Max-Planck-Institut für Mathematik
Bonn

On Baker's explicit abc-conjecture

by

Kwok Chi Chim
Tarlok N. Shorey
Sneh Bala Sinha

Max-Planck-Institut für Mathematik
Preprint Series 2018 (51)

On Baker's explicit *abc*-conjecture

Kwok Chi Chim
Tarlok N. Shorey
Sneh Bala Sinha

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Institute of Analysis and Number Theory
Graz University of Technology
Kopernikusgasse 24/II
8010 Graz
Austria

National Institute of Advanced Studies
Bangalore 560012
India

Indian Institute of Sciences
Bangalore 560012
India

MPIM 18-51

ON BAKER'S EXPLICIT *abc*-CONJECTURE

KWOK CHI CHIM, TARLOK N. SHOREY, AND SNEH BALA SINHA

Dedicated to the memory of Professor Alan Baker.

Abstract. We derived from Baker's explicit *abc*-conjecture that (1.1) implies that $c < N^{1.72}$ for $N \geq 1$ and $c < 32N^{1.6}$ for $N \geq 1$. This sharpens an estimate of Laishram and Shorey. We also show that it implies $c < \frac{6}{5}N^{1+G(N)}$ for $N \geq 3$ and $c < \frac{6}{5}N^{1+G_1(N)}$ for $N \geq 297856$ where $G(N)$ and $G_1(N)$ are explicitly given positive valued decreasing functions of N tending to zero as N tends to infinity given by (1.4) and (1.6), respectively. Finally we give applications of our estimates on the greatest prime factor of product of consecutive positive integers, triples of consecutive powerful integers and generalized Fermat equation.

1. Introduction

The well known *abc*-conjecture was formulated by Joseph Oesterlé [7] and David Masser [4] in 1988. It states that

Conjecture 1.1. *For any given $\epsilon > 0$, there exists a number K_ϵ depending only on ϵ such that if*

$$a + b = c \tag{1.1}$$

where a, b and c are relatively prime positive integers, then

$$c \leq K_\epsilon \Big(\prod_{p|abc} p \Big)^{1+\epsilon}$$

where the product is taken over all primes p dividing abc.

The name *abc*-conjecture derives from letters a, b, c that are used in the statement. There are several works on *abc*-conjecture and its variations.

For a positive integer ν, we define the radical $N(\nu)$ of ν by the product of primes dividing ν and $\omega(\nu)$ for the number of distinct prime divisors of ν. The letter p always denote a prime number in this paper except in Theorem 1.6 and its proof. We denote the radical of abc by

$$N = N(abc) = \prod_{p|abc} p \tag{1.2}$$

unless otherwise specified. Further we write $\omega = \omega(N)$ for the number of distinct prime divisors of N. We see when $\omega \in \{0, 1\}$ or N is odd then (1.1) does not hold. Therefore we always have $\omega \geq 2$ unless $(a, b, c) = (1, 1, 2)$ and N is even. We understand that $\log_2 x = \log\log x$ for $x \geq 2$ and $\log_3 x = \log\log\log x$ for $x \geq 3$. We observe that Conjecture 1.1 is not explicit in the

---

2010 *Mathematics Subject Classification.* 11D75, 11J25, 11D41.

Key words and phrases. *abc*-conjecture, Explicit conjecture, Fermat's equation, consecutive integers.

sense that K_ϵ is not explicit. Alan Baker [1] in 2004 formulated the following explicit version of Conjecture 1.1.

Conjecture 1.2. *Let a, b and c be relatively prime positive integers satisfying* (1.1) *with $N > 2$. Then*

$$c < \frac{6}{5}N\frac{(\log N)^\omega}{\omega!} \tag{1.3}$$

where $N = N(abc)$ and $\omega = \omega(N)$.

We refer to Conjecture 1.1 as *abc*-conjecture and Conjecture 1.2 as explicit version of *abc*-conjecture. For integer $N > 2$, let

$$A(N) = \log_2 N - \log_3 N, A_1(N) = A(N) + \log A(N) - 1.076869$$

and

$$G(N) = \frac{1 + \log A(N)}{A(N)}. \tag{1.4}$$

Further we define $G(x) = G([x])$ for $x > 2$. We observe that $G(N)$ is positive valued function that tends to zero as N tends to infinity. It is decreasing if $A^{'}(N)\log A(N) > 0$ which is the case when $N \geq 16$ since

$$A^{'}(N) = \frac{1}{N\log N}\Big(1 - \frac{1}{\log_2 N}\Big). \tag{1.5}$$

Thus $G(N)$ is decreasing for $N \geq 16$. Further for integer $N \geq 40$, let

$$G_1(N) = \frac{1 + \log A_1(N)}{A_1(N)} \tag{1.6}$$

and $G_1(x) = G_1([x])$ for $x \geq 40$. We observe that $G_1(N)$ is positive for $N \geq 574$ and tends to zero as N tends to infinity. Further $G_1(N)$ is decreasing if $A_1^{'}(N)\log A_1(N) > 0$. Let $N \geq 297856$. Then $A_1(N) > 1$. Further $A(N) > 0$ and $A^{'}(N) > 0$ by (1.5). Since

$$A_1^{'}(N) = A^{'}(N) + \frac{A^{'}(N)}{A(N)} = \frac{A^{'}(N)}{A(N)}(1 + A(N)),$$

we see that $A_1^{'}(N)\log A_1(N) > 0$. Hence $G_1(N)$ is decreasing whenever $N \geq 297856$.

We compare these functions. For this, we observe that the function $F(x) = \frac{1+\log x}{x}$ is decreasing for $x > 1$ and

$$1 < A(N) < A_1(N) \quad \text{for} \quad N \geq 1.5 \times 10^{36}$$

since $A(N) > e^{1.076869}$ for $N \geq 1.5 \times 10^{36}$. Therefore

$$G(N) = F(A(N)) \geq F(A_1(N)) = G_1(N) \quad \text{for} \quad N \geq 1.5 \times 10^{36} \tag{1.7}$$

and similarily we derive that

$$G(N) \leq G_1(N) \text{ for } 297856 \leq N \leq 10^{36}.$$

Conjecture 1.2 implies the following sharper and explicit version of *abc*-conjecture in which we allow ϵ to be a function of N tending to zero as N tends to infinity.

Theorem 1.3. *Let a, b and c be relatively prime positive integers satisfying* (1.1). *Then* (1.3) *implies that*

$$c < \frac{6}{5}N^{1+G(N)} \text{ for } N > 2 \tag{1.8}$$

and

$$c < \frac{6}{5}N^{1+G_1(N)} \text{ for } N \geq 297856. \tag{1.9}$$

On the other hand, Stewart and Tijdeman [9] showed that there are infinitely many relatively prime positive integers a, b, c satisfying (1.1) such that for $\delta > 0$, we have

$$c > N^{1+\frac{4-\delta}{\sqrt{\log N}\log\log N}}.$$

Laishram and Shorey [3] showed that Conjecture 1.2 implies that for $N > 2$, we have

$$c < N^{1+\theta} \quad \text{with} \quad \theta = \frac{3}{4}. \tag{1.10}$$

Further they also derived under Conjecture 1.2 that for $0 < \theta < 3/4$, (1.10) holds when $N \geq N_\theta$ where N_θ is an effectively computable number depending only on θ. Theorem 1.3 provides a value of N_θ for every $0 < \theta < 1$ determined by an explicitly given function; we do not have to compute for every θ. Now we prove the following Theorem with a sharper exponent than (1.10).

Theorem 1.4. *Let a, b and c be relatively prime positive integers satisfying* (1.1). *Then* (1.3) *implies that for $N > 2$, we have*

$$c < N^{1.72}. \tag{1.11}$$

Further

$$c < 10N^{1.62991} \tag{1.12}$$

and

$$c < 32N^{1.6}. \tag{1.13}$$

E. Reyssat [13] considered (1.1) with $a = 2, b = 3^{10} \times 109, c = 23^5$ and $N = 15042$. This implies $c > N^{1.62991}$ which we may compare with (1.12).

The following theorem gives the comparison among bounds of c and it follows immediately from (1.11), (1.13), (1.9).

Theorem 1.5. *Let a, b and c be relatively prime positive integers satisfying* (1.1). *Then* (1.3) *implies that*

$$c < \begin{cases} N^{1.72} & \text{if } N > 2 \\ 32N^{1.6} & \text{if } N \geq 10^{12.55} \\ \frac{6}{5}N^{1+G_1(N)} & \text{if } N \geq 10^{80.53}. \end{cases}$$

Remark. Note that $N^{1.72} > 32N^{1.6}$ for $N \geq 10^{12.55}$ and $32N^{1.6} > \frac{6}{5}N^{1+G_1(N)}$ for $N \geq 10^{80.53}$.

The result can be applied to give an explicit bound for the magnitude of solutions of the generalized Fermat equation. Let $(p, q, r) \in \mathbb{Z}_{\geq 2}$ with $(p, q, r) \neq (2, 2, 2)$. The equation

$$x^p + y^q = z^r, \quad (x, y, z) = 1 \text{ with integers } x > 0, y > 0, z > 0 \tag{1.14}$$

is called the *generalized Fermat equation*. We consider (1.14) with $p \geq 3, q \geq 3, r \geq 3$. For solving (1.14), there is no loss of generality in assuming $x > 1$, $y > 1$ and $z > 1$ since otherwise (1.14) is completely solved by Mihăilescu [5].

Let $[p, q, r]$ denote all permutations of the ordered triple (p, q, r). Let

$$Q = \{[3, 5, p] : 7 \leq p \leq 23, p \text{ prime}\} \cup \{[3, 4, p] : p \text{ prime}\}.$$

Then Laishram and Shorey [3] proved that (1.14) with $x > 1, y > 1, z > 1, p \geq 3, q \geq 3, r \geq 3$ implies that $[p, q, r] \in Q$ such that

$$\max(x^p, y^q, z^r) < e^{1758.3353}$$

whenever (1.3) holds. We sharpen the above result as follows. Let

$$Q_1 = \{[3, 5, p] : 7 \leq p \leq 19\} \cup \{[3, 4, p] : p \geq 11\}$$

where p is a prime number. Then

Theorem 1.6. *Assume* (1.3)*. Then* (1.14) *with* $x > 1$, $y > 1$, $z > 1$, $p \geq 3, q \geq 3$ *and* $r \geq 3$ *implies that* $[p, q, r] \in Q_1$*. Further for each* $[p, q, r] \in Q_1$*, we have the following upper bound for* $\max(x^p, y^q, z^r)$*.*

$[p, q, r]$	*Upper bound for* $\max(x^p, y^q, z^r)$
$[3, 4, p], p \geq 37$	8.1×10^{75}
$[3, 4, 31]$	1.3×10^{123}
$[3, 4, 29]$	4.3×10^{130}
$[3, 4, 23]$	1.2×10^{167}
$[3, 4, 19]$	9.8×10^{217}
$[3, 4, 17]$	1.2×10^{263}
$[3, 4, 13]$	1.5×10^{481}
$[3, 4, 11]$	2.2×10^{599}

$[p, q, r]$	*Upper bound for* $\max(x^p, y^q, z^r)$
$[3, 5, 19]$	1.6×10^{61}
$[3, 5, 17]$	6.7×10^{69}
$[3, 5, 13]$	3.9×10^{107}
$[3, 5, 11]$	3.9×10^{155}
$[3, 5, 7]$	6.6×10^{645}

Next we give some applications of our theorems to powerful numbers. An integer ν is called powerful if $\nu > 0$ and $p^2 | \nu$ whenever $p | \nu$ for every prime p. Golomb [2] proved in 1970 that there are infinitely many pairs of consecutive powerful integers and there exists no four (or more) consecutive powerful integers. Erdős conjectured that there does not exist three consecutive powerful integers. Trudgian [12] proved, under Conjecture 1.2, that $t < 10^{20000}$ whenever $(t - 1, t, t+1)$ is a triple of consecutive powerful integers. Mollin and Walsh [6] obtained the following results. Assume $t - 1, t, t + 1$ are powerful. Put

$$P = t, \quad Q = (t - 1)(t + 1) = my^2$$

where m is squarefree. Then $m \equiv 7 \pmod 8$ and (t, y) is a solution of $x^2 - my^2 = 1$. For the case when $m = 7$, Mollin and Walsh [6] proved that

$$t > 10^{10^8}. \tag{1.15}$$

Hence, together with the result by Trudgian [12], there is no triple $(t - 1, t, t + 1)$ of consecutive powerful integers such that $t^2 - 7y^2 = 1$. By following the arguments given in Mollin and Walsh

[6], we have checked that if $m = 7$ is replaced by $m \in \{15, 23, 31, 39, 47, 55, 87\}$, then (1.15) can be replaced by

$$t > 10^{3\times 10^{13}}.$$

Therefore, combining with the result by Trudgian [12], there is no triple $(t-1, t, t+1)$ of consecutive powerful integers such that $t^2 - my^2 = 1$ with $m \in \{7, 15, 23, 31, 39, 47, 55, 87\}$.

Next, we prove the following result on triples of $(a+kd, a+(k+1)d, a+(k+2)d)$ of consecutive powerful integers in arithmetic progression.

Theorem 1.7. *Let $a > 0$, $d > 0$ and $k \geq 0$ be integers such that $(a, d) = 1$. Assume that $a+kd$, $a+(k+1)d$ and $a+(k+2)d$ are all powerful integers. Then (1.3) implies the following:*
(1). *Let $\varepsilon > 0$. There exists an effectively computable number k_0 depending only on ε such that for $k \geq k_0$, we have*

$$a_{k+1} < (1.2d)^{2+\varepsilon}. \tag{1.16}$$

(2). *We have*

$$a_{k+1} < \max\{2.31 \times 10^{158} d^{2666}, 10^{51075}\}. \tag{1.17}$$

If $(t-1, t, t+1)$ is a triple of powerful integers, then $\frac{N(t,(t^2-1))}{t^{3/2}} < 1$. In the next result we show that $\frac{N(t,(t^2-1))}{t^{3/2}} > 1$ for all sufficiently large t whenever (1.3) holds.

Theorem 1.8. *If $t > 10^{51075}$, then (1.3) implies that*

$$N > t^{1.52}$$

where N is the square free part of $t(t^2-1)$.

For an integer $\nu > 1$, we denote by $P(\nu)$ the greatest prime factor of ν. For $n \geq 1$ and $k \geq 2$, we write

$$P(n, k) = n(n+1)\cdots(n+k-1).$$

If $n \leq k^{3/2}$ and n is sufficiently large, we see from the results on difference between consecutive primes that $P(n, k) \geq n$. Therefore we always suppose that $n > k^{3/2}$. It is, perhaps, conjectured by Erdős that

$$P(n, k) > (1-\epsilon)k \log n \quad \text{for} \quad k \geq k_0 = k_0(\epsilon).$$

It remains open even after assuming *abc*-conjecture. Shorey and Tijdeman [11] proved that there exists a number k_1 depending only on ϵ such that for integers n and $k \geq 2$ with $n \geq k^{3/2}$, we have

$$P(n, k) > \Big(\frac{1}{2} - \epsilon\Big) k \log n \quad \text{for} \quad k \geq k_1$$

under *abc*-conjecture. We derive from Theorem 1.3 the following effective sharpening of the above inequality.

Theorem 1.9. *Assume Conjecture 1.2. There exist effectively computable absolute positive constants k_2 and k_3 such that for integers n and $k \geq k_2$ with $n \geq k^{3/2}$, we have*

$$P(n, k) > \Big(\frac{1}{2} - k_3 G_2(n)\Big) k \log n$$

where $G_2(n) = \left(\frac{\log_3 n}{\log_2 n}\right)^{1/2}$.

We use SAGE for calculation and, in particular, for extracting values of a, b, c that fulfill specified conditions to come to the conclusion that (1.11) holds for $5 \leq \omega \leq 9$ when proving Theorem 1.4.

2. Preliminaries

For any real number $x > 0$, let $\theta(x) = \sum_{p \leq x} \log p$. In 1983, G. Robin [8] proved the following lemma for $\theta(x)$.

Lemma 2.1. *Let p_n be the nth prime. Then*

$$\theta(p_n) \geq n\Big(\log n + \log_2 n - 1.076869\Big) \text{ for } n > 1. \tag{2.1}$$

Lemma 2.2. *For $N \geq 4$, the function $g(x) = (\frac{e \log N}{x})^x$ is increasing in $1 \leq x < \log N$.*

Proof. To show $g(x)$ is increasing, we see the positivity of its derivative. Let $u = e \log N$. We have

$$g(x) = \left(\frac{u}{x}\right)^x = e^{x \log(u/x)}.$$

Now

$$\begin{aligned} g'(x) &= e^{x \log(u/x)}\Big(\log(u/x) + x(x/u)(-u/x^2)\Big) \\ &= e^{x \log(u/x)}\Big(\log(u/x) - 1\Big). \end{aligned}$$

Thus $g'(x) > 0$ if $e \log N = u > ex$. Hence $g(x)$ is increasing in $1 \leq x < \log N$. □

Lemma 2.3. *Let $\omega = \omega(N) \geq 13$. Then*

$$\log N > \omega \log \omega.$$

Proof. Let $N = Q_1 Q_2 \cdots Q_\omega$ where $Q_1 < Q_2 < \cdots < Q_\omega$ are prime numbers. Now if p_i denotes the ith prime, then we have

$$N = \prod_{i=1}^{\omega} Q_i \geq \prod_{i=1}^{\omega} p_i.$$

This gives

$$\log N \geq \sum_{i=1}^{\omega} \log p_i = \theta(p_\omega).$$

Therefore it suffices to show that $\theta(p_\omega) > \omega \log \omega$ for $\omega \geq 13$. This follows by Lemma 2.1 for $\omega \geq 19$ since $\log_2 \omega - 1.07869$ is positive. Further we check that $\theta(p_\omega) > \omega \log \omega$ for $13 \leq \omega \leq 18$ by direct computation. □

Lemma 2.4. *Assume that $\log N > \omega \log \omega$. Then*

$$\omega < \frac{\log N}{A(N)}.$$

Proof. Let $\log N > \omega \log \omega$. Then we have

$$\omega < \frac{\log N}{\log \omega}. \tag{2.2}$$

Let

$$\omega > \frac{\log N}{\log_2 N}.$$

Then

$$\log \omega > \log_2 N - \log_3 N = A(N). \tag{2.3}$$

By combining (2.2), (2.3) and $A(N) < \log \log N$, we get $\omega < \frac{\log N}{A(N)}$. $\square$

Lemma 2.5. *The equation* (1.1) *with* (1.3) *implies that* $c < \frac{6}{5}N^{1+G(N)}$ *for* $\log N > \omega \log \omega$ *where* $G(N)$ *is given by* (1.4).

Proof. Let $N < 16$. Then $\omega = 2$ and $N = 2p$ with $p \in \{3, 5, 7\}$. Now we re-write (1.1) as $2^x - p^y = \pm 1$ where $x \geq 1$ and $y \geq 1$ are integers. We may suppose that $x > 1$ and $y > 1$ otherwise the assertion follows. Mihăilescu [5] proved that Catalan equation $x^p - y^q = 1$ with $p > 1, q > 1$ has unique integral solution $(x, y, p, q) = (3, 2, 2, 3)$ and this implies that the solutions of (1.1) are given by $(a, b, c) \in \{(8, 1, 9), (1, 8, 9)\}$ and the assertion follows for each of these triplets.

Thus we may assume that $N \geq 16$. Let $\log N > \omega \log \omega$. Since $\omega! \geq \omega^\omega e^{-\omega}$ by induction on ω, we derive from (1.3) that

$$c < \frac{6}{5}N\frac{(\log N)^\omega}{\omega!} \leq \frac{6}{5}N\Big(\frac{e \log N}{\omega}\Big)^\omega. \tag{2.4}$$

Since $A(N) > 1$ for $N \geq 16$, we derive from Lemma 2.4 that

$$\omega < \frac{\log N}{A(N)} < \log N.$$

Then Lemma 2.2 implies that

$$\Big(\frac{e \log N}{\omega}\Big)^\omega \leq (eA(N))^{\frac{\log N}{A(N)}} = N^{G(N)}.$$

Thus, by (2.4) , we get

$$c < \frac{6}{5}N^{1+G(N)}.$$

$\square$

Corollary 2.6. *The equation* (1.1) *with* (1.3) *implies that* $c < \frac{6}{5}N^{1+G(N)}$ *for* $\omega \geq 13$ *where* $G(N)$ *is given by* (1.4).

Proof. The assertion follows from Lemma 2.3 and 2.5. $\square$

Lemma 2.7. *The equation* (1.1) *with* (1.3) *implies that* $c < \frac{6}{5}N^{1+G(N)}$ *for* $N > 2$.

Proof. By Corollary 2.6 and Lemma 2.5, we have to consider $2 \leq \omega \leq 12$ and $\log N \leq \omega \log \omega$. Let $\omega = 2$. Then $6 \leq N \leq 4$ which is not possible. Let $\omega = 3$. Then $N \leq 27$ which is not possible since the product of the first three prime numbers is equal to 30. Thus $\omega \geq 4$ and $N \geq 210$. Therefore $G(N)$ is decreasing. We check that $G(10^{23}) > \frac{3}{4}$ and therefore $G(N) > \frac{3}{4}$ for $N \leq 10^{23}$ since $G(N)$ is decreasing. Hence the assertion follows for $N \leq 10^{23}$ by (1.10). Thus we may assume that $N > 10^{23}$. Then $\omega^\omega \geq N > 10^{23}$ which implies that $\omega > 12$. This is a contradiction. $\square$

For given $0 < \theta < 1$, $m \geq 2$ and $K > 0$, let

$$f(x) = \frac{(\log x)^m}{m!} - Kx^\theta.$$

Then

$$g(x) = x^{1-\theta}(m-1)!f'(x) = \frac{(\log x)^{m-1}}{x^\theta} - K\theta(m-1)!$$

and

$$g'(x) = \frac{(\log x)^{m-2}}{x^{1+\theta}}\big(m - 1 - \theta \log x\big). \tag{2.5}$$

Then we have the following Lemma.

Lemma 2.8. *Assume that there exist positive numbers x_0 and x_1 with $1 < x_1 \leq x_0$ such that*

$$f(x_0) < 0, \ g(x_0) < 0 \ \textit{and} \ \ g'(x_1) < 0. \tag{2.6}$$

Then $f(x) < 0$ for $x \geq x_0$.

Proof. Since $g'(x_1) < 0$, we see from (2.5) that $g'(x) < 0$ for $x \geq x_1$. Therefore g is a decreasing function for $x \geq x_1$. Then, since $g(x_0) < 0$ and $x_0 \geq x_1$, we derive that $g(x) < 0$ for $x \geq x_0$ which implies that $f'(x) < 0$ for $x \geq x_0$. Thus $f(x)$ is decreasing for $x \geq x_0$. Hence the assertion follows since $f(x_0) < 0$. $\square$

Lemma 2.9. *Let a, b and c be relatively prime positive integers satisfying* (1.1)*. Then* (1.3) *implies that*

$$c < 32N^{1.6} \ \textit{for} \ \ N > 2.$$

Proof. Following the same proof as in [3, Theorem 1], we have $\omega_1 = \omega_\epsilon = 42$ for $\epsilon = 0.6$ such that

$$\epsilon \geq \frac{1 + \log X_0(i)}{X_0(i)} \text{ for } i \geq \omega_1 \text{ and } \frac{i!\Theta(p_i)^\epsilon}{\theta(p_i)^i} > \sqrt{2\pi i} \ \text{ for } \ i \geq \omega_\epsilon \tag{2.7}$$

holds. Here $X_0(i) = \log i + \log_2 i - 1.076869$ and $\frac{i!N^\epsilon}{(\log N)^i} > \frac{i!\Theta(p_i)^\epsilon}{\theta(p_i)^i}$. We check that for $35 \leq \omega < 42$, we have

$$\frac{\omega!\Theta(p_\omega)^\epsilon}{\theta(p_\omega)^\omega} > \frac{6}{5}. \tag{2.8}$$

Then

$$\frac{(\log N)^\omega}{\omega!} < \frac{5}{6}N^{0.6} \ \text{ for } \ N > 2, \omega \geq 35$$

and the assertion follows from (1.3). Let $2 \leq \omega \leq 34$. We check that, for all ω, we may choose x_0, x_1 as in Lemma 2.8 with $x_1 = x_0 = \prod_{p \leq p_\omega} p$, $K = 80/3$ and $\theta = 0.6$ so that (2.6) is satisfied. Thus $f(x) < 0$ for $x \geq x_0$. Therefore $f(N) < 0$ since $N \geq \prod_{p \leq p_\omega} p = x_0$. Hence Lemma 2.9 follows. $\square$

Lemma 2.10. *Let a, b and c be relatively prime positive integers satisfying* (1.1). *Then* (1.3) *implies that*

$$c < 10N^{1.62991} \textit{ for } \; N > 2.$$

Proof. Let $\epsilon = 0.62991$. As in Lemma 2.9, we have $\omega_1 = 33$, $\omega_\epsilon = 32$ such that (2.7) holds. We check that for $26 \leq \omega < 32$, we have (2.8). Therefore $c < 10N^{1.62991}$ for $N > 2$ with $\omega \geq 26$. Let $2 \leq \omega \leq 25$. We may choose $x_1 = x_0 = \prod_{p \leq p_\omega} p$ with $K = 25/3$ and $\theta = 0.62991$ in Lemma 2.8, we get $f(x) < 0$ for $x \geq x_0$ which implies that $f(N) < 0$ for $N \geq \prod_{p \leq p_\omega} p = x_0$. Hence Lemma 2.10 follows. □

3. **Proof of Theorem 1.3**

By Lemma 2.7, we have (1.8). Now by (1.7), we have

$$c < \frac{6}{5}N^{1+G(N)} \leq \frac{6}{5}N^{1+G_1(N)} \quad \text{for } \; 297856 \leq N \leq 10^{36}.$$

Therefore we may assume that $N > 10^{36}$. By Lemma 2.1 with $n = \omega$, we have

$$\omega \leq \frac{\log N}{\log \omega + \log_2 \omega - 1.076869}. \tag{3.1}$$

Let

$$\omega \geq \frac{\log N}{\log_2 N}.$$

Then $\log \omega \geq A(N), \log_2 \omega \geq \log A(N)$. Thus (3.1) gives $\omega \leq \frac{\log N}{A_1(N)}$. Therefore

$$\omega \leq \max\Big(\frac{\log N}{\log_2 N}, \frac{\log N}{A_1(N)}\Big) < \frac{\log N}{A_1(N)} < \log N. \tag{3.2}$$

since $A_1(N) \leq \log_2 N - 1.076869 < \log_2 N$ and $A_1(N) > 1$ by $N \geq 297856$. Then we derive from (1.3), (3.2) and Lemma 2.2 that

$$c < \frac{6}{5}N\Big(\frac{e \log N}{\omega}\Big)^\omega \leq \frac{6}{5}N(eA_1(N))^{\frac{\log N}{A_1(N)}} = \frac{6}{5}N^{1+G_1(N)}.$$

4. **Proof of Theorem 1.4**

The assertions (1.12) and (1.13) follows from Lemma 2.10 and Lemma 2.9, respectively. We proceed with the proof of assertion (1.11).

As in Lemma 2.9, we have $\omega = 18$ and $\omega_\epsilon = 17$ for $\epsilon = 0.72$ such that (2.7) holds. We check that for $10 \leq \omega < 17$, we have (2.8). Thus we get

$$\frac{(\log N)^\omega}{\omega!} < \frac{5}{6}N^{0.72} \quad \text{for } \; N > 2, \omega \geq 10.$$

Let $\omega \leq 9$. We apply Lemma 2.8 with $x_1 = x_0, K = 5/6$ and $\theta = 0.72$. Then N's lies in the range $\Big[\prod_{p \leq p_\omega} p, x_0\Big)$.

We observe that for $\omega \leq 4$, we may choose $x_1 = x_0 = \prod_{p \leq p_\omega} p$ so that (2.6) is satisfied. Then (1.11) follows by Lemma 2.8 with $K = 5/6$.

For $5 \leq \omega \leq 9$, we choose $x_1 = x_0$ as given in Table 1 so that they satisfy (2.6) and we extract all square free N with $\omega(N) = \omega$ that lie in the range $\Big[\prod_{p \leq p_\omega} p, x_0\Big)$. Hence we obtain Table 1.

TABLE 1. Data for $5 \le \omega \le 9$.

ω	$\prod_{p \le p_\omega} p$	$x_0,\ x_1$	$N \in \left[\prod_{p \le p_\omega} p, x_0\right)$
5	2310	4100	2310, 2730, 3570, 3990.
6	30030	8.79×10^4	30030, 39270, 43890, 46410, 51870, 53130, 62790, 66990, 67830, 71610, 72930, 79170, 81510, 82110, 84630, 85470.
7	510510	1.51×10^6	510510, 570570, 690690, 746130, 870870, 881790, 903210, 930930, 1009470, 1067430, 1111110, 1138830, 1193010, 1217370, 1231230, 1272810, 1291290, 1345890, 1360590, 1385670, 1411410, 1438710, 1452990, 1504230.
8	9.69969×10^6	2.45×10^7	9699690, 11741730, 13123110, 14804790, 15825810, 16546530, 17160990, 17687670, 18888870, 20030010, 20281170, 20930910, 21111090, 21411390, 21637770, 21951930, 23130030, 23393370, 23993970.
9	2.2309287×10^8	3.91×10^8	223092870, 281291010, 300690390, 340510170, 358888530, 363993630, 380570190.

By (1.3), for each $N = Q_1 Q_2 \cdots Q_\omega$ where $Q_1, Q_2, \ldots, Q_\omega$ are distinct primes and $5 \le \omega \le 9$, it suffices to restrict $c \in \left[N^{1.72}, \frac{6}{5} N^{\frac{(\log N)^\omega}{\omega!}}\right)$ otherwise (1.11) holds. We perform searching of c with SAGE by identifying all integers falling in this interval having only prime factors in the set $\{Q_1, \ldots, Q_\omega\}$. This can be done as follows: We write $Q_1^{\gamma_1} \cdots Q_\omega^{\gamma_\omega}$ where $\gamma_1, \ldots, \gamma_\omega$ are non-negative integers and estimate

$$0 \le \gamma_i \le \left[\frac{\log\left(\frac{6}{5} N^{\frac{(\log N)^\omega}{\omega!}} Q_1^{-\gamma_1} \cdots Q_{i-1}^{-\gamma_{i-1}}\right)}{\log Q_i} \right] \qquad \text{for } 1 \le i \le \omega.$$

After all γ_i's are determined, we take $c = Q_1^{\gamma_1} \cdots Q_\omega^{\gamma_\omega}$ if $Q_1^{\gamma_1} \cdots Q_\omega^{\gamma_\omega} \in \left[N^{1.72}, \frac{6}{5} N^{\frac{(\log N)^\omega}{\omega!}}\right)$ is satisfied. For each c with $\mathrm{rad}(c) < N$, we construct all possible choices of a satisfying $a < b$, which we may assume without loss of generality, so that $a < \frac{c}{2}$ and the property that a has only prime factors in $\{Q_1, \ldots, Q_\omega\}$ and $(a, c) = 1$. Similar to the case of obtaining c, we let $a = Q_1^{\mu_1} \cdots Q_\omega^{\mu_\omega}$ where $\mu_1, \ldots, \mu_\omega$ are non-negative integers and we estimate

$$0 \le \mu_i \le \begin{cases} \left[\frac{\log\left(\frac{c}{2} Q_1^{-\mu_1} \cdots Q_{i-1}^{-\mu_{i-1}}\right)}{\log Q_i} \right], & \text{if } \gamma_i = 0, \\ 0 \qquad\qquad , & \text{if } \gamma_i > 0 \end{cases}$$

for $1 \le i \le \omega$. Then for each pair of (c, a) obtained with $\mathrm{rad}(ac) < N$, we construct the corresponding b by (1.1). We note that $(a, b, c) = 1$. We check that for each case there does not exist any a, b, c such that the radical of abc is equal to N. Besides, it is clear that if $\mathrm{rad}(c) = N$ or $\mathrm{rad}(ac) = N$, then there exists no relatively prime positive integers a, b, c satisfying (1.1) with $\mathrm{rad}(abc) = N$. Hence (1.11) holds.

To illustrate, for $\omega = 5, N = 3990 = 2\times3\times5\times7\times19$, the only c extracted is $1562500 = 2^2\times5^8$. There are a total of 117 a's each having only prime factors in $\{2, 3, 5, 7, 19\}$ and is relatively prime to c. For $\omega = 7, N = 1504230 = 2 \times 3 \times 5 \times 7 \times 13 \times 19 \times 29$ the only c's extracted are 42168581000, 42169420800, 42174006784, 42174732915, 42176295000, 42178070844, 42182400000, 42185786580 and 42185937500. For each c in the above list, the number of corresponding a's having only prime factors in $\{2, 3, 5, 7, 13, 19, 29\}$ and is relatively prime to c is $22, 54, 599, 181, 10, 71, 186, 147$ and 115 respectively.

Table 2 lists the number of c extracted for some selected cases of ω and N.

TABLE 2. Number of c extracted in selected cases of ω and N.

ω	N	Number of c extracted
5	2310	32
	3570	9
6	30030	631
	85470	18
7	510510	4565
	1452990	183

ω	N	Number of c extracted
8	9699690	25548
	23993970	648
9	223092870	98273
	380570190	4885

5. **Proof of Theorem 1.6**

We may assume that each of p, q, r is either 4 or an odd prime. Let $[p, q, r]$ denote all permutations of the ordered triple (p, q, r). An account of earlier results has been mentioned in [3]. Hence we may suppose (p, q, r) is different from those values. We may assume that $x > 1, y > 1, z > 1$. Then

$$x < z^{r/p}, \quad y < z^{r/q}.$$

We observe that $N(x^py^qz^r) = N(xyz)$ and we always write $N = N(xyz)$ in the proof of Theorem 1.6. Then by using (1.11), we get

$$z^r < N^{1.72} \le (xyz)^{1.72} < z^{1.72(1+r/p+r/q)},$$

implying

$$\frac{1}{1.72} < \frac{1}{p} + \frac{1}{q} + \frac{1}{r}.$$

Thus we need to consider $(p, q, r) \in Q_1$ and $[3, 3, p]$ for $p > 10^9$. For $N < 297856$, we apply (1.11) to get

$$\max(x^p, y^q, z^r) < N^{1.72} < 297856^{1.72} < 2.7 \times 10^9.$$

Therefore we may assume that $N \ge 297856$. We deduce the upper bound for each case of $[p, q, r]$ separately. We present the proof of $[3, 4, p]$ with $p \ge 37$ as follows. Let $N > e^{107.07}$ where we observe that $\prod_{p\le p_{30}} p < e^{107.07}$. By following the proof as in [3, Theorem 1], we have $\omega_1 = 31$, $\omega_\epsilon = 30$ for $\epsilon = 173/271$ such that (2.7) holds and

$$z^r < \frac{6}{5\sqrt{2\pi\omega_\epsilon}} N^{1+\epsilon} \le (xyz)^{1+\epsilon}.$$

Then

$$z^r < z^{(1+\epsilon)(1+r/p+r/q)}$$

implying

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r} > \frac{1}{1+\epsilon} = \frac{271}{444} = \frac{1}{3}+\frac{1}{4}+\frac{1}{37}.$$

This is a contradiction. Therefore we may suppose that $N < e^{107.07}$. By (1.13), we have

$$\max(x^p, y^q, z^r) < 32N^{1.6} < 32e^{107.07(1.6)} < 8.1 \times 10^{75}.$$

The proof of $[3,3,p]$ with $p > 10^9$ is similar. In this case, we argue with $\epsilon = \frac{999999997}{2000000003}$, $\omega_1 = 129$, and $\omega_\epsilon = 128$ to conclude that $N < \prod_{p \le p_{128}} p < e^{686.163}$ and then we derive

$$\max(x^p, y^q, z^r) < 32e^{686.163(1.6)} < 2 \times 10^{478}$$

which is not possible since $\max(x^p, y^q, z^r) \ge 2^{10^9}$.

Let $[p,q,r] = [3,5,7]$. First we consider $N \ge e^{1004.763}$. We apply [3, Theorem 1] with $\epsilon = 34/71$. We observe that $\omega_\epsilon = 175$ and $\prod_{p \le p_{175}} p < e^{1004.763}$. Therefore, by [3, Theorem 1], we have

$$z^r < N^{1+\epsilon} \le (xyz)^{1+\epsilon} < z^{(1+\epsilon)(1+r/p+r/q)}.$$

This implies that

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r} > \frac{1}{1+\epsilon} = \frac{71}{105} = \frac{1}{3}+\frac{1}{5}+\frac{1}{7}$$

which is a contradiction. Therefore we may suppose that $N < e^{1004.763}$. Now we apply Theorem 1.5 repetitively to obtain upper bound for z^r as follows:

(1) For $N < 10^{12.55}$,

$$z^r < N^{1.72} < 10^{12.55(1.72)} < 3.9 \times 10^{21}.$$

(2) For $10^{12.55} \le N < 10^{80.53}$,

$$z^r < 32N^{1.6} < 32\left(10^{80.53}\right)^{1.6} < 2.3 \times 10^{130}.$$

(3) For $10^{80.53} \le N < e^{900}$, we use $G_1(10^{80.53}) \le 0.61771$ to get

$$z^r < \frac{6}{5}N^{1+G_1(10^{80.53})} < \frac{6}{5}e^{900(1.61771)} < e^{1457}.$$

(4) For $e^{900} \le N < e^{984}$, we use $G_1(e^{900}) \le 0.49781$ to get

$$z^r < \frac{6}{5}N^{1+G_1(e^{900})} < \frac{6}{5}e^{984(1.49781)} < e^{1475}.$$

(5) For $e^{984} \le N < e^{1004.763}$, we observe that $\prod_{p \le p_{172}} p < e^{984}$. By following the proof as in [3, Theorem 1] with $\epsilon = 0.48$, $\omega_1 = 173$ and $\omega_\epsilon = 172$, we get

$$z^r < \frac{6}{5\sqrt{2\pi\omega_\epsilon}}N^{1+\epsilon} < e^{1004.763(1.48)} < e^{1488}.$$

Now we combine all the above estimates. We get

$$\max(x^p, y^q, z^r) < e^{1488} < 6.6 \times 10^{645}.$$

The proof of $[3,4,11]$ is similar. In this case, we suppose $N < e^{928.667}$ by following the proof of [3, Theorem 1] with $\epsilon = 43/89$ and observing that $\omega_\epsilon = 164$, $\prod_{p \le p_{164}} p < e^{928.667}$. We apply Theorem 1.5 repetitively to obtain

$$\max(x^p, y^q, z^r) < e^{1380} < 2.2 \times 10^{599}.$$

We now present the proof of the case $[3,5,19]$ with $r = 3$. We first suppose that $z < 1.21 \times 10^{15} := Z_{[3,5,19]}$. By (1.13),

$$\begin{aligned} z^r &< 32N^{1.6} \le 32(xyz)^{1.6} < 32z^{1.6(1+r/p+r/q)} < 32{Z_{[3,5,19]}}^{1.6(1+3/5+3/19)} \\ &< 8.5 \times 10^{43} := A_{[3,5,19]}. \end{aligned}$$

Next, suppose that $z \ge Z_{[3,5,19]}$. From (1.9) we have

$$\begin{aligned} z^r &< \frac{6}{5}(xyz)^{1+G_1(N)} < \frac{6}{5}z^{(1+r/p+r/q)(1+G_1(N))} \\ &< z^{0.00525+r(1/r+1/p+1/q)(1+G_1(N))}, \end{aligned}$$

giving

$$\frac{1}{1+G_1(N)} < \frac{r}{r-0.00525}\left(\frac{1}{r}+\frac{1}{p}+\frac{1}{q}\right) = \frac{3}{3-0.00525}\left(\frac{167}{285}\right). \tag{5.1}$$

If $N \ge 2 \times 10^{37} := N_{[3,5,19]}$, we use the fact that G_1 is decreasing to get $G_1(N) \le 0.7036 := G_1(N_{[3,5,19]})$. Then $\frac{1}{1+G_1(N)}$ exceeds the right hand side of (5.1). Thus, we may assume $N < 2 \times 10^{37}$ and hence

$$z^r < 32\big(2 \times 10^{37}\big)^{1.6} < 1.6 \times 10^{61} := B_{[3,5,19]}.$$

For $r = 5$ and $r = 19$, the proofs are similar with the corresponding parameters $Z_{[3,5,19]}$, $A_{[3,5,19]}$, $G_1(N_{[3,5,19]})$, $N_{[3,5,19]}$ and $B_{[3,5,19]}$ as shown in Table 4. Hence we conclude

$$\max(x^p, y^q, z^r) < 1.6 \times 10^{61} := C_{[3,5,19]}.$$

The proofs for the remaining cases of $[p,q,r]$ can deduced similarly. The results for all cases of $[p,q,r]$ are shown in Table 3 and Table 4.

TABLE 3. Upper bound for $\max(x^p, y^q, z^r)$ for $[3,4,p]$ $(p \ge 37)$, $[3,5,7]$ and $[3,4,11]$.

$[p,q,r]$	Upper bound for $\max(x^p, y^q, z^r)$
$[3,4,p], p \ge 37$	8.1×10^{75}
$[3,5,7]$	6.6×10^{645}
$[3,4,11]$	2.2×10^{599}

TABLE 4. Upper bound for $\max(x^p, y^q, z^r)$ for the remaining cases of $[p, q, r]$.

$[p,q,r]$	r	$Z_{[p,q,r]}$	$A_{[p,q,r]}$	$G_1(N_{[p,q,r]})$	$N_{[p,q,r]}$	$B_{[p,q,r]}$	$C_{[p,q,r]}$
	3	1.21×10^{15}	8.5×10^{43}	0.7036	2×10^{37}	1.6×10^{61}	
$[3,5,19]$	5	1.12×10^{9}	8.5×10^{43}	0.7036	2×10^{37}	1.6×10^{61}	1.6×10^{61}
	19	241	8.7×10^{43}	0.7036	2×10^{37}	1.6×10^{61}	
	3	6.8×10^{21}	3.7×10^{63}	0.6867	5×10^{42}	6.7×10^{69}	
$[3,5,17]$	5	1.26×10^{13}	3.7×10^{63}	0.6867	5×10^{42}	6.7×10^{69}	6.7×10^{69}
	17	7125	3.7×10^{63}	0.6867	5×10^{42}	6.7×10^{69}	
	3	5.2×10^{29}	3.6×10^{88}	0.6372	2×10^{66}	3.9×10^{107}	
$[3,5,13]$	5	6.8×10^{17}	3.7×10^{88}	0.6372	2×10^{66}	3.9×10^{107}	3.9×10^{107}
	13	7.21×10^{6}	3.6×10^{88}	0.6372	2×10^{66}	3.9×10^{107}	
	3	7.9×10^{44}	1.1×10^{136}	0.601	2×10^{96}	3.9×10^{155}	
$[3,5,11]$	5	8.7×10^{26}	1.1×10^{136}	0.601	2×10^{96}	3.9×10^{155}	3.9×10^{155}
	11	1.8×10^{12}	1.5×10^{136}	0.601	2×10^{96}	3.9×10^{155}	
	3	4.72×10^{40}	4.9×10^{121}	0.6234	10^{76}	1.3×10^{123}	
$[3,4,31]$	4	3.2×10^{30}	4.9×10^{121}	0.6234	10^{76}	1.3×10^{123}	1.3×10^{123}
	31	8635	5×10^{121}	0.6234	10^{76}	1.3×10^{123}	
	3	3.4×10^{42}	4.3×10^{127}	0.6176	5×10^{80}	4.3×10^{130}	
$[3,4,29]$	4	7.9×10^{31}	4.3×10^{127}	0.6176	5×10^{80}	4.3×10^{130}	4.3×10^{130}
	29	25065	4.1×10^{127}	0.6176	5×10^{80}	4.3×10^{130}	
	3	1.3×10^{48}	1.9×10^{146}	0.5945	3×10^{103}	1.2×10^{167}	
$[3,4,23]$	4	1.2×10^{36}	1.8×10^{146}	0.5945	3×10^{103}	1.2×10^{167}	1.2×10^{167}
	23	1.9×10^{6}	2.2×10^{146}	0.5945	3×10^{103}	1.2×10^{167}	
	3	1.4×10^{58}	1.1×10^{179}	0.5717	2×10^{135}	9.8×10^{217}	
$[3,4,19]$	4	4.1×10^{43}	1.1×10^{179}	0.5717	2×10^{135}	9.8×10^{217}	9.8×10^{217}
	19	1.52×10^{9}	1.1×10^{179}	0.5717	2×10^{135}	9.8×10^{217}	
	3	3×10^{74}	1.2×10^{231}	0.5567	3×10^{163}	1.2×10^{263}	
$[3,4,17]$	4	7.2×10^{55}	1.2×10^{231}	0.5567	3×10^{163}	1.2×10^{263}	1.2×10^{263}
	17	1.4×10^{13}	1.4×10^{231}	0.5567	3×10^{163}	1.2×10^{263}	
	3	1.3×10^{110}	3.1×10^{350}	0.5142	6×10^{299}	1.5×10^{481}	
$[3,4,13]$	4	3.8×10^{82}	2.9×10^{350}	0.5142	6×10^{299}	1.5×10^{481}	1.5×10^{481}
	13	2.6×10^{25}	3.5×10^{350}	0.5142	6×10^{299}	1.5×10^{481}	

6. **Proof of Theorem 1.7**

Let a_k, a_{k+1} and a_{k+2} be powerful integers where

$$a_{k+i} = a + (k+i)d \quad \text{for } 0 \le i \le 2.$$

We denote $M = N(a_k a_{k+1} a_{k+2})$ and $M_1 = N(d a_k a_{k+1} a_{k+2})$. Note that

$$2a_{k+1} = a_k + a_{k+2} \tag{6.1}$$

and $a_k \equiv a_{k+2} \pmod 2$. First, we obtain a lower bound for M and M_1 in terms of a_k by using (1.13). We consider the cases $2 \nmid a_k$ and $2 | a_k$ separately.

Case 1. $2 \nmid a_k$. Then $(2a_{k+1}, a_k) = 1$ implying $(2a_{k+1}, a_k, a_{k+2}) = 1$. Thus, by (1.13) after taking $a = a_k$, $b = a_{k+2}$ and $c = 2a_{k+1}$ in (6.1), we obtain

$$2a_{k+1} < 32\left(N(2a_k a_{k+1} a_{k+2})\right)^{1.6} \le 98M^{1.6}.$$

Case 2. $2|a_k$. Then $2|a_{k+2}$ so that from (6.1), we have

$$a_{k+1} = \frac{a_k}{2} + \frac{a_{k+2}}{2} \tag{6.2}$$

where $a_{k+1}, \frac{a_k}{2}, \frac{a_{k+2}}{2} \in \mathbb{Z}$ and $\left(a_{k+1}, \frac{a_k}{2}, \frac{a_{k+2}}{2}\right) = 1$. We observe that d is odd since $(a, d) = 1$ and therefore a_{k+1} is odd. This time, by taking $a = \frac{a_k}{2}$, $b = \frac{a_{k+2}}{2}$ and $c = a_{k+1}$ in (6.2) we obtain from (1.13) that

$$a_{k+1} < 32\left(N\left(\frac{1}{4}a_k a_{k+1} a_{k+2}\right)\right)^{1.6} \le 32M^{1.6}.$$

Hence, in both cases, we get

$$a_{k+1} < 49M^{1.6},$$

which implies that

$$M_1 \ge M > \left(\frac{a_{k+1}}{49}\right)^{1/1.6}. \tag{6.3}$$

Next, we note that

$$a_k a_{k+2} = a_{k+1}^2 - d^2 < a_{k+1}^2$$

and $(d^2, a_k a_{k+2}, a_{k+1}^2) = 1$. Assume

$$M \ge 297856. \tag{6.4}$$

Then (1.9) holds. Since G_1 is decreasing we have $G_1(M_1) \le G_1(M)$. By applying (1.9) with $a = a_k a_{k+2}$, $b = d^2$ and $c = a_{k+1}^2$, we obtain

$$a_{k+1}^2 < \frac{6}{5}M_1^{1+G_1(M_1)} \le \frac{6}{5}M_1^{1+G_1(M)}.$$

Further

$$M_1 \le N(d)M \le N(d)\left(a_k a_{k+1} a_{k+2}\right)^{1/2} < d a_{k+1}^{3/2}$$

since a_k, a_{k+1} and a_{k+2} are powerful. Thus we get

$$a_{k+1}^2 < \frac{6}{5}\left(d a_{k+1}^{3/2}\right)^{1+G_1(M)},$$

that is

$$a_{k+1} < 1.2^{\frac{2}{1-3G_1(M)}} d^{\frac{2(1+G_1(M))}{1-3G_1(M)}} \tag{6.5}$$

implying

$$a_{k+1} < (1.2d)^{\frac{2(1+G_1(M))}{1-3G_1(M)}}. \tag{6.6}$$

(1). Let $\varepsilon > 0$. We take $\varepsilon_1 = \frac{\varepsilon}{8+3\varepsilon}$. We may assume that $k \ge k_0$ where k_0 is a sufficiently large effectively computable number depending only on ε such that from (6.3) the assumption (6.4) is satisfied and $G_1(M) < \varepsilon_1$ using the fact that G_1 is decreasing. From (6.6) we have

$$a_{k+1} < (1.2d)^{\frac{2(1+G_1(M))}{1-3G_1(M)}} < (1.2d)^{\frac{2(1+\varepsilon_1)}{1-3\varepsilon_1}} = (1.2d)^{2+\varepsilon}.$$

(2). Suppose on the contrary that (1.17) does not hold. Then we have

$$a_{k+1} \ge \max\{2.31 \times 10^{158} d^{2666}, 10^{51075}\}. \tag{6.7}$$

Applying (6.7) to (6.3), we have

$$M_1 \geq M > \left(\frac{a_{k+1}}{49}\right)^{1/1.6} \geq e^{73500}$$

so that the assumption (6.4) is satisfied. Further, we derive that $G_1(M_1) \leq G_1(M) \leq 0.333$ by (1.6), Now we derive from (6.5) to give

$$a_{k+1} < 1.2^{2000} d^{2666} < 2.31 \times 10^{158} d^{2666}.$$

This is a contradiction.

7. **Proof of Theorem 1.8**

We assume (1.3) and write

$$t^2 = (t^2 - 1) + 1.$$

By (1.1) with $a = 1$, $b = t^2 - 1$ and $c = t^2$ and (1.13), we have

$$10^{2\times 51075} < t^2 < 32N^{1.6} \tag{7.1}$$

which implies that $N > 10^{63842}$. Then

$$G_1(N) < 0.317315. \tag{7.2}$$

Thus we obtain a sharper upper bound for t^2 and we can revise (7.1) to give

$$10^{2\times 51075} < t^2 < \frac{6}{5} N^{1.317315}. \tag{7.3}$$

This time we have $N > 10^{77544}$. Then, by following as above, we obtain $G_1(N) < 0.313229$ and $N > 10^{77785}$. Then

$$G_1(N) < 0.313165. \tag{7.4}$$

Finally we apply (1.9) and (7.4) to derive that

$$t^2 < \frac{6}{5} N^{1.313165}$$

which implies that

$$N > 0.87 t^{1.523037} > t^{1.52}.$$

8. **Proof of Theorem 1.9**

The proof is on the same lines as in Shorey and Tijdeman [11] which we refer in our proof without reference. We do not fix ϵ but allow it to be a function of n. Let k_2 be a sufficiently large absolute constant and we shall choose it later suitably. We put $\epsilon = k_2 G_2(n)$. Assume that

$$P(n, k) < \left(\frac{1}{2} - \epsilon\right) k \log n.$$

Then we proceed as in [11]. We choose $A_{i_1}, A_{i_2}, B_{i_1}, B_{i_2}$ as in [11] and apply Theorem 1.3 in place of abc-conjecture. We obtain

$$n < c_1 k^{\frac{1}{7}\epsilon} n^{1-\frac{2\epsilon}{3}}.$$

We denote by c_2, c_3, c_4, c_5 absolute constants. The above inequality implies

$$\epsilon^2 \log n < c_2 \log k.$$

Further Shorey [10] proved that

$$P(n,k) > c_3 k \log k \frac{\log_2 k}{\log_3 k}.$$

By combining the preceding two inequalities, we get

$$P(n,k) > c_4 \epsilon^2 k \log n \frac{\log_2 n}{\log_3 n} = c_4 k_2^2 k \log n. \tag{8.1}$$

Finally we take k_2 such that $k_2 > c_4^{-1/2}$ and fix it to conclude that $P(n,k) > k \log n$.

Acknowledgements. The authors would like to thank Samrat Kadge* for his help in sage programming. The first author is supported by the Austrian Science Fund (FWF) under the projects P26114 and W1230. The second author is supported by INSA Senior Scientist award. Part of the work in this paper was done when the first and second authors visited the Max Plank Institute for Mathematics in Bonn. They would like to thank the MPIM for the invitation, support, hospitality and computer facilities. The third author would like to thank SERB.

* samkadge@gmail.com

References

[1] A. Baker, Experiments on the *abc*-conjecture, *Publ. Math. Debrecen* **65** (2004), 253–260.

[2] S. W. Golomb, Powerful numbers, *Amer. Math. Monthly* **77** (1970), 848–855.

[3] Shanta Laishram and T.N. Shorey, Baker's explicit *abc*-conjecture and applications, *Acta Arith.*, no. 4, **155** (2012), 419–429.

[4] D. W. Masser, Note on a conjecture of Szpiro, *in Les pinceaux de courbes elliptiques*, Semin., Paris/Fr. 1988, Astérisque **183** (1990), 19–23.

[5] Preda Mihăilescu, Primary Cyclotomic Units and a proof of Catalan Conjecture, *J. Reine Angew Math.*, **572** (2004), 167–195.

[6] R. A. Mollin and P. G. Walsh, A note on powerful numbers, quadratic fields and the Pellian, *C. R. Math. Rep. Acad. Sci. Canada* **8** (1986), no. 2, 109–114.

[7] J. Oesterlé, Nouvelles approches du "théorème" de Fermat. (New approaches to Fermat's "theorem"), *Séminaire Bourbaki*, Vol. 1987/88. *Astérisque* No. 161–162 (1988), Exp. No. 694, 4, 165–186 (1989).

[8] G Robin, Estimation de la fonction de Tchebychef θ sur le k-ième nombre premier et grandes valeurs de la fonction $\omega(n)$ nombre de diviseurs premiers de n, *Acta Arith.* **42** (1983), no. 4, 367–389.

[9] C. L. Stewart and R. Tijdeman, On the Oesterlé-Masser Conjecture, *Monatshefte Für Mathematik* **102** (1986), 251–257.

[10] T. N. Shorey, On gaps between numbers with a large prime factor II, *Acta Arith.*, **25** (1974), 365–373.

[11] T. N. Shorey, Rob Tijdeman, Arithmetic properties of blocks of consecutive integers, *in From arithmetic to zeta functions*, Springer (2016), 455–471.

[12] Trudgian, Tim, Baker's explicit ABC conjecture implies there is no hat-trick of powerful numbers, *American Mathematical Monthly*, **121** (2016).

[13] Mathematical Institute, Universiteit Leiden. (2018). Bart de Smit - ABC triples - by quality. `http://www.math.leidenuniv.nl/~desmit/abc/index.php?set=2`

K. C. Chim
Institute of Analysis and Number Theory, Graz University of Technology
Kopernikusgasse 24/II, A-8010 Graz, Austria
E-mail address: chim@math.tugraz.at

T. N. Shorey
National Institute Of Advanced Studies, Bangalore 560012, India
E-mail address: shorey@math.iitb.ac.in

Sneh Bala Sinha
Indian Institute Of Sciences, Bangalore 560012, India
E-mail address: 24.sneh@gmail.com