that $\{\rho_\bullet = 0\} = \bullet$ and $d\rho_\bullet$ is non-vanishing on $\bullet$ and $\rho_\bullet \geq 0$. We can take $\rho_{\mathrm{ff}} = \overline{\rho}$ as in (3.12). Note also that $x$ vanishes at lf, $\mathrm{ff}_1$, and ff, and although it is not a boundary defining function of any of these three boundary hypersurfaces, for *any* choice of boundary defining functions $\rho_{\mathrm{lf}}, \rho_{\mathrm{ff}_1}$, an $\rho_{\mathrm{ff}}$, it holds that $f := x/\rho_{\mathrm{lf}}\rho_{\mathrm{ff}_1}\rho_{\mathrm{ff}}$ is a smooth, positive function on $M^2_{\mathrm{heat}}$. It follows, for example by setting $\widetilde{\rho_{\mathrm{lf}}} = f \cdot \rho_{\mathrm{lf}}$ that one can choose these three boundary defining functions so that

$$\rho_{\mathrm{lf}}\rho_{\mathrm{ff}}\rho_{\mathrm{ff}_1} = x, \quad \rho_{\mathrm{rf}}\rho_{\mathrm{ff}}\rho_{\mathrm{ff}_1} = \widetilde{x}. \tag{3.19}$$

The same argument applies to $\rho$, which vanishes on $\mathrm{ff}_1$ and ff, i.e. we can take

$$\rho_{\mathrm{ff}_1}\rho_{\mathrm{ff}} = \rho, \text{ i.e. } \rho_{\mathrm{ff}} = \rho/\overline{\rho}.$$

In Theorem 3.7 we will show that the heat kernel lifts to be polyhomogeneous on $M^2_{\mathrm{heat}}$.

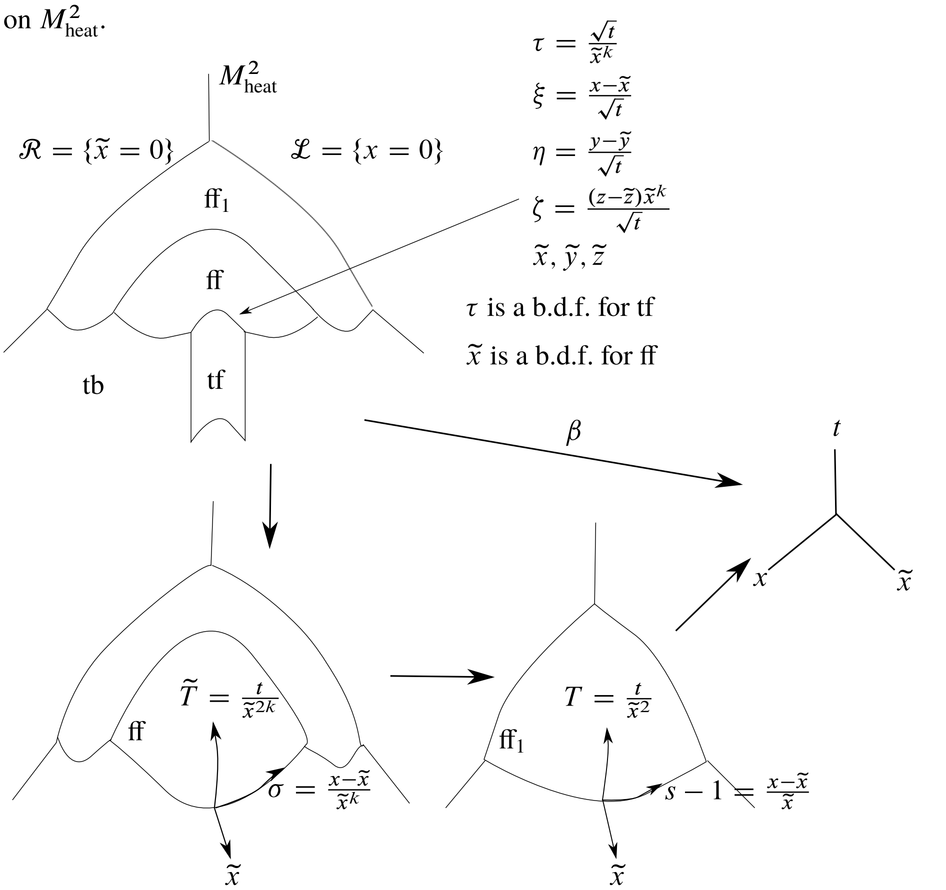


Figure 1. The heat double space (upper left) and the various intermediate blowups together with their blow down maps.

**3.2. Model operators.** The blown up space $M^2_{\mathrm{heat}}$ is useful in the construction of a parametrix for the heat equation in part because the operator $\partial_t + \Delta$ (more specifically $t(\partial_t + \Delta)$) behaves nicely at the three introduced boundary hypersurfaces ff, $\mathrm{ff}_1$, and tf; in particular, the first steps in the parametrix construction involve finding the right asymptotic behavior for the heat kernel so that the heat equation (1.14) is satisfied *at least to leading order at* ff, $\mathrm{ff}_1$, and tf.

Thus, we consider the operator $\Delta$ acting on the left spacial factor of $M \times M \times [0,\infty)_t$, and the pullback $\beta^*(t(\partial_t + \Delta))$ to $M^2_{\mathrm{heat}}$, and show that this restricts to an operator at tf. To be precise, fix a point $p \in M$ and consider the fiber $\mathrm{tf}_p = \pi^{-1}(p)$ where

$$\pi\colon \mathrm{tf} \longrightarrow \mathrm{diag}_M = M$$

is the projection onto the diagonal (or more concretely it is $\beta|_{\mathrm{tf}}$). In the interior of tf, i.e. away from the intersection with ff, this is standard [30], so we concern ourselves only with an open neighborhood of the intersection of tf with ff. Indeed, working locally over the base in both spacial factors, consider a subset of tf of the form $\{(\widetilde{x}, \widetilde{y}, \widetilde{z}, \xi, \eta, \zeta, \tau) : (\widetilde{x}, \widetilde{y}, \widetilde{z}) \in \mathcal{O}\}$. Now note

$$\sqrt{t}\partial_x = \partial_\xi, \quad \sqrt{t}\partial_y = \partial_\eta, \quad \frac{\sqrt{t}}{x^k}\partial_z = \partial_\zeta + O(\tau), \tag{3.20}$$

and

$$t\partial_t = \frac{1}{2}\left(\tau\partial_\tau - R\right), \tag{3.21}$$

where $R = \xi\partial_\xi + \eta \cdot \partial_\eta + \zeta\partial_\zeta$ is the radial vector field on the fiber. Letting

$$\pi_L, \pi_R\colon M \times M \times [0,\infty)_t \longrightarrow M$$

denote the projections onto the left and right $M$ factors, and End $\longrightarrow M \times M$ the endomorphism bundle, whose fiber at $(p,q) \in M^\circ \times M^\circ$ is $\mathrm{End}(\Lambda^*_q; \Lambda^*_p)$, for $t > 0$, the heat kernel *restricted to the interior* will be a smooth section of this bundle. To study the heat kernel at the boundary we use the incomplete cusp edge forms and the corresponding endomorphism bundle $\mathrm{End}({}^{\mathrm{ice}}\Lambda^*)$ back to $M \times M \times [0,\infty)_t$ and then to $M^2_{\mathrm{heat}}$ via the blowdown $\beta$. As usual, restricting to the spacial diagonal gives the "little endomorphism" bundle

$$\mathrm{End}({}^{\mathrm{ice}}\Lambda^*)|_{\mathrm{diag}(M)} \simeq \mathrm{end}({}^{\mathrm{ice}}\Lambda^*),$$

where $\mathrm{end}({}^{\mathrm{ice}}\Lambda^*) \longrightarrow M$ is the endomorphism bundle of the exterior algebra of $M$. The restriction to the time face, $\beta^* \mathrm{End}|_{\mathrm{tf}}$, is isomorphic to the pullback of $\mathrm{end}(\Lambda^*_p)$ to the tangent bundle of $M$ via the projection map.

Writing $w = (x,y,z)$, $\widetilde{w} = (\widetilde{x}, \widetilde{y}, \widetilde{z})$, sections of $\beta^*\mathrm{End}$ near the fiber of tf over $p$ can be written

$$\alpha = \sum_{I,J} a_{IJ}\, dw_I \otimes \partial\widetilde{w}_J, \tag{3.22}$$

where $I, J$ run over all multi-indices and $\partial\widetilde{w}_J$ is dual to $dw_J$, and here $a_{IJ} = a_{IJ}(w, \widetilde{w}, t)$. We claim that, writing sections of $\beta^*\mathrm{End}$ near tf as sections of $\beta^*\mathrm{End}|_{\mathrm{tf}} \simeq \Lambda^*(M) \otimes \Lambda(M)$,

$$t(\partial_t + \Delta) = \Big(\frac{1}{2}\left(\tau\partial_\tau - R\right) + \sigma(\Delta)\Big) \otimes \mathrm{Id} + O(\tau), \tag{3.23}$$

where $\sigma(\Delta)$ is a constant coefficient differential operator in the coordinates $\Xi = (\xi, \eta, \zeta)$ depending on the metric $g$ at $p = (\tilde{x}, \tilde{y}, \tilde{z})$, namely

$$\sigma(\Delta) = (d_\Xi + \star_{g(p)}^{-1} d_\Xi \star_{g(p)})^2, \tag{3.24}$$

acting on differential forms on the vector space ${}^{\text{ice}}\Lambda^*_p(M)$ with metric $g(p)$. Indeed, let $w$ be geodesic normal coordinates. In the interior of tf away from ff we have coordinates $\Xi = (w - \tilde{w})/\sqrt{t}, \tilde{w}, \sqrt{t}$. Then

$$t(\partial_t + \Delta)\alpha = \Big(t(\partial_t + \Delta)\sum_{I,J} a_{IJ} dw_I\Big) \otimes \partial\tilde{w}_J$$

and moreover

$$\begin{aligned} \star dw_I \otimes \partial\tilde{w}_J &= (\star dw_I) \otimes \partial\tilde{w}_J \\ &= \pm dw_{I^c} \otimes \partial\tilde{w}_J + O(w - \tilde{w}) \\ &= \pm(d\tilde{w}_{I^c} + \sqrt{t} d\Xi_{I^c}) \otimes \partial\tilde{w}_J + O(\sqrt{t}\,\Xi) \\ &= (\star_{g(p)} d\tilde{w}_I) \otimes \partial\tilde{w}_J + O(\tilde{t}). \end{aligned} \tag{3.25}$$

Similarly, letting the exterior derivative act on the left gives

$$d(a dw_I \otimes \partial\tilde{w}_J) = (\partial_{\Xi_i} a d\tilde{w}_i \wedge \tilde{w}_I) \otimes \partial\tilde{w}_J.$$

To motivate our construction of the heat kernel further, in a neighborhood of tf let $\gamma$ be a section of End with the property that $\gamma|\operatorname{diag}_M = \operatorname{Id}$ on the form bundles, and consider the section of $\beta^*$ End on $M^2_{\text{heat}}$ of the form

$$K(p, q, t) = \frac{1}{(2\pi t)^{n/2}} e^{-G(p,q)/2t}\gamma, \tag{3.26}$$

such that $G(p, q)$ satisfies that $\beta^*(G(p, q)/t)|_{\text{tf}} = \|\Xi\|^2_g$, that is, that $G(p, q)/t$ restricts to the metric function on tf. Such a form $\gamma$ and function $G$ can be constructed but we neither prove nor use this; we merely use it as motivation. It is straightforward to check that for any smooth compactly supported form $\alpha$

$$\begin{aligned} \lim_{t\to 0} \int_M K(p, q, t)\alpha(q)\, \mathrm{dVol}_q &= (4\pi)^{-n/2} \int_M e^{-\|\Xi\|^2_{g(p)}/4} \alpha(p) \sqrt{g(p)} |d\Xi| \\ &= \alpha(p), \end{aligned} \tag{3.27}$$

and in fact the convergence takes place in $L^2$. (In fact, such an endomorphism $\gamma$ can be constructed easily by taking the identity map on ${}^{\text{ice}}\Lambda^*$ over $M$, pulling this back via $\beta$ to $\beta^* \text{End}|_{\text{tf}}$ and extending off smoothly in a neighborhood. On each exterior algebra $\Lambda^*_p M$, the identity can be expressed in terms of a basis $e_i$ with dual basis $e_i^*$ as $\sum_I e_I \otimes e_I^*$. In a neighborhood of tf $\cap$ ff we can take the basis $e_i$ to be $dx$, $dy_i$, $x^k dz_\alpha$, i.e. we can take the $e_i$ to be a basis of forms for ${}^{\text{ice}}\Lambda^*$ all the way down to $x = 0$.)

Working in the coordinates (3.17), since $t^{-n/2} = \tau^{-n}\widetilde{x}^{-nk}$, the Taylor expansion of the heat kernel at tf should take the form

$$\frac{1}{(4\pi)^{n/2}\widetilde{x}^{kn}}\tau^{-n}\sum_{j=0}^{\infty}\tau^{j}b_{j}, \tag{3.28}$$

where the $b_j = b_j(\widetilde{x}, \widetilde{y}, \widetilde{z}, \xi, \eta, \zeta)$ are sections of $\beta^*\,\mathrm{End}$, which we again write in a neighborhood of $\mathrm{tf} \cap \mathrm{ff}$ as sections of $\mathrm{End}({}^{\mathrm{ice}}\Lambda^*)$ pulled back to the fibers of tf. Writing each $b_j$ as a finite sum of terms of the form

$$\alpha \otimes g^{-1}\beta, \tag{3.29}$$

where $\alpha$ and $\beta$ are sections of ${}^{\mathrm{ice}}\Lambda^*$ and $g^{-1}$ indicates taking the dual vector field, we see that by (3.23) we have,

$$\left(\left(\frac{n}{2} - \frac{1}{2}R + \sigma(\Delta)\right) \otimes \mathrm{Id}\right)b_0 = \left(\left(\frac{n}{2} - \frac{1}{2}R + \begin{pmatrix}\Delta_{\Xi} & 0\\ 0 & \Delta_{\Xi}\end{pmatrix}\right) \otimes \mathrm{Id}\right)b_0. \tag{3.30}$$

The only solution to this equation which gives the identity operator at $t = 0$ is

$$b_0 = e^{-||\Xi||^2/4} \times \mathrm{Id}\,. \tag{3.31}$$

The procedure of solving for the remaining $b_j$ is standard [30, Chapter 7]; letting the Laplacian act on this expansion we show that on each term $a_j$ it acts fiberwise like a constant coefficient, second order elliptic differential operator plus the radial vector field plus a constant corresponding to the order of the term in the expansion. We have the following

**Lemma 3.1.** *There exist sections $b_j$ of $\mathcal{A}_{\mathrm{phg}}(\mathrm{End}|_{\mathrm{tf}})$ satisfying*

$$b_j = e^{-\|\Xi\|^2/4}\widetilde{b}_j(\widetilde{x}, \widetilde{y}, \widetilde{z}, \xi, \eta, \zeta),$$

*where $\widetilde{b}_j$ is a polynomial in $\xi, \eta, \zeta$ and a polyhomogeneous section of* End *over* tf, *such that for any distribution $H'$ in $\mathcal{A}^{\mathrm{phg}}(\mathrm{End})$ with asymptotic expansion near* tf *given by* (3.28) *we have*

$$t(\partial_t + \Delta)H' = O(\tau^{\infty}),$$

*i.e. $t(\partial_t + \Delta)H'$ vanishes to infinite order at the blown up $t = 0$ diagonal, and, moreover, the asymptotic sum of the $b_j$ exists and yields such an $H'$.*

The existence of a distribution $H'$ as in Lemma 3.1 is only a first step in constructing a parametrix for the heat kernel. We will discuss the rest of the process in §3.3.

A useful double check of the order of blowup of the heat kernel at ff is the following. Near $\mathrm{ff} \cap \mathrm{tf}$ we have

$$\delta(x - \widetilde{x})\delta(z - \widetilde{z})\delta(y - \widetilde{y}) = \delta(\xi\tau\widetilde{x}^{k})\delta(\eta\tau\widetilde{x}^{k})\delta(\zeta\tau) = \frac{1}{\tau^{n}\widetilde{x}^{(n-f)k}}\delta(\xi)\delta(\eta)\delta(\zeta).$$

Since $\mathrm{Id} = \lim_{t\searrow 0} H \, \mathrm{dVol}_g \sim \lim_{t\searrow 0} H \widetilde{x}^{kf} d\widetilde{x} d\widetilde{y} d\widetilde{z}$, we confirm that $H$ should have order $-nk$ at ff. In fact, we can deduce more; considering $\widetilde{x}^{kn} H|_{\mathrm{ff}}$, on the interior of ff we can use coordinates in (3.15), we get that

$$\delta(x-\widetilde{x})\delta(y-\widetilde{y})\delta(z-\widetilde{z}) = \widetilde{x}^{-(n-f)k}\delta(\sigma)\delta(\widetilde{\eta})\delta(z-\widetilde{z}), \tag{3.32}$$

which means that, on the face ff, we expect that the restriction $\widetilde{x}^{nk} H|_{\mathrm{ff}}$ will be given by $\delta(\sigma)\delta(\widetilde{\eta})\delta(z-\widetilde{z})$ *at least as the time variable* $\widetilde{T} = t/\widetilde{x}^k$ *goes to zero*, as that is the region in which the action of $H$ is definitively approximated by the identity. On the other hand, $\widetilde{x}$ commutes with the heat operator $\partial_t + \Delta$. As we will see in (3.35), $t(\partial_t + \Delta)$ restricts to an operator on ff and defines a fiber-wise heat type operator on ff, so we expect to have

$$t(\partial_t + \Delta)|_{\mathrm{ff}}(\widetilde{x}^{nk} H)|_{\mathrm{ff}} = 0. \tag{3.33}$$

This, together with (3.32), implies that an ansatz for the heat kernel should include that *on each fiber of* ff, $\widetilde{x}^{nk} H|_{\mathrm{ff}}$ *is the fundamental solution to the induced heat equation on the fiber*, more precisely, it is the solution which equals $\delta_{\sigma=0}\delta_{\widetilde{\eta}=0} \mathrm{Id}_Z$ at time equals zero. The induced heat equations are translation invariant in $\sigma$ and $\widetilde{\eta}$, thus induced by convolution operators, and the heat kernels we speak of are the convolution kernels in $\sigma$ and $\widetilde{\eta}$.

As for the blowup at $\mathrm{ff}_1$, as we will see below, the operator acts as a modified heat operator in $\partial_x$ and $Y$ on the bundle of fiber harmonic forms, so in the coordinates in (3.11) we will have

$$\delta(x-\widetilde{x})\delta(y-\widetilde{y})\delta(z-\widetilde{z}) = \frac{1}{\widetilde{x}^{1+b}}\delta(s-1)\delta(z-\widetilde{z})\delta(\eta). \tag{3.34}$$

In this case, $t(\partial_t + \Delta)$ only admits a restriction to $\mathrm{ff}_1$ on the fiber-harmonic forms $\mathcal{H}$, on which $\delta(z-\widetilde{z})$ becomes projections $\Pi_{Z,y}$ onto the kernel of $\Delta_{\partial M/Y}$. Thus we expect that $\widetilde{x}^{1+b+kf} H|_{\mathrm{ff}_1}$ on fiber harmonic forms is given by the convolution kernel for the heat kernel in $\eta$, times the dilation invariant kernel for the heat kernel in $s$ with limit $\delta_{s=1}$ at time 0.

We now compute the asymptotic behavior of $t(\partial_t + \Delta)$ at the faces ff and $\mathrm{ff}_1$. First we will work at ff.

**Proposition 3.2** (The model problem on ff). *The operator*

$$N_{\mathrm{ff}}(t(\partial_t + \Delta^g)) = t(\partial_t + \Delta^g)|_{\mathrm{ff}}$$

*acts fiberwise on* ff*, and is expressed in the coordinates in* (3.15) *by*

$$N_{\mathrm{ff}}(t(\partial_t + \Delta^g)) = \widetilde{T}\left(\partial_{\widetilde{T}} + \begin{pmatrix} -\partial_\sigma^2 + \Delta_\eta + \Delta_{\partial M/Y} & 0 \\ 0 & -\partial_\sigma^2 + \Delta_\eta + \Delta_{\partial M/Y} \end{pmatrix}\right) \tag{3.35}$$

*on the fiber above* $y \in Y$*. Here* $\Delta_\eta$ *is the constant coefficient Hodge–Laplacian on the tangent space* $T_yY$ *with translation invariant metric* $h(y)$*, and* $\Delta_{\partial M/Y}$ *is the Hodge–Laplacian on* $(Z, \boldsymbol{k}_y)$*.*

The situation is more delicate at $\mathrm{ff}_1$. As we will see in §3.3, near $\mathrm{ff}_1$, it will suffice to consider $t(\partial_t + \Delta)$ restricted to fiber harmonic forms. Thus let $\gamma \in x^s \mathcal{H}$ (see (2.11)) and so by assumption $\delta_{\partial M/Y}\gamma, d_{\partial M/Y}\gamma = 0$. From (2.21) it follows that for such fiber harmonic forms,

$$\Delta\gamma = \widetilde{\Delta}_0\gamma + x^{-k}\eth_{\partial M/Y} P\gamma + O(x^{s-1}), \tag{3.36}$$

where $\widetilde{\Delta}_0$ acts on forms decomposed as in (2.14), as

$$\widetilde{\Delta}_0 = -\partial_x^2 - \frac{kf}{x}\partial_x + \Delta_H + \begin{pmatrix} k\,\mathbf{N}(1-k(f-\mathbf{N}))x^{-2} & -2kx^{-k-1}d_{\partial M/Z} \\ -2kx^{-k-1}\delta_{\partial M/Y} & k(f-\mathbf{N})(1-k\,\mathbf{N})x^{-2} \end{pmatrix}.$$

The term $x^{-k}\eth_{\partial M/Y}P$ acts on polyhomogeneous forms as operators of order $x^{-k}$, and thus in the heat operator $t(\partial_t + \Delta)$ there are term behaving like $tx^{-k}$ (on fiber harmonic forms) but *$t/x^{-k}$ is not a bounded function at* $\mathrm{ff}_1$! On the other hand, if we project back to the fiber harmonic forms we kill these terms; concretely, with $\Pi_{\mathcal{H}}$ the fiber harmonic projector in (2.12), we have

$$\Pi_{\mathcal{H}}\Delta\Pi_{\mathcal{H}} = \widetilde{\Delta}_0 + x^{-1}\widetilde{E}' \tag{3.37}$$

where $\widetilde{E}' \in \mathrm{Diff}^2_{\mathrm{b,phg}}$ (see (2.17)), and thus does not decrease the order of vanishing of polyhomogeneous distributions. Defining

$$P_{A,B} := -\partial_s^2 - \frac{A}{s}\partial_s + \frac{B}{s^2}. \tag{3.38}$$

and

$$\alpha(\mathbf{N}) := kf, \quad \beta(\mathbf{N}) := k\,\mathbf{N}(1-k(f-\mathbf{N})), \quad \gamma(\mathbf{N}) = k(f-\mathbf{N})(1-k\,\mathbf{N}), \tag{3.39}$$

we have the following.

**Proposition 3.3** (Heat operator on fiber harmonic forms at $\mathrm{ff}_1$). *Restricted to the fiber harmonic forms $\mathcal{H}$ as defined through* (2.11)*,*

$$N_{\mathrm{ff}_1}(t(\partial_t + \Delta^g)) := \Pi_{\mathcal{H}} t(\partial_t + \Delta)\Pi_{\mathcal{H}}|_{\mathrm{ff}_1} \tag{3.40}$$

*restricts to the face* $\mathrm{ff}_1$ *in the coordinates* (3.11) *as*

$$N_{\mathrm{ff}_1}(t(\partial_t + \Delta^g)) = t'\left(\partial_{t'} + \begin{pmatrix} P_{\alpha(\mathbf{N}),\beta(\mathbf{N})} + \Delta_\eta & 0 \\ 0 & P_{\alpha(\mathbf{N}),\gamma(\mathbf{N})} + \Delta_\eta \end{pmatrix}\right). \tag{3.41}$$

**Remark 3.4.** Analysis of the fiber harmonic forms is necessary in particular because the structure of the operator $\Delta^g$ is such that, off of the fiber harmonic forms, the leading order term is $x^{-2k}\Delta_{\partial M/Y}$, while restricted to the fiber harmonic forms the leading order term drops in order. Indeed, if it weren't for the presence of the term $x^{-k}\widetilde{P}$ in (2.21), which presents complications in the analysis, on fiber harmonic forms $\Delta^g$ would be given by to leading order by $\widetilde{\Delta}_0$. Indeed, the need for the two different regimes represented by the boundary hypersurfaces ff and $\mathrm{ff}_1$ is exactly this change in asymptotic order of the operator on and off the fiber harmonic forms. Correspondingly, we will see below in the proof of Lemma 3.5 that the operator $t(\partial_t + \Delta)$ restricted to ff has a fundamental solution which vanishes at $\mathrm{ff}_1$ to infinite order *off the fiber harmonic forms*.

The heat equation for the regular singular ODEs in (3.38) has been studied in detail. To such an operator there corresponds a pair of indicial roots given by the order of vanishing of homogeneous solutions, specifically $P_{A,B}(s^\ell) = 0$ if and only if

$$\ell = \frac{-(A-1) \pm \sqrt{(A-1)^2 + 4B}}{2}. \tag{3.42}$$

The numbers $\ell$ give important information about the operator $P_{A,B}$, in particular they give the order of vanishing of the Green's function at $s = 0$. The operators that will arise in our work are those in the matrices in (3.41). We define the indicial set

$$\begin{aligned}\Lambda &= \bigcup_{\mathbf{N}=1}^{f} \left\{ \frac{-(\alpha-1) \pm \sqrt{(\alpha-1)^2 + 4\beta}}{2}, \frac{-(\alpha-1) \pm \sqrt{(\alpha-1)^2 + 4\gamma}}{2} \right\} \\ &= \bigcup_{\mathbf{N}=1}^{f} \big\{ -(kf-1)/2 \pm |k(\mathbf{N} - f/2) + 1/2|, \\ &\qquad\qquad -(kf-1)/2 \pm |k(\mathbf{N} - f/2) - 1/2| \big\}.\end{aligned} \tag{3.43}$$

Letting

$$\nu^2 = B + \left(\frac{A-1}{2}\right)^2 > 0 \tag{3.44}$$

where $\nu > 0$, from [36, Vol. 2, Eqn. 8.60] there is a fundamental solution $H_{A,B}(s, \widetilde{s}, t)$

$$(\partial_t + P_{A,B}) H_{A,B}(s, \widetilde{s}, t) = 0 \quad \text{and} \quad H \to \mathrm{Id} \text{ as } t \to 0 \text{ on } L^2(s^A ds). \tag{3.45}$$

Indeed, one has the explicit formula

$$H_{A,B}(s, \widetilde{s}, t) = (s\widetilde{s})^{-(A-1)/2} \frac{1}{2t} e^{-(s^2+\widetilde{s}^2)/4t} I_\nu\left(\frac{s\widetilde{s}}{2t}\right) \tag{3.46}$$

where $I_\nu$ is the modified Bessel function of order $\nu$ of the first kind [1, Chap. 9].

As discussed below (3.34), at the face $\mathrm{ff}_1$ we expect the heat kernel to be of order $\widetilde{x}^{-1-b-kf}$. Thus we expect to have

$$0 = t(\partial_t + \Delta) H = \frac{1}{\widetilde{x}^{1+b+kf}} (t(\partial_t + \Delta))(\widetilde{x}^{1+b+kf} H), \tag{3.47}$$

and since $\Pi_{\mathcal{H}} t(\partial_t + \Delta) \Pi_{\mathcal{H}}$ defines a differential operator on section of $\mathcal{H} \otimes \overline{\mathcal{H}}^*$ restricted to $\mathrm{ff}_1$, we include in our ansatz for the fundamental solution (1.14), and indeed prove in Theorem 3.7 below, that there is a fundamental solution $H$ satisfying that $\widetilde{x}^{1+b+kf} H$ has a smooth restriction to $\mathrm{ff}_1$, and writing

$$N_{\mathrm{ff}_1}(H) := (\widetilde{x}^{1+b+kf} H)|_{\mathrm{ff}_1}, \text{ we have } N_{\mathrm{ff}_1}(t(\partial_t + \Delta))N_{\mathrm{ff}_1}(H) = 0. \tag{3.48}$$

Furthermore, again as discussed below (3.34), it is sensible to include in the ansatz for $H$ that $N_{\mathrm{ff}_1}(H)$ is the fundamental solution for the model operator $N_{\mathrm{ff}_1}(t(\partial_t + \Delta))$, meaning specifically that $N_{\mathrm{ff}_1}(H)$ is a section of the restriction of the sub-bundle $\mathrm{End}(\mathcal{H})$ to $\mathrm{ff}_1$ and is given using the fundamental solutions to the model heat equations $H_{A,B}$ from (3.45)–(3.46). Specifically, we will have as an ansatz that $N_{\mathrm{ff}_1}(H) = \kappa_{\mathrm{ff}_1}$, where

$$\kappa_{\mathrm{ff}_1,y} := \begin{pmatrix} H_{\alpha,\beta}(s,1,t') & 0 \\ 0 & H_{\alpha,\gamma}(s,1,t') \end{pmatrix} (4\pi t')^{-b/2} e^{-|\eta|^2_y/4t'}, \tag{3.49}$$

where $\alpha$, $\beta$, $\gamma$ are as in (3.39), and in particular continue to be *operators depending on the fiber form degree* $\mathbf{N}$. The distribution $N_{\mathrm{ff}_1}(H)$ is polyhomogeneous on $\mathrm{ff}_1$, and the leading order behavior at $s = 0$ satisfies that for $0 < c \leq t' \leq C < \infty$, for some smooth $a(t'), b(t')$,

$$H_{\alpha,\beta}(s,1,t') \sim s^{-(kf-1)/2} a(t') s^{\nu(\alpha,\beta)}, \quad H_{\alpha,\gamma}(s,1,t') \sim s^{-(kf-1)/2} b(t') s^{\nu(\alpha,\gamma)} \tag{3.50}$$

with $\nu$ as in (3.44)

$$\begin{aligned} \nu(\alpha,\beta) &= \begin{cases} k(f/2-\mathbf{N}) - 1/2 & \text{if } \mathbf{N} < f/2, \\ k(\mathbf{N} - f/2) + 1/2 & \text{if } \mathbf{N} \geq f/2, \end{cases} \\ \nu(\alpha,\gamma) &= \begin{cases} k(f/2-\mathbf{N}) + 1/2 & \text{if } \mathbf{N} \leq f/2, \\ k(\mathbf{N} - f/2) - 1/2 & \text{if } \mathbf{N} > f/2, \end{cases} \end{aligned} \tag{3.51}$$

and thus by (3.50) on $\mathrm{ff}_1$ in the region $0 < c \leq t' \leq C < \infty$,

$$\kappa_{\mathrm{ff}_1} = O(s^{\overline{\nu}}), \quad \text{where } \overline{\nu}(\mathbf{N}) = \begin{cases} -k\mathbf{N} & \text{if } \mathbf{N} < f/2, \\ -k\mathbf{N} + 1 & \text{if } \mathbf{N} = f/2, \\ -k(f-\mathbf{N}) & \text{if } \mathbf{N} > f/2. \end{cases} \tag{3.52}$$

In words, each $P_{\alpha,\beta}$ has two indicial roots, the order of $H_{\alpha,\beta}$ for fixed $\widetilde{s}, t > 0$ is the larger of these two, and $p$ is the *smaller* of the leading orders of $H_{\alpha,\beta}$ and $H_{\alpha,\gamma}$.

The behavior of the heat kernel at $\mathrm{ff}_1$ also shows what to expect at the left face, the lift of $x = 0$. There we should just have the projection onto the fiber harmonic forms times the leading order behavior of the $H_{\alpha,\beta}$ and $H_{\alpha,\gamma}$, acting appropriately on ${}^{\mathrm{ice}}\Lambda^*$, times the lifted heat kernel of the base $Y$. Indeed, we expect

$$\Pi_{\mathcal{H}} H \Pi_{\mathcal{H}} \simeq \kappa := \begin{pmatrix} H_{\alpha(\mathbf{N}),\beta(\mathbf{N})}(x,\widetilde{x},t) & 0 \\ 0 & H_{\alpha(\mathbf{N}),\gamma(\mathbf{N})}(x,\widetilde{x},t) \end{pmatrix} H_Y, \tag{3.53}$$

where $H_Y$ is the heat kernel on $(Y, h)$ lifted to the tubular neighborhood $\mathcal{U}$ in (2.1) via the projection and $\kappa$ acts on sections of the bundle of fiber harmonic forms $\mathcal{H}$ with its grading by fiber form degree $\mathbf{N}$ (see §2). In fact, with $\overline{\nu} = \overline{\nu}(\mathbf{N})$ the fiber degree dependent weight in (3.52), this $\kappa$ defines a section near the corner $\partial M \times \partial M$ of the endomorphism bundle of the vertical harmonic forms:

$$\kappa|_{\mathcal{U}\times\mathcal{U}} \in x^{\widetilde{\nu}(\mathbf{N})} C^\infty\big(\mathcal{U}\times\mathcal{U}; \oplus_{\mathbf{N}=0}^{f} \mathcal{H}^{\mathbf{N}} \otimes (\widetilde{\mathcal{H}}^{\mathbf{N}})^*\big) \tag{3.54}$$

where $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, respectively, the pullbacks of the fiber harmonic form bundle (defined on a neighborhood $\mathcal{U}$ of the boundary) via the left and right projections of $\mathcal{U}\times\mathcal{U}$ onto $\mathcal{U}$. This is all cooked up so that

$$\widetilde{x}^{1+b+kf}\kappa|_{\mathrm{ff}_1} = \kappa_{\mathrm{ff}_1}, \tag{3.55}$$

indeed, extracting matrix components from the definition of $\kappa$, using $\alpha \equiv kf$, and writing $x/\widetilde{x} = s, t/\widetilde{x}^2 = t'$ gives

$$\begin{aligned}
&\widetilde{x}^{1+b+kf} H_{\alpha(\mathbf{N}),\beta(\mathbf{N})}(x,\widetilde{x},t) H_Y \\
&= \widetilde{x}^{1+b+kf}(x\widetilde{x})^{-(kf-1)/2}\frac{1}{2t}e^{-(x^2+\widetilde{x}^2)/4t} I_\nu\left(\frac{x\widetilde{x}}{2t}\right)\frac{1}{t^{b/2}}e^{\mathrm{dist}(y,\widetilde{y})^2/4t}(1+O(t^{1/2})) \\
&= (x/\widetilde{x})^{-(kf-1)/2}\frac{1}{2t/\widetilde{x}^2}e^{-(x^2+\widetilde{x}^2)/4t} I_\nu\left(\frac{x\widetilde{x}}{2t}\right)\frac{1}{(t/\widetilde{x}^2)^{b/2}}e^{\mathrm{dist}(y,\widetilde{y})^2/4t}(1+O(t^{1/2})) \\
&= s^{-(kf-1)/2}\frac{1}{t'^2}e^{-(s^2+1)/4t'} I_\nu\left(\frac{s}{2t'}\right)\frac{1}{(t)'^{b/2}}e^{|\eta|^2)^2/4t'}(1+O(\rho^{\mathrm{ff}_1})),
\end{aligned}$$

which implies (3.55). Below, we mean by $\kappa$ a form which restricts to $\mathcal{U}\times\mathcal{U}$ to be as above and extends to all of $M \times M$ polyhomogeneously with the same index set as $\kappa$. (This is easily arranged, and the index set of $\kappa$ is well defined since $\mathcal{U}\times\mathcal{U}$ intersects all bhs's of $M \times M$.)

As discussed below (3.33), on the face ff, we expect that the heat kernel will have leading asymptotic $\widetilde{x}^{-nk}$, so we expect and prove that

$$N_{\mathrm{ff}}(H) := (\widetilde{x}^{nk}H)|_{\mathrm{ff}} \implies N_{\mathrm{ff}}(t(\partial_t+\Delta))N_{\mathrm{ff}}(H) = 0. \tag{3.56}$$

Again, we will set $N_{\mathrm{ff}}(H)$ equal to a fundamental solution to the heat equation, namely, using the decomposition in (3.41), we expect to have $N_{\mathrm{ff}}(H) = \kappa_{\mathrm{ff}}$ where

$$\kappa_{\mathrm{ff},y}(\sigma,\eta,z,z',\widetilde{T}) := \mathrm{Id}_{2\times 2}(4\pi\widetilde{T})^{-(b+1)/2}e^{-(\sigma^2+|\eta|^2_{h_1})/4\widetilde{T}} H_{Z,y}, \tag{3.57}$$

where $H_{Z,y} = H_{Z,y}(z,z',\widetilde{T})$ is the heat kernel for $\Delta_{\partial M/Y}$.

**3.3. Parametrix construction and the asymptotic behavior of the heat kernel.** We now construct, and describe in detail the asymptotic behavior of, a parametrix for the heat kernel. Then, using a Neumann series argument and composition properties for operators which are formally similar to our parametrix (established in Appendix B), we upgrade this to a description of the heat kernel itself.

To begin our discussion of the parametrix construction for the heat kernel, let us briefly recall the notion of an *index set*, which by definition is a $\mathcal{E}(\bullet) = \{(\gamma, p)\} \subset \mathbb{C} \times \mathbb{N}$ associated with each face $\bullet \in \{\mathrm{lf}, \mathrm{rf}, \mathrm{tb}, \mathrm{ff}_1, \mathrm{ff}, \mathrm{tf}\}$ such that:

(i) each half-plane $\operatorname{Re}\gamma < C$ contains only finitely many $\gamma$;

(ii) for each $\gamma$, there is a number $P(\gamma) \in \mathbb{N}_0$ such that $(\gamma, p) \in \mathcal{E}(\bullet)$ for every $0 \le p \le P(\gamma)$ and $(\gamma, p) \notin \mathcal{E}(\bullet)$ if $p > P(\gamma)$;

(iii) if $(\gamma, p) \in \mathcal{E}(\bullet)$, then $(\gamma + j, p) \in \mathcal{E}(\bullet)$ for all $j \in \mathbb{N}$.

We recall the full definition of polyhomogeneity in Appendix A, but roughly speaking, we call a differential form $\alpha$ *polyhomogeneous with index family*

$$\mathcal{E} = \{\mathcal{E}(\bullet) \mid \bullet \in \{\mathrm{lf}, \mathrm{rf}, \mathrm{tb}, \mathrm{ff}_1, \mathrm{ff}, \mathrm{tf}\}\}$$

if it has an expansion at each boundary hypersurface $\bullet$ with exponents determined by the corresponding index set $\mathcal{E}(\bullet)$ and coefficient functions which are themselves polyhomogeneous (with exponents determined by $\mathcal{E}$). For example, smooth functions on $M^2_{\mathrm{heat}}$ are polyhomogeneous with indicial set satisfying $\mathcal{E}(\bullet) = \mathbb{Z} \times \{0\}$ for all $\bullet$, and if a polyhomogeneous function vanishes to infinite order at a particular boundary hypersurface $\bullet$, then it is polyhomogeneous with an index set $\mathcal{E}$ satisfying $\mathcal{E}(\bullet) = \varnothing$. We define

$$\inf \mathcal{E}(\bullet) = \inf\{\operatorname{Re}\gamma \mid (\gamma, p) \in \mathcal{E}(\bullet)\}.$$

Our first step is to establish the existence of a polyhomogeneous distribution whose behavior at the various boundary hypersurfaces matches the behavior we expect from the heat kernel.

**Lemma 3.5.** *There exists a distribution $K_1 \in \mathcal{A}_{\mathrm{phg}}(M^2_{\mathrm{heat}}; \beta^*(\mathrm{End}))$, polyhomogeneous with respect to an index set $\mathcal{E}$ satisfying the following properties:*

(1) *$K_1$ satisfies (3.28) for the indicated $b_j$. In particular,*

$$\mathcal{E}(\mathrm{tf}) = \mathbb{N} - \dim(M), \quad \mathcal{E}(\mathrm{tb}) = \varnothing.$$

*and thinking of $K_1(t)$ as an operator on differential forms on $M$ for each $t$,*

$$K_1(t)\alpha \to \alpha \text{ in } L^2 \text{ as } t \to 0. \tag{3.58}$$

(2) *At the faces* ff *and* $\mathrm{ff}_1$, *we have* $\inf \mathcal{E}(\mathrm{ff}_1) \ge -1 - b - kf$ *and* $\inf \mathcal{E}(\mathrm{ff}) \ge -kn$; *more precisely*

$$(\tilde{x}^{1+b+kf} K_1)|_{\mathrm{ff}_1} = \kappa_{\mathrm{ff}_1}, \quad (\tilde{x}^{kn} K_1)|_{\mathrm{ff}} = \kappa_{\mathrm{ff}}, \tag{3.59}$$

*with $\kappa_{\mathrm{ff}_1}$ and $\kappa_{\mathrm{ff}}$ the model heat kernels defined in (3.49) and (3.57).*

(3) *In the neighborhood of the corner defined by* $\mathcal{U} \times \mathcal{U}$ *(for our fixed tubular neighborhood* $\mathcal{U}$ *of* $\partial M$*), in an open neighborhood of* lf *and* rf*,*

$$K_1 \equiv \kappa, \tag{3.60}$$

*where* $\kappa$ *is as in* (3.53).

*Moreover for the behavior at the codimension* 2 *face* $\mathrm{lf} \cap \mathrm{rf}$*, the leading order behavior is the product of that at* lf *and* rf*, i.e.* $K_1 = O((\rho_{\mathrm{lf}}\rho_{\mathrm{rf}})^{-kf/2+1})$*. In particular,*

$$\inf \mathcal{E}(\mathrm{lf}) \geq -\frac{kf}{2} + 1. \tag{3.61}$$

*Furthermore,* $K_1$ *can be taken fiber harmonic in a neighborhood of* $\mathrm{ff}_1$*.*

*Proof.* Proving the existence of a polyhomogeneous distribution with prescribed leading order behavior at the boundary hypersurfaces of a manifold with corners boils down to showing that certain matching conditions hold at the intersections of the bhs's. For example, for smooth functions on a manifold with corners, a set of functions $f_i : H_i \longrightarrow \mathbb{R}$ admits an extension to a smooth function $u$ (i.e. $u|_{H_i} = f_i$) if and only if $f_i|_{H_i \cap H_j} = f_j|_{H_i \cap H_j}$ for all $i, j$ with $H_i \cap H_j \neq \varnothing$. For the convenience of the reader we include the general matching condition in Lemma A.1 below, and we verify these now.

Such a $K'$ will exist by Lemma A.1 in Appendix A provided the hypotheses are satisfied, meaning that the following matching conditions hold. We must find a set $\{\rho_\bullet\}$ of boundary defining functions for the boundary hypersurfaces, $\bullet$ = lf, rf, tf, tb, ff, $\mathrm{ff}_1$ of $M^2_{\mathrm{heat}}$ such that

$$\begin{aligned} \kappa_{\mathrm{ff}} &= \frac{1}{(4\pi)^{n/2}} \tau^{-n} \sum_{j \in \mathbb{N}} \tau^j \widetilde{b}_j|_{\mathrm{ff}}, \\ \widetilde{x}^{kn} \kappa_{\mathrm{ff}_1} &= \widetilde{x}^{1+b+kf} \kappa_{\mathrm{ff}} \text{ on } \mathrm{ff} \cap \mathrm{ff}_1, \end{aligned} \tag{3.62}$$

and that $\kappa_{\mathrm{ff}}, \kappa_{\mathrm{ff}_1}$ and the $b_j$ vanish to infinite order at tb. Indeed, in the notation of Lemma A.1 we have $\kappa_1 = (\rho_{\mathrm{ff}_1}/\widetilde{x})^{1+b+kf} \kappa_{\mathrm{ff}_1}$ and $\kappa_2 = (\rho_{\mathrm{ff}}/\widetilde{x})^{kn} \kappa_{\mathrm{ff}}$, and the matching conditions in terms of $\kappa_1$ and $\kappa_2$ in Lemma A.1 are exactly (3.62). We use boundary defining functions $\rho_{\mathrm{ff}} = \overline{\rho}$, $\rho_{\mathrm{ff}_1} = \rho/\overline{\rho}$ for the faces ff and $\mathrm{ff}_1$ defined in (3.12) and (3.9). Finally, we use $\tau$ in (3.17) as $\rho_{\mathrm{tf}}$; though it is not valid at $\mathrm{tb} \cap \mathrm{tf}$, all the distributions in question will vanish to infinite order there and there will be no conditions to check.

The first matching condition in (3.62) follows easily since the coefficients of the expansion of $\kappa_{\mathrm{ff}_1}$ are determined by the same differential equation which determines the $b_j$, and the coefficients in both expansions are uniquely determined by their being equal to polynomials times Gaussians on the fibers of $\mathrm{tf} \cap \mathrm{ff}$.

Finally we check that the second condition in (3.62) holds. First we consider $\kappa_{\mathrm{ff}_1} = \kappa_{\mathrm{ff}_1,\widetilde{y}}(s,1,\eta,t')$ above the point $\widetilde{y} \in Y$ (i.e. restricted to $\mathrm{ff}_{1\widetilde{y}}$). In the polar coordinates in (3.12) and using the boundary defining functions above (3.19), we have

$$s = \frac{\overline{\psi}_x \rho_{\mathrm{ff}}^{k-1}}{\rho_{\mathrm{lf}}} + 1, \quad t' = \frac{\overline{\phi}_t \rho_{\mathrm{ff}}^{2(k-1)}}{\rho_{\mathrm{lf}}^2}, \quad \eta = \frac{\overline{\psi}_y \rho_{\mathrm{ff}}^{k-1}}{\rho_{\mathrm{lf}}}. \tag{3.63}$$

Using [1, Eqn. 9.7.1], we have that the modified Bessel function satisfies

$$I_\nu(z) = (e^{-z}/\sqrt{2\pi z})(1 + O(1/z)),$$

and thus

$$\rho_{\mathrm{ff}}^{kn}(\rho_{\mathrm{lf}}\rho_{\mathrm{ff}})^{-1-b-kf}\kappa_{\mathrm{ff}_1,\widetilde{y}} = \frac{\rho_{\mathrm{lf}}^{-kf}}{(4\pi\overline{\phi}_t)^{(b+1)/2}} e^{-(|\overline{\psi}_{\widetilde{x}}^2+|\overline{\psi}_y|_{\widetilde{y}}^2)/4\overline{\phi}_t}(1 + O(\rho_{\mathrm{ff}})). \tag{3.64}$$

On the other hand, above each base point $\widetilde{y} \in Y$, $\kappa_{\mathrm{ff},\widetilde{y}}(\sigma,\eta',z,z',\widetilde{T})$ can be written using separation of variables with respect to the spectrum of $\Delta_{\partial M/Y}$. Indeed, since $H_{Z,y}$ has discrete spectrum, it is standard that $H_{Z,y}(z,\widetilde{z},t) = \Pi_0 + E$, where $\Pi_0$ is projection onto the kernel of $\Delta_{\partial M/Y}$ and $|E| < e^{-\lambda_0 t}$ as $t \to \infty$, $\lambda_0$ being the smallest non-zero eigenvalue of $\Delta_{\partial M/Y}$. Thus

$$\kappa_{\mathrm{ff},y} = (2\pi\widetilde{T})^{-(b+1)/2} e^{-(\sigma^2+|\eta'|_{h_1}^2)/2}\Pi_0 + E', \tag{3.65}$$

where $E'$ is exponentially decaying. Now we have

$$\widetilde{T} = \overline{\phi}_t \rho_{\mathrm{ff}_1}^{-2(k-1)}\rho_{\mathrm{lf}}^{-2k}, \quad \eta' = \overline{\phi}_t \rho_{\mathrm{ff}_1}^{-(k-1)}\rho_{\mathrm{lf}}^{-k}, \quad \sigma = \overline{\psi}_x \rho_{\mathrm{ff}_1}^{-(k-1)}\rho_{\mathrm{lf}}^{-k}, \tag{3.66}$$

and thus

$$\rho_{\mathrm{ff}_1}^{1+b+kf}(\rho_{\mathrm{lf}}\rho_{\mathrm{ff}_1})^{-kn}\kappa_{\mathrm{ff}} = \frac{\rho_{\mathrm{lf}}^{-kf}}{(4\pi\overline{\phi}_t)^{(b+1)/2}} e^{-(|\overline{\psi}_{\widetilde{x}}^2+|\overline{\psi}_y|_{\widetilde{y}}^2)/4\overline{\phi}_t}, \tag{3.67}$$

so the matching condition at $\mathrm{ff} \cap \mathrm{ff}_1$ holds.

On the other hand, $\widetilde{x}^{1+b+kf}\kappa|_{\mathrm{ff}_1} = \kappa_{\mathrm{ff}_1}$ by (3.55). Since we have not yet prescribed $K'$ near lf and rf, we may set $K'$ equal to $\kappa$ in an open neighborhoods of $\mathrm{lf} \cap \mathrm{ff}_1$ and $\mathrm{rf} \cap \mathrm{ff}_1$ and the compatibility condition will be satisfied there. □

Next we correct this distribution $K_1$ by adding terms to it, so that the resulting distribution $K$ satisfies appropriate decay estimates for the error $(\partial_t + \Delta)K$. Our distribution $K$ will have the same asymptotic properties at the boundary hypersurfaces enumerated in Lemma 3.5 as $K_1$ does, except that (3) must be modified to include error terms of order $O(\rho_{\mathrm{lf}}^k)$.

**Proposition 3.6.** *There exists a polyhomogeneous section $K \in \mathcal{A}_{\mathrm{phg}}(M^2_{\mathrm{heat}}; \beta^*(\mathrm{End}))$ satisfying properties* (1) *and* (2) *of the distribution $K_1$ in Lemma 3.5, and satisfying* (3) *with the exception of* (3.60)*, instead satisfying*

$$K = \kappa\big(1 + O(\rho_{\mathrm{lf}}^k)\big) \tag{3.68}$$

*in a neighborhood of* lf*,nn such that the "error" $Q := t(\partial_t + \Delta)K$ is polyhomogeneous with index set $\mathcal{E}'$ satisfying*

$$\begin{gathered}\inf \mathcal{E}'(\mathrm{ff}_1)) \geq -1 - b - kf + 1, \quad \inf \mathcal{E}'(\mathrm{ff}) \geq -kn + 1, \\ \mathcal{E}'(\mathrm{lf}) = \mathcal{E}'(\mathrm{tf}) = \mathcal{E}'(\mathrm{tb}) = \varnothing.\end{gathered} \tag{3.69}$$

*Proof.* Taking a distribution $K_1$ provided by Lemma 3.5, we study $t(\partial_t + \Delta)K_1$. Automatically we have that $t(\partial_t + \Delta)K_1$ vanishes to infinite order at tf and tb, as follows from Lemma 3.1. Furthermore, $t(\partial_t + \Delta)K_1$ vanishes to order $-kn+1$ at ff by (3.35) and the fact that the leading order term $\kappa_{\mathrm{ff}}$ there solves the model problem.

At $\mathrm{ff}_1$ things are again more delicate. Recall that $K_1 = O(\rho_{\mathrm{ff}}^{-1-b-kf})$ at $\mathrm{ff}_1$, where $\rho_{\mathrm{ff}_1}$ is the boundary defining function for $\mathrm{ff}_1$ in, e.g. $\rho_{\mathrm{ff}_1} = \rho/\overline{\rho}$ with $\rho$ as in (3.9) and $\overline{\rho}$ as in (3.12). Since $K_1$ is fiber harmonic near $\mathrm{ff}_1$, by (2.12) and (3.36) we have

$$\begin{aligned}\Delta K_1 &= \widetilde{\Delta}_0 K_1 + x^{-2k}\Delta_{\partial M/Y} K_1 + x^{-k}\eth_{\partial M/Y} P K_1 + x^{-1}\widetilde{E} K_1 \\ &= \widetilde{\Delta}_0 K_1 + x^{-k}\eth_{\partial M/Y} P K_1 + x^{-1}\widetilde{E} K_1 + x^{-k}\eth_{\partial M/Y} P K' \\ &\qquad + O\big(\rho_{\mathrm{ff}_1}^{-1-b-kf+2k}\big).\end{aligned}$$

Furthermore, by (3.37) we have that $\Pi_{\mathcal{H}} t(\partial_t + \Delta)\Pi_{\mathcal{H}} K_1$ is order $-1-b-kf+1$ since its leading order term solves the model problem.

We assert the existence of a polyhomogeneous distribution $A$ of order $-1-b-kf+k$ such that $t(\partial_t + \Delta)(K_1 - A)$ itself vanishes to order $-1-b-kf+1$ at $\mathrm{ff}_1$. Indeed, since the leading order term in $t(\partial_t + \Delta)$ is $tx^{-2k}\Delta_{\partial M/Y}$, and since by (2.13) we can find $B$ such that

$$\Delta_{\partial M/Y} B = \eth_{\partial M/Y} P K' + \eth_{\partial M/Y} P K_1, \tag{3.70}$$

where $B$ is polyhomogeneous with asymptotic expansion determined by the expansion of the right hand side, in particular $B = O(\rho_{\mathrm{ff}_1}^{-1-b-kf})$. We take $A = x^k B$ and thus obtain, with $\widetilde{P}$ as in (2.21),

$$\begin{aligned}t(\partial_t + \Delta)(K_1 - x^k B) &= t(\partial_t + \widetilde{\Delta}_0)(K_1 - x^k B) + tx^{-1}\widetilde{E}(K_1 - x^k B) \\ &\qquad - tx^{-k}\widetilde{P}x^k B + tO\big(\rho_{\mathrm{ff}_1}^{-1-b-kf+2k}\big) \\ &= t(\partial_t + \widetilde{\Delta}_0)(K_1 - x^k B) + tO\big(\rho_{\mathrm{ff}_1}^{-1-b-kf}\big) \\ &\qquad + tx^{-1}O\big(\rho_{\mathrm{ff}_1}^{-1-b-kf}\big) + tO\big(\rho_{\mathrm{ff}_1}^{-1-b-kf}\big) \\ &= O\big(\rho_{\mathrm{ff}_1}^{-1-b-kf+1}\big).\end{aligned}$$

Since the expansion of $B$ at ff has the same order as $K_1$, the distribution

$$K_2 = K_1 - x^k B$$

has all of the desired properties of $K$ in the statement of the proposition except that $(\partial_t + \Delta)K_2$ is not rapidly decreasing at lf. Note that, since $\rho_{\mathrm{ff}_1}^{1+b+kf} K_1 = O(s^{\overline{\nu}(\mathbf{N})})$ where $\overline{\nu}$ is the (fiber degree dependent) order of $\kappa$ computed in (3.52). We claim that $B$ also satisfies $B = O(\rho_{\mathrm{lf}}^{\overline{\nu}})$. Indeed, $B$ is determined by solving (3.70), i.e. by inverting an elliptic differential operator on the space orthogonal to its cokernel; by the basic elliptic regularity estimate, for any $m \in \mathbb{R}$ there is a $C$ such that

$$\Delta_{\partial M/B} u = \eth f \implies \|u\|_{H^m(\partial M/B)} \leq C\|f\|_{H^{m-1}(\partial M/B)}.$$

Thus if $f$ is a parametrized family satisfying $f = O_{H^{m-1}}(\rho_{\mathrm{lf}}^{\overline{\nu}})$ then $u = O_{H^m}(\rho_{\mathrm{lf}}^{\overline{\nu}})$, and the same goes for $B$, and since $m$ can be taken arbitrarily large the claim follows by Sobolev embedding.

To deal with the expansion at lf we argue along similar lines, but there we iterate the argument to get a parametrix $K$ with $(\partial_t + \Delta)K$ vanishing to infinite order at lf. (We work in the interior of lf though the arguments at the intersection of lf and $\mathrm{ff}_1$ are the same in the projective coordinates

$$s' = x/\widetilde{x}, \eta' = (y - \widetilde{y})/\widetilde{x}, \tau' = t/\widetilde{x}^2$$

together with $z, \widetilde{x}, \widetilde{y}, \widetilde{z}$.) Recall that $K_1 \equiv \kappa$ near lf and thus $K_2 = \kappa - x^k B$ near $\mathrm{ff}_1$. Again with $\widetilde{P}$ as in (2.21), we have

$$\begin{aligned}(\partial_t + \Delta)K_2 &= x^{-k}\widetilde{P}\kappa + x^{k-2}\widetilde{E}\kappa - x^{-k}\Delta_{\partial M/Y}B - x^{-k}\widetilde{P}x^k B + O(x^{\overline{\nu}+k}) \\ &= x^{-k}\eth_{\partial M/Y}P\kappa - x^{-k}\Delta_{\partial M/Y}B + O(x^{\overline{\nu}+k-2}),\end{aligned} \tag{3.71}$$

where $\overline{\nu}$ is the leading order power of $\kappa$ computed in (3.52). As in the argument at $\mathrm{ff}_1$, since the RHS of (3.71) manifestly gives that $\Pi_{\mathcal{H}}((\partial_t + \Delta)K_2) = O(x^{\overline{\nu}+k-2})$, by (2.13) there is distribution $A_0$ such that

$$x^{\overline{\nu}}\Delta_{\partial M/Y}A_0 = \eth_{\partial M/Y}P\kappa - \Delta_{\partial M/Y}B + O(x^{\overline{\nu}+k}).$$

Here the factor $x^{\overline{\nu}}$ in front makes it so that $A_0$ is $O(1)$. Thus

$$(\partial_t + \Delta)(K_2 - x^{\overline{\nu}+k}A_0) = O(x^{\overline{\nu}+k-2}) - x^{-k}\widetilde{P}x^{\overline{\nu}+k}A_0 = O(x^{\overline{\nu}}).$$

We will now solve away iteratively to decrease the order of the error. For this we assume for the moment that we are given, for some $q > \overline{\nu} + \epsilon$, *any* distribution $A_1 = x^q\widetilde{A}_1 + O(x^{q+\epsilon})$ with $\widetilde{A}_1$ smooth and non-vanishing up to the boundary as an ice-form. First, we find a distribution $B_1$ so that

$$x^q A_2 := (\partial_t + \Delta)(x^{q+2k}B_1) - A_1$$

is fiber harmonic. We can do this by solving

$$(I - \Pi_{\mathcal{H}})A_1 = \Delta_{\partial M/Y} B_1 + O(x^k)$$

as in (2.13), where $\Pi_{\mathcal{H}}$ is the projection onto the fiber harmonic forms, since then

$$(\partial_t + \Delta)x^{q+2k} B_1 = x^q \Delta_{\partial M/Y} B_1 + O(x^{q+k}).$$

We then construct a term $C_1$ with $(\partial_t + \Delta)x^{q+2}C_1 \approx A_2$, as follows. Decomposing $A_2 = A_2^1 + A_2^2 dx$ with $\iota(dx)A_2^1 = 0$, write

$$C_1 = \big((-(q+2)^2 - (\alpha-1)(q+2) + \beta)^1 A_2^{-1}, (-(q+2)^2 - (\alpha-1)(q+2) + \gamma)^{-1} A_2^2\big),$$

then

$$\begin{pmatrix} P_{\alpha,\beta} & 0 \\ 0 & P_{\alpha,\gamma} \end{pmatrix} x^{q+2} C_1 = x^q A_2.$$

(The numbers we divided by above are non-zero, since the indicial roots of $P_{\alpha,\beta}$ and $P_{\alpha,\gamma}$ are bounded above by $\overline{\nu} - \epsilon$, as explained below (3.52).) For this $C_1$ we have

$$\begin{aligned} x^q A_2 - (\partial_t + \Delta)x^{q+2}C_1 &= x^q A_2 - \widetilde{\Delta}_0 x^{q+2} C_1 + x^{-k} \widetilde{P}' x^{q+2} C_1 + O(x^{q+2}) \\ &= O(x^{q+\delta}) + x^{-k} \widetilde{P}' x^{q+2} C_1 + O(x^{q+2+k-2}), \end{aligned}$$

where $q+\delta$ can be taken to be the order of the subsequent term in the expansion of $A_2$ where $\widetilde{\Delta}_0$ is in (3.36) and $\widetilde{P}$ is as in (2.21), and thus by (2.13) we see that the left hand side lies in the image of $\Delta_{\partial M/Y}$ to order $x^k$. We can thus find a distribution $D_1$ such that

$$\begin{aligned} &x^q A_2 - (\partial_t + \Delta)(x^{q+2}C_1 - x^{q+2+k} D_1) \\ &\qquad = O(x^{q+1}) - x^{q+2-k} \Delta_{\partial M/Y} D_1 + x^k \widetilde{P}' x^{q+2} C_1 \\ &\qquad = O(x^{q+1}), \end{aligned}$$

which gives

$$(\partial_t + \Delta)(x^q(x^{2k}B_1 - x^2 C_1 + x^{2+k} D_1)) = x^q A_1 + O(x^{q+\delta}). \tag{3.72}$$

It is straightforward to check that the added terms do not increase the order of blowup at $\mathrm{ff}_1$. Thus we can kill off the leading order term of $x^q A$, and in fact can kill off all terms iteratively by this process. (If there are log terms present the argument is analogous and left to the reader.)

From the previous two paragraphs, it follows that we can find a distribution $K'$ such that $K := K_2 - K'$ satisfies the requirements of the lemma, specifically such that $t(\partial_t + \Delta)K$, in addition to having the same leading order asymptotics at tf and ff and $\mathrm{ff}_1$ that $t(\partial_t + \Delta)K_2$ has, also vanishes to infinite order at lf. Indeed, since we can solve away terms to obtain errors of succesivly decreasing order, taking the Borel sum [31] of these distributions gives $K'$. □

Finally we establish our main structure theorem for the heat kernel.

**Theorem 3.7.** *There exists a section* $H \in \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(M^2_{\mathrm{heat}}; \beta^* \operatorname{End})$ *satisfying all of the properties of the distributions* $K$ *from Proposition 3.6, and which is a fundamental solution to the heat equation, meaning that in the interior of* $M^2_{\mathrm{heat}}$, $(\partial_t + \Delta)H = 0$, *while the operator* $H_t$ *defined initially on forms* $\alpha \in C^\infty_c(M; \Omega^*(M))$ *by*

$$H_t\alpha(w) = \int_M H(w, \widetilde{w}, t)\alpha(\widetilde{w})\, \mathrm{dVol}_{\widetilde{w}} \tag{3.73}$$

*extends to a bounded map of* $L^2(\Omega^*(M), d\,\mathrm{Vol}_g)$, *and for such* $\alpha$ $H_t\alpha \to \alpha$ *as* $t \to 0$ *in* $L^2$.

We will prove the theorem now modulo arguments in Appendix B.

*Proof.* Consider the parametrix $K$ whose existence is established in Proposition 3.6. This $K$ satisfies all but one of the properties of the $H$ in the theorem, namely $(\partial_t + \Delta)K$ is not equal to zero. (Indeed, the statement about convergence to the identity in (3.73) follows from the behavior of $K$ at tf described in (3.28).

We now invert error $Q = t(\partial_t + \Delta)K$ from Proposition 3.6 via a Neumann series. To be precise, it will be convenient to think of distributional kernels $A(p, p', t)$ on $M \times M \times \mathbb{R}^+$ acting on $C^\infty_c(M^\circ \times (0,\infty))$ by operating as convolution kernels in the time variable, so for $\phi \in C^\infty_c(M^\circ \times (0,\infty))$ by

$$(A \star \phi)(p, t) := \int_M \int_0^t A(p, p', t-s)\phi(p', s)\, ds\, \mathrm{dVol}_{p'}\,. \tag{3.74}$$

Then

$$(\partial_t + \Delta)\widetilde{K} = I + t^{-1}Q, \tag{3.75}$$

and the right hand side can be inverted via a Neumann series, i.e.

$$(\mathrm{Id} + t^{-1}Q)(I + Q') = \mathrm{Id},$$

where $Q' = \sum_{j=1}^\infty (-1)^j (t^{-1}Q)^j$ and $(t^{-1}Q)^j = t^{-1}Q \star \cdots \star t^{-1}Q$, $j$-times. We then show that

$$H := K(I + Q')$$

satisfies all of the properties claimed in the theorem, but now it is automatic that $(\partial_t + \Delta)H = 0$; what it will remain to show is that $K(I + Q')$ continues to satisfy the properties of $K$ from Proposition 3.6.

We use Proposition B.5 below to analyze the summands $(t^{-1}Q)^j$. Note that $t^{-1}$ is a polyhomogeneous distribution on $M^2_{\mathrm{heat}}$; indeed $t = \rho_{\mathrm{tb}}\rho^2_{\mathrm{tf}}\rho^2_{\mathrm{ff}_1}\rho^{2k}_{\mathrm{ff}}a$ with $a \in C^\infty(M^2_{\mathrm{heat}})$ with $a > c \geq 0$. Thus $t^{-1}Q$ is polyhomogeneous with index set $\mathcal{E}''$ given by shifting the index set $\mathcal{E}'$ of $Q$ from Proposition 3.6 by appropriate integers, namely

$$\begin{gathered} \inf \mathcal{E}''(\mathrm{ff}_1)) \geq -2 - b - kf, \quad \inf \mathcal{E}''(\mathrm{ff}) \geq -kn - 2k + 1, \\ \mathcal{E}''(\mathrm{lf}) = \mathcal{E}''(\mathrm{tf}) = \mathcal{E}''(\mathrm{tb}) = \varnothing. \end{gathered} \tag{3.76}$$

Proposition B.5 then implies that $(t^{-1}Q)^j$ is polyhomogeneous with index set $\mathcal{E}^{(j)}$ satisfying, for any $\epsilon > 0$,

$$\inf \mathcal{E}^{(j)}(\mathrm{ff}_1)) \geq j(1-\epsilon) - 3 - b - kf, \quad \inf \mathcal{E}^{(j)}(\mathrm{ff}) \geq -kn - 2k + 1, \quad (3.77)$$

in addition to $\mathcal{E}^{(j)}(\mathrm{lf}) = \mathcal{E}^{(j)}(\mathrm{tf}) = \mathcal{E}^{(j)}(\mathrm{tb}) = \varnothing$. There for the $(t^{-1}Q)^j$ admit a Borel sum, i.e. a sum $Q' = \sum_{j=1}^{\infty}(-1)^j(t^{-1}Q)^j$ with the property that the difference of a partial sum up to $j = N$ with $Q'$ is polyhomogeneous and vanishes at each face to the order of $(t^{-1}Q)^j$ at each bhs. Moreover, as discussed in [5, 30], this series is convergent in $C^\infty$, indeed the uniform bounds in [5, Theorem 2.23] hold in this setting, and the infinite order of vanishing of $t^{-1}Q$ at lf is preserved in the sum, i.e. $Q'$ vanishes also to infinite order there. The form of the distributional kernel $H = K(I + Q')$ is analyzed as in [28]. There it is shown that polyhomogeneous with the index set $\mathcal{E}$ satisfying the properties of Theorem 3.7. □

## 4. Spectral and Hodge theoretic properties of the Hodge–Laplacian

In this section we deduce the main theorems from the introduction. We begin with a detailed analysis of the polyhomogeneous forms in the maximal domain.

**4.1. Polyhomogeneous forms in $\mathcal{D}_{\max}$ and $\mathcal{D}_{\min}$.** Recall the definition of $\mathcal{D}_{\max}$ and $\mathcal{D}_{\min}$ from the introduction, and the space $\mathcal{A}_{\mathrm{phg}}({}^{\mathrm{ice}}\Lambda^*)$ of polyhomogeneous ice-forms (below denoted simply by $\mathcal{A}_{\mathrm{phg}}$) discussed in Appendix A. We also recall that the incomplete cusp edge manifold $(M, g_{\mathrm{ice}})$ is assumed to satisfy the Witt condition (1.4) and that the metric $g_{\mathrm{ice}}$ takes the form

$$g_{\mathrm{ice}} = dx^2 + x^{2k} g_Z + \pi^* g_Y + \tilde{g},$$

where the exponent $k \geq 3$.

We determine conditions which assure that a given polyhomogeneous differential form $\gamma \in \mathcal{A}_{\mathrm{phg}}$ is contained in the maximal domain $\mathcal{D}_{\max}$ of $\Delta^g$. This will be used to show, with an additional assumption on the index set of a phg form, that

$$\gamma \in \mathcal{D}_{\max} \cap \mathcal{A}_{\mathrm{phg}} \implies \gamma \in \mathcal{D}_{\min} \cap \mathcal{A}_{\mathrm{phg}}. \quad (4.1)$$

Let $\gamma \in \mathcal{A}_{\mathrm{phg}}$ be contained in the maximal domain, i.e. we assume that $\gamma \in L^2$ and $\Delta^g \gamma \in L^2$. Let $\gamma = x^s \tilde{\gamma}$ where $\tilde{\gamma} = \tilde{\gamma}_0(y,z) + \mathcal{O}(x^\epsilon)$. Here notation such as $\mathcal{O}(x^\epsilon)$ indicates that the differential form $\gamma$ is locally a combination of basis forms

$$dy_I \wedge x^{k\mathbf{N}} dz_A \quad \text{and} \quad dx \wedge x^{k\mathbf{N}} dy_I \wedge x^{k\mathbf{N}} dz_A,$$

where $I$ and $A$ are multi-indices on the base and fiber, respectively. with coefficient functions which are bounded by $cx^\epsilon$ pointwise in norm when $x \searrow 0$, and $\tilde{\gamma}_0$ is a form on $M$ whose coefficient functions are independent of $x$. Let us determine the possible range of values $s$. From (2.7)–(2.8) it follows that in a neighborhood of the boundary, the volume form of the cusp edge metric $g$ is

$$\mathrm{dVol}_g = x^{kf} \rho \, dx \wedge dy \wedge dz,$$

where $\rho = a(y,z) + \mathcal{O}(x^k)$ and $a$ is a non-vanishing positive function. It follows that

$$x^s \widetilde{\gamma} \in L^2(M,g) \iff s > -\frac{1}{2}(kf+1). \tag{4.2}$$

We begin by analyzing the indicial roots of $\Delta^g$, specifically we find the order of vanishing of fiber harmonic homogeneous forms in the kernel of $\Delta^g$. By Proposition 2.3, the leading order part of $\Delta^g$ restricted to fiber harmonic forms is

$$\Pi_{\mathcal{H}} \Delta_0^g \Pi_{\mathcal{H}} \sim := \begin{pmatrix} P_{\alpha(\mathbf{N}),\beta(\mathbf{N})} & 0 \\ 0 & P_{\alpha(\mathbf{N}),\gamma(\mathbf{N})} \end{pmatrix},$$

with $P_{\alpha(\mathbf{N}),\beta(\mathbf{N})}$, $P_{\alpha(\mathbf{N}),\gamma(\mathbf{N})}$ the operators, depending on fiber degree, defined in (3.38)–(3.39) We note that

$$P_{\alpha(f-\mathbf{N}),\beta(f-\mathbf{N})} = P_{\alpha(\mathbf{N}),\gamma(\mathbf{N})} \quad (\mathbf{N} = 0, \dots, f). \tag{4.3}$$

Using (3.43), a straightforward calculation shows that $P_{\alpha(\mathbf{N}),\beta(\mathbf{N})} x^s = 0$ if

$$s \in \bigcup_{N=1}^{f} \big\{1 - k(f-N), -k(f-N), -kN, 1-kN\big\}.$$

It in addition satisfies condition (4.2) if

$$s = \begin{cases} -k\,\mathbf{N} & \text{and} \quad \mathbf{N} < \frac{1}{2}(f + \frac{1}{k}), \\ 1 - k\,\mathbf{N} & \text{and} \quad \mathbf{N} < \frac{1}{2}(f + \frac{3}{k}), \\ -k(f-\mathbf{N}) & \text{and} \quad \mathbf{N} > \frac{1}{2}(f - \frac{1}{k}), \\ 1-k(f-\mathbf{N}) & \text{and} \quad \mathbf{N} > \frac{1}{2}(f - \frac{3}{k}). \end{cases} \tag{4.4}$$

**Proposition 4.1.** *Suppose the differential form* $\gamma = (\gamma^1, \gamma^2) = (x^{s_1}\widetilde{\gamma}^1, x^{s_2}\widetilde{\gamma}^2) \in \mathcal{A}_{\mathrm{phg}}$ *and that* $\gamma = \widetilde{\gamma}_0^j(y,z) + \mathcal{O}(x^\epsilon)$ *is contained in the maximal domain* $\mathcal{D}_{\max}$. *(Thus the leading order term is assumed not to have a logarithm, as is a priori allowed for* phg*-distributions.) Then each* $s_j$ *is an indicial root of* $P_{\alpha(\mathbf{N}),\beta(\mathbf{N})}$ *for some* $0 \le \mathbf{N}_j \le f$ *or* $s_j > \frac{1}{2}(-kf+3)$. *In either case,* $s_j \ge \frac{1}{2}(-kf+3)$.

*Proof.* Recall from Proposition 2.3 the decomposition $\Delta = \Delta_0 + x^{-k}\widetilde{P} + x^{-1}\widetilde{E}$ and write

$$\Delta_0 = \begin{pmatrix} P_{\alpha(\mathbf{N}),\beta(\mathbf{N})} & 0 \\ 0 & P_{\alpha(\mathbf{N}),\gamma(\mathbf{N})} \end{pmatrix} + \begin{pmatrix} \frac{1}{x^{2k}}\Delta_{\partial M/Y} + \Delta_H & -2kx^{-k-1} d_{\partial M/Z} \\ -2kx^{-k-1}\delta_{\partial M/Y} & \frac{1}{x^{2k}}\Delta_{\partial M/Y} + \Delta_H \end{pmatrix}. \tag{4.5}$$

In view of the symmetry (4.3) it suffices to consider the image of the the component $\gamma^1 = x^{s_1}\widetilde{\gamma}^1$ under $\Delta^g$. The discussion naturally falls into several cases.

(1) The form $\widetilde{\gamma}_0^1$ is *not* fiber harmonic. Then the lowest nonvanishing term in (4.5) is $x^{-2k+s_1}\Delta_{\partial M/Y}\widetilde{\gamma}_0^1$, which is contained in $L^2$ if and only if

$$s_1 > \frac{1}{2}\Big(4k - \frac{1}{2}kf - 1\Big).$$

(2) The form $\widetilde{\gamma}_0^1$ is fiber harmonic. We then consider the following subcases.

(2.a) $s_1$ is an indicial root of $P_{\alpha(\mathbf{N}_1),\beta(\mathbf{N}_1)}$ and hence equals the number in (4.4).

(2.b) $s_1$ is not an indicial root of $P_{\alpha(\mathbf{N}_1),\beta(\mathbf{N}_1)}$, i.e. $P_{\alpha(\mathbf{N}_1),\beta(\mathbf{N}_1)}(x^{s_1}\widetilde{\gamma}^1) \neq 0$. We claim that at least one of the following two statements holds true:

- The polyhomogeneous expansion of $\widetilde{\gamma}^1$ contains a term $\widetilde{\gamma}_\ell^1$ of order $\mathcal{O}(x^\delta)$ where $\delta - 2k < s_1 - 2$ and $\widetilde{\gamma}_\ell^1$ is not fibre harmonic.
- The lowest nonvanishing term in the first component of $\Delta^g\gamma$ is of order $x^{s_1-2}$.

If this claim holds true we conclude that the lowest nonvanishing term in the first component of $\Delta^g\gamma$ is of order at most $x^{s_1-2}$. To prove the claim, assume that the first statement is false. Then the second one must hold true as is clear from the form of the Laplacian $\Delta_0$ in (4.5). To be specific, collecting the terms of order $x^{s_1-2}$ in the first component of $\Delta_0\gamma$ we obtain

$$\begin{aligned} P_{\alpha(\mathbf{N}_1),\beta(\mathbf{N}_1)}(x^{s_1}\widetilde{\gamma}^1) + x^{-2k}\Delta_{\partial M/Y}\tau^1 + x^{-k-1}d_{\partial M/Y}\tau^2 \\ + x^{-k}\eth_{\partial M/Y}P\tau^3 + x^{-k}P\eth_{\partial M/Y}\tau^4 \end{aligned} \tag{4.6}$$

for suitable differential forms $\tau^1,\dots,\tau^4$ of orders

$$\tau^1 = O(x^{s_1+2k-2}), \quad \tau^2 = O(x^{s_1+k-1}), \quad \text{and} \quad \tau^j = O(x^{s_1+k-2}) \quad (j=3,4).$$

By Hodge theory, the term $\eth_{\partial M/Y}\tau^4$ vanishes, since otherwise a nonvanishing term $x^{-2k}\Delta_{\partial M/Y}\tau^4$ would occur, which is of order strictly less than $s_1 - 2$, contradicting our initial assumption. Considering the remaining three terms in (4.6) it follows from Hodge theory and the assumption that $\widetilde{\gamma}_0^1$ is fibre harmonic that the sum

$$x^{-2k}\Delta_{\partial M/Y}\tau^1 + x^{-k-1}d_{\partial M/Y}\tau^2 + x^{-k}\eth P\tau^3 \tag{4.7}$$

is orthogonal, over each fibre, to $P_{\alpha(\mathbf{N}_1),\beta(\mathbf{N}_1)}(x^{s_1}\widetilde{\gamma}^1)$. Hence we conclude that the nonzero term $P_{\alpha(\mathbf{N}_1),\beta(\mathbf{N}_1)}(x^{s_1}\widetilde{\gamma}^1)$ cannot cancel with the sum (4.7). It follows that the second statement is true, whence the claim.

The asserted statement follows by inspection of each of the above cases. In case (1) it follows from

$$s_1 > \frac{1}{2}\Big(4k - \frac{1}{2}kf - 1\Big) > \frac{1}{2}(-kf + 3),$$

using that $k \geq 3$. In case (2.b) the lowest nonvanishing term in $\Delta^g \gamma$ is of order at most $s_1 - 2$. Since $\gamma \in \mathcal{D}_{\max}$ it follows from (4.2) that

$$s_1 - 2 > -\frac{1}{2}(kf + 1) \iff s_1 > -\frac{1}{2}(kf + 3).$$

In case (2.a), the form $\widetilde{\gamma}_0^1$ is fibre harmonic and therefore by the Witt condition $\mathbf{N} \neq \frac{f}{2}$. We continue by discussing each of the four possible cases in (4.4) separately. Suppose first that $s_1 = -k\,\mathbf{N}$ and $\mathbf{N} < \frac{1}{2}(f + \frac{1}{k})$. If $f$ is even this implies that the integer $\mathbf{N} \leq \frac{f}{2} - 1$ (here we use the Witt condition) and consequently

$$s_1 \geq -k\Big(\frac{f}{2} - 1\Big) \geq -\frac{kf}{2} + \frac{3}{2},$$

since $k \geq 3$. If $f$ is odd then $\mathbf{N} \leq \frac{f}{2} - \frac{1}{2}$ and

$$s_1 \geq -k\Big(\frac{f}{2} - \frac{1}{2}\Big) \geq -\frac{kf}{2} + \frac{3}{2},$$

where the last inequality follows again from the assumption $k \geq 3$. Similarly, in the case $s_1 = -k(f - \mathbf{N})$ and $\mathbf{N} > \frac{1}{2}(f - \frac{3}{k})$ it follows that if $f$ is even that $\mathbf{N} \geq \frac{f}{2} + 1$ (using the Witt condition). This implies the estimate

$$s_1 \geq -k\Big(f - \frac{f}{2} - 1\Big) \geq -\frac{kf}{2} + \frac{3}{2},$$

using that $k \geq 3$. If $f$ is odd then we conclude that $\mathbf{N} \geq \frac{f}{2} + \frac{1}{2}$ and hence

$$s_1 \geq -k\Big(f - \frac{f}{2} - \frac{1}{2}\Big) \geq -\frac{kf}{2} + \frac{3}{2},$$

using again that $k \geq 3$. The conclusion in the remaining two cases where $s_1 = 1 - k\,\mathbf{N}$ or $s_1 = 1 - k(f - \mathbf{N})$ follows analogously. $\square$

**Lemma 4.2.** *Assume $k \geq 3$. Then $\mathcal{D}_{\min} \cap \mathcal{A}_{\mathrm{phg}} = \mathcal{D}_{\max} \cap \mathcal{A}_{\mathrm{phg}}$.*

*Proof.* It suffices to prove the inclusion $\mathcal{D}_{\max} \cap \mathcal{A}_{\mathrm{phg}} \subseteq \mathcal{D}_{\min} \cap \mathcal{A}_{\mathrm{phg}}$. For $\varepsilon > 0$ we define the logarithmic cutoff function $\chi_\varepsilon \colon [0, \infty) \to [0, 1]$ by

$$\chi_\varepsilon(x) := \begin{cases} 0, & x \leq \varepsilon^2, \\ -\frac{\log(x/\varepsilon^2)}{\log(\varepsilon)}, & \varepsilon^2 < x < \varepsilon, \\ 1, & x \geq \varepsilon. \end{cases}$$

For $\varepsilon^2 < x < \varepsilon$ it satisfies

$$\chi_\varepsilon'(x) = -\frac{1}{\log(\varepsilon) x} \quad \text{and} \quad \chi_\varepsilon''(x) = \frac{1}{\log(\varepsilon) x^2}. \tag{4.8}$$

Let $\gamma \in \mathcal{D}_{\max} \cap \mathcal{A}_{\rm phg}$ and set $\gamma_\varepsilon = \chi_\epsilon \gamma$. Then

$$\Delta^g \gamma_\varepsilon = \chi_\varepsilon \Delta^g \gamma - (\partial_x^2 \chi_\varepsilon)\gamma - (\partial_x \chi_\varepsilon)(\partial_x \gamma) - \frac{kf}{x}(\partial_x \chi_\varepsilon)\gamma + A^j(\partial_x \chi_\varepsilon)(\partial_{z_j}\gamma) + B^i(\partial_x \chi_\varepsilon)(\partial_{y_i}\gamma), \quad (4.9)$$

where $A^j = \mathcal{O}(x^k)$ and $B^i = \mathcal{O}(x^{2k})$ are bounded functions with that order of decay in $x$. We show that

$$\|\Delta^g \gamma_\varepsilon - \Delta^g \gamma\|_{L^2(M,g)} \to 0 \quad \text{as } \varepsilon \to 0, \quad (4.10)$$

hence establishing that $\gamma \in \mathcal{D}_{\min}$. It is clear that

$$\|\chi_\varepsilon \Delta^g \gamma - \Delta^g \gamma\|_{L^2(M,g)} \to 0 \quad \text{as } \varepsilon \to 0,$$

and thus it suffices to consider the next three terms in (4.9) and to show that

$$\frac{1}{\log^2(\varepsilon)} \int_{\varepsilon^2}^{\varepsilon} \frac{1}{x^4}|\gamma|^2 x^{kf}\, dx + \frac{1}{\log^2(\varepsilon)} \int_{\varepsilon^2}^{\varepsilon} \frac{1}{x^2}|\partial_x \gamma|^2 x^{kf}\, dx + \frac{k^2 f^2}{\log^2(\varepsilon)} \int_{\varepsilon^2}^{\varepsilon} \frac{1}{x^4}|\gamma|^2 x^{kf}\, dx \quad (4.11)$$

converges to 0 as $\varepsilon \to 0$. Let $\gamma = x^s \tilde{\gamma}$ for some $\tilde{\gamma} = \mathcal{O}(1)$. A short calculation shows that each integrand in (4.11) is of order $x^{-1+\delta}$ for some $\delta > 0$ and hence converges to 0 as $\varepsilon \to 0$ if

$$s > -\frac{kf}{2} + \frac{3}{2}. \quad (4.12)$$

In the borderline case $s = -\frac{kf}{2} + \frac{3}{2}$ we still get convergence since then the first integral in (4.11) becomes

$$\frac{1}{\log^2(\varepsilon)} \int_{\varepsilon^2}^{\varepsilon} \frac{1}{x}\, dx = \frac{1}{\log^2(\varepsilon)}(\log(\varepsilon) - \log(\varepsilon^2)) = -\frac{1}{\log(\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0,$$

and analogously for the second and third integral. Hence

$$s \geq -\frac{kf}{2} + \frac{3}{2} \implies \gamma \in \mathcal{D}_{\min}$$

for any $\gamma = x^s \tilde{\gamma} \in \mathcal{D}_{\max} \cap \mathcal{A}_{\rm phg}$. On the other hand, Proposition 4.1 shows that

$$\gamma = x^s \tilde{\gamma} \in \mathcal{D}_{\max} \cap \mathcal{A}_{\rm phg} \implies s \geq -\frac{kf}{2} + \frac{3}{2},$$

and hence the claim follows. □

### 4.2. Spectral theory.

*Proof of Theorem 1.4.* Let $H \in \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(M^2_{\mathrm{heat}}; \beta^* \operatorname{End})$ be as in 3.7; we will show that such $H$ has the properties stated in Theorem 1.4.

First note that, as we show in a moment, $H$ maps into $\mathcal{D}_{\min}$ and hence the following three properties hold:

- $(\partial_t + \Delta_g)H = 0$,
- $\lim_{t\to 0} H_t = \mathrm{Id}$,
- $H(L^2) \subset \mathcal{D}_{\mathcal{F}}$, where $\mathcal{D}_{\mathcal{F}}$ is the Friedrichs domain.

From [28, p. 21], these three properties characterize the Friedrichs heat kernel of $\Delta$. Hence $H_t$ is automatically symmetric.

Since $H_t$ and $\partial_t H_t$ are formally self-adjoint (i.e. symmetric), to show that they are self-adjoint it suffices to show that they are compact operators. But indeed they are, as follows from [34, Thm. VI.23–24] together with

$$H_t, \partial_t H_t \in L^2(\operatorname{End}; M \times M),$$

where, given a smooth section $A$ of End, then $A \in L^2(\operatorname{End}; M \times M)$ if

$$\int \|A(p,q)\|^2_{\operatorname{End}} \, d\mathrm{Vol}_M(p)\, d\mathrm{Vol}_M(q) < \infty.$$

For $t > 0$, $H_t$ is given by an $L^2$ integral kernel, so is a compact operator; indeed, by (3.68), the index set $\mathcal{F}$ of $H_t \in \mathcal{A}_{\mathrm{phg}}(M \times M)$ restricted to $t > 0$ constant is $\mathcal{F}(\mathrm{lf}) = \mathcal{E}(\mathrm{lf})$ and $\mathcal{F}(\mathrm{rf}) = \mathcal{E}(\mathrm{rf})$, for $\mathcal{E}$ the index family of $H$. From (3.61), these satisfy the lower bound

$$\inf \mathcal{F}(\mathrm{lf}), \inf \mathcal{F}(\mathrm{rf}) \geq -\frac{kf}{2} + 1 \tag{4.13}$$

(meaning $H_t$ is a bounded endomorphism) and

$$d\mathrm{Vol}_M(p)\, d\mathrm{Vol}_M(q) \simeq x^{kf} \tilde{x}^{kf}\, dx\, d\tilde{x}\, dy\, d\tilde{y}\, dz\, d\tilde{z}, \tag{4.14}$$

so the kernel of $H_t$ is square integrable.

It remains to establish (1.4), i.e. that $H_t(\alpha) \in \mathcal{D}_{\min}$ for every $\alpha \in L^2$. In fact, $H_t(\alpha)$ is a polyhomogeneous distribution with index set $\mathcal{E}(\mathrm{lf})$. This is straightforward: writing the expansion of $H_t$ at $x = 0$ up to some order $N$ we have

$$H_t = \sum_{\substack{(s,p)\in\mathcal{E}(\mathrm{lf}) \\ |s|\leq N}} x^s \log^p(x) a_{s,p}(y, z, \tilde{w}) + E_N,$$

where $\tilde{w} = (\tilde{x}, \tilde{y}, \tilde{z})$, and the coefficients $a_{s,p}$ are polyhomogeneous endomorphisms on the manifold with boundary $\partial M \times M$ and $E_N$ is a polyhomogeneous endomorphism on $M \times M$ with $E_N = o(x^N)$. Thus

$$H_t(\alpha) = \int_M \Big( \sum_{\substack{(s,p)\in\mathcal{E}(\mathrm{lf}) \\ \Re s \leq N}} x^s \log^p(x) a_{s,p}(z,y,\widetilde{w})\alpha(\widetilde{w}) + E_N \alpha(\widetilde{w}) \Big) \, \mathrm{dVol}_g(\widetilde{w}). \tag{4.15}$$

For example by [26, Proposition 3.20], since the $x^{-N}E_N$ are given by a polyhomogeneous integral kernel, they define bounded maps of $L^2$, and the conormality estimates (see (A.3)–(A.4)) follow by differentiating $x^{-N}E_N$. The integrals coming from the partial expansion terms are finite and give the expansion coefficients of $H_t(\alpha)$. This shows that $H_t(\alpha) \in \mathcal{A}_{\mathrm{phg}}$, and moreover that the leading order term has no logarithmic factor. Thus, In view of Lemma 4.2 it suffices to prove $H_t(\alpha) \in \mathcal{D}_{\max}$ in order to conclude that $H_t(\alpha) \in \mathcal{D}_{\min}$. But indeed, $\inf \mathcal{E}(\mathrm{lf})$ satisfies the lower bound (4.13), hence it follows that the lowest order term in the polyhomogeneous expansion (4.15) is of order at least $-\frac{kf}{2} + 1$ which by (4.2) is sufficient to conclude $H_t(\alpha) \in L^2$. Because $H_t$ is a fundamental solution of the heat equation, it follows that $\Delta^g H_t(\alpha) = -\partial_t H_t(\alpha)$ which by the same argument is contained in $L^2$ since $\partial_t H_t$ has the same index set as $H_t$ for $t > 0$. □

It now follows that the fundamental solution $H_t$ from Theorem 3.7 is in fact the heat kernel in the following sense.

**Proposition 4.3.** *The heat kernel* $\exp(-t\Delta^g)$ *defined by applying the spectral theorem to the self-adjoint operator* $(\Delta^g, \mathcal{D})$ *has Schwartz kernel equal to the fundamental solution* $H_t$ *in Theorem 3.7, meaning*

$$(e^{-t\Delta^g}\alpha)(w) = \int_M H_t(w, \widetilde{w}, t)\alpha(\widetilde{w}) \, \mathrm{dVol}_g(\widetilde{w}).$$

Using this we may finish the proof of Theorem 1.1.

*Proof of Theorem 1.1.* As discussed below the statement of Theorem 1.4, Theorem 1.4 itself establishes essential self-adjointness of $\Delta^g$. It remains to prove that the spectrum is discrete, but this follows immediately from the spectral theorem and the fact that $H_t$ is a compact operator (hence has discrete spectrum.)

Moreover, the Weyl asymptotic formula in (1.6) follows from the standard heat kernel argument in [36, §8.3] together with the heat trace asymptotics in Corollary 4.4. □

**Corollary 4.4** (Heat trace asymptotics). *For each* $t > 0$, *the fundamental solution* $H_t$ *in Theorem 3.7 is trace class and satisfies that* $F(t) := \operatorname{Tr} H_t$ *is a polyhomogenous conormal distribution on* $\mathbb{R}^+$ *satisfying*

$$F(t) = t^{-n/2}\,\mathrm{Vol}(M,g) + \Big( \sum_{j=1}^{f=\dim Z} a_j t^{-n/2+j/2} \Big) + c_0 t^{-(b+1)/2+1/2k} + O(t^{-(b+1)/2+1/2k+\epsilon}). \tag{4.16}$$

The proof of Corollary 4.4, which uses Theorem 3.7 and Melrose's pushforward theorem, is deferred to Section A.2 below.

**4.3. Harmonic forms and Hodge theory.** We begin our discussion of Hodge theory by pointing out that elements $\gamma \in L^2$ satisfying $\Delta^g \gamma = 0$, admit asymptotic expansions at the boundary of $M$. Indeed, for such forms $\gamma$, by the spectral theorem and the fact that $H_t$ is the heat kernel (Corollary 4.3), we see that

$$\gamma = H_t \gamma = \int_M H_t(w, \widetilde{w}, t)\gamma(\widetilde{w})\, \mathrm{dVol}_g(\widetilde{w}). \tag{4.17}$$

By the proof of Theorem 1.4, specifically (4.15), we have the following.

**Lemma 4.5.** *Assume that* $\gamma \in \ker(\Delta^g : L^2 \longrightarrow L^2)$*. Then* $\gamma$ *is polyhomogeneous conormal and* $\gamma = \mathcal{O}(1)$*, i.e. is bounded in norm.*

Lemma 4.5 allows us to conclude that the $L^2$ kernel of $\Delta^g$ is equal to the Hodge cohomology in (1.7).

**Lemma 4.6.** *Notation as above,* $\mathcal{H}_{L^2}(M, g) = \ker(\Delta^g : L^2 \longrightarrow L^2)$.

*Proof.* If $\gamma \in \mathcal{H}_{L^2}(M, g)$ then $\gamma$ is in the maximal domains of both $d$ and $\delta$, and so for smooth compactly supported $\beta$,

$$\langle \Delta^g \gamma, \beta\rangle_{L^2} := \langle \gamma, \Delta^g \beta\rangle_{L^2} = \langle \gamma, d\delta\beta\rangle_{L^2} + \langle \gamma, \delta d\beta\rangle_{L^2} = 0 + 0 = 0,$$

so $\gamma \in \ker(\Delta^g : L^2 \longrightarrow L^2)$. On the other hand, if $\gamma \in \ker(\Delta^g : L^2 \longrightarrow L^2)$, then by Lemma 4.5 we can integrate by parts to obtain

$$0 = \langle \Delta^g \gamma, \gamma\rangle_{L^2} = \|d\gamma\|^2_{L^2} + \|\delta\gamma\|^2_{L^2},$$

so $\gamma \in \mathcal{H}^*_{L^2}(M, g)$. □

We can now follow the arguments in [18,20] to prove Theorem 1.2 above. Before we begin we recall some facts about intersection cohomology, a cohomology theory that applies to stratified spaces. We do not attempt to make a full explanation of it here, but mention only that there is in fact a family of intersection cohomology groups for our stratified space $X$ defined in (1.8) (obtained by collapsing the boundary of $\partial M$ over the base $Y$) depending on a function $\mathfrak{p}: \mathbb{N} \longrightarrow \mathbb{N}$ called the "perversity," which is non-decreasing and whose values matter only on the codimensions of the strata of $X$. Here we have only one singular stratum, $Y \subset X$, the image of the boundary $\partial M$ via the projection onto $X$, and its codimension is $f + 1$, where $\dim Z = f$. The "upper middle degree" perversity $\overline{\mathfrak{m}}$ is a special example of a perversity, which satisfies

$$\overline{\mathfrak{m}}(f+1) = \begin{cases} (f-1)/2 & \text{if } f \text{ is odd,} \\ f/2 - 1 & \text{if } f \text{ is even.} \end{cases} \tag{4.18}$$

The "lower middle perversity" $\underline{\mathfrak{m}}$ differs from $\overline{\mathfrak{m}}$ only when $f$ is even, in which case $\underline{\mathfrak{m}}(f+1) = f/2$. As we will rely on the spectral sequence arguments from [18, 20] during the proof, we will only need to study the intersection cohomology locally, specifically on a basis of open sets of $X$. Concretely, from [18], for canonical neighborhoods $U = V \times C_1(Z)$ as in (2.2) with contractible $V$, we have

$$IH^p_{\mathfrak{p}}(U) = \begin{cases} H^p(Z) & \text{if } p < f - 1 - \mathfrak{p}(f+1), \\ \{0\} & \text{if } p \geq f - 1 - \mathfrak{p}(f+1). \end{cases}$$

From the Witt condition (1.4), we see that

$$IH^p_{\overline{\mathfrak{m}}}(U) = IH^p_{\underline{\mathfrak{m}}}(U) = \begin{cases} H^p(Z) & \text{if } p < f/2, \\ \{0\} & \text{if } p \geq f/2, \end{cases} \tag{4.19}$$

regardless of the parity of $f$.

*Proof of Theorem 1.2.* Although Theorem 1.2 describes a relationship between the Hodge cohomology and the intersection cohomology, to prove it we go through the standard route and use the intermediary of $L^2$-cohomology. Thus consider the chain complex

$$\cdots \longrightarrow L^2_d\Omega^{p-1}(M,g) \longrightarrow L^2_d\Omega^p(M,g) \longrightarrow L^2_d\Omega^{p+1}(M,g) \longrightarrow \cdots, \tag{4.20}$$

where $L^2_d\Omega^p(M,g)$ is the maximal domain of the exterior derivative $d$, specifically

$$L^2_d\Omega^p(M,g) = \big\{\alpha \in L^2\Omega^p(M,g) : d\alpha \in L^2\Omega^{p+1}(M,g)\big\}.$$

Then the $L^2$-cohomology is the quotient

$$L^2H^p(M,g) = \frac{\{\alpha \in L^2_d\Omega^p(M,g) : d\alpha = 0\}}{\{d\eta : \eta \in L^2_d\Omega^{p-1}(M,g)\}}.$$

As explained in [20, p. 6], it suffices to show that

$$L^2H^p(M,g) \simeq IH^p_{\overline{\mathfrak{m}}}(X;\mathbb{R}), \tag{4.21}$$

for then the $L^2$-cohomology is finite dimensional, which implies that the range of $d$ (and thus its adjoint $\delta$) is closed. From [18, §2.1] it then follows using the Kodaira decomposition theorem that $\mathcal{H}^p_{L^2}(M,g)$ is isomorphic to $L^2H^p(M,g)$ and thus by (4.21) Theorem 1.2 holds.

Thus it suffices to prove (4.21), and for this we also follow the arguments in [20, pp. 5–6], where it is explained that it suffices to show that for canonical neighborhoods $U = V \times C_1(Z)$ as in (2.2) with contractible $V$, the local chain complex

$$\cdots \longrightarrow L^2_d\Omega^{p-1}(U,g) \longrightarrow L^2_d\Omega^p(U,g) \longrightarrow L^2_d\Omega^{p+1}(U,g) \longrightarrow \cdots, \tag{4.22}$$

satisfies

$$L^2H^p(U,g) \simeq IH^p_{\overline{\mathfrak{m}}}(U). \tag{4.23}$$

Here $L^2H^p(U,g)$ is defined as above with $U$ replacing $M$. The intersection cohomology groups for $\overline{\mathfrak{m}}$ are computed in (4.19), and thus we need only to analyze the groups on the left. To see (4.23) we use the Künneth formula of Zucker, [40, Corollary 2.34], whose assumptions are satisfied here by the fact that the exterior derivative on $Z$ is closed on its maximal domain. Thus, in the notation of [20, p. 5], we have

$$L^2H^p(U,g) = \bigoplus_{i=0}^{1} WH^i\big((0,1), dx^2, k(p-i-f/2)\big) \otimes H^{p-i}(Z;\mathbb{R}), \tag{4.24}$$

where $WH^i((0,1), dx^2, a)$ is the cohomology of the complex

$$0 \longrightarrow \big(x^a L^2\Omega^0((0,1), dx^2)\big) \xrightarrow{d} x^a L^2\Omega^1((0,1), dx^2) \longrightarrow 0, \tag{4.25}$$

where the space on the left is the maximal domain of $d$ on $x^a L^2\Omega^0((0,1), dx^2)$. Again from [20] (via [18]),

$$\begin{aligned} &WH^1((0,1), dx^2, a) = 0 && \text{if } a \neq 1/2,\\ &WH^0((0,1), dx^2, a) = \mathbb{R} && \text{if } a < 1/2, \text{ and}\\ &\{0\} && \text{if } a \geq 1/2. \end{aligned}$$

When $i = 1$, $k(p-i-f/2) \neq 1/2$ since $k > 1$, so the $i = 1$ terms do not contribute. When $i = 0$, we have $k(p-i-f/2) = k(p-f/2)$ which satisfies

$$k(p-f/2) < 1/2 \text{ if } p \leq f/2 \quad \text{and} \quad k(p-f/2) > 1/2 \text{ if } p > f/2.$$

Using the Witt condition then gives

$$L^2H^p(U,g) = \begin{cases} H^p(Z) & \text{if } p < f/2,\\ \{0\} & \text{if } p \geq f/2, \end{cases} \tag{4.26}$$

matching (4.19) and completing the proof. □

We now discuss the proof of Theorem 1.3. As the spaces in the theorem are incomplete cusp edge spaces in a neighborhood of the divisor by [32], our results would apply to these spaces, if not for the fact that moduli spaces such as these have interior orbifold points. This is not a problem, since, as in [21] we may lift to a finite cover with no such points. One can then work on the space $C^\infty_{c,\mathrm{orb}}(\mathcal{M}_{1,1})$ of functions which near each orbifold point are smooth when lifted to a local finite cover resolving the singularity. Constructing a heat kernel on the lift and averaging over the group action then gives a fundamental solution to the heat kernel downstairs which has all the desired properties. We leave the details of this simple extension to the reader.

## A. Manifolds with corners

In this section we recall some of the facts about distributions on manifolds with corners (mwc's) used in this paper. This material is due largely to Melrose, and the reader is referred to his book [30] for more details. See also [19].

The objects considered here, for example the ice-metrics, have polyhomogeneous regularity, which we define now. The sheaf of *polyhomogeneous conormal* (or polyhomogeneous, or simply phg) functions $\mathcal{A}_{\text{phg}}(X)$ is defined as follows. First, an index set $\mathcal{E}$ on a manifold with corners $X$ is an association to each boundary hypersurface $H$ of $X$ a set

$$\begin{aligned}&\mathcal{E}(H) \subset \mathbb{C} \times \mathbb{N} \text{ satisfying that the subset}\\&\{(z,p) \in \mathcal{E}(H) : \operatorname{Re} z < c\} \text{ is finite for all } c \in \mathbb{R}.\end{aligned} \tag{A.1}$$

Given an index set $\mathcal{E}$, for a boundary face $F = \cap_{i=1}^{\delta} H_i$ for boundary hypersurfaces $H_i$, define the subset $\mathcal{E}(F) \subset \mathbb{C}^p \times \mathbb{N}^p$ by $(z,p) = (z_1,\dots,z_\delta,p_1,\dots,p_\delta) \in \mathcal{E}(F)$ if and only if $(z_i,p_i) \in \mathcal{E}(H_i)$. We define the Frechet space $\mathcal{A}_{\text{phg}}^{\mathcal{E}}(X)$ as follows. We write $u \in \mathcal{A}_{\text{phg}}^{\mathcal{E}}(X)$ if and only if for each boundary face $F = \cap_{i=1}^{\delta} H_i$, writing $\rho_i$ for a boundary defining function of $H_i$, $u$ satisfies

$$u \sim \sum_{(z,p)\in\mathcal{E}(F)} a_{z,p}\rho^z \log^p \rho \tag{A.2}$$

where

$$\rho^z = \prod_{i=1}^{\delta} \rho_i^{z_i}, \quad \log^p \rho = \prod_{i=1}^{\delta} \log^{p_i} \rho_i,$$

and the symbol $\sim$ means that

$$E_N = u - \sum_{\substack{(z,p)\in\mathcal{E}(F)\\ \operatorname{Re} z_i < N\ \forall i}} a_{z,p}\rho^z \log^p \rho. \tag{A.3}$$

Here $E_N$ is a smooth function on the interior of $X$ which is $O(|\rho|^N)$, where

$$|\rho| = (\rho_1^2 + \cdots + \rho_\delta^2)^{1/2}.$$

Moreover, $E_N$ is conormal, meaning that if $\mathcal{V}_b = \mathcal{V}_b(X)$ denotes the set of smooth vector fields on $X$ that are tangent to all boundary hypersurfaces, then

$$|\rho|^{-N}\mathcal{V}_b^k E \subset L^\infty. \tag{A.4}$$

Note that if a phg function $u$ vanishes to infinite order at $H$, then $u$ is polyhomogeneous with index set $\mathcal{E}$ satisfying $\mathcal{E}(H) = \varnothing$.

**Lemma A.1.** *Let $X$ denote a mwc, $\mathcal{M}(X) = \{H_i\}_{i \in \mathcal{I}}$ its boundary hypersurfaces, and for each $i \in \mathcal{I}$, let $\rho_i$ denote a boundary defining function of $H_i$. Given a smooth vector bundle $E \longrightarrow X$, if $\kappa_i$ are polyhomogeneous sections on $H_i$, then provided*

$$\rho_i^{c_i} \kappa_j|_{H_i \cap H_j} = \rho_j^{c_j} \kappa_i|_{H_i \cap H_j} \tag{A.5}$$

*there exists a polyhomogeneous conormal distribution $K$ on $X$ satisfying*

$$\rho_i^{c_i} K|_{H_i} = \kappa_i. \tag{A.6}$$

*Assume moreover that at a particular boundary hypersurface which we take to be $H_1$, that we are given an index set $F_1 \subset \mathbb{C} \times \mathbb{N}$ and polyhomogeneous sections $b_{j,p} \in \mathcal{A}_{\mathrm{phg}}(E|_{H^1}; H_1)$. Then given functions $\kappa_i$ on $H_i$, $i \neq 1$, there exists a distribution $K$ satisfying (A.6) for $i \neq 1$ and such that*

$$K \sim \sum_{s,p \in F_1} \rho_1^s \log^p(\rho_1) b_{s,p} \tag{A.7}$$

*provided (A.5) holds for $i, j \neq 1$ and furthermore for $i \neq 1$*

$$\kappa_i \sim \rho_i^{c_i} \sum_{s,p \in F_1} \rho_1^s \log^p(\rho_1) b_{s,p}|_{H_i}. \tag{A.8}$$

**Remark A.2.** (1) Note the converse; if $K = \rho_i^{-c_i} \rho_j^{-c_j} a$ for some positive function $a$ near $H_i \cap H_j$ then setting $\rho_l^{c_l} K|_{H_l} = \kappa_l$ for $l = i, j$, we have $\rho_j^{c_j} \kappa_i = \rho_i^{c_i} \kappa_j$ on $H_i \cap H_j$.

(2) The matching condition (A.5) implies further matching conditions on multifold intersections, e.g. it implies that

$$\rho_i^{c_i} \rho_j^{c_j} \kappa_l = \rho_i^{c_i} \rho_l^{c_l} \kappa_j = \rho_l^{c_l} \rho_j^{c_j} \kappa_i \text{ on } H_i \cap H_j \cap H_l.$$

(3) The second matching condition (A.8) merely says that the desired data on a bhs $H_i$ has the same asymptotic expansion at $H_1$ as the the desired distribution restricted to $H_i$.

*Proof.* Denote the number of boundary hypersurfaces of $X$ by $m = |\mathcal{M}|$. There is a number $\delta$ and boundary defining functions $\rho_i$ such that the set $\{\rho_i < \delta\}$ is diffeomorphic as mwc's to $H_i \times [0, \delta)$. Without loss of generality we take $\delta = 1$. Following the remark, for a collection of bhs' $H_{i_1}, \dots, H_{i_p}$, the distribution

$$\kappa_{i_1 \dots i_p} = \Big( \prod_{i \neq i_k} \rho^{c_i} \Big) \kappa_{i_k}|_{\rho_{i_1} = \cdots = \rho_{i_p} = 0}$$

is well-defined independently of the choice of $i_k \in \{1, \dots, m\}$.

Let $\chi(x)$ be a cutoff function with $\chi \equiv 1$ for $x \leq 1/3$ and $\chi \equiv 0$ for $x \geq 2\epsilon/3$. For the distribution $K$ we may take

$$K = \sum_{p=1}^{m} (-1)^{p-1} \sum_{1 \leq i_1 < \cdots < i_p \leq m} \kappa_{i_1 \ldots i_p} \Big( \prod_{j \in \{i_1,\ldots,i_p\}} \chi(\rho_j) \rho_j^{-c_j} \Big).$$

For example if $m = 2$ then

$$K = \chi(\rho_1)\rho_1^{-c_1}\kappa_1 + \chi(\rho_2)\rho_2^{-c_2}\kappa_2 - \chi(\rho_1)\chi(\rho_2)\rho_1^{-c_1}\rho_2^{-c_2}\kappa_{12}.$$

Note that each term in the sum defining $K$ defines a polyhomogeneous conormal distribution on all of $X$, as the distribution $\kappa_{i_1 \ldots i_p}$ is defined on a neighborhood of $H_{i_1} \cap \cdots \cap H_{i_p}$ off which the product $\prod_{j \in \{i_1,\ldots,i_p\}} \chi(\rho_j)$ vanishes.

Letting $A_{i_1 \ldots i_p}$ be the term corresponding term in the definition of $K$, note that if $i \notin \{i_1, \ldots, i_p\}$ then $\rho_i^{c_i} A_{i_1 \ldots i_p} = \rho_i^{c_i} A_{i_1 \ldots i \ldots i_p}|_{\rho_i = 0}$. Fixing $i$, multiplying by $\rho^{c_i} K$ and restricting to $\rho_i = 0$ gives

$$\begin{aligned} &\rho_i^{c_i} K|_{\rho_i=0} \\ &= \kappa_i + \sum_{p}^{m-1} (-1)^{p-1} \rho_i^{c_i} \Big( \sum_{\substack{1 \leq i_1 < \cdots < i_p \leq m \\ i \notin \{i_1 \ldots i_p\}}} A_{i_1 \ldots i_p} - \sum_{\substack{1 \leq i_1 < \cdots < i_{p+1} \leq m \\ i \in \{i_1 \ldots i_{p+1}\}}} A_{i_1 \ldots i_{p+1}} \Big)|_{\rho_i=0} \\ &= \kappa_i, \end{aligned}$$

which establishes (A.6).

We now prove the final statement of the lemma. Let $\chi$ be the cutoff function defined above. First, we claim that under the stated assumptions there exists a distribution $K'$ supported in $\{\rho_1 \leq 1\}$ satisfying both (A.7) (with $K$ replaced by $K'$) and that

$$\rho_i^{c_i} K'|_{H_i} = \chi(\rho_1)\kappa_i \tag{A.9}$$

for each $i \neq 1$. To see this, take any distribution $K''$ supported in $\{\rho_1 \leq 1\}$ satisfying (A.7), and note that $a_i := \rho_i^{c_i} K''|_{H_i} - \chi(\rho_1)\kappa_i = O(\rho_1^{\infty})$. By the support condition, the distribution $K' = K'' - \sum_{i \neq 1} \chi(\rho_i) a_i$ is defined globally, has the same asymptotic expansion at $H_1$ as $K''$, and satisfies (A.9). This $K'$ will play the role of $\chi(\rho_1)\rho^{-c_1}\kappa_1$ from the previous paragraph. Concretely, for $1 < i_1 < \cdots < i_p \leq m$, let $a_{i_1 \ldots i_p} = (\Pi_{j \in \{i_1,\ldots,i_p} \rho_j^{c_j} K')|_{H_{i_1} \cap \cdots \cap H_{i_p}}$. Then we may take

$$\begin{aligned} K = &\sum_{p=1}^{m} (-1)^{p-1} \sum_{1 < i_1 < \cdots < i_p \leq m} \kappa_{i_1 \ldots i_p} \Big( \prod_{j \in \{i_1,\ldots,i_p\}} \chi(\rho_j)\rho_j^{-c_j} \Big) \\ &+ K' + \sum_{p=1}^{m} (-1)^{p-1} \sum_{1 < i_1 < \cdots < i_p \leq m} a_{i_1 \ldots i_p} \Big( \prod_{j \in \{i_1,\ldots,i_p\}} \chi(\rho_j)\rho_j^{-c_j} \Big). \end{aligned}$$

Again, for example if $m = 2$ then

$$K = K' + \chi(\rho_2)\rho_2^{-c_2}\kappa_2 - (\rho_2^{c_1}K')|_{H_2}\rho_2^{-c_2}\chi(\rho_2).$$

The given expression for $K$ can be directly checked to satisfy (A.6) and (A.7). □

**A.1. Melrose's pushforward theorem.** Given a map $\beta\colon X \longrightarrow Y$ between manifolds with corners, if $\mathcal{M}(\bullet)$ with $\bullet = X, Y$ denotes the space of boundary hypersurfaces, then $\beta$ is a *b-map* if it is smooth and if for each $H \in \mathcal{M}(Y)$ with $\rho_H$ a boundary defining function for $H$ then

$$\beta^*\rho_H = a\Pi_{H'_j\in\mathcal{M}(X)}\rho_{H'_1}^{e(H'_1,H)}\rho_{H'_1}^{e(H'_2,H)}\dots\rho_{H'_1}^{e(H'_N,H)},$$

where $a \in C^\infty(X)$ is non-vanishing and $N$ is the number of boundary hypersurfaces of $Y$ and the $e(H', H)$ are non-negative integers. This means foremost that $\rho_H$ pulls back to a smooth function, and the numbers $e(H', H)$ simply keep track of the order of vanishing of $\beta^*\rho_H$ at each face of $X$. The function

$$e\colon \mathcal{M}(X) \times \mathcal{M}(Y) \longrightarrow \mathbb{N}_0 \tag{A.10}$$

is the *exponent matrix* of $\beta$, and $e(H', H) > 0$ means $H'$ maps into $H$ via $\beta$.

If a b-map has a few additional properties then it pushes forward polyhomogeneous distributions (more accurately, densities) to polyhomogeneous distributions and their index sets change in a way dictated by the exponent matrix. Note that it follows from the definition of a b-map that every boundary face $F$ of $X$ (meaning an intersection of boundary hypersurfaces), can be associated to a face $\overline{\beta}(F)$ of $Y$ defined to be the unique face with $\beta(x) \in \overline{\beta}(F)^\circ$ for every $x \in F^\circ$. A b-map $\beta\colon X \longrightarrow Y$ is a *b-fibration* if:

- $\beta$ does not increase the codimension of faces, i.e. for each boundary face $F$ of $X$, the associated face $\overline{\beta}(F)$ in $Y$ satisfies that $\operatorname{codim}(F) \leq \operatorname{codim}(\overline{\beta}(F))$.
- Restricted to the interior of any face $F^\circ$, $\beta\colon F^\circ \longrightarrow (\overline{\beta}(F))^\circ$ is a fibration of open manifolds in the standard sense.

According to a theorem of Melrose [29] which we state below, a b-fibration pushes forward phg densities to phg densities in a manner we describe now. First, on a manifold with corners we choose a non-vanishing b-density $\mu$, meaning a section of $|\Lambda|^n({}^bT^*X)$, the density bundle of the b-cotangent bundle. The b-tangent bundle ${}^bTX$ is the bundle whose smooth sections are $\mathcal{V}_b$, the vector fields tangent to the boundary. The bundle ${}^bT^*X$ is the dual bundle of ${}^bTX$, and near a face $F = \cap_{i=1}^{\delta} H_i$ where $\rho_i$ are bdf's and $y$ are coordinates on $F$ then, the sections of ${}^bT^*X$ take the form

$$\sum_i \xi_i \frac{d\rho_i}{\rho_i} + \eta\, dy.$$

It follows that near any intersection $F = \cap_{j \in J} H_{j_q}$ of boundary hypersurfaces for $J \subset \mathcal{J}$ where $\mathcal{J}$ indexes $\mathcal{M}(X)$ (i.e. any face of $X$) that a non-vanishing b-density takes the form

$$\mu = \left| a \frac{dy \prod_{j \in J} d\rho_j}{\prod_{j \in J} \rho_j} \right| \tag{A.11}$$

for some smooth non-vanishing function $a$ on $X$. A polyhomogeneous b-density $u \in \mathcal{A}^{\mathcal{E}}_{\text{phg}}(X) \otimes |\Lambda|^n({}^bT^*X)$ can be written as $f\mu$ for a phg function $f$ and the index set of $u$ is by definition the index set of $f$.

**Theorem A.3** (Melrose [29]). *Let $u \in \mathcal{A}^{\mathcal{E}}_{\text{phg}}(X) \otimes |\Lambda|^n({}^bT^*X)$ be a polyhomogeneous b-density on $X$ with index set $\mathcal{E}$, let $f : X \longrightarrow Y$ be a b-fibration with exponent matrix e, and define the pushforward $f_*u$ to be the distribution on smooth functions $v \in C^\infty_{comp}(Y)$ acting by $\langle f_*u, v\rangle_Y = \langle u, f^*v\rangle_X$. Then provided for each $H \in \mathcal{M}(X)$ we have*

$$e(H, H') = 0\ \forall H' \in \mathcal{M}(Y) \implies \mathcal{E}(H) > 0, \tag{A.12}$$

*then $f_*u \in \mathcal{A}^{\mathcal{E}'}_{\text{phg}}(Y) \otimes |\Lambda|^n({}^bT^*Y)$ where*

$$\mathcal{E}'(H) = \overline{\bigcup}_{H'} \Big\{ \Big( \frac{z}{e(H', H)}, p \Big) : (z, p) \in \mathcal{E}(H) \Big\},$$

*with the (extended) union taken over $H'$ with $e(H', H) > 0$.*

The extended union, defined in [30], contains the standard union and possibly more log terms.

**A.2. Heat trace asymptotics.** We now use Theorem A.3 to prove the heat trace formula in Corollary 4.4 above. The heat trace is equal to

$$\operatorname{Tr}(e^{-t\Delta}) = \int_M H_t(w, w)\, \mathrm{dVol}_g = \sigma_*((\iota^* H_t)\, \mathrm{dVol}), \tag{A.13}$$

where $\iota : M \times [0, \infty) \longrightarrow M \times M \times [0, \infty)$ is the diagonal inclusion and $\sigma : M \times [0, \infty) \longrightarrow [0, \infty)$ is the projection onto the right factor. The natural space here on which to consider $H_t$ is $M^2_{\text{heat}}$, and thus to evaluate this pushforward we must see how $\sigma$ and $\iota$ act on the natural blown up spaces. The following may be easily verified.

(1) The closure

$$(M^2_{\text{heat}})_\Delta := \operatorname{cl}(\iota(M^\circ \times (0, \infty)))$$

is a manifold with corners with 4 boundary hypersurfaces, sf, $\text{ff}_1{}^d$, $\text{ff}^d$, $\text{tf}^d$, equal to the intersection of $\operatorname{cl}(\iota(M^\circ \times (0, \infty)))$ with $\text{rf} \cap \text{lf}$, $\text{ff}_1$, ff, and tf, respectively

(2) The map $\sigma$ extends form the interior $M^\circ \times (0, \infty)$ to a b-fibration

$$\sigma : (M^2_{\text{heat}})_\Delta \longrightarrow [0, \infty)$$

with exponent matrix

$$e_\sigma(\mathrm{sf}) = 0, e_\sigma(\mathrm{ff_1}{}^d) = 2, \quad e_\sigma(\mathrm{ff}^d) = 2k, e_\sigma(\mathrm{tf}^d) = 2.$$

To apply the pushforward theorem, we note that the volume density

$$\mu = \left| \mathrm{dVol}_g \frac{dt}{t} \right| = x^{kf+1} \frac{|dxdydzdt|}{xt}$$

is equal on $(M^2_{\mathrm{heat}})_\Delta$ to

$$\mu = a\,(\rho_{\mathrm{sf}}\rho_{\mathrm{ff_1}{}^d}\rho_{\mathrm{ff}^d})^{kf+1}\mu_0,$$

where $\mu_0$ is a non-vanishing b-density on $(M^2_{\mathrm{heat}})_\Delta$. Thus $(\iota^* H)\mu$ is phg on $(M^2_{\mathrm{heat}})_\Delta$ with index family $\mathcal{E}^d$ satisfying

$$\inf \mathcal{E}^d(\mathrm{sf}) = 3, \quad \inf \mathcal{E}^d(\mathrm{ff_1}{}^d) = -b,$$
$$\inf \mathcal{E}^d(\mathrm{ff}^d) = k(f-n)+1, \quad \mathcal{E}^d(\mathrm{tf}^d) = \{-n, -n+1, \dots\}.$$

Note that $\mathrm{Tr}\, e^{-t\Delta}\frac{dt}{t} = \sigma_*((\iota^* H_t)\mu)$. The integrability condition (A.12) must be checked only for sf and thus holds by Theorem 3.7, and we apply the pushforward theorem to obtain that $\mathrm{Tr}\, e^{-t\Delta}$ is polyhomogeneous with index set

$$\{(\zeta_1/2, p_1): (\zeta_1, p_1) \in \mathcal{E}^d(\mathrm{ff_1}{}^d)\} \,\overline{\cup}\, \{(\zeta_2/(2k), p_2): (\zeta_2, p_2) \in \mathcal{E}^d(\mathrm{ff}^d)\}$$
$$\overline{\cup}\, \{(\zeta_3/2, p_3): (\zeta_3, p_3) \in \mathcal{E}^d(\mathrm{tf}^d)\}.$$

In particular,

$$F(t) = \Big(\sum_{j=0}^{f} a_j t^{-n/2+j/2}\Big) + c_0 t^{-(b+1)/2+1/(2k)} + O(t^{-(b+1)/2+1/(2k)+\epsilon}),$$

for some $\epsilon > 0$. As discussed in [28, Section 3.3], the heat kernel in fact lies in an even calculus and thus the terms for odd $j$ in this sum are equal to 0, giving the trace formula (4.16). The fact that the leading order term is the volume is standard.

## B. Triple space

We will now analyze composition properties for "Volterra" type convolution operators as described in (3.74). To do so, following [16, 30], we construct a "triple space," which we denote by $M^3_{\mathrm{heat}}$, which is designed specifically to accommodate the process of composing operators which have the structure of the error terms in (3.75). The structure of our triple space is analogous to that constructed by Grieser and Hunsicker in [16], with slightly different homogeneities and with the added complication that there are time variables involved.

Note that, given $A_i$, $i = 1, 2$, we want is to make sense of the integral

$$\int_M \int_0^{t'} A_1(w, w', t') A_2(w', \widetilde{w}, t - t') \,\mathrm{dVol}_g(w') dt'. \tag{B.1}$$

Define the wedge

$$W := \{t - t' \geq 0\} \subset \mathbb{R}_t^+ \times \mathbb{R}_{t'}^+, \tag{B.2}$$

and define the left, center, and right projections

$$\begin{aligned}
\pi_L \colon M \times M \times M \times W &\longrightarrow M \times M \times [0, \infty)_{\mathrm{t}}, \\
(w, w', \widetilde{w}, t, t') &\longmapsto (w, w', t'), \\
\pi_C \colon M \times M \times M \times W &\longrightarrow M \times M \times [0, \infty)_{\mathrm{t}}, \\
(w, w', \widetilde{w}, t, t') &\longmapsto (w, \widetilde{w}, t), \\
\pi_R \colon M \times M \times M \times W &\longrightarrow M \times M \times [0, \infty)_{\mathrm{t}}, \\
(w, w', \widetilde{w}, t, t') &\longmapsto (w', \widetilde{w}, t - t').
\end{aligned} \tag{B.3}$$

Then, formally, the integral in (B.1) says that the integral kernel of $A_1 A_2$ (as an operator acting by convolution in time) is

$$(A_1 A_2)(w, \widetilde{w}, t) = (\pi_C)_* (\pi_L^* A_1)(\pi_R^* A_2), \tag{B.4}$$

where $(\pi_C)_*$ denotes the pushforward, i.e. the integral along the fibers of $\pi_C$ (which, by the way we have set up the problem, requires the choice of a metric on the fibers which we come to shortly.) Analysis of (B.4) becomes tractable if the space $M^3 \times W$ is blown up so that the pushforward theorem described in §A.1 applies.

Note that $M^3 \times W$ is a manifold with corners with 5 boundary hypersurfaces

$$\begin{aligned}
&\mathrm{L} = \{x = 0\}, \quad \mathrm{C} = \{x' = 0\}, \quad \mathrm{R} = \{\widetilde{x} = 0\}, \\
&\mathrm{tb}_1' = \{t' = 0\}, \quad \mathrm{tb}_2' = \{t - t' = 0\}.
\end{aligned}$$

It is easy to check that, in the language of Appendix A, the maps $\pi_\bullet$ with $\bullet \in \{L, C, R\}$ are b-maps from $M^3 \times W$ to $M^2 \times [0, \infty)_t$ and the exponent matrices are also easy to compute,

$$e_{\pi_L}(\bullet, \bullet') = \begin{cases} 1 & \bullet = \mathrm{L}, \quad \bullet' = \mathrm{lf}, \\ 1 & \bullet = \mathrm{C}, \quad \bullet' = \mathrm{rf}, \\ 1 & \bullet = \mathrm{tb}_1', \quad \bullet' = \mathrm{tb}, \\ 0 & \text{otherwise}, \end{cases} \quad e_{\pi_C}(\bullet, \bullet') = \begin{cases} 1 & \bullet = \mathrm{L}, \quad \bullet' = \mathrm{lf}, \\ 1 & \bullet = \mathrm{R}, \quad \bullet' = \mathrm{rf}, \\ 0 & \text{otherwise}, \end{cases} \tag{B.5}$$

$$e_{\pi_R}(\bullet,\bullet') = \begin{cases} 1 & \bullet = \mathrm{C}, \quad \bullet' = \mathrm{lf}, \\ 1 & \bullet = \mathrm{R}, \quad \bullet' = \mathrm{rf}, \\ 1 & \bullet = \mathrm{tb}_2', \quad \bullet' = \mathrm{tb}, \\ 0 & \text{otherwise}. \end{cases} \tag{B.6}$$

We blowup $M^3 \times W$ to form a space $\widetilde{\beta}\colon M^3_{\mathrm{heat}} \longrightarrow M^3 \times W$ in a sequence of steps as follows.

First, consider the three pullbacks of the submanifold

$$\mathcal{B}_0 = \{x = \widetilde{x}, y = \widetilde{y}, \widetilde{t} = 0\} \subset M^2 \times [0,\infty)_{\widetilde{t}}$$

defined in (3.6)

$$\pi_L^{-1}(\mathcal{B}_0), \quad \pi_C^{-1}(\mathcal{B}_0), \quad \pi_R^{-1}(\mathcal{B}_0). \tag{B.7}$$

These three sets intersect pair-wise in the triple intersection:

$$\pi_L^{-1}(\mathcal{B}_0) \cap \pi_C^{-1}(\mathcal{B}_0) = \pi_C^{-1}(\mathcal{B}_0) \cap \pi_R^{-1}(\mathcal{B}_0) = \pi_L^{-1}(\mathcal{B}_0) \cap \pi_R^{-1}(\mathcal{B}_0) = \mathcal{S}, \tag{B.8}$$

where

$$\mathcal{S} = \{x = x' = \widetilde{x} = t = t' = y - y' = y' - \widetilde{y} = 0\}. \tag{B.9}$$

We blowup the set $\mathcal{S}$, with appropriate homogeneities, specifically letting

$$M^3_{\mathrm{heat},0} = [M^3 \times W; \mathcal{S}]_{\text{q-hom}}, \tag{B.10}$$

with $t \sim x^2 \sim (x')^2 \sim \widetilde{x}^2 \sim |y-y'|^2 \sim |y'-\widetilde{y}|^2$, and let $\widetilde{\beta}_0\colon M^3_{\mathrm{heat},0} \longrightarrow M^3 \times W$ denote the blowdown map. Call the introduced boundary hypersurface $\mathrm{ff_1}^{\cap}$. Near to $\mathrm{ff_1}^{\cap}$, we have polar coordinates

$$\begin{aligned} \rho_\cap &= \big(t + x^2 + (x')^2 + \widetilde{x}^2 + |y-y'|^2 + |y'-\widetilde{y}|^2\big)^{1/2}, \\ \phi^\cap &= \Big(\frac{t'}{\rho_\cap^2}, \frac{t-t'}{\rho_\cap^2}, \frac{x}{\rho_\cap}, \frac{x'}{\rho_\cap}, \frac{\widetilde{x}}{\rho_\cap}, \frac{y-y'}{\rho_\cap}, \frac{y'-\widetilde{y}}{\rho_\cap}\Big) \\ &=: \big(\phi^\cap_{t'}, \phi^\cap_{t-t'}, \phi^\cap_x, \phi^\cap_{x'}, \phi^\cap_{\widetilde{x}}, \phi^\cap_{y-y'}, \phi^\cap_{y'-\widetilde{y}}\big), \text{ along with } y', z, z', \widetilde{z}. \end{aligned} \tag{B.11}$$

The asymmetry of the $y, y', \widetilde{y}$ in the coordinates is spurious in the sense that if one defines $\phi^\cap_{y-\widetilde{y}} = (y-\widetilde{y})/\rho_\cap$, then any two of the $\phi^\cap_{y-y'}, \phi^\cap_{y'-\widetilde{y}}$ can be used in $\phi^\cap$ by redefining $\rho_\cap$ using e.g. $|y-y'|^2$ and $|y-\widetilde{y}|^2$ (and then using $\phi^\cap_{y-y'}, \phi^\cap_{y'-\widetilde{y}}$). Either set of coordinates is defined in a collar neighborhood of $\mathrm{ff_1}^{\cap}$.

We then blowup the closures of the lifts

$$\mathcal{S}_\bullet := \mathrm{cl}((\pi_\bullet \circ \widetilde{\beta}_0)^{-1}(\mathcal{B}_0) \setminus \mathrm{ff_1}^{\cap}),$$

i.e. the rest of the lifts of the $\mathcal{B}_0$ via the three projections, where $\bullet \in \{L, C, R\}$. These are disjoint subsets and we blow them up in any order, setting

$$M^3_{\mathrm{heat},1} = [M^3_{\mathrm{heat},0}; \cup_{\bullet = L,C,R} \mathcal{S}_\bullet]_{\text{q-hom}}, \tag{B.12}$$

with the appropriate homogeneities, e.g. for $\mathcal{S}_L$ we have $t' \sim x^2 \sim (x')^2 \sim |y-y'|^2$. Again, we have a blowdown map

$$\widetilde{\beta}_1 \colon M^3_{\mathrm{heat},1} \longrightarrow M \times M \times M \times W. \tag{B.13}$$

The new faces we call $\mathrm{ff}_1^\bullet$ with $\bullet \in \{L, C, R\}$. Coordinates at $\mathrm{ff}_1^L$ can be determined as follows. Note that $\mathcal{S}_L$ is given in the coordinates (B.11) by $\phi^\cap_{t'} = \phi^\cap_x = \phi^\cap_{x'} = \phi^\cap_{y-y'} = 0$, and that in a neighborhood of $\mathcal{S}_L$ away from $\mathrm{ff}_1{}^\cap$, $\phi^\cap_{t'} \sim t'$. Thus, to match homogeneities with the blowups of the double space, we want to blow this up so that the following give polar coordinates near the intersection of $\mathrm{ff}_1^L$ and $\mathrm{ff}_1{}^\cap$:

$$\begin{aligned} \rho^L &= \big(\phi^\cap_{t'} + (\phi^\cap)^2_x + (\phi^\cap)^2_{x'} + |(\phi^\cap)_{y-y'}|^2\big)^{1/2}, \\ \phi^L &= \Big(\frac{\phi^\cap_{t'}}{(\rho^L)^2}, \frac{\phi^\cap_x}{\rho^L}, \frac{\phi^\cap_{x'}}{\rho^L}, \frac{\phi^\cap_{y-y'}}{\rho^L}\Big) \\ &=: \big(\phi^L_{t'}, \phi^L_x, \phi^L_{x'}, \phi^L_{y-y'}\big), \text{ along with } y', z, z', \widetilde{z}, \rho_\cap, \phi^\cap_{\widetilde{x}}, \phi^\cap_{y'-\widetilde{y}}, \phi^\cap_{t-t'} \end{aligned} \tag{B.14}$$

with functions as in (B.11). It is also possible to use simpler projective coordinates, as we will see below. Coordinates near $\mathrm{ff}_1^R$ can be derived similarly by switching $\phi^\cap_{t'}$ with $\phi^\cap_{t-t'}$ and $\phi^\cap_x$ with $\phi^\cap_{\widetilde{x}}$. The situation at $\mathrm{ff}_1^C$ is slightly different since, writing $\phi^\cap_t = \phi^\cap_{t'} + \phi^\cap_{t-t'}$, the pullback of $\phi^\cap_t$ on $M^2_{\mathrm{heat},1}$ via $\pi_C$ vanishes at $\phi^\cap_{t'} = 0 = \phi^\cap_{t-t'}$, and thus $\mathcal{S}_C$ is codimension 1 higher than $\mathcal{S}_\bullet$ for $\bullet = L, R$.

Here we blowup so that the following give coordinates

$$\begin{aligned} \rho^C &= \big(\phi^\cap_t + (\phi^\cap)^2_x + (\phi^\cap)^2_{\widetilde{x}} + |\phi^\cap_{y-\widetilde{y}}|^2\big)^{1/2}, \\ (\phi^\cap)^C &= \Big(\frac{\phi^\cap_{t'}}{(\rho^C)^2}, \frac{\phi^\cap_{t-t'}}{(\rho^C)^2}, \frac{\phi^\cap_x}{\rho^C}, \frac{\phi^\cap_{\widetilde{x}}}{\rho^C}, \frac{\phi^\cap_{y-\widetilde{y}}}{\rho^C}\Big) \\ &=: \big(\phi^C_{t'}, \phi^C_{t-t'}, \phi^C_x, \phi^C_{\widetilde{x}}, \phi^C_{y-\widetilde{y}}\big), \text{ along with } y', z, z', \widetilde{z}, \rho_\cap, \phi^\cap_{x'}, \phi^\cap_{y'-\widetilde{y}}. \end{aligned} \tag{B.15}$$

**Lemma B.1.** *With terminology as in Appendix A.1, the maps $\pi_\bullet$ extend from the interior to b-maps*

$$\widetilde{\pi}_\bullet \colon M^3_{\mathrm{heat},1} \longrightarrow M^2_{\mathrm{heat},1} \tag{B.16}$$

*for $\bullet \in \{L, C, R\}$ with exponent matrices $e_{\widetilde{\pi}_\bullet}$ satisfying*

$$\begin{aligned} &e_{\widetilde{\pi}_\bullet}(\mathrm{ff}_1^\cap, \mathrm{ff}_1) = 1, \quad e_{\widetilde{\pi}_\bullet}(\mathrm{ff}_1^{\bullet'}, \mathrm{ff}_1) = \delta_{\bullet,\bullet'}, \quad e_{\widetilde{\pi}_C}(\mathrm{ff}_1^L, \mathrm{lf}) = 1, \quad e_{\pi_C}(\mathrm{ff}_1^R, \mathrm{rf}) = 1, \\ &e_{\widetilde{\pi}_R}(\mathrm{ff}_1^L, \mathrm{lf}) = 1, \quad e_{\widetilde{\pi}_L}(\mathrm{ff}_1^R, \mathrm{rf}) = 1, \quad e_{\widetilde{\pi}_R}(\mathrm{ff}_1^C, \mathrm{tb}) > 0, \quad e_{\widetilde{\pi}_L}(\mathrm{ff}_1^C, \mathrm{tb}) > 0, \end{aligned} \tag{B.17}$$

*where $\delta_{\bullet,\bullet'} = 1$ if $\bullet = \bullet'$ and zero otherwise. When $\bullet \in \{\mathrm{L}, \mathrm{C}, \mathrm{R}, \mathrm{tb}'_1, \mathrm{tb}'_2\}$, i.e. when it is the pullback of a boundary hypersurface of $M \times M \times M \times W$ via the blowdown map, then the exponent matrix satisfies (B.5) with $\widetilde{\pi}$ replacing $\pi$.*

*Moreover, $\widetilde{\pi}_C$ is a b-fibration in the sense of Appendix A.1.*

**Remark B.2.** The significance of the inequalities in (B.17) involving tb is that all the distributions under consideration vanish to infinite order at tb, and thus the pullbacks of these distributions via $\pi_R$ will vanish to infinite order at $\mathrm{ff}_1^C$, and the same for $\pi_L$.

*Proof.* We verify the lemma for for $\widetilde{\pi}_C$ and leave the other nearly identical calculations to the reader. That $\widetilde{\pi}_C$ extends to a b-map follows easily by writing the pulling back the coordinates in (3.9) and writing them in terms of those in (B.11). In particular, note that the pullback

$$\widetilde{\pi}_C^* \rho = \rho_\cap \rho^C, \tag{B.18}$$

so the exponent matrix claim holds. The rest of the definitions of b-fibration are easy to check. □

**Remark B.3.** The extended map $\widetilde{\pi}_L$ is *not* a b-fibration as it maps the interior of $\mathrm{ff}_1^C$ to the interior of the face $\mathrm{tb} \cap \mathrm{lf}$ due to the fact that $t = 0$ on $W$ implies that $t' = 0$ also, thus the map increases the codimension of a face. The same holds for $\widetilde{\pi}_R$, i.e. $\widetilde{\pi}_R(\mathrm{ff}_1^C) \subset \mathrm{tb} \cap \mathrm{rf}$.

Next we must blowup the lifts of $\mathcal{B}_1$ in (3.10). Since by (B.17), $\widetilde{\pi}_\bullet$ only maps $\mathrm{ff}_1^{\bullet'}$ to $\mathrm{ff}_1$ if $\bullet = \bullet'$, any of the pair-wise intersections is again equal to the triple intersection

$$\mathcal{S}' = \widetilde{\pi}_L^{-1}(\mathcal{B}_1) \cap \widetilde{\pi}_C^{-1}(\mathcal{B}_1) = \widetilde{\pi}_C^{-1}(\mathcal{B}_1) \cap \widetilde{\pi}_R^{-1}(\mathcal{B}_1) = \widetilde{\pi}_L^{-1}(\mathcal{B}_1) \cap \widetilde{\pi}_R^{-1}(\mathcal{B}_1).$$

Indeed, each is a subset of $\mathrm{ff}_1^\cap$, and in the polar coordinates defined on the interior of $\mathrm{ff}_1^\cap$, using the definition of $\mathcal{B}_1$ in (3.10)

$$\mathcal{S}' = \{\rho = \phi_{t'}^\cap = \phi_{t-t'}^\cap = 0, \phi_x^\cap = \phi_{x'}^\cap = \phi_{\widetilde{x}}^\cap, \phi_{y-y'}^\cap = \phi_{y-\widetilde{y}}^\cap = 0\}, \tag{B.19}$$

with no restrictions on $y', z, z', \widetilde{z}$. We form a space $[M^3_{\mathrm{heat},1}; \mathcal{S}']_{\text{q-hom}}$ with appropriate homogeneities. To understand this space, note first that near $\mathcal{S}'$ we can use projective coordinates on $\mathrm{ff}_1{}^\cap$, concretely we can take for example $\widetilde{x}$ to be a boundary defining function of $\mathrm{ff}_1 \cap$ and coordinates $\widetilde{x}$, $t'/\widetilde{x}^2$, $(t - t')/\widetilde{x}^2$, $x/\widetilde{x}$, $x'/\widetilde{x}$, $(y - y')/\widetilde{x}$, $(y' - \widetilde{y})/\widetilde{x}$ to replace the polar coordinates in (B.11). Then the homogeneities are determined by those in the ff blowdown of the double space, and one has coordinates

$$\begin{aligned}
\overline{\rho}_\cap &= \Big(\widetilde{x}^{2(k-1)} + \frac{t}{\widetilde{x}^2} + \Big(\frac{x-\widetilde{x}}{\widetilde{x}}\Big)^2 + \Big(\frac{x'-\widetilde{x}}{\widetilde{x}}\Big)^2 + \Big(\frac{|y-\widetilde{y}|}{\widetilde{x}}\Big)^2 + \Big(\frac{|y'-\widetilde{y}|}{\widetilde{x}}\Big)^2\Big)^{1/2(k-1)}, \\
\overline{\phi} &:= (\overline{\phi}_{\widetilde{x}}, \overline{\phi}_{t'}, \overline{\phi}_{t-t'}, \overline{\phi}_{x-\widetilde{x}}, \overline{\phi}_{x'-\widetilde{x}}, \overline{\phi}_{y-\widetilde{y}}, \overline{\phi}_{y'-\widetilde{y}}) \\
&= \Big(\frac{\widetilde{x}}{\overline{\rho}_\cap}, \frac{t'}{\widetilde{x}^2 \overline{\rho}_\cap^{2(k-1)}}, \frac{t-t'}{\widetilde{x}^2 \overline{\rho}_\cap^{2(k-1)}} \frac{x-\widetilde{x}}{\widetilde{x}\overline{\rho}_\cap^{(k-1)}}, \frac{x'-\widetilde{x}}{\widetilde{x}\overline{\rho}_\cap^{(k-1)}} \frac{y-\widetilde{y}}{\widetilde{x}\overline{\rho}_\cap^{(k-1)}}, \frac{y'-\widetilde{y}}{\widetilde{x}\overline{\rho}_\cap^{(k-1)}}\Big) \\
&\qquad \text{along with } \widetilde{y}, z, z', \widetilde{z}.
\end{aligned} \tag{B.20}$$

One can also take coordinates in which $x, x', \widetilde{x}$ are permuted, and the same with $y$, $y', \widetilde{y}$.

We let $\mathrm{ff}^{\cap}$ denote the introduced boundary hypersurface.

The lifts of the $\widetilde{\pi}_{\bullet}^{-1}(\mathcal{B}_1)$ minus their intersections now have disjoint closures. For example, we have

$$\widetilde{\pi}_{\mathrm{L}}^{-1}(\mathcal{B}_1) \cap \mathrm{ff}_1{}^{\cap} \setminus \mathrm{ff}^{\cap} = \{\overline{\phi}_{t'} = \overline{\phi}_{x-x'} = \overline{\phi}_{y-y'} = 0\},$$

while

$$\widetilde{\pi}_{\mathrm{C}}^{-1}(\mathcal{B}_1) \cap \mathrm{ff}_1{}^{\cap} \setminus \mathrm{ff}^{\cap} = \{\overline{\phi}_{t} = \overline{\phi}_{x-\widetilde{x}} = \overline{\phi}_{y-\widetilde{y}} = 0\},$$

where $\overline{\phi}_t = \overline{\phi}_{t'} + \overline{\phi}_{t-t'}$ and $\overline{\phi}_{x-\widetilde{x}} = \overline{\phi}_{x-x'} + \overline{\phi}_{x'-\widetilde{x}}$ and for $\widetilde{\pi}_{\mathrm{R}}$ we have $\overline{\psi}_{x'} = \psi_{t-t'} = 0$, $\psi_{y'} = 0$; since $|\overline{\phi}| = 1$, these sets are disjoint. Furthermore, the pullbacks satisfy that

$$\widetilde{\pi}_{\bullet}^{-1}(\mathcal{B}_1) \cap \mathrm{ff}_1^{\bullet'} = \delta_{\bullet,\bullet'},$$

for $\bullet, \bullet' \in \{R, C, L\}$, and each intersection is straightforward to write down, e.g. with coordinates as in (B.15),

$$\widetilde{\pi}_{\mathrm{C}}^{-1}(\mathcal{B}_1) \cap \mathrm{ff}_1^{C} = \big\{\rho^C = \phi^C_{t'} = \phi^C_{t-t'} = \phi^C_{x} - \phi^C_{\widetilde{x}} = 0, \phi^C_{y-\widetilde{y}} = 0\big\}.$$

We will blowup first the $\widetilde{\pi}_{\bullet}^{-1}(\mathcal{B}_1) \cap \mathrm{ff}_1^{\cap}$ and then the $\widetilde{\pi}_{\bullet}^{-1}(\mathcal{B}_1) \cap \mathrm{ff}_1^{\bullet}$ with for $\bullet \in \{L, C, R\}$.

In the interior of $\mathrm{ff}_1^{\bullet}$ with $\bullet \in \{L, R\}$ the blowups of the pullbacks of $\mathcal{B}_1$ are particularly easy to understand as there we can just pullback the projective coordinates in (3.14) and use these together with the other unaffected coordinates to obtain projective coordinates e.g. near $\pi_{\mathrm{L}} \circ \widetilde{\beta}_0^{-1}(\mathcal{B}_1) \cap \mathrm{ff}_1^{L}$ valid near the interior of the introduced boundary hypersurface.

$$x', \quad \sigma = \frac{s-1}{(x')^{k-1}} = \frac{x-x'}{(x')^k}, \quad \eta' = \frac{y-y'}{(x')^k}, \quad T' = \frac{t'}{(x')^{2k}}, \tag{B.21}$$

together with $\widetilde{w}, t$ on the introduced boundary hypersurface. In the interior of $\mathrm{ff}_1^{C}$, one needs only to remember that the vanishing of the pullback of the $\phi_t$ coordinate implies the vanishing of both $\phi_{t'}$ and $\phi_{t-t'}$. One can use $\widetilde{x}$ as a boundary defining function and then two projective time coordinates $T' = t'/\widetilde{x}^{2k}$ and $\widetilde{T} = (t-t')/\widetilde{x}^{2k}$. In the interior of $\mathrm{ff}_1^{\cap}$ but away from $\mathrm{ff}^{\cap}$, we want the same homogeneities, but now the pullback of $\widetilde{x'}$ in the interior of $\mathrm{ff}^{\cap}$ is proportional to $\rho^{\cap}$ and in the interior of the introduced blowup we will have coordinates as in (B.21) with all the functions replaced by their $\phi$ counterparts, e.g. $x'$ replaced by $\phi_{x'}$ and $\frac{y-\widetilde{y}}{(x')^k}$ replaced by $\psi_{y-y'}/\phi_{x'}$.

We focus at the intersection $\mathrm{ff}_1{}^{\cap} \cap \mathrm{ff}_1{}^{\bullet}$, first with $\bullet = C$. Near $\mathcal{S}_C$, we can simplify things slightly by using projective coordinates, derived from (B.15) by noting that $\phi_{\widetilde{x}}^{\cap}$ is non-zero at $\mathrm{ff}_1{}^{\cap} \cap \mathrm{ff}_1{}^{\bullet} \cap \mathcal{S}_C$ and can thus be used as a boundary defining function. Specifically, take

$$\widetilde{\mathcal{X}} = \phi_{\widetilde{x}}^{\cap}, \quad \mathcal{X} = \frac{x}{\widetilde{x}}, \quad \mathcal{T} = \frac{t}{\widetilde{x}^2}, \quad \mathcal{Y} = \frac{y - \widetilde{y}}{\widetilde{x}},$$

together with the other (non-polar) coordinates in (B.15). Blowing up to introduce a face $\mathrm{ff}^{\cap,C}$, have

$$\begin{aligned} \mathcal{P} &= \big(\mathcal{T} + \rho_{\cap}^{2(k-1)} + (\mathcal{X}-1)^2 + |\mathcal{Y}|^2\big)^{1/(2(k-1))}, \\ \Psi &= \big(\mathcal{T}/\mathcal{P}^{2(k-1)}, \rho_{\cap}/\mathcal{P}, (\mathcal{X}-1)/\mathcal{P}^{k-1}, \mathcal{Y}/\mathcal{P}^{k-1}\big), \end{aligned}$$

but it follows that $\mathcal{S}_1 \cap \mathrm{ff}_1{}^C$ intersects $\mathrm{ff}^{\cap,C}$ at $\Psi = (0,1,0,0)$ and thus $\rho^{\cap}$ can be used as a boundary defining function. Again working near $\mathcal{S}_C$ we can take $\rho_{\cap}$ as a boundary defining function for $\mathrm{ff}^{\cap,C}$ and use projective coordinates $\rho_{\cap}$, $\mathcal{T}/\rho_{\cap}^{2(k-1)}$, $(\mathcal{X}-1)/\rho_{\cap}^{k-1}$, $\mathcal{Y}/\rho_{\cap}^{k-1}$. Using these we blow up $\mathcal{S}_1 \cap \mathrm{ff}_1{}^C$ with

$$\begin{aligned} \bar{\mathcal{P}} &= \Big(\frac{\mathcal{T}}{\rho_{\cap}^{2(k-1)}} + (\phi_{\widetilde{x}}^{\cap})^{2(k-1)} + \frac{(\mathcal{X}-1)^2}{\rho_{\cap}^{k-1}} + \frac{|\mathcal{Y}|^2}{\rho_{\cap}^{k-1}}\Big)^{1/(2(k-1))}, \\ \overline{\Psi} &= \Big(\frac{\mathcal{T}}{(\bar{\mathcal{P}}\rho_{\cap})^{2(k-1)}}, \frac{\phi_{\widetilde{x}}^{\cap}}{\bar{\mathcal{P}}}, \frac{\mathcal{X}-1}{(\bar{\mathcal{P}}\rho_{\cap})^{k-1}}, \frac{\mathcal{Y}}{(\bar{\mathcal{P}}\rho_{\cap})^{k-1}}\Big) \\ &= \Big(\frac{t}{\widetilde{x}^2(\bar{\mathcal{P}}\rho_{\cap})^{2(k-1)}}, \frac{\widetilde{x}}{\bar{\mathcal{P}}\rho_{\cap}}, \frac{x-\widetilde{x}}{\widetilde{x}(\bar{\mathcal{P}}\rho_{\cap})^{k-1}}, \frac{y-\widetilde{y}}{\widetilde{x}(\bar{\mathcal{P}}\rho_{\cap})^{k-1}}\Big), \end{aligned}$$

and this is the final blowup of $\mathcal{S}_C$. The blowups for $\mathcal{S}_L, \mathcal{S}_R$ are similar and left to the reader.

**Proposition B.4** (Incomplete cusp edge heat triple space). *The above construction yields a space and blowdown map*

$$\widetilde{\beta}\colon M^3_{\mathrm{heat}} \longrightarrow M \times M \times M \times W, \tag{B.22}$$

*such that the maps $\widetilde{\pi}_\bullet$ from (B.16) extend to b-maps $\overline{\pi}_\bullet\colon M^3_{\mathrm{heat}} \longrightarrow M^2_{\mathrm{heat},2}$ with exponent matrix satisfying (B.5), (B.17) (with $\pi$ and $\widetilde{\pi}$ replaced by $\overline{\pi}$), and*

$$\begin{gathered} e_{\overline{\pi}_\bullet}(\mathrm{ff}^{\cap},\mathrm{ff}) = 1, \quad e_{\overline{\pi}_\bullet}(\mathrm{ff}^{\cap,\bullet'},\mathrm{ff}) = e_{\overline{\pi}_\bullet}(\mathrm{ff}^{\bullet'},\mathrm{ff}) = \delta_{\bullet,\bullet'}, \\ e_{\overline{\pi}_L}(\mathrm{ff}^{\cap,R},\mathrm{ff}_1) = e_{\overline{\pi}_R}(\mathrm{ff}^{\cap,L},\mathrm{ff}_1) = e_{\overline{\pi}_C}(\mathrm{ff}^{\cap,R},\mathrm{ff}_1) = e_{\overline{\pi}_C}(\mathrm{ff}^{\cap,L},\mathrm{ff}_1) = 1, \\ e_{\overline{\pi}_L}(\mathrm{ff}^R,\mathrm{rf}) = e_{\overline{\pi}_R}(\mathrm{ff}^L,\mathrm{lf}) = e_{\overline{\pi}_C}(\mathrm{ff}^R,\mathrm{rf}) = e_{\overline{\pi}_C}(\mathrm{ff}^L,\mathrm{lf}) = 1, \\ e_{\overline{\pi}_L}(\mathrm{ff}^{\cap,C},\mathrm{tb}), e_{\overline{\pi}_L}(\mathrm{ff}^C,\mathrm{tb}), e_{\overline{\pi}_R}(\mathrm{ff}^{\cap,C},\mathrm{tb}), e_{\overline{\pi}_R}(\mathrm{ff}^C,\mathrm{tb}) \geq 1. \end{gathered} \tag{B.23}$$

*Moreover, apart from components of $e_{\overline{\pi}_\bullet}(\mathrm{ff}^{\cap,C},\bullet')$ and $e_{\overline{\pi}_\bullet}(\mathrm{ff}^C,\bullet')$ with $\bullet \in \{L,R\}$, all other components are zero.*

*Again, $\overline{\pi}_C$ is a b-fibration.*

*Proof.* Again, we focus on $\overline{\pi}_C$. To check that $\widetilde{\pi}_C$ extends to a $b$-map, we pull back the polar coordinates $\overline{\rho}, \overline{\phi}, \widetilde{y}, z, \widetilde{z}$, from (3.12) defined at ff in $M^2_{\mathrm{heat},2}$. First, we compute

$$\begin{aligned}\widetilde{\pi}_C^* \overline{\rho} &= \widetilde{\pi}_C^*\big(\big((t/\widetilde{x}^2) + \widetilde{x}^{2(k-1)} + (s-1)^2 + (|y-\widetilde{y}|/\widetilde{x})^2\big)^{1/2(k-1)}\big)\\ &= \big(\mathcal{T} + \widetilde{x}^{2(k-1)} + (\mathcal{X}-1)^2 + |\mathcal{Y}|^2\big)^{1/2(k-1)}\\ &= \overline{\mathcal{P}}\rho_{\cap},\end{aligned}$$

and then note that

$$\begin{aligned}\widetilde{\pi}_C^* \overline{\phi} &= \widetilde{\pi}_C^*\Big(\frac{t}{\widetilde{x}^2\overline{\rho}^{2(k-1)}}, \frac{\widetilde{x}}{\overline{\rho}}, \frac{x-\widetilde{x}}{\widetilde{x}\overline{\rho}^{(k-1)}}, \frac{y-\widetilde{y}}{\widetilde{x}\overline{\rho}^{(k-1)}}\Big)\\ &= \Big(\frac{t}{\widetilde{x}^2(\overline{\mathcal{P}}\rho_{\cap})^{2(k-1)}}, \frac{\widetilde{x}}{\overline{\mathcal{P}}\rho_{\cap}}, \frac{x-\widetilde{x}}{\widetilde{x}(\overline{\mathcal{P}}\rho_{\cap})^{(k-1)}}, \frac{y-\widetilde{y}}{\widetilde{x}(\overline{\mathcal{P}}\rho_{\cap})^{(k-1)}}\Big)\\ &= \overline{\Psi}.\end{aligned}$$

This establishes both claims for $\overline{\pi}_C$. The $R, L$ case are left to the reader. □

**Proposition B.5.** *For $i = 1, 2$, let $A_i \in \mathcal{A}^{\mathcal{E}_i}_{\mathrm{phg}}(M^2_{\mathrm{heat},2})$ with the index sets $\mathcal{E}_i$ satisfying*

*$\mathcal{E}_i(\mathrm{ff}_1) \geq -3-b-kf$, $\mathcal{E}_i(\mathrm{ff}) \geq -kn-2k$, $\mathcal{E}_i(\mathrm{lf}) = \mathcal{E}_i(\mathrm{tb}) = \varnothing$, and $\mathcal{E}_i(\mathrm{rf})$ satisfying (3.68). Then*

$$A_3 := \iint_0^t A_1(w, w', t')A_2(w', \widetilde{w}, t-t')\,\mathrm{dVol}_{w'}\,dt'$$

*lies in $\mathcal{A}^{\mathcal{E}_3}_{\mathrm{phg}}(M^2_{\mathrm{heat},2})$ where for any $\epsilon > 0$,*

$$\begin{aligned}\inf \mathcal{E}_3(\mathrm{ff}_1) &\geq \inf \mathcal{E}_1(\mathrm{ff}_1) + \inf \mathcal{E}_2(\mathrm{ff}_1) + 3 + b + kf - \epsilon,\\ \inf \mathcal{E}_3(\mathrm{ff}) &\geq \inf \mathcal{E}_1(\mathrm{ff}) + \inf \mathcal{E}_2(\mathrm{ff}) + kn + 2k - \epsilon.\end{aligned} \tag{B.24}$$

**Remark B.6.** The constants $kn+2k$ and $3+b+kf$ in (B.24) should be interpreted, for instance in the case of ff, as saying that the (Volterra type) composition of two operators given by Schwartz kernels as in the theorem has Schwartz kernel whose leading order asymptotic behavior at ff increases *relative to the rate* $-kn-2k$, in particular if both the composed operators grow like $-kn-2k$ then so does the composition. These are, incidentally, the exact rates of blowup of the heat kernel times $t^{-1}$ at the faces ff and $\mathrm{ff}_1$, and furthermore the fact that the errors $t^{-1}Q$ in (3.75) vanish one order faster than $t^{-1}H$ means, as described above, that taking powers makes them vanish at increasing rates at both ff and $\mathrm{ff}_1$.

*Proof.* We write $A_3$ as the pushforward of a b-density and then apply the Pushforward theorem from Section A.1. First we define a non-vanishing b-density $\mu_0$ on $M \times M \times M \times W$ as follows. We let $\nu$ be a non-vanishing b-density on $M$ satisfying $\nu = a|\frac{dx dy dz}{x}|$ for a smooth nonvanishing function $a$ near the boundary, and consider

$$\mu_0 = \nu \, \nu' \, \widetilde{\nu} \left| \frac{dt' dt}{t'(t-t')} \right|$$

where $\nu', \widetilde{\nu}$ are equal to $\nu$ in the primed and tilded coordinates, respectively. Since the blowdown map $\widetilde{\beta}$ from (B.22) is a b-map, $\widetilde{\beta}^* \mu_0$ is a b-density on $M^3_{\text{heat}}$, and one checks that

$$\widetilde{\beta}^* \mu_0 = G \overline{\mu}_0, \tag{B.25}$$

for a non-vanishing b-density $\overline{\mu}_0$ on $M^3_{\text{heat}}$ and $G \in C^\infty(M^3_{\text{heat}})$ satisfying that for some non-vanishing smooth function $G'$,

$$G = G' (\rho_{\text{ff}_1 L} \rho_{\text{ff}_1 C} \rho_{\text{ff}_1 R})^b \rho^{2b}_{\text{ff}_1 \cap} (\rho_{\text{ff} L} \rho_{\text{ff} C} \rho_{\text{ff} R})^{kb+k-1} \times (\rho_{\text{ff}\cap, L} \rho_{\text{ff}\cap, C} \rho_{\text{ff}\cap, R})^{(k+1)b+k-1} \rho^{2kb+2(k-1)}_{\text{ff}\cap}.$$

Then we can write the desired pushforward as a pushforward of a b-density, specifically

$$\begin{aligned} A_3 \left( \nu \, \widetilde{\nu} \, |\frac{dt}{t}| \right) &= (\pi_C)_* \big( \pi_L^* A_1 \pi_R^* A_2 \cdot ((t'/t)(t-t')) F(w') \mu_0 \big) \\ &= (\overline{\pi}_C)_* \big( \overline{\pi}_L^* A_1 \overline{\pi}_R^* A_2 \cdot \widetilde{\beta}^* ((t'/t)(t-t') F(w') \mu_0) \big) \end{aligned} \tag{B.26}$$

where $F$ is the function defined by $\mathrm{dVol}_g = F \nu$ and in particular

$$F = a \, x^{kf+1},$$

where $a$ is a non-vanishing polyhomogeneous function on $M$, and $\nu, \widetilde{\nu}$ are the pullbacks of the density $\nu$ above to the left and right spacial factors of $M \times M \times \mathbb{R}^+$. To find the asymptotics of $A_3$ itself we must compute the asymptotics of the densities on the left hand side of (B.26); Letting $\beta_2$ again denote the blowdown map

$$M^2_{\text{heat},2} \longrightarrow M \times M \times [0, \infty)$$

in (3.13), we check that

$$\beta_2^* \left( (\pi'_L)^* \nu \, (\pi'_R)^* \nu \frac{dt}{t} \right) = \rho^b_{\text{ff}_1} \rho^{bk+k-1}_{\text{ff}} \mu_2,$$

where $\mu_2$ is a non-vanishing b-density on $M^2_{\text{heat},2}$. Thus from (B.26), if the distribution

$$(\overline{\pi}_C)_* \big( \overline{\pi}_L^* A_1 \overline{\pi}_R^* A_2 \cdot \widetilde{\beta}^* ((t'/t)(t-t') F(w') \mu_0) \big)$$

is polyhomogeneous with index set $\mathcal{E}_3'$ then $A_3$ is phg with index set $\mathcal{E}_3$ satisfying

$$\mathcal{E}_3(\mathrm{ff}_1) = \mathcal{E}_3'(\mathrm{ff}_1) - b, \quad \mathcal{E}_3(\mathrm{ff}) = \mathcal{E}_3'(\mathrm{ff}) - (kb + k - 1), \tag{B.27}$$

and $\mathcal{E}_3(\bullet) = \mathcal{E}_3'(\bullet)$ otherwise.

Thus the index family of $\pi_L^* A_1 \pi_R^* A_2 \cdot \widetilde{\beta}^*((t'/t)(t-t')F(w'))$ must be determined. To determine $\pi_L^* A_1$, we see that, at a bhs $H$ of $M^3_{\mathrm{heat}}$, the index set of $\pi_L^* A_1$ is simply the index set of $A_1$ at the bhs $H'$ of $M^2_{\mathrm{heat},2}$ at which $H$ is incident. Thus from our work above we see that $\pi_L^* A_1$ has index set $\overline{\mathcal{E}}_1$ satisfying

$$\begin{aligned}
\overline{\mathcal{E}}_1(\mathrm{L}) &= \mathcal{E}_1(\mathrm{lf}) = \varnothing, \\
\overline{\mathcal{E}}_1(\mathrm{C}) = \overline{\mathcal{E}}_1({\mathrm{ff}_1}^R) = \overline{\mathcal{E}}_1(\mathrm{ff}^R) &= \mathcal{E}_1(\mathrm{rf}), \\
\overline{\mathcal{E}}_1(\mathrm{tb}_1') = \overline{\mathcal{E}}_1({\mathrm{ff}_1}^C) = \overline{\mathcal{E}}_1(\mathrm{ff}^{\cap,C}) = \overline{\mathcal{E}}_1(\mathrm{ff}^C) &= \mathcal{E}_1(\mathrm{tb}) = \varnothing, \\
\overline{\mathcal{E}}_1({\mathrm{ff}_1}^\cap) = \overline{\mathcal{E}}_1({\mathrm{ff}_1}^L) = \overline{\mathcal{E}}_1(\mathrm{ff}^{\cap,R}) &= \mathcal{E}_1(\mathrm{ff}_1), \\
\overline{\mathcal{E}}_1(\mathrm{ff}^\cap) = \overline{\mathcal{E}}_1(\mathrm{ff}^{\cap,L}) = \overline{\mathcal{E}}_1(\mathrm{ff}^L) &= \mathcal{E}_1(\mathrm{ff}), \\
\overline{\mathcal{E}}_1(\mathrm{R}) &= \mathbb{Z},
\end{aligned}$$

the last line coming from the fact that $\pi_L^* A_1$ is independent of $\widetilde{x}$, in particular is smooth up to R. The index set $\overline{\mathcal{E}}_2$ of $\pi_R^* A_2$ has the same expression in terms of $\mathcal{E}_2$ but with all 'R's switched with 'L's, all lf's with rf's, and all 1's with 2's (except of course for the 1 in the subscript of $\mathrm{ff}_1$). For example, (c.f. the second line above) $\overline{\mathcal{E}}_2(\mathrm{C}), \overline{\mathcal{E}}_2({\mathrm{ff}_1}^L), \overline{\mathcal{E}}_2(\mathrm{ff}^L)$ are all equal to $\mathcal{E}_2(\mathrm{lf})$, which is assumed to be $\varnothing$. If we define the operation $\mathcal{E}_1 \oplus \mathcal{E}_2$ on index sets to denote the index set whose elements are sums of elements of the two index sets, It follows that $\pi_L^* A_1 \, \pi_R^* A_2$ is polyhomogeneous with index set $\mathcal{F}$ satisfying

$$\begin{gathered}
\left\{\mathcal{F}(\mathrm{C}), \mathcal{F}(\mathrm{L}), \mathcal{F}({\mathrm{ff}_1}^L), \mathcal{F}(\mathrm{ff}^L), \mathcal{F}(\mathrm{tb}_1'), \mathcal{F}(\mathrm{tb}_2'), \mathcal{F}({\mathrm{ff}_1}^C), \mathcal{F}(\mathrm{ff}^{\cap,C}), \mathcal{F}(\mathrm{ff}^C)\right\} = \varnothing, \\
\mathcal{F}({\mathrm{ff}_1}^\cap) = \mathcal{E}_1(\mathrm{ff}_1) \oplus \mathcal{E}_2(\mathrm{ff}_1), \quad \mathcal{F}(\mathrm{ff}^\cap) = \mathcal{E}_1(\mathrm{ff}) \oplus \mathcal{E}_2(\mathrm{ff}), \quad \mathcal{F}(\mathrm{R}) = \mathcal{E}_2(\mathrm{rf}), \\
\mathcal{F}({\mathrm{ff}_1}^R) = \mathcal{E}_1(\mathrm{rf}) \oplus \mathcal{E}_2(\mathrm{ff}_1), \quad \mathcal{F}(\mathrm{ff}^R) = \mathcal{E}_1(\mathrm{rf}) \oplus \mathcal{E}_2(\mathrm{ff}), \\
\mathcal{F}(\mathrm{ff}^{\cap,R}) = \mathcal{E}_1(\mathrm{ff}_1) \oplus \mathcal{E}_2(\mathrm{ff}), \quad \mathcal{F}(\mathrm{ff}^{\cap,L}) = \mathcal{E}_1(\mathrm{ff}) \oplus \mathcal{E}_2(\mathrm{ff}_1).
\end{gathered} \tag{B.28}$$

We now compute the asymptotics of the term

$$\widetilde{\beta}^*(((t'/t)(t-t'))F(w')\mu_0) = \widetilde{\beta}^*(((t'/t)(t-t'))F(w'))G\overline{\mu}_0$$

with $G$ in (B.25). First, write

$$\widetilde{\beta}^*((t(t-t'))F(w')) = \pi_L^*(\mathrm{t})\pi_R^*(\mathrm{t})\pi_R^*(F),$$

where $F$ is thought of as a function of the left factor of $M \times M \times [0,\infty)$. Recalling $\rho$, $\overline{\rho}$ from (3.9) and (3.12), respectively, and letting $a$ denote a polyhomogeneous function which is smooth and non-vanishing up to boundary hypersurfaces $\bullet$ for which $\mathcal{F}(\bullet) \neq \varnothing$ (and whose value will change from line to line), we compute

$$\begin{aligned}
\widetilde{\beta}^*\Big(\frac{t'(t-t')}{t}F(w')\Big) &= a\,\frac{\overline{\pi}_L^*(\rho^2\overline{\rho}^{2k})\overline{\pi}_R^*(\rho^2\overline{\rho}^{2k})}{\overline{\pi}_C^*(\rho^2\overline{\rho}^{2k})}\overline{\pi}_R^*((\rho_{\mathrm{lf}}\rho\overline{\rho})^{kf+1}) \\
&= a\,\frac{(\rho_{\mathrm{ff}_1\cap}\rho_{\mathrm{ff}_1L}\rho_{\mathrm{ff}\cap,R})^2(\rho_{\mathrm{ff}\cap}\rho_{\mathrm{ff}_1\cap,L}\rho_{\mathrm{ff}L})^{2k}}{(\rho_{\mathrm{ff}_1\cap}\rho_{\mathrm{ff}_1C}\rho_{\mathrm{ff}\cap,R}\rho_{\mathrm{ff}\cap,L})^2(\rho_{\mathrm{ff}\cap}\rho_{\mathrm{ff}_1\cap,C}\rho_{\mathrm{ff}C})^{2k}} \\
&\quad\times(\rho_{\mathrm{ff}_1\cap}\rho_{\mathrm{ff}_1R}\rho_{\mathrm{ff}\cap,L})^2(\rho_{\mathrm{ff}\cap}\rho_{\mathrm{ff}_1\cap,R}\rho_{\mathrm{ff}R})^{2k} \\
&\quad\times(\rho_C\rho_{\mathrm{ff}_1L}\rho_{\mathrm{ff}L}\rho_{\mathrm{ff}_1\cap}\rho_{\mathrm{ff}_1R}\rho_{\mathrm{ff}\cap,L}\rho_{\mathrm{ff}\cap}\rho_{\mathrm{ff}\cap,R}\rho_{\mathrm{ff}R})^{kf+1} \\
&= a\,(\rho_{\mathrm{ff}_1L})^2(\rho_{\mathrm{ff}_1\cap,L}\rho_{\mathrm{ff}L})^{2k}(\rho_{\mathrm{ff}_1\cap}\rho_{\mathrm{ff}_1R})^2(\rho_{\mathrm{ff}\cap}\rho_{\mathrm{ff}_1\cap,R}\rho_{\mathrm{ff}R})^{2k} \\
&\quad\times(\rho_C\rho_{\mathrm{ff}_1L}\rho_{\mathrm{ff}L}\rho_{\mathrm{ff}_1\cap}\rho_{\mathrm{ff}_1R}\rho_{\mathrm{ff}\cap,L}\rho_{\mathrm{ff}\cap}\rho_{\mathrm{ff}\cap,R}\rho_{\mathrm{ff}R})^{kf+1} \\
&= a\,(\rho_{\mathrm{ff}_1\cap}\rho_{\mathrm{ff}_1R})^2(\rho_{\mathrm{ff}\cap}\rho_{\mathrm{ff}\cap,L}\rho_{\mathrm{ff}\cap,R}\rho_{\mathrm{ff}R})^{2k} \\
&\quad\times(\rho_{\mathrm{ff}_1R}\rho_{\mathrm{ff}_1\cap}\rho_{\mathrm{ff}\cap}\rho_{\mathrm{ff}\cap,R}\rho_{\mathrm{ff}\cap,L}\rho_{\mathrm{ff}R})^{kf+1}.
\end{aligned}$$

Putting this all together, we see that $\overline{\pi}_L^*A_1\overline{\pi}_R^*A_2\cdot\widetilde{\beta}^*(((t'/t)(t-t'))F(w')\mu_0)$ is polyhomogeneous with index set $\overline{\mathcal{F}}$

$$\begin{aligned}
\overline{\mathcal{F}}(\mathrm{ff}_1{}^{\cap}) &= \mathcal{E}_1(\mathrm{ff}_1)\oplus\mathcal{E}_2(\mathrm{ff}_1)+(3+kf+2b),\\
\overline{\mathcal{F}}(\mathrm{ff}^{\cap}) &= \mathcal{E}_1(\mathrm{ff})\oplus\mathcal{E}_2(\mathrm{ff})+(1+2k+kf+2kb),\\
\overline{\mathcal{F}}(\mathrm{R}) &= \mathcal{E}_2(\mathrm{rf})\\
\overline{\mathcal{F}}(\mathrm{ff}_1{}^{R}) &= \mathcal{E}_1(\mathrm{rf})\oplus\mathcal{E}_2(\mathrm{ff}_1)+(3+kf+b),\\
\overline{\mathcal{F}}(\mathrm{ff}^{R}) &= \mathcal{E}_1(\mathrm{rf})\oplus\mathcal{E}_2(\mathrm{ff})+(1+2k+kf+kb)\\
\overline{\mathcal{F}}(\mathrm{ff}^{\cap,R}) &= \mathcal{E}_1(\mathrm{ff}_1)\oplus\mathcal{E}_2(\mathrm{ff})+(1+2k+kf+(k+1)b),\\
\overline{\mathcal{F}}(\mathrm{ff}^{\cap,L}) &= \mathcal{E}_1(\mathrm{ff})\oplus\mathcal{E}_2(\mathrm{ff}_1)+(1+2k+kf+(k+1)b),
\end{aligned} \tag{B.29}$$

and $\overline{\mathcal{F}}(\bullet)=\varnothing$ for all other values of $\bullet$.

Now we apply Theorem A.3 to analyze

$$(\overline{\pi}_C)_*\big(\pi_L^*A_1\pi_R^*A_2\cdot((t'/t)(t-t'))F(w')\mu_0\big)$$

from (B.26). To check that the conditions of the theorem hold, we first recall that $\overline{\pi}_C$ is a b-fibration. Also, note that

$$e_{\overline{\pi}_C}(\mathrm{C},H') = e_{\overline{\pi}_C}(\mathrm{tb}_1',H') = e_{\overline{\pi}_C}(\mathrm{tb}_2',H') = 0$$

for all $H'\in\mathcal{M}(M^2_{\mathrm{heat},2})$, and so we must check the integrability condition there, but by below (B.29) we have

$$\overline{\mathcal{F}}(\mathrm{C}) = \overline{\mathcal{F}}(\mathrm{tb}_1') = \overline{\mathcal{F}}(\mathrm{tb}_2') = \varnothing,$$

so the integrability condition holds. Thus $A_3 (\pi'_L)^* \nu \, (\pi'_R)^* \nu$ is phg on $M^2_{\mathrm{heat},2}$ with index set $\mathcal{E}'_3$ satisfying

$$
\begin{aligned}
\mathcal{E}'_3(\mathrm{lf}) &= \overline{\mathcal{F}}(\mathrm{L}) \,\overline{\cup}\, \overline{\mathcal{F}}(\mathrm{ff_1}^L) \,\overline{\cup}\, \overline{\mathcal{F}}(\mathrm{ff}^L) = \varnothing, \\
\mathcal{E}'_3(\mathrm{rf}) &= \overline{\mathcal{F}}(\mathrm{R}) \,\overline{\cup}\, \overline{\mathcal{F}}(\mathrm{ff_1}^R) \,\overline{\cup}\, \overline{\mathcal{F}}(\mathrm{ff}^R), \\
\mathcal{E}'_3(\mathrm{ff_1}) &= \overline{\mathcal{F}}(\mathrm{ff_1}^\cap) \,\overline{\cup}\, \overline{\mathcal{F}}(\mathrm{ff_1}^\cap) \,\overline{\cup}\, \overline{\mathcal{F}}(\mathrm{ff}^{\cap,L}) \,\overline{\cup}\, \overline{\mathcal{F}}(\mathrm{ff}^{\cap,R}), \\
\mathcal{E}'_3(\mathrm{ff}) &= \overline{\mathcal{F}}(\mathrm{ff}^\cap) \,\overline{\cup}\, \overline{\mathcal{F}}(\mathrm{ff}^{\cap,C}) \,\overline{\cup}\, \overline{\mathcal{F}}(\mathrm{ff}^C) = \overline{\mathcal{F}}(\mathrm{ff}^\cap), \\
\mathcal{E}'_3(\mathrm{tb}) &= \varnothing,
\end{aligned}
\tag{B.30}
$$

where we used from below (B.29) that various bhs's have infinite order vanishing. From this we see that the bounds in Proposition B.5 hold, in particular that for any $\epsilon > 0$,

$$
\begin{aligned}
\inf \mathcal{E}'_3(\mathrm{ff_1}) &\geq \inf \mathcal{E}_1(\mathrm{ff_1}) + \inf \mathcal{E}_2(\mathrm{ff_1}) + 3 + kf + 2b - \epsilon, \\
\mathcal{E}'_3(\mathrm{ff}) &= \inf \mathcal{E}_1(\mathrm{ff}) + \inf \mathcal{E}_2(\mathrm{ff}) + 1 + k + kn - \epsilon,
\end{aligned}
$$

and thus by (B.27) the actual index set $\mathcal{E}_3$ of $A_3$ satisfies (B.24), and the proof is complete. □

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