Gauge Symmetries and Renormalization

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December 31, 2019

Abstract

The preservation of gauge symmetries to the quantum level induces symmetries between renormalized Green's functions. These symmetries are known by the names of Ward-Takahashi and Slavnov-Taylor identities. On a perturbative level, these symmetries can be implemented as Hopf ideals in the Connes-Kreimer renormalization Hopf algebra. In this article, we generalize the existing literature to the most general case by first motivating these symmetries on a generic level and then proving that they indeed generate Hopf ideals, where we also include the more involved cases of super- and non-renormalizable local QFTs. Finally, we provide a criterion for their validity on the level of renormalized Feynman rules.

1 Introduction

In classical physics, Noether's Theorem relates symmetries to conserved quantities. In the context of classical gauge theories this theorem states, that gauge invariance corresponds to charge conservation. Thus, when quantizing a gauge theory, it is desirable to remain some sort of quantum gauge invariance in order to lift charge conservation to the quantum level. This leads to the so-called Ward-Takahashi identities [1, 2] in Quantum Electrodynamics and the so-called Slavnov-Taylor identities in Yang-Mills theories [3, 4, 5],[1] to which we will refer as quantum gauge symmetries in order to avoid name conflicts, c.f. [6, Footnote on page 93]. In particular, the classical gauge symmetries induce symmetries on the Z-factors, when multiplicative renormalization is considered. On a perturbative level, these symmetries can be implemented as symmetries inside the renormalization Hopf algebra [7], which turn out to be Hopf ideals [8, 9, 10, 11]. The aim of this article is to generalize these results for the most general case of gauge symmetries and furthermore to include also super- and non-renormalizable QFTs, as the existing literature focuses on renormalizable QFTs only, culminating in Theorem 5.4. Finally, we prove conditions for the unrenormalized Feynman rules and the renormalization scheme which guarantee, that these Hopf ideals are in the kernel of the counterterm-map or even the renormalized Feynman rules in Theorem 6.4. A consequence of this result is, that the Corolla polynomial for Yang-Mills theory is well-defined without reference to a particular renormalization scheme, c.f. [12, 13, 14, 15, 16, 17]. The analysis of this article is motivated by perturbative Quantum General Relativity, which demands this generality, c.f. [18]. For the sake of completeness, we quote important definitions, lemma and remarks from [18] in parts or even complete, as noted in their corresponding headings.

This article is organised as follows: We start in Section 2 with a brief introduction to Hopf algebraic renormalization, stating the necessary definitions and conventions. Then, in Section 3

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[1] Actually, Slavnov-Taylor identities were first discovered by Gerard 't Hooft in [3].

we study combinatorial properties of the superficial degree of divergence allowing to state our results not only for the cases of renormalizable local QFTs, but also for the cases of super- and non-renormalizable local QFTs. Next, in Section 4 we reprove and generalize known coproduct and antipode identities. Then, in Section 5 we show, that quantum gauge symmetries induce Hopf ideals inside the renormalization Hopf algebra. Finally, in Section 7 we conclude our investigations and provide an outlook into further projects.

2 Preliminaries of Hopf algebraic renormalization

The aim of this section is to briefly recall the necessary definitions and set the corresponding notations. We refer the reader to [18] for a more detailed treatment using the same notations and conventions. Furthermore, we also point out the original references for Hopf algebraic renormalization [19, 20, 21, 22] and in particular the original references for Hopf algebraic renormalization of Quantum Gauge Theories [7, 8, 9, 10]. We consider $\mathcal{Q}$ to be a local QFT and denote the Hopf algebra associated to $\mathcal{Q}$ via $\mathcal{H}_\mathcal{Q}$, c.f. [18, Section 3.3], which is a Hopf algebra over $\mathbb{Q}$ generated via the set of all one-particle irreducible (1PI) Feynman graphs $\mathcal{G}_\mathcal{Q}$.[2] Then, the (associated) renormalization Hopf algebra organizes the structure of subdivergences and allows the definition of renormalized Feynman rules via an algebraic Birkhoff decomposition. This mathematical formulation of the renormalization operation allows for a precise analysis thereof. In particular, symmetries compatible with the treatment of subdivergences generate Hopf ideals in the corresponding renormalization Hopf algebra. Next, in Section 6 we check the validity of these symmetries on the level of renormalized Feynman rules, providing criteria for the unrenormalized Feynman rules and the renormalization scheme.

Definition 2.1 ((Feynman) graphs and related notions)**.** A graph $G := (V, E, \beta)$ is given via a set of vertices V, a set of edges $E = E_0 \amalg E_1$, where E_0 is the subset of unoriented and E_1 is the subset of oriented edges,[3] and a morphism

$$\beta : \quad E \hookrightarrow (V \times V \times \mathbb{Z}_2) \, , \quad e \mapsto \begin{cases} (v_1, v_2; 0) & \text{if } e \in E_0 \\ (v_i, v_t; 1) & \text{if } e \in E_1 \end{cases} \, , \tag{1}$$

mapping edges to tuples of vertices together with their binary orientation information; if the edge is oriented, the order of the vertices is first initial then terminal. Furthermore, we introduce the set of half-edges

$$H := \left\{ h_v \cong (v, e) \,\middle|\, v \in \beta\left(e\right) \text{ for } v \in V \text{ and } e \in E \right\} \, , \tag{2}$$

where $v \in \beta\left(e\right)$ means, that the vertex v is attached to the edge e. Moreover, we introduce the set of corollas

$$C := \left\{ c_v \cong \big(v, \{h_v\}\big) \,\middle|\, v \in V \text{ and } h_v \in H \right\} \, , \tag{3}$$

which is given as the set of tuples of a vertex and the set of its attached half-edges. Given a graph G, the corresponding sets are denoted via $V \equiv V\left(G\right) \equiv G^{[0]}$, $E \equiv E\left(G\right) \equiv G^{[1]}$, $H \equiv H\left(G\right) \equiv G^{[1/2]}$ and $C \equiv C\left(G\right)$, where we omit the dependence of the graph G only if there is no ambiguity possible. Finally, a Feynman graph $\Gamma := (G, \mathbf{p})$ is a graph G together with a coloring function $\mathbf{p} \colon E\left(G\right) \to \mathbf{P}_\mathcal{Q}$, where $\mathbf{P}_\mathcal{Q} \cong \mathcal{R}_\mathcal{Q}^{[1]}$ denotes the set of particle-types of the local QFT $\mathcal{Q}$. We also address the above constructions directly for Feynman graphs Γ without reference to its underlying graph G. We remark, that in the context of Feynman graphs the

[2]In the mathematical literature such graphs as called bridge-free.

[3]The nomenclature is chosen in accordance with supergeometry, where the even part is commutative and the odd part is anticommutative.

vertex set V is a multiset over the set $\mathcal{R}_{\mathcal{Q}}^{[0]}$ and the edge set E is a multiset over the set $\mathcal{R}_{\mathcal{Q}}^{[1]}$, i.e. vertices represent fundamental interactions and edges correspond to particle types of $\mathcal{Q}$.

Definition 2.2 (Notations for sets of amplitudes, residues, Feynman graphs and the associated renormalization Hopf algebra, c.f. [18])**.** Given a local QFT $\mathcal{Q}$, we denote the set of its amplitudes via $\mathcal{A}_{\mathcal{Q}}$ and the set of its residues via $\mathcal{R}_{\mathcal{Q}}$. Furthermore, we denote the set of its Feynman graphs via $\mathcal{G}_{\mathcal{Q}}$ and its associated renormalization Hopf algebra in the sense of [18, Subsection 3.3] via $\mathcal{H}_{\mathcal{Q}}$, which is a Hopf algebra over $\mathbb{Q}$ generated by the set $\mathcal{G}_{\mathcal{Q}}$.

Definition 2.3 (Sets of summands and connected components for graphs)**.** Let $\mathcal{Q}$ be a local QFT, $\mathcal{G}_{\mathcal{Q}}$ the set of its Feynman graphs, $\mathcal{H}_{\mathcal{Q}}$ its associated renormalization Hopf algebra and $\mathfrak{G} \in \mathcal{H}_{\mathcal{Q}}$ an element therein. We are interested in the decomposition of $\mathfrak{G}$ into the generators of $\mathcal{H}_{\mathcal{Q}}$, i.e. the elements in the set $\mathcal{G}_{\mathcal{Q}}$. Therefore, we denote by $\mathcal{S}(\mathfrak{G})$ the set of its summands, grouped into tuples of prefactors $\alpha_\mathrm{s} \in \mathbb{Q}$ and graphs $\mathfrak{G}_\mathrm{s} \in \mathcal{H}_{\mathcal{Q}}$ of unit norm, such that

$$\mathfrak{G} \cong \sum_{\{\alpha_\mathrm{s}, \mathfrak{G}_\mathrm{s}\} \in \mathcal{S}(\mathfrak{G})} \alpha_\mathrm{s} \mathfrak{G}_\mathrm{s} \, . \tag{4}$$

Furthermore, we denote for each normed summand $\mathfrak{G}_\mathrm{s} \in \mathcal{S}(\mathfrak{G})$ by $\mathcal{C}(\mathfrak{G}_\mathrm{s})$ the set of its connected components (where we include the identity $\mathbb{I}$, if convenient), such that

$$\mathfrak{G}_\mathrm{s} \cong \coprod_{\mathfrak{G}_\mathrm{c} \in \mathcal{C}(\mathfrak{G}_\mathrm{s})} \mathfrak{G}_\mathrm{c} \tag{5}$$

with $\mathfrak{G}_\mathrm{c} \in \mathcal{G}_{\mathcal{Q}}$ and $b_0(\mathfrak{G}_\mathrm{c}) = 1$ for all $\mathfrak{G}_\mathrm{c} \in \mathcal{C}(\mathfrak{G}_\mathrm{s})$ (and $\mathfrak{G}_\mathrm{c} \neq \mathbb{I}$ in the latter case, as $b_0(\mathbb{I}) = 0$)[4], c.f. Definition 2.5. In particular, we have

$$\mathfrak{G} \cong \sum_{\{\alpha_\mathrm{s}, \mathfrak{G}_\mathrm{s}\} \in \mathcal{S}(\mathfrak{G})} \alpha_\mathrm{s} \left(\coprod_{\mathfrak{G}_\mathrm{c} \in \mathcal{C}(\mathfrak{G}_\mathrm{s})} \mathfrak{G}_\mathrm{c} \right) . \tag{6}$$

Definition 2.4 (Projection to divergent graphs)**.** Let $\mathcal{Q}$ be a local QFT, $\mathcal{H}_{\mathcal{Q}}$ its associated renormalization Hopf algebra and $\mathfrak{G} \in \mathcal{H}_{\mathcal{Q}}$ an element therein. Then, we define the projection to divergent graphs via

$$\mathscr{D} : \quad \mathcal{H}_{\mathcal{Q}} \to \mathcal{H}_{\mathcal{Q}} \, , \quad \mathfrak{G} \mapsto \sum_{\substack{\{\alpha_\mathrm{s}, \mathfrak{G}_\mathrm{s}\} \in \mathcal{S}(\mathfrak{G}) \\ \omega(\mathfrak{G}_\mathrm{c}) \geq 0 \; \forall \; \mathfrak{G}_\mathrm{c} \in \mathcal{C}(\mathfrak{G}_\mathrm{s})}} \alpha_\mathrm{s} \mathfrak{G}_\mathrm{s} \, , \tag{7}$$

i.e. we keep the summands of $\mathfrak{G}$, if all of its connected components are divergent. We remark, that this projection map is additive and multiplicative by definition. Furthermore, we also use the following notation:

$$\overline{\mathcal{H}_{\mathcal{Q}}} := \operatorname{Im}(\mathscr{D}) \tag{8a}$$

and

$$\overline{\mathfrak{G}} := \mathscr{D}(\mathfrak{G}) \tag{8b}$$

This definition will be useful for combinatorial Green's functions $\mathfrak{X}^r$ and combinatorial charges $\mathfrak{Q}^v$ and products thereof in the context of Hopf subalgebras for multiplicative renormalization.

[4] This follows from $b_0(\mathbb{I}) = b_0(\mathbb{I}\mathbb{I}) = b_0(\mathbb{I}) + b_0(\mathbb{I})$, which implies $b_0(\mathbb{I}) = 0$.

Definition 2.5 (Zeroth and first Betti number, internal and external coupling- and residue multi-index)**.** Let $\mathcal{Q}$ be a local QFT, $\mathcal{R}_\mathcal{Q}$ its residue set, $\mathbf{q}_\mathcal{Q}$ its physical coupling constants set,[5] $\mathcal{H}_\mathcal{Q}$ its associated renormalization Hopf algebra and $\mathfrak{G} \in \mathcal{H}_\mathcal{Q}$ an element therein. We equip the elements in the set of physical coupling constants $\mathbf{q}_\mathcal{Q}$ and the elements in the set of vertex residues $\mathcal{R}_\mathcal{Q}^{[0]}$ with an arbitrary numbering, such that we are able to refer to coupling constant j or vertex residue k. Then we assign to each element $\mathfrak{G}_\mathrm{s} \in \mathcal{S}(\mathfrak{G})$ with $\mathfrak{G} \in \mathcal{H}_\mathcal{Q}$ two numbers and four multi-indices in the following way: The first number with values in $\mathbb{N}_0$ is the zeroth Betti number, counting the connected components, and is given via

$$b_0(\mathfrak{G}_\mathrm{s}) := \operatorname{Dim}_\mathbb{Q}\left(H_0(\mathfrak{G}_\mathrm{s}, \mathbb{Q})\right) . \tag{9}$$

The second number with values in $\mathbb{N}_0$ is the first Betti number, counting the loops,[6] and is given via

$$b_1(\mathfrak{G}_\mathrm{s}) := \operatorname{Dim}_\mathbb{Q}\left(H_1(\mathfrak{G}_\mathrm{s}, \mathbb{Q})\right) . \tag{10}$$

The first multi-index with values in $\mathbb{Z}^{\#\mathbf{q}_\mathcal{Q}}$, to which we refer to as internal coupling multi-index, counts the number of coupling constants associated to each vertex of $\mathfrak{G}_\mathrm{s}$

$$\left(\operatorname{IntCpl}(\mathfrak{G}_\mathrm{s})\right)_j := \left(\text{Number of } v \in V(\mathfrak{G}_\mathrm{s}) \text{ with } \operatorname{Cpl}(v) = q_j \in \mathbf{q}_\mathcal{Q}\right) . \tag{11}$$

The second multi-index with values in $\mathbb{Z}^{\#\mathbf{q}_\mathcal{Q}}$, to which we refer to as external coupling multi-index, counts the number of coupling constants associated to the vertex graph residues of the connected components of $\mathfrak{G}_\mathrm{s}$

$$\left(\operatorname{ExtCpl}(\mathfrak{G}_\mathrm{s})\right)_j := \Big(\text{Number of } \mathfrak{G}_\mathrm{c} \in \mathcal{C}(\mathfrak{G}_\mathrm{s}) \text{ with } \operatorname{Cpl}\left(\operatorname{Res}(\mathfrak{G}_\mathrm{c})\right) = q_j \in \mathbf{q}_\mathcal{Q}\Big) . \tag{12}$$

The third multi-index with values in $\mathbb{Z}^{\#\mathcal{R}_\mathcal{Q}^{[0]}}$, to which we refer to as internal residue multi-index, counts the number of vertex types of $\mathfrak{G}_\mathrm{s}$

$$\left(\operatorname{IntRes}(\mathfrak{G}_\mathrm{s})\right)_k := \Big(\text{Number of } v \in V(\mathfrak{G}_\mathrm{s}) \text{ of type } v_k \in \mathcal{R}_\mathcal{Q}^{[0]}\Big) . \tag{13}$$

The fourth multi-index with values in $\mathbb{Z}^{\#\mathcal{R}_\mathcal{Q}^{[0]}}$, to which we refer to as external residue multi-index, counts the number of vertex types of the vertex graph residues of the connected components of $\mathfrak{G}_\mathrm{s}$

$$\left(\operatorname{ExtRes}(\mathfrak{G}_\mathrm{s})\right)_k := \Big(\text{Number of } \mathfrak{G}_\mathrm{c} \in \mathcal{C}(\mathfrak{G}_\mathrm{s}) \text{ with } \operatorname{Res}(\mathfrak{G}_\mathrm{c}) = v_k \in \mathcal{R}_\mathcal{Q}^{[0]}\Big) . \tag{14}$$

Definition 2.6 (Connectedness and gradings of the renormalization Hopf algebra, quoted from [18])**.** Let $\mathcal{Q}$ be a local QFT, $\mathcal{R}_\mathcal{Q}$ the set of its residues and $\mathcal{H}_\mathcal{Q}$ its associated renormalization Hopf algebra, c.f. [18, Subsection 3.3]. We denote restrictions to any of these three gradings $\mathbf{G}$ via

$$\left(\mathcal{H}_\mathcal{Q}\right)_\mathbf{G} := \mathcal{H}_\mathcal{Q}\Bigg|_\mathbf{G} , \tag{15}$$

and we omit the brackets, if no lower index is present. Then, we consider the following three gradings of $\mathcal{H}_\mathcal{Q}$ as a Hopf algebra, which are further refinements of each other: The first grading comes from the first Betti number, which we refer to as loop number and denote it via L or l, yielding the decomposition

$$\mathcal{H}_\mathcal{Q} = \bigoplus_{L=0}^\infty \left(\mathcal{H}_\mathcal{Q}\right)_L . \tag{16}$$

[5] The set of coupling constants which appear in the definition of the local QFT $\mathcal{Q}$, e.g. in its Lagrange density, c.f. Definition 2.13 for the proper definition.

[6] In the mathematical literature, this is usually referred to as cycles.

The second grading comes from the difference of the internal and external coupling multi-indices, i.e. via assigning to each element $\mathfrak{G}_\mathrm{s} \in \mathcal{S}(\mathfrak{G})$ with $\mathfrak{G} \in \mathcal{H}_\mathcal{Q}$ the multi-index

$$\operatorname{CplGrd}(\mathfrak{G}_\mathrm{s}) := \operatorname{IntCpl}(\mathfrak{G}_\mathrm{s}) - \operatorname{ExtCpl}(\mathfrak{G}_\mathrm{s}) \, , \tag{17}$$

which we refer to as coupling-grading multi-index and denote it via $\mathbf{C}$ or $\mathbf{c}$, yielding the decomposition

$$\mathcal{H}_\mathcal{Q} = \bigoplus_{\mathbf{C}=-\infty}^{\infty} (\mathcal{H}_\mathcal{Q})_\mathbf{C} \, . \tag{18}$$

Finally, the third grading comes from the difference of the internal and external residue multi-indices, i.e. via assigning to each element $\mathfrak{G}_\mathrm{s} \in \mathcal{S}(\mathfrak{G})$ with $\mathfrak{G} \in \mathcal{H}_\mathcal{Q}$ the multi-index

$$\operatorname{ResGrd}(\mathfrak{G}_\mathrm{s}) := \operatorname{IntRes}(\mathfrak{G}_\mathrm{s}) - \operatorname{ExtRes}(\mathfrak{G}_\mathrm{s}) \, , \tag{19}$$

which we refer to as residue-grading multi-index and denote it via $\mathbf{R}$ or $\mathbf{r}$, yielding the decomposition

$$\mathcal{H}_\mathcal{Q} = \bigoplus_{\mathbf{R}=-\infty}^{\infty} (\mathcal{H}_\mathcal{Q})_\mathbf{R} \, . \tag{20}$$

Clearly, $(\mathcal{H}_\mathcal{Q})_{L=0} \cong (\mathcal{H}_\mathcal{Q})_{\mathbf{C}=\mathbf{0}} \cong (\mathcal{H}_\mathcal{Q})_{\mathbf{R}=\mathbf{0}} \cong \mathbb{Q}$, and thus $\mathcal{H}_\mathcal{Q}$ is connected in all three gradings. Statements which are valid for any of these three gradings, such as Equation (15), are formulated with the symbols $\mathbf{G}$ and $\mathbf{g}$.

Remark 2.7. The numbers and multi-indices from Definition 2.5 and the gradings from Definition 2.6 are compatible with the multiplication of $\mathcal{H}_\mathcal{Q}$ via addition, but not with the addition of $\mathcal{H}_\mathcal{Q}$, as summands can live in different gradings.

Lemma 2.8 (Vertex, edge, half-edge and corolla sets depent only on residue and residue–grading)**.** *Given a Feynman graph $\Gamma \in \mathcal{G}_\mathcal{Q}$, then its vertex set $V(\Gamma)$, its half-edge set $H(\Gamma)$, its edge set $E(\Gamma)$ and its corolla set $C(\Gamma)$ depend only on its residue* $\operatorname{Res}(\Gamma)$ *and its residue-grading multi-index* $\operatorname{ResGrd}(\Gamma)$*, i.e. we can define well-defined sets $V(r,\mathbf{r})$, $H(r,\mathbf{r})$, $E(r,\mathbf{r})$ and $C(r,\mathbf{r})$ such that we have $V(r,\mathbf{r}) \cong V(\Gamma)$, $H(r,\mathbf{r}) \cong H(\Gamma)$, $E(r,\mathbf{r}) \cong E(\Gamma)$ and $C(r,\mathbf{r}) \cong C(\Gamma)$ for all $\Gamma \in \mathcal{G}_\mathcal{Q}$ with* $\operatorname{Res}(\Gamma) = r$ *and* $\operatorname{ResGrd}(\Gamma) = \mathbf{r}$.

Proof. Given $\Gamma \in \mathcal{G}_\mathcal{Q}$, then by definition its vertex set $V(\Gamma)$ is a multiset over $\mathcal{R}_\mathcal{Q}^{[0]}$ and thus bijective to its internal residue multi-index $\operatorname{IntRes}(\Gamma)$. Furthermore, we can reconstruct $\operatorname{IntRes}(\Gamma)$ from $\operatorname{Res}(\Gamma)$ and $\operatorname{ResGrd}(\Gamma)$ using the definition, Equation (19), i.e.

$$V(r,\mathbf{r}) \cong \operatorname{ResGrd}(\Gamma) + \operatorname{ExtRes}(\Gamma) \, , \tag{21}$$

while noting, that $\operatorname{ExtRes}(\Gamma)$ is given for connected Feynman graphs $\Gamma \in \mathcal{G}_\mathcal{Q}$ with $\operatorname{Res}(\Gamma) \in \mathcal{R}_\mathcal{Q}^{[0]}$ as the multi-index having a one for the corresponding vertex residue and zeros else, i.e.

$$\big(\operatorname{ExtRes}(\Gamma)\big)_k = \begin{cases} 1 & \text{if } \operatorname{Res}(\Gamma) = v_k \in \mathcal{R}_\mathcal{Q}^{[0]} \\ 0 & \text{else} \end{cases} \tag{22}$$

and for Feynman graphs $\Gamma \in \mathcal{G}_\mathcal{Q}$ with $\operatorname{Res}(\Gamma) \in \left(\mathcal{A}_\mathcal{Q} \setminus \mathcal{R}_\mathcal{Q}^{[0]}\right)$ as the zero-multi-index, i.e.

$$\operatorname{ExtRes}(\Gamma) = \mathbf{0} \, . \tag{23}$$

Thus we have have shown, that the set $V(r, \mathbf{r})$ is well-defined. Moreover, we can obtain the half-edge set $H(r, \mathbf{r})$ from $\operatorname{Res}(\Gamma)$ and $V(r, \mathbf{r})$ via

$$H(r, \mathbf{r}) := \left\{ h_v \in \coprod_{v \in V(r, \mathbf{r})} H(v) \right\} \setminus H\big(\operatorname{Res}(\Gamma)\big) \; , \tag{24}$$

i.e. we consider to each vertex $v \in V(r, \mathbf{r})$ the set of half-edges attached to it, take its disjoint union and then remove the set of external half-edges of Γ. Finally, we obtain the edge set $E(r, \mathbf{r})$ from the half-edge set $H(r, \mathbf{r})$ using the equivalence relation $\sim$ which identifies two half-edges to a single edge, if they are of the same particle type, i.e.

$$E(r, \mathbf{r}) := H(r, \mathbf{r}) / \sim \; . \tag{25}$$

Finally, we obtain the corolla set $C(r, \mathbf{r})$ from the vertex set $V(r, \mathbf{r})$, as we can associate to each vertex the set of half-edges attached to it, i.e.

$$C(r, \mathbf{r}) \cong V(r, \mathbf{r}) \; . \tag{26}$$

■

Definition 2.9 ((Restricted) combinatorial Green's functions, quoted from [18])**.** Let $\mathcal{Q}$ be a local QFT, $\mathcal{A}_\mathcal{Q}$ the set of its amplitudes and $\mathcal{G}_\mathcal{Q}$ the set of its Feynman graphs. Given an amplitude $r \in \mathcal{A}_\mathcal{Q}$, we set

$$\mathfrak{x}^r := \sum_{\substack{\Gamma \in \mathcal{G}_\mathcal{Q} \\ \operatorname{Res}(\Gamma) = r}} \frac{1}{\operatorname{Sym}(\Gamma)} \Gamma \tag{27}$$

and then define the total combinatorial Green's function with amplitude r as the following sum:

$$\mathfrak{X}^r := \begin{cases} \mathbb{I} + \mathfrak{x}^r & \text{if } r \in \mathcal{R}_\mathcal{Q}^{[0]} \\ \mathbb{I} - \mathfrak{x}^r & \text{if } r \in \mathcal{R}_\mathcal{Q}^{[1]} \\ \mathfrak{x}^r & \text{else, i.e. } r \in \big(\mathcal{A}_\mathcal{Q} \setminus \mathcal{R}_\mathcal{Q}\big) \end{cases} \tag{28}$$

Furthermore, we denote the restriction of $\mathfrak{X}^r$ to one of the gradings $\mathbf{g}$ from Definition 2.6 via

$$\mathfrak{X}^r_\mathbf{g} := \mathfrak{X}^r \bigg|_\mathbf{g} \; . \tag{29}$$

Remark 2.10 (Quoted from [18]). We remark, that restricted combinatorial Green's functions are in the literature often denoted via $c^r_\mathbf{g}$ and differ by a minus sign from our definition. Our convention is such, that they are given as the restriction of the total combinatorial Green's function to the corresponding grading, which yield minus signs for non-empty propagator graphs.

Definition 2.11 ((Restricted) combinatorial Charges)**.** Let $v \in \mathcal{R}_\mathcal{Q}^{[0]}$ be a vertex residue, then we define its combinatorial charge $\mathfrak{Q}^v$ via

$$\mathfrak{Q}^v := \frac{\mathfrak{X}^v}{\prod_{e \in E(v)} \sqrt{\mathfrak{X}^e}} \, , \tag{30}$$

where $E(v)$ denotes the set of all edges attached to the vertex v. Furthermore, we denote the restriction of $\mathfrak{Q}^v$ to one of the gradings $\mathbf{g}$ from Definition 2.6 via

$$\mathfrak{Q}^v_\mathbf{g} := \mathfrak{Q}^v \bigg|_\mathbf{g} \; . \tag{31}$$

Definition 2.12 ((Restricted) products of combinatorial charges)**.** Let $v \in \mathcal{R}_{\mathcal{Q}}^{[0]}$ be a vertex residue, $\mathfrak{Q}^v$ its combinatorial charge and $\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}}$ a multi-index of vertex residues. Then we define the exponent of combinatorial charges via the multi-index $\mathbf{r}$ via

$$\mathfrak{Q}^{\mathbf{r}} := \prod_{k=1}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}} \left(\mathfrak{Q}^{v_k} \right)^{\mathbf{r}_k} . \tag{32}$$

Furthermore, we define the restriction to the grading $\mathbf{g}$ via

$$\mathfrak{Q}_{\mathbf{g}}^{\mathbf{r}} := \left. \left(\prod_{k=1}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}} \left(\mathfrak{Q}^{v_k} \right)^{\mathbf{r}_k} \right) \right|_{\mathbf{g}} . \tag{33}$$

Definition 2.13 (Sets of combinatorial and physical charges, projection map)**.** Let $\mathcal{Q}$ be a local QFT. Then, we denote via $\mathbf{Q}_{\mathcal{Q}}$ and $\mathbf{q}_{\mathcal{Q}}$ the sets of combinatorial and physical charges, respectively. We associate to each vertex residue of $\mathcal{Q}$ a combinatorial charge and obtain the physical charges from the definition of $\mathcal{Q}$, e.g. via its Lagrange density. Furthermore, we define the set-theoretic projection map $\operatorname{Cpl} \colon \mathbf{Q}_{\mathcal{Q}} \twoheadrightarrow \mathbf{q}_{\mathcal{Q}}$, mapping the combinatorial charge to its associated physical charge.[7]

Definition 2.14 (Set of superficially divergent subgraphs of a Feynman graph)**.** Let $\mathcal{Q}$ be a local QFT and $\Gamma \in \mathcal{G}_{\mathcal{Q}}$ a Feynman graph of $\mathcal{Q}$. Then we denote by $\mathcal{D}(\Gamma)$ the set of superficially divergent subgraphs of Γ, i.e.

$$\mathcal{D}(\Gamma) := \left\{ \mathbb{I} \subseteq \gamma \subseteq \Gamma \;\middle|\; \omega(\gamma_{\mathrm{c}}) \geq 0 \text{ and } \operatorname{Res}(\gamma_{\mathrm{c}}) \in \mathcal{R}_{\mathcal{Q}} \text{ for all } \gamma_{\mathrm{c}} \in \mathcal{C}(\gamma) \right\} . \tag{34}$$

We remark, that the condition $\operatorname{Res}(\gamma_{\mathrm{c}}) \in \mathcal{R}_{\mathcal{Q}}$ for all $\gamma_{\mathrm{c}} \in \mathcal{C}(\gamma)$ ensures the well-definedness of the renormalization Hopf algebra, c.f. [18, Subsection 3.3].

Definition 2.15 (Set of superficially divergent insertable graphs to a Feynman graph)**.** Let $\mathcal{Q}$ be a local QFT and $\Gamma \in \mathcal{G}_{\mathcal{Q}}$ a Feynman graph of $\mathcal{Q}$. Then we denote by $\mathcal{I}(\Gamma)$ the set of superficially divergent graphs insertable into Γ, i.e.

$$\begin{aligned} \mathcal{I}(\Gamma) := \Big\{ \gamma \in \mathcal{H}_{\mathcal{Q}} \;\Big|\; & \operatorname{ExtRes}(\gamma) \leq \operatorname{IntRes}(\Gamma) \text{ and } \omega(\gamma_{\mathrm{c}}) \geq 0 \text{ for all } \gamma_{\mathrm{c}} \in \mathcal{C}(\gamma) \\ & \text{and } \operatorname{Res}(\gamma_{\mathrm{p}}) \in E(\Gamma) \text{ for all } \gamma_{\mathrm{p}} \in \mathcal{P}(\gamma) \Big\} , \end{aligned} \tag{35}$$

where $\mathcal{P}(\gamma) \subseteq \mathcal{C}(\gamma)$ denotes the connected components of γ which are propagator graphs.

Definition 2.16 (Insertion factors)**.** Let $\mathcal{Q}$ be a local QFT, $\gamma \in \overline{\mathcal{H}_{\mathcal{Q}}}$ be a divergent element in the associated renormalization Hopf algebra of $\mathcal{Q}$ and $\Gamma, \Gamma' \in \mathcal{G}_{\mathcal{Q}}$ be Feynman graphs of $\mathcal{Q}$. Then, we denote by $\operatorname{Ins}_{\mathrm{Aut}}(\gamma \triangleright \Gamma; \Gamma')$ the number of ways to insert γ into Γ, such that the insertion is automorphic to Γ', which is zero, if either $\gamma \notin \mathcal{I}(\Gamma)$ or if there is no insertion possible, which is automorphic to Γ'. Furthermore, we set $\operatorname{Ins}(\gamma \triangleright \Gamma)$ to be the number of ways to insert γ into Γ, which is zero, if $\gamma \notin \mathcal{I}(\Gamma)$. Moreover, as the last definition depends only on the sets $V(r, \mathbf{r})$ and $E(r, \mathbf{r})$, which, due to Lemma 2.8, can be reconstructed from $\operatorname{Res}(\Gamma)$ and $\operatorname{ResGrd}(\Gamma)$, we set $\operatorname{Ins}^{r,\mathbf{r}}(\gamma)$ to be the number of ways to insert γ into Feynman graphs Γ with $\operatorname{Res}(\Gamma) = r$ and $\operatorname{ResGrd}(\Gamma) = \mathbf{r}$. Finally, we set for all $\Gamma \in \mathcal{G}_{\mathcal{Q}}$ the three factors $\operatorname{Ins}_{\mathrm{Aut}}(\mathbb{I} \triangleright \Gamma; \Gamma) = \operatorname{Ins}(\mathbb{I} \triangleright \Gamma) = \operatorname{Ins}^{r,\mathbf{r}}(\mathbb{I}) := 1$.

[7] This map is the set-theoretic restriction of the renormalized Feynman rules, which map combinatorial charges to the Feynman integrals corresponding to their renormalized physical charges.

Proposition 2.17. *Given the situation of Definition 2.15, we have for all Feynman graphs* $\Gamma \in \mathcal{G}_\mathcal{Q}$

$$\sum_{\gamma \in \mathcal{I}(\Gamma)} \frac{\operatorname{Ins}(\gamma \triangleright \Gamma)}{\operatorname{Sym}(\gamma)} \gamma = \frac{\prod_{v \in V(\Gamma)} \overline{\mathfrak{X}}^v}{\prod_{e \in E(\Gamma)} \overline{\mathfrak{X}}^e} \, , \tag{36}$$

i.e. the following objects are isomorphic:

$$\mathcal{I}(\Gamma) \cong \frac{\prod_{v \in V(\Gamma)} \overline{\mathfrak{X}}^v}{\prod_{e \in E(\Gamma)} \overline{\mathfrak{X}}^e} \tag{37}$$

Proof. We can insert in each vertex $v \in V(\Gamma)$ at most one superficially divergent vertex correction γ^v with $\operatorname{Res}(\gamma^v) = v$, i.e. a summand of $\overline{\mathfrak{X}}^v$. Furthermore, we can insert in each edge $e \in E(\Gamma)$ arbitrary many superficially divergent edge corrections $\gamma^e = \coprod_i \gamma_i^e$ with $\operatorname{Res}\left(\gamma_j^e\right) = e$ for all j, i.e. a summand of $1/\overline{\mathfrak{X}}^e$, where the fraction is understood formally via the geometric series $1/(1-x) = \sum_{k=0}^\infty x^k$.[8] Finally, the prefactor $\operatorname{Ins}(\gamma \triangleright \Gamma)$ corresponds to the number of equivalent vertices and edges of Γ. ■

Definition 2.18 (Set of superficially divergent insertable graphs for residue and residue-grading)**.** Let $\mathcal{Q}$ be a local QFT, and $\mathcal{H}_\mathcal{Q}$ its (associated) renormalization Hopf algebra, $r \in \mathcal{A}_\mathcal{Q}$ an amplitude of $\mathcal{Q}$ and $\mathbf{r} \in \mathbb{Z}^{\#\mathcal{R}_\mathcal{Q}^{[0]}}$, $\mathbf{r} \neq \mathbf{0}$, a residue-grading multi-index. Using Lemma 2.8 and Proposition 2.17, we can define the set of superficially divergent graphs insertable into Feynman graphs $\Gamma \in \mathcal{G}_\mathcal{Q}$ with residue $\operatorname{Res}(\Gamma) = r$ and residue-grading $\operatorname{ResGrd}(\Gamma) = \mathbf{r}$ via

$$\mathcal{I}^{r,\mathbf{r}} :\cong \frac{\prod_{v \in V(r,\mathbf{r})} \overline{\mathfrak{X}}^v}{\prod_{e \in E(r,\mathbf{r})} \overline{\mathfrak{X}}^e} \, , \tag{38}$$

where we use again the isomorphism from Equation (37) of Proposition 2.17.

Proposition 2.19. *Given the situation of Definition 2.18, we have for all amplitudes $r \in \mathcal{A}_\mathcal{Q}$ and residue-grading multi-indices $\mathbf{r} \in \mathbb{Z}^{\#\mathcal{R}_\mathcal{Q}^{[0]}}$*

$$\sum_{\gamma \in \mathcal{I}^{r,\mathbf{r}}} \frac{\operatorname{Ins}^{r,\mathbf{r}}(\gamma)}{\operatorname{Sym}(\gamma)} \gamma = \begin{cases} \overline{\mathfrak{X}}^r \overline{\mathfrak{Q}}^\mathbf{r} & \textit{if } r \in \mathcal{R}_\mathcal{Q} \\ \prod_{e \in E(r)} \sqrt{\overline{\mathfrak{X}}^e} \overline{\mathfrak{Q}}^\mathbf{r} & \textit{else, i.e. } r \in \left(\mathcal{A}_\mathcal{Q} \setminus \mathcal{R}_\mathcal{Q}\right) \end{cases} \tag{39}$$

i.e. the following objects are isomorphic:

$$\mathcal{I}^{r,\mathbf{r}} \cong \begin{cases} \overline{\mathfrak{X}}^r \overline{\mathfrak{Q}}^\mathbf{r} & \textit{if } r \in \mathcal{R}_\mathcal{Q} \\ \prod_{e \in E(r)} \sqrt{\overline{\mathfrak{X}}^e} \overline{\mathfrak{Q}}^\mathbf{r} & \textit{else, i.e. } r \in \left(\mathcal{A}_\mathcal{Q} \setminus \mathcal{R}_\mathcal{Q}\right) \end{cases} \tag{40}$$

Proof. The numerator of the right hand side of Equation (38) of Definition 2.18 can be expressed as follows:[9]

$$\prod_{v \in V(r,\mathbf{r})} \overline{\mathfrak{X}}^v = \begin{cases} \overline{\mathfrak{X}}^r \overline{\mathfrak{X}}^\mathbf{r} & \text{if } r \in \mathcal{R}_\mathcal{Q}^{[0]} \\ \overline{\mathfrak{X}}^\mathbf{r} & \text{else, i.e. } r \in \left(\mathcal{A}_\mathcal{Q} \setminus \mathcal{R}_\mathcal{Q}^{[0]}\right) \end{cases} \, , \tag{41a}$$

[8] Since there is no topology on $\mathcal{H}_\mathcal{Q}$, there is no notion of convergence for the formal expression $1/\overline{\mathfrak{X}}^e$. Furthermore, we remark that this viewpoint is the reason for the minus sign in the combinatorial Greens function for propagators, c.f. Equation (28) of Definition 2.9.

[9] The two cases emerge due to the residue-grading, which treats Feynman graphs with vertex residues differently in order to obtain a valid grading of the renormalization Hopf algebra, c.f. Equation (19) of Definition 2.6.

where the notation $\overline{\mathfrak{X}}^{\mathbf{r}} := \prod_{k=1}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}} \left(\overline{\mathfrak{X}}^{v_k} \right)^{\mathbf{r}_k}$ is analogous to Equation (32) of Definition 2.12. Furthermore, the denominator of the right hand side of Equation (38) of Definition 2.18 can be expressed as follows:

$$
\begin{aligned}
\frac{\mathbb{I}}{\prod_{e \in E(r,\mathbf{r})} \overline{\mathfrak{X}}^e} &= \begin{cases} \dfrac{\mathbb{I}}{\prod_{v \in V(r,\mathbf{r})} \left(\prod_{e \in E(v)} \sqrt{\overline{\mathfrak{X}}^e} \right)} & \text{if } r \in \mathcal{R}_{\mathcal{Q}}^{[0]} \\ \dfrac{\prod_{e_1 \in E(r)} \sqrt{\overline{\mathfrak{X}}^{e_1}}}{\prod_{v \in V(r,\mathbf{r})} \left(\prod_{e_2 \in E(v)} \sqrt{\overline{\mathfrak{X}}^{e_2}} \right)} & \text{else, i.e. } r \in \left(\mathcal{A}_{\mathcal{Q}} \setminus \mathcal{R}_{\mathcal{Q}}^{[0]} \right) \end{cases} \\
&= \begin{cases} \frac{\overline{\mathfrak{Q}}^{\mathbf{r}}}{\overline{\mathfrak{X}}^{\mathbf{r}}} & \text{if } r \in \mathcal{R}_{\mathcal{Q}}^{[0]} \\ \frac{\overline{\mathfrak{X}}^r \overline{\mathfrak{Q}}^{\mathbf{r}}}{\overline{\mathfrak{X}}^{\mathbf{r}}} & \text{if } r \in \mathcal{R}_{\mathcal{Q}}^{[1]} \\ \dfrac{\prod_{e \in E(r)} \sqrt{\overline{\mathfrak{X}}^e} \overline{\mathfrak{Q}}^{\mathbf{r}}}{\overline{\mathfrak{X}}^{\mathbf{r}}} & \text{else, i.e. } r \in \left(\mathcal{A}_{\mathcal{Q}} \setminus \mathcal{R}_{\mathcal{Q}} \right) \end{cases}
\end{aligned}
\tag{41b}
$$

Multiplying Equation (41a) with Equation (41b), we obtain

$$
\frac{\prod_{v \in V(r,\mathbf{r})} \overline{\mathfrak{X}}^v}{\prod_{e \in E(r,\mathbf{r})} \overline{\mathfrak{X}}^e} = \begin{cases} \overline{\mathfrak{X}}^r \overline{\mathfrak{Q}}^{\mathbf{r}} & \text{if } r \in \mathcal{R}_{\mathcal{Q}} \\ \prod_{e \in E(r)} \sqrt{\overline{\mathfrak{X}}^e} \overline{\mathfrak{Q}}^{\mathbf{r}} & \text{else, i.e. } r \in \left(\mathcal{A}_{\mathcal{Q}} \setminus \mathcal{R}_{\mathcal{Q}} \right) \end{cases} . \tag{42}
$$

Finally, the prefactor $\operatorname{Ins}^{r,\mathbf{r}} (\gamma)$ corresponds to the number of equivalent vertices and edges of graphs with residue r and residue-grading multi-index $\mathbf{r}$. ■

Lemma 2.20. *Given the situation of Definition 2.16, we have for all Feynman graphs $\Gamma \in \mathcal{G}_{\mathcal{Q}}$ and their corresponding divergent subgraphs $\gamma \in \mathcal{D} (\Gamma)$*

$$
\frac{1}{\operatorname{Sym} (\Gamma)} = \frac{\operatorname{Ins}_{\operatorname{Aut}} \left(\gamma \triangleright \Gamma / \gamma ; \Gamma \right)}{\operatorname{Sym} (\gamma) \operatorname{Sym} \left(\Gamma / \gamma \right)} . \tag{43}
$$

Proof. Let $\Gamma \in \mathcal{G}_{\mathcal{Q}}$ be a Feynman graph. Then, by definition, we have

$$
\operatorname{Sym} (\Gamma) = \# \operatorname{Aut}_{\operatorname{Int}} (\Gamma) , \tag{44}
$$

where $\operatorname{Aut}_{\operatorname{Int}} (\Gamma)$ denotes automorphisms of Γ fixing its external legs. Thus, for a given divergent subgraph $\gamma \in \mathcal{D} (\Gamma)$, we have

$$
\operatorname{Sym} (\gamma) \operatorname{Sym} \left(\Gamma / \gamma \right) = \# \operatorname{Aut}_{\operatorname{Int}} (\gamma) \# \operatorname{Aut}_{\operatorname{Int}} \left(\Gamma / \gamma \right) , \tag{45}
$$

which counts all automorphisms of Γ / γ times those of γ, fixing both their external legs. However, this might be a multiple of $\operatorname{Sym} (\Gamma)$ due to the existence of automorphisms in $\operatorname{Aut}_{\operatorname{Int}} \left(\Gamma / \gamma \right)$, which correspond to symmetries that are only present in the quotient graph Γ / γ and get spoiled when the insertion of γ is considered. This is precisely given via the number $\operatorname{Ins}_{\operatorname{Aut}} \left(\gamma \triangleright \Gamma / \gamma ; \Gamma \right)$, as it counts the number of equivalent insertions of γ into Γ / γ to obtain Γ. Thus, we obtain

$$
\frac{1}{\operatorname{Sym} (\Gamma)} = \frac{\operatorname{Ins}_{\operatorname{Aut}} \left(\gamma \triangleright \Gamma / \gamma ; \Gamma \right)}{\operatorname{Sym} (\gamma) \operatorname{Sym} \left(\Gamma / \gamma \right)} , \tag{46}
$$

as claimed. ■

Definition 2.21 (Convolution product, quoted from [18])**.** Let A be an algebra and C a coalgebra. Then using the multiplication m_A on A and the comultiplication Δ_C on C, we can turn the k-module $\mathrm{Hom}_{k-\mathsf{Mod}}(C, A)$ of k-linear maps from C to A into a k-algebra by defining the convolution product $\star$ for given $f, g \in \mathrm{Hom}_{k-\mathsf{Mod}}(C, A)$ via

$$f \star g := m_A \circ (f \otimes g) \circ \Delta_C \,. \tag{47}$$

Obviously, this definition extends trivially if A or C possesses additionally a bi- or Hopf algebra structure, since it only requires a coalgebra structure in the source algebra and an algebra structure in the target algebra. It is commutative, if C is cocommutative and A is commutative.

Definition 2.22 (Augmentation ideal, quoted from [18])**.** Given a bi- or a Hopf algebra B, then the kernel of the coidentity

$$\mathrm{Aug}\left(\mathcal{H}_\mathcal{Q}\right) := \mathrm{Ker}\left(\hat{\mathbb{I}}\right) \tag{48}$$

is an ideal, called the augmentation ideal. Furthermore, we denote the projector to it via $\mathscr{P}$, i.e.

$$\mathscr{P} : \quad \mathcal{H}_\mathcal{Q} \twoheadrightarrow \mathrm{Aug}\left(\mathcal{H}_\mathcal{Q}\right) \,, \quad \mathfrak{G} \mapsto \sum_{\substack{\{\alpha_\mathrm{s}, \mathfrak{G}_\mathrm{s}\} \in \mathcal{S}(\mathfrak{G}) \\ \hat{\mathbb{I}}(\mathfrak{G}_\mathrm{c}) = 0 \; \forall \; \mathfrak{G}_\mathrm{c} \in \mathcal{C}(\mathfrak{G}_\mathrm{s})}} \alpha_\mathrm{s} \mathfrak{G}_\mathrm{s} \,. \tag{49}$$

Definition 2.23 ((Renormalized) Feynman rules, regularization and renormalization schemes and the counterterm map)**.** Let $\mathcal{Q}$ be a local QFT, $\mathcal{H}_\mathcal{Q}$ its associated renormalization Hopf algebra and $\Omega_\mathcal{Q}$ its algebra of Feynman differential forms, defined via

$$\Omega_\mathcal{Q} := \bigoplus_{i=1}^\infty \Omega \left(\bigotimes_\mathbb{R}^i \mathbb{M}^{1,3} \right) \,, \tag{50}$$

i.e. direct sums of differential forms on $\mathbb{R}$-linear tensor products of Minkowski spaces of loop momenta with signature $(1, 3)$. Then, the Feynman rules are a character, i.e. an algebra morphism

$$\Phi : \quad \mathcal{H}_\mathcal{Q} \to \Omega_\mathcal{Q} \,, \quad \Gamma \mapsto I_\Gamma \,, \tag{51}$$

where I_Γ is the Feynman differential form of the Feynman integral of the Feynman graph Γ. Furthermore, we introduce a regularization scheme $\mathscr{E}$ as the map[10]

$$\mathscr{E} : \quad \Omega_\mathcal{Q} \hookrightarrow {\Omega_\mathcal{Q}}^\varepsilon := \Omega_\mathcal{Q}\left[\left[\varepsilon\right]\right] \tag{52}$$

mapping a differential form I to the one-parameter family I^ε, which is defined such that $I^0 \equiv I$ and such that I^ε is integrable over its complete domain for $\varepsilon \neq 0$. Moreover, we introduce a renormalization scheme as a linear projection map inducing a splitting on ${\Omega_\mathcal{Q}}^\varepsilon$ via

$$\mathscr{R} : \quad {\Omega_\mathcal{Q}}^\varepsilon \twoheadrightarrow {\Omega_\mathcal{Q}}^\varepsilon_- := \mathrm{Im}\left(\mathscr{R}\right) \tag{53}$$

which, to ensure locality of the counterterm, needs to be a Rota-Baxter operator of weight $\lambda = -1$, i.e. fulfill

$$m_{{\Omega_\mathcal{Q}}^\varepsilon} \circ \left(\mathscr{R} \otimes \mathscr{R}\right) + \mathscr{R} \circ m_{{\Omega_\mathcal{Q}}^\varepsilon} = \mathscr{R} \circ m_{{\Omega_\mathcal{Q}}^\varepsilon} \circ \left(\mathscr{R} \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathscr{R}\right) \,, \tag{54}$$

such that $({\Omega_\mathcal{Q}}^\varepsilon, \mathscr{R})$ is a Rota-Baxter algebra of weight $\lambda = -1$.[11] In particular, it induces the splitting

$${\Omega_\mathcal{Q}}^\varepsilon \cong {\Omega_\mathcal{Q}}^\varepsilon_+ \oplus {\Omega_\mathcal{Q}}^\varepsilon_- \tag{55}$$

[10]There exist renormalization schemes, such as kinematic renormalization schemes, which allow for a renormalization on the level of Feynman differential forms. In these cases a regularization is not necessary and thus the regularization step can be omitted, replacing ${\Omega_\mathcal{Q}}^\varepsilon$ by $\Omega_\mathcal{Q}$ in the following.

[11]We remark, that the multiplication $m_{{\Omega_\mathcal{Q}}^\varepsilon}$ is commutative, as the dimensions of the Minkowski spaces of loop momenta are 4 and thus in particular even.

with $\Omega_{\mathcal{Q}_+}^\varepsilon := \operatorname{CoKer}(\mathscr{R})$ and $\Omega_{\mathcal{Q}_-}^\varepsilon := \operatorname{Im}(\mathscr{R})$. Then, we can introduce renormalized Feynman rules via

$$\Phi_{\mathscr{R}} : \quad \mathcal{H}_{\mathcal{Q}} \to \Omega_{\mathcal{Q}_+}^\varepsilon , \quad \Gamma \mapsto \operatorname*{Lim}_{\varepsilon \mapsto 0} \left(S_{\mathscr{R}}^{\Phi_{\mathscr{E}}^\varepsilon} \star \Phi \right) (\Gamma) \tag{56}$$

with the counterterm map, given recursively via the normalization $S_{\mathscr{R}}^{\Phi_{\mathscr{E}}^\varepsilon} (\mathbb{I}) = \mathbf{1}_{\Omega_{\mathcal{Q}}{}^\varepsilon}$ and

$$S_{\mathscr{R}}^{\Phi_{\mathscr{E}}^\varepsilon} : \quad \operatorname{Aug}(\mathcal{H}_{\mathcal{Q}}) \to \Omega_{\mathcal{Q}_-}^\varepsilon , \quad \Gamma \mapsto -\mathscr{R} \left[S_{\mathscr{R}}^{\Phi_{\mathscr{E}}^\varepsilon} \star \left(\Phi_{\mathscr{E}}^\varepsilon \circ \mathscr{P} \right) \right] (\Gamma) \tag{57}$$

else, where $\mathscr{P} \colon \mathcal{H}_{\mathcal{Q}} \twoheadrightarrow \operatorname{Aug}(\mathcal{H}_{\mathcal{Q}})$ is the projector onto the augmentation ideal from Definition 2.22. We remark, that the renormalized Feynman rules $\Phi_{\mathscr{R}}$ and the counterterm map $S_{\mathscr{R}}^{\Phi_{\mathscr{E}}^\varepsilon}$ correspond to the algebraic Birkhoff decomposition of the Feynman rules Φ with respect to the splitting of $\Omega_{\mathcal{Q}}{}^\varepsilon$, induced via the renormalization scheme $\mathscr{R}$, c.f. [23, 24] for further reading in this direction. Finally, we remark, that the above discussion can be also lifted to the algebra of meromorphic functions $\mathcal{M}^\varepsilon := \mathbb{C}\big[\varepsilon^{-1}, \varepsilon\big]\big]$, when using a suitable regularization scheme $\mathscr{E}$ and then integrating the regularized Feynman differential forms again its complete domain to obtain the corresponding meromorphic function $f_\Gamma^\varepsilon := \int_{\varepsilon \neq 0} I_\Gamma^\varepsilon$ for fixed external momentum configurations which do not correspond to a Landau singularity.

Definition 2.24 (Hopf subalgebras for multiplicative renormalization, quoted from [18])**.** Let $\mathcal{Q}$ be a local QFT, $\mathcal{R}_{\mathcal{Q}}$ its weighted residue set, $\mathcal{H}_{\mathcal{Q}}$ its associated renormalization Hopf algebra and $\mathfrak{X}_{\mathbf{G}}^r \in \mathcal{H}_{\mathcal{Q}}$ its restricted Green's functions, where $\mathbf{G}$ and $\mathbf{g}$ denotes one of the gradings from Definition 2.6. We are interested in Hopf subalgebras which correspond to multiplicative renormalization, i.e. Hopf subalgebras of $\mathcal{H}_{\mathcal{Q}}$ such that the coproduct factors on the restricted combinatorial Green's functions for all multi-indices $\mathbf{G}$ in the following way:

$$\Delta \left(\mathfrak{X}_{\mathbf{G}}^r \right) = \sum_{\mathbf{g}} \mathfrak{P}_{\mathbf{g}} \left(\mathfrak{X}_{\mathbf{G}}^r \right) \otimes \mathfrak{X}_{\mathbf{G}-\mathbf{g}}^r , \tag{58}$$

where $\mathfrak{P}_{\mathbf{g}} \left(\mathfrak{X}_{\mathbf{G}}^r \right) \in \mathcal{H}_{\mathcal{Q}}$ is a polynomial in graphs such that each summand has multi-index $\mathbf{g}$.[12]

Remark 2.25 (Hopf subalgebras and multiplicative renormalization, quoted from [18]). Given the situation of Definition 2.23 and suppose, that the renormalization Hopf algebra $\mathcal{H}_{\mathcal{Q}}$ possesses Hopf subalgebras in the sense of Definition 2.24. Then we can calculate the Z-factor for a given residue $r \in \mathcal{R}_{\mathcal{Q}}$ via

$$Z_{\mathscr{E},\mathscr{R}}^r (\varepsilon) := S_{\mathscr{R}}^{\Phi_{\mathscr{E}}^\varepsilon} \left(\mathfrak{X}^r \right) . \tag{59}$$

More details in this direction can be found in [24, 26] (using a different notation).

Remark 2.26 (Hopf subalgebras and different gradings, quoted from [18]). Furthermore, we remark that the existence of the Hopf subalgebras from Definition 2.24 depends crucially on the grading $\mathbf{g}$. In particular, for the grading induced by the first Betti number these Hopf subalgebras exist if and only if the local QFT has only one vertex, for the coupling-constant grading if and only if the local QFT has for each vertex a different coupling constant and always for the residue-grading, as we will see in Sections 4 and 5.

Lemma 2.27 (Finite renormalization schemes)**.** *The image of the renormalized Feynman rules* $\operatorname{Im}(\Phi_{\mathscr{R}})$ *maps to convergent integral expressions, if the cokernel* $\operatorname{CoKer}(\mathscr{R})$ *of the corresponding renormalization scheme consists only of convergent integral expressions.*

[12] There exist closed expressions for the polynomials $\mathfrak{P}_{\mathbf{g}} \left(\mathfrak{X}_{\mathbf{G}}^r \right)$ as we will see in Section 4, which were first introduced in [25].

Proof. The Theorem about algebraic Birkhoff decompositions, first observed in this context in [20], states in this context, that

$$\Phi_{\mathscr{R}} : \quad \mathcal{H}_{\mathcal{Q}} \to \operatorname{CoKer}(\mathscr{R}) \subseteq \Omega_{\mathcal{Q}} \tag{60}$$

and

$$S^{\Phi_{\mathscr{E}}}_{\mathscr{R}} : \quad \mathcal{H}_{\mathcal{Q}} \to \operatorname{Im}(\mathscr{R}) \subseteq \Omega_{\mathcal{Q}} \, , \tag{61}$$

and thus $\operatorname{Im}(\Phi_{\mathscr{R}})$ consists of finite integral expressions, if $\operatorname{CoKer}(\mathscr{R})$ does. ∎

Definition 2.28 (Physical renormalization schemes)**.** A renormalization scheme $\mathscr{R} \in \operatorname{End}(\Omega_{\mathcal{Q}})$ is called physical, if both its kernel $\operatorname{Ker}(\mathscr{R})$ and its cokernel $\operatorname{CoKer}(\mathscr{R})$ consist only of convergent integral expressions. In particular, we demand that

$$\operatorname{Im}(\Phi \circ \mathscr{D}) \subseteq \operatorname{CoIm}(\mathscr{R}) \, , \tag{62}$$

i.e. the image of superficially divergent graphs under the Feynman rules is a subset of the coimage of a physical renormalization scheme.

Remark 2.29. Definition 2.28 is motivated by the fact, that we want renormalization schemes in physics to be finite and furthermore such, that it removes the divergent contribution of a Feynman graph via itself.

3 A superficial argument

In this section we study combinatorial properties of the superficial degree of divergence. This combinatorial number, associated to each Feynman graph, gives a measure of the divergence of the associated Feynman integral. In fact, a result of Weinberg [27] states, that the ultraviolet divergence of a Feynman integral is bounded via a polynomial of degree $n \in \mathbb{Z}$, if the associated superficial degree of divergence of the corresponding Feynman graph and all of its subgraphs are less than n; in particular it is finite, if $n < 0$. The aim of this section is now to give an alternative characterization of (super-/non-)renormalizability of a local QFT in terms of weights of corollas, as defined below. With this criterion on hand, we consider local QFTs with more than one interaction term; here, it is in principle possible, that the different interaction terms mix up the classifications of (super-/non-)renormalizability, if the weights of their corollas differ. The results are then Lemma 3.4, stating, that for a given residue, the superficial degree of divergence depends affine linearly on the residue-grading of the Feynman graph. Furthermore, we show in Corollary 3.8 that coproduct and antipode identities are compatible with the restriction to divergent graphs, if the corresponding local QFT is one-loop divergent. This result is in particular useful for the following sections, where we want to state our results not only for the renormalizable cases, but also include the more involved super- and non-renormalizable cases, or even mixes thereof.

Definition 3.1 (Weight of residues)**.** Let $\mathcal{Q}$ be a local QFT with residue set $\mathcal{R}_{\mathcal{Q}}$. We introduce a weight function

$$\omega : \quad \mathcal{R}_{\mathcal{Q}} \to \mathbb{Z} \, , \, , \quad r \mapsto \operatorname{Deg}_p \big(\Phi(r) \big) \, , \tag{63}$$

which maps a residue $r \in \mathcal{R}_{\mathcal{Q}}$ to the degree of the corresponding Feynman rule, viewed as a polynomial in momenta (or, in position space, derivatives).

Definition 3.2 (Superficial degree of divergence)**.** Let $\mathcal{Q}$ be a local QFT with weighted residue set $\mathcal{R}_\mathcal{Q}$ and Feynman graph set $\mathcal{G}_\mathcal{Q}$. We turn $\mathcal{G}_\mathcal{Q}$ into a weighted set as well by declaring the function

$$\omega : \quad \mathcal{G}_\mathcal{Q} \to \mathbb{Z} \, , \quad \Gamma \mapsto d b_1 (\Gamma) + \sum_{v \in V(\Gamma)} \omega (v) + \sum_{e \in E(\Gamma)} \omega (e) \, , \tag{64}$$

where d is the dimension of spacetime of $\mathcal{Q}$ and $b_1 (\Gamma)$ the first Betti number of the Feynman graph $\Gamma \in \mathcal{G}_\mathcal{Q}$. Then, the weight $\omega (\Gamma)$ of a Feynman graph $\Gamma \in \mathcal{G}_\mathcal{Q}$ is called the superficial degree of divergence of Γ. A Feynman graph $\Gamma \in \mathcal{G}_\mathcal{Q}$ is called superficially divergent if $\omega (\Gamma) \geq 0$ and superficially convergent if $\omega (\Gamma) < 0$. Finally, we additionally set $\omega (\mathbb{I}) = 0$ for convenience.

Definition 3.3 (Alternative definition for the superficial degree of divergence)**.** Given the situation of Definition 3.2 and a vertex residue $v \in \mathcal{R}_\mathcal{Q}^{[0]}$, we define the weight of corollas

$$\varpi (v) \equiv \omega (c_v) := \omega (v) + \frac{1}{2} \sum_{e \in E(v)} \omega (e) \, , \tag{65}$$

such that the superficial degree of divergence can be equivalently calculated via

$$\omega : \quad \mathcal{G}_\mathcal{Q} \to \mathbb{Z} \, , \quad \Gamma \mapsto d b_1 (\Gamma) + \sum_{v \in V(\Gamma)} \varpi (v) - \frac{1}{2} \sum_{e \in E\left(\operatorname{Res}(\Gamma)\right)} \omega (e) \, . \tag{66}$$

Lemma 3.4 (Alternative definition for the superficial degree of divergence)**.** *Given the situation of Definition 3.3, the superficial degree of divergence can be expressed via*

$$\begin{aligned} \omega : \quad \mathcal{G}_\mathcal{Q} \to \mathbb{Z} \, , \quad \Gamma \mapsto & d \left(1 - \delta_{\operatorname{Res}(\Gamma) \in \mathcal{R}_\mathcal{Q}^{[0]}} \right) - \frac{1}{2} \sum_{e \in E\left(\operatorname{Res}(\Gamma)\right)} \omega (e) \\ & + \sum_{i = 1}^{\# \mathcal{R}_\mathcal{Q}^{[0]}} \left(d \left(\frac{1}{2} \operatorname{Val} (v_i) - 1 \right) + \varpi (v_i) \right) \big(\operatorname{ResGrd} (\Gamma) \big)_i \, , \end{aligned} \tag{67}$$

where

$$\delta_{\operatorname{Res}(\Gamma) \in \mathcal{R}_\mathcal{Q}^{[0]}} = \begin{cases} 1 & \textit{if } \operatorname{Res} (\Gamma) \in \mathcal{R}_\mathcal{Q}^{[0]} \\ 0 & \textit{else, i.e. } \operatorname{Res} (\Gamma) \in \left(\mathcal{A}_\mathcal{Q} \setminus \mathcal{R}_\mathcal{Q}^{[0]} \right) \end{cases} \, . \tag{68}$$

In particular, the superficial degree of divergence of a Feynman graph $\Gamma \in \mathcal{G}_\mathcal{Q}$ depends only on its residue and its residue-grading, i.e. $\omega (\Gamma) \equiv \omega \big(\operatorname{Res} (\Gamma) , \operatorname{ResGrd} (\Gamma) \big)$, and the dependence on its residue-grading is affine linear.[13]

Proof. We start with Equation (66) from Definition 3.3: First we rewrite the first Betti number using the Euler characteristic[14]

$$b_1 (\Gamma) = b_0 (\Gamma) - \# V (\Gamma) + \# E (\Gamma) \, , \tag{69}$$

[13] If $\mathcal{Q}$ is sQGSc, c.f. Definition 3.11, then the dependence on the residue-grading can be furthermore replaced by its coupling-grading, and if $\mathcal{Q}$ is even uniweighted, again c.f. Definition 3.11, then the dependence on the residue-grading can be furthermore replaced by its first Betti number.

[14] For the Euler characteristic, Equation (69), we need to either ignore the external half-edges or assume that they are attached to external vertices.

where $b_0(\Gamma) = 1$, as $\Gamma \in \mathcal{G}_\mathcal{Q}$ is connected. Then we express $\# V(\Gamma)$ in terms of $\operatorname{ResGrd}(\Gamma)$ and $\operatorname{Res}(\Gamma)$ via

$$\begin{aligned}
\# V(\Gamma) &= \sum_{i=1}^{\# \mathcal{R}_\mathcal{Q}^{[0]}} \big(\operatorname{IntRes}(\Gamma)\big)_i \\
&= \sum_{i=1}^{\# \mathcal{R}_\mathcal{Q}^{[0]}} \Big(\big(\operatorname{ResGrd}(\Gamma)\big)_i + \big(\operatorname{ExtRes}(\Gamma)\big)_i\Big) \\
&= \sum_{i=1}^{\# \mathcal{R}_\mathcal{Q}^{[0]}} \Big(\big(\operatorname{ResGrd}(\Gamma)\big)_i\Big) + \delta_{\operatorname{Res}(\Gamma) \in \mathcal{R}_\mathcal{Q}^{[0]}} \, ,
\end{aligned} \tag{70}$$

where in the last equality we have used that $\Gamma \in \mathcal{G}_\mathcal{Q}$ is connected, and we express $\# E(\Gamma)$ in terms of $\operatorname{ResGrd}(\Gamma)$ via

$$\begin{aligned}
\# E(\Gamma) &= \frac{1}{2} \sum_{i=1}^{\# \mathcal{R}_\mathcal{Q}^{[0]}} \operatorname{Val}(v_i) \big(\operatorname{IntRes}(\Gamma)\big)_i - \frac{1}{2} \sum_{i=1}^{\# \mathcal{R}_\mathcal{Q}^{[0]}} \operatorname{Val}(v_i) \big(\operatorname{ExtRes}(\Gamma)\big)_i \\
&= \frac{1}{2} \sum_{i=1}^{\# \mathcal{R}_\mathcal{Q}^{[0]}} \operatorname{Val}(v_i) \big(\operatorname{ResGrd}(\Gamma)\big)_i
\end{aligned} \tag{71}$$

Thus, we obtain

$$b_1(\Gamma) = 1 - \delta_{\operatorname{Res}(\Gamma) \in \mathcal{R}_\mathcal{Q}^{[0]}} + \frac{1}{2} \sum_{i=1}^{\# \mathcal{R}_\mathcal{Q}^{[0]}} \big(\operatorname{Val}(v_i) - 2\big) \big(\operatorname{ResGrd}(\Gamma)\big)_i \, . \tag{72}$$

Plugging the above results into Equation (66) from Definition 3.3 yields the claimed equation. ■

Definition 3.5 (One-loop divergent QFTs)**.** Let $\mathcal{Q}$ be a local QFT. We call $\mathcal{Q}$ one-loop divergent, if for each Feynman graph $\Gamma \in \mathcal{G}_\mathcal{Q}$ with $\operatorname{Res}(\Gamma) = r$ and $\omega(\Gamma) \geq 0$ there exists a Feynman graph $\gamma \in \mathcal{G}_\mathcal{Q}$ with $\operatorname{Res}(\gamma) = r$, $\omega(\gamma) \geq 0$ and $b_1(\gamma) = 1$.

Remark 3.6. Definition 3.5 is trivially satisfied for super-renormalizable and renormalizable QFTs. However, it is the obstacle for non-renormalizable QFTs to enjoy the simple coproduct and antipode identities from Corollary 3.8. An example of a non-renormalizable local QFT that is not one-loop divergent is given in Example 3.7.

Example 3.7. Let ψ_4^4 be a local QFT, described via the Lagrange density

$$\mathcal{L}_{\psi_4^4} := \overline{\psi} \not{\partial}^2 \psi + \left(\overline{\psi} \not{\partial} \psi\right) \left(\overline{\psi} \psi\right) + \mathcal{L}_{\psi_4^4}^{\text{Ren-Int}} \, , \tag{73}$$

where $\psi \in \Gamma\left(\Sigma \mathbb{M}^{1,3}\right)$ is a spinor field on the four-dimensional Minkowski spacetime $\mathbb{M}^{1,3}$ with signature $(1,3)$, $\not{\partial}$ the Dirac operator on $\Sigma \mathbb{M}^{1,3}$ and $\mathcal{L}_{\psi_4^4}^{\text{Ren-Int}}$ is the Lagrange density containing all interaction terms needed for renormalization, c.f. [18, Solution 3.38.]. Then, ψ_4^4 is non-renormalizable and the one-loop ten-point graph is superficially convergent, whereas all higher ten-point graphs are superficially divergent. Thus, ψ_4^4 is not one-loop divergent.

Corollary 3.8. *Let $\mathcal{Q}$ be a one-loop divergent local QFT. Then, the coproduct and the antipode are compatible with the restriction to divergent graphs from Definition 2.4. More precisely, given a multi-index $\mathbf{g}$ corresponding to some grading of $\mathcal{H}_{\mathcal{Q}}$ (c.f. Definition 2.6) compatible with the superficial degree of divergence (c.f. Footnote 13) and elements in the Hopf algebra $\mathfrak{G}, \mathfrak{h}(\mathbf{g}), \mathfrak{H}_{\mathbf{g}} \in \mathcal{H}_{\mathcal{Q}}$,[15] such that the following coproduct and antipode identities hold*[16]

$$\Delta(\mathfrak{G}) = \sum_{\mathbf{g}} \mathfrak{h}(\mathbf{g}) \otimes \mathfrak{H}_{\mathbf{g}} \tag{74}$$

which implies

$$\Delta\left(\overline{\mathfrak{G}}\right) = \sum_{\mathbf{g}} \mathfrak{h}(\mathbf{g}) \otimes \overline{\mathfrak{H}}_{\mathbf{g}} \tag{75}$$

and similarly

$$S(\mathfrak{G}) = -\sum_{\mathbf{g}} S\left(\mathfrak{h}(\mathbf{g})\right) \mathscr{P}\left(\mathfrak{H}_{\mathbf{g}}\right) \tag{76}$$

which implies

$$S\left(\overline{\mathfrak{G}}\right) = -\sum_{\mathbf{g}} S\left(\mathfrak{h}(\mathbf{g})\right) \mathscr{P}\left(\overline{\mathfrak{H}}_{\mathbf{g}}\right) . \tag{77}$$

Proof. We check the argument first on the level of generators of the renormalization Hopf algebra, i.e. single Feynman graphs. The result extends then general elements in $\mathcal{H}_{\mathcal{Q}}$, i.e. sums and products thereof, by the linearity and multiplicativity of the coproduct and the antipode. Thus, let $\Gamma \in \mathcal{G}_{\mathcal{Q}}$ be a Feynman graph. Then, we have

$$\Delta(\Gamma) = \sum_{\gamma \in \mathcal{D}(\Gamma)} \gamma \otimes \Gamma/\gamma \tag{78}$$

which implies

$$\Delta\left(\overline{\Gamma}\right) = \sum_{\gamma \in \mathcal{D}\left(\overline{\Gamma}\right)} \gamma \otimes \overline{\Gamma/\gamma} \tag{79}$$

and similarly

$$S(\Gamma) = -\sum_{\gamma \in \mathcal{D}(\Gamma)} S(\gamma) \mathscr{P}\left(\Gamma/\gamma\right) \tag{80}$$

which implies

$$S\left(\overline{\Gamma}\right) = -\sum_{\gamma \in \mathcal{D}\left(\overline{\Gamma}\right)} S(\gamma) \mathscr{P}\left(\overline{\Gamma/\gamma}\right) , \tag{81}$$

if $\mathcal{Q}$ is one-loop divergent. This is a consequence of Lemma 3.4: As the superficial degree of divergence depends only on the residue and affine linearly on a compatible grading it either increases or decreases with the grading $\mathbf{g}$. As $\operatorname{Grd}\left(\Gamma/\gamma\right) \leq \operatorname{Grd}(\Gamma)$, the only problem occurs when the superficial degree of convergence increases with increasing grading, i.e. for non-renormalizable local QFTs: Then it may happen, that there exists a bound $\mathbf{g}^r$ such that for a given residue r all graphs with $\operatorname{Grd}\left(\Gamma/\gamma\right) < \mathbf{g}^r$ are convergent and thus the quotient Γ/γ will be in the kernel of $\mathscr{D}$, i.e. $\overline{\Gamma/\gamma} = 0$. However, this is excluded by the assumption of $\mathcal{Q}$ being one-loop divergent and thus finishes the proof. ■

[15] The notation is chosen to emphasize, that $\mathfrak{h}(\mathbf{g})$ is a function of $\mathbf{g}$, whereas $\mathfrak{H}_{\mathbf{g}}$ is the restriction of an $\mathfrak{H}$ to $\left(\mathcal{H}_{\mathcal{Q}}\right)_{\mathbf{g}}$.

[16] We remark, that by definition of the coproduct and the antipode we have here $\mathfrak{h}(\mathbf{g}) \equiv \overline{\mathfrak{h}(\mathbf{g})}$.

Remark 3.9. All local QFTs of physical interest are one-loop divergent, in particular all QFTs from the Standard Model (as they are renormalizable) and Quantum General Relativity, viewed as an effective QFT, (a simple combinatorial argument shows, that the superficial degree of divergence of a Feynman graph is independent of its residue and depends only on its loop number, and is in particular already divergent for one-loop Feynman graphs, c.f. [28, 29]). In the case of local QFTs which are not one-loop divergent Equations (75) and (77) need to be corrected to (for simplicity, we assume that $\mathfrak{G}^r \in \mathcal{H}_{\mathcal{Q}}$ is a sum of products of Feynman graphs with residue $r \in \mathcal{R}_{\mathcal{Q}}$ — otherwise the correcting sums depend also on the different residues and also products between them need to be considered)

$$\Delta\left(\overline{\mathfrak{G}}^r\right) = \sum_{\mathbf{g}} \mathfrak{h}(\mathbf{g}) \otimes \overline{\mathfrak{H}}_{\mathbf{g}}^r + \sum_{\mathbf{g} \leq \mathbf{g}^r} \mathfrak{h}(\mathbf{g}) \otimes \mathfrak{H}_{\mathbf{g}}^r \tag{82}$$

and

$$S\left(\overline{\mathfrak{G}}^r\right) = - \sum_{\mathbf{g}} S\left(\mathfrak{h}(\mathbf{g})\right) \mathscr{P}\left(\overline{\mathfrak{H}}_{\mathbf{g}}^r\right) - \sum_{\mathbf{g} \leq \mathbf{g}^r} S\left(\mathfrak{h}(\mathbf{g})\right) \mathscr{P}\left(\mathfrak{H}_{\mathbf{g}}^r\right) , \tag{83}$$

where $\mathbf{g}^r$ is the critical multi-index mentioned in the proof of Corollary 3.8.

Corollary 3.10 (Weight of corollas and renormalizability)**.** *A local QFT $\mathcal{Q}$ is renormalizable if and only if for all $v \in \mathcal{R}_{\mathcal{Q}}^{[0]}$ the weights of the corresponding corollas are*

$$\varpi(v) = d\left(1 - \frac{1}{2}\operatorname{Val}(v)\right) , \tag{84}$$

super-renormalizable if and only if the weight is smaller and non-renormalizable if and only if the weight is bigger.

Proof. Before presenting the actual argument, we recall, that a local QFT $\mathcal{Q}$ is called super-renormalizable, if, for a fixed residue r, the superficial degree of divergence decreases with increasing grading, is called renormalizable, if the superficial degree of divergence is independent of the grading and is called non-renormalizable, if the superficial degree of divergence increases with increasing grading. Using Equation (67) from Lemma 3.4, we obtain the claimed bound

$$\varpi(v) = d\left(1 - \frac{1}{2}\operatorname{Val}(v)\right) , \tag{85}$$

as for this value the superficial degree of divergence of a Feynman graph is independent of its grading. ■

Definition 3.11 (sQGSc and uniweighted local QFTs)**.** Let $\mathcal{Q}$ be a local QFT and let $v, w \in \mathcal{R}_{\mathcal{Q}}^{[0]}$ be vertex residues. Consider the QGS equivalence relation from Definition 5.1. Then we call $\mathcal{Q}$ superficially QGS compatible (sQGSc), if and only if the weight of corollas is compatible with this equivalence relation, i.e. if and only if for all $\{v, m; w, n\} \in \mathrm{QGS}_{\mathcal{Q}}$ the following equation holds:

$$m\varpi(v) = n\varpi(w) \tag{86}$$

If furthermore the weight of corollas for all vertices $v, w \in \mathcal{R}_{\mathcal{Q}}^{[0]}$ depends only on the valence of the vertices, i.e.

$$\frac{1}{\left(\operatorname{Val}(v) - 2\right)}\varpi(v) = \frac{1}{\left(\operatorname{Val}(w) - 2\right)}\varpi(w) \tag{87}$$

holds, we call $\mathcal{Q}$ uniweighted.

Remark 3.12. Being sQGSc is a necessary criterion for the validity of QGS on the level of Feynman rules, though not a sufficient one.

Proposition 3.13 (QGR-SM is sQGSc)**.** *Quantum General Relativity coupled to the Standard Model or any physical sub-QFT thereof is sQGSc.*

Proof. We start with considering only the pure gravitational part, i.e. gravitons and graviton-ghosts: The Feynman rules of Quantum General Relativity (QGR) are such, that each vertex $v \in \mathcal{R}^{[0]}_{\text{QGR}}$ has weight $\omega(v) = 2$ and each edge $e \in \mathcal{R}^{[1]}_{\text{QGR}}$ has weight $\omega(e) = -2$, as the corresponding Feynman rules are quadratic and inverse quadratic in momenta, respectively. Thus, the corolla-weight of a vertex $v \in \mathcal{R}^{[0]}_{\text{QGR}}$ is

$$\varpi(v) = 2 - \operatorname{Val}(v) \, . \tag{88}$$

Given another vertex $w \in \mathcal{R}^{[0]}_{\text{QGR}}$, we calculate

$$\begin{aligned} \frac{1}{\big(\operatorname{Val}(v) - 2\big)} \varpi(v) &= \frac{2 - \operatorname{Val}(v)}{\operatorname{Val}(v) - 2} \\ &= -1 \\ &= \frac{2 - \operatorname{Val}(w)}{\operatorname{Val}(w) - 2} \\ &= \frac{1}{\big(\operatorname{Val}(w) - 2\big)} \varpi(w) \, , \end{aligned} \tag{89}$$

showing, that the pure Quantum General Relativity part is uniweighted and thus in particular sQGSc. Furthermore, the pure Standard Model part is renormalizable and thus in particular sQGSc, due to Proposition 3.10, as we have for all $v, w \in \mathcal{R}^{[0]}_{\text{SM}}$

$$\begin{aligned} \frac{1}{\big(\operatorname{Val}(v) - 2\big)} \varpi(v) &= \frac{d\big(1 - 1/2 \operatorname{Val}(v)\big)}{\big(\operatorname{Val}(v) - 2\big)} \\ &= -\frac{d}{2} \\ &= \frac{d\big(1 - 1/2 \operatorname{Val}(w)\big)}{\big(\operatorname{Val}(w) - 2\big)} \\ &= \frac{1}{\big(\operatorname{Val}(w) - 2\big)} \varpi(w) \, , \end{aligned} \tag{90}$$

showing, that the pure Standard Model part is uniweighted and thus in particular sQGSc. Finally, we consider the mixed parts, i.e. Standard Model residues with a positive number of gravitons attached to it. It follows from the corresponding Feynman rules, that the weights of these corollas depends only on the Standard Model residue and is independent of the number of gravitons attached to it. Equivalently, increasing the number of gravitons by gluing a three-valent graviton vertex using a graviton propagator to such a residue also leaves the weight of the corolla unchanged (as the net difference is $2 - 2 = 0$), which finishes the proof. ■

4 Coproduct and antipode identities

In this section, we state coproduct and antipode identities. These coproduct identities are known in the literature in the case of renormalizable local QFTs [9, 25, 30]. In this section, we reprove these identities and generalize them to super-renormalizable and non-renormalizable local QFTs.

Lemma 4.1 (Coproduct and antipode identities)**.** *Coproduct identities, in particular Proposition 4.2, 4.3 and 4.4, are equivalent to recursive antipode identities. More precisely, given a multi-index* $\mathbf{g}$ *corresponding to some grading of* $\mathcal{H}_Q$ *(c.f. Definition 2.6) and elements in the Hopf algebra* $\mathfrak{G}, \mathfrak{h}(\mathbf{g}), \mathfrak{H}_\mathbf{g} \in \mathcal{H}_Q$,[17] *such that the following coproduct identity holds*

$$\Delta(\mathfrak{G}) = \sum_\mathbf{g} \mathfrak{h}(\mathbf{g}) \otimes \mathfrak{H}_\mathbf{g}\,, \tag{91}$$

then this is equivalent to the following recursive antipode identity

$$S(\mathfrak{G}) = -\sum_\mathbf{g} S\big(\mathfrak{h}(\mathbf{g})\big)\, \mathscr{P}\big(\mathfrak{H}_\mathbf{g}\big)\ . \tag{92}$$

Proof. This follows immediately from the definition of the coproduct and the recursive definition of the antipode, which are defined on generators $\Gamma \in \mathcal{H}_Q$ via

$$\Delta(\Gamma) := \sum_{\gamma \in \mathcal{D}(\Gamma)} \gamma \otimes \Gamma/\gamma \tag{93}$$

and

$$S(\Gamma) := -\left(S \star \mathscr{P}\right)(\Gamma)\ , \tag{94}$$

and are then linearly and multiplicatively extended, such that

$$S \star \mathrm{Id} \equiv \mathbb{I} \circ \hat{\mathbb{I}} \equiv \mathrm{Id} \star S \tag{95}$$

holds.[18] ■

Proposition 4.2 (Coproduct identities for (divergent or restricted) combinatorial Green's functions)**.** *Let* Q *be a one-loop divergent local QFT,* $\mathcal{H}_Q$ *its associated renormalization Hopf algebra,* $r \in \mathcal{R}_Q$ *a residue and* $\mathbf{R} \in \mathbb{Z}^{\#\mathcal{R}_Q^{[0]}}$ *a residue-grading multi-index. Then, we have the following coproduct identities for Green's functions:*

$$\Delta\left(\mathfrak{X}^r\right) = \sum_{\mathbf{r} \in \mathbb{Z}^{\#\mathcal{R}_Q^{[0]}}} \overline{\mathfrak{X}}^r \overline{\mathfrak{Q}}^\mathbf{r} \otimes \mathfrak{X}^r_\mathbf{r}\,, \tag{96a}$$

$$\Delta\left(\mathfrak{X}^r_\mathbf{R}\right) = \sum_{\mathbf{r} \in \mathbb{Z}^{\#\mathcal{R}_Q^{[0]}}} \left(\overline{\mathfrak{X}}^r \overline{\mathfrak{Q}}^\mathbf{r}\right)\bigg|_{\mathbf{R}-\mathbf{r}} \otimes \mathfrak{X}^r_\mathbf{r}\,, \tag{96b}$$

$$\Delta\left(\overline{\mathfrak{X}}^r\right) = \sum_{\mathbf{r} \in \mathbb{Z}^{\#\mathcal{R}_Q^{[0]}}} \overline{\mathfrak{X}}^r \overline{\mathfrak{Q}}^\mathbf{r} \otimes \overline{\mathfrak{X}}^r_\mathbf{r} \tag{96c}$$

and

$$\Delta\left(\overline{\mathfrak{X}}^r_\mathbf{R}\right) = \sum_{\mathbf{r} \in \mathbb{Z}^{\#\mathcal{R}_Q^{[0]}}} \left(\overline{\mathfrak{X}}^r \overline{\mathfrak{Q}}^\mathbf{r}\right)\bigg|_{\mathbf{R}-\mathbf{r}} \otimes \overline{\mathfrak{X}}^r_\mathbf{r}\,. \tag{96d}$$

[17] The notation is chosen to emphasize, that $\mathfrak{h}(\mathbf{g})$ is a function of $\mathbf{g}$, whereas $\mathfrak{H}_\mathbf{g}$ is the restriction of an $\mathfrak{H}$ to $\left(\mathcal{H}_Q\right)_\mathbf{g}$.

[18] We remark, that S is the $\star$-inverse to Id and $\mathbb{I} \circ \hat{\mathbb{I}}$ the $\star$-identity.

Proof. Equations (116a) and (116b) follow by the following computations, using the linearity of the coproduct, Proposition 2.19 and Lemma 2.20:

$$\begin{aligned}
\Delta\left(\mathfrak{X}^r\right) &= \sum_{\substack{\Gamma \in \mathcal{G}_{\mathcal{Q}} \\ \operatorname{Res}(\Gamma) = r}} \frac{1}{\operatorname{Sym}(\Gamma)} \Delta(\Gamma) \\
&= \sum_{\substack{\Gamma \in \mathcal{G}_{\mathcal{Q}} \\ \operatorname{Res}(\Gamma) = r}} \sum_{\gamma \in \mathcal{D}(\Gamma)} \frac{1}{\operatorname{Sym}(\Gamma)} \gamma \otimes \Gamma / \gamma \\
&= \sum_{\substack{\Gamma \in \mathcal{G}_{\mathcal{Q}} \\ \operatorname{Res}(\Gamma) = r}} \sum_{\gamma \in \mathcal{D}(\Gamma)} \left(\frac{\operatorname{Ins}_{\operatorname{Aut}}\left(\gamma \triangleright \Gamma/\gamma; \Gamma\right)}{\operatorname{Sym}(\gamma)} \gamma \right) \otimes \left(\frac{1}{\operatorname{Sym}\left(\Gamma/\gamma\right)} \Gamma/\gamma \right) \\
&= \sum_{\substack{\mathfrak{G} \in \mathcal{G}_{\mathcal{Q}} \\ \operatorname{Res}(\mathfrak{G}) = r}} \left(\sum_{\mathfrak{g} \in \mathcal{I}(\mathfrak{G})} \frac{\operatorname{Ins}\left(\mathfrak{g} \triangleright \mathfrak{G}\right)}{\operatorname{Sym}(\mathfrak{g})} \mathfrak{g} \right) \otimes \left(\frac{1}{\operatorname{Sym}(\mathfrak{G})} \mathfrak{G} \right) + \overline{\mathfrak{X}}^r \otimes \mathbb{I} \\
&= \sum_{\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}}} \left(\sum_{\mathfrak{g} \in \mathcal{I}^{r, \mathbf{r}}} \frac{\operatorname{Ins}^{r, \mathbf{r}}(\mathfrak{g})}{\operatorname{Sym}(\mathfrak{g})} \mathfrak{g} \right) \otimes \left(\sum_{\substack{\mathfrak{G} \in \mathcal{G}_{\mathcal{Q}} \\ \operatorname{Res}(\mathfrak{G}) = r \\ \operatorname{ResGrd}(\mathfrak{G}) = \mathbf{r}}} \frac{1}{\operatorname{Sym}(\mathfrak{G})} \mathfrak{G} \right) + \overline{\mathfrak{X}}^r \otimes \mathbb{I} \\
&= \sum_{\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}}} \overline{\mathfrak{X}}^r \overline{\mathfrak{Q}}^{\mathbf{r}} \otimes \mathfrak{X}^r_{\mathbf{r}}
\end{aligned} \tag{97a}$$

and

$$\begin{aligned}
\Delta\left(\mathfrak{X}^r_{\mathbf{R}}\right) &= \left(\Delta\left(\mathfrak{X}^r\right)\right)\Big|_{\mathbf{R}} \\
&= \left(\sum_{\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}}} \overline{\mathfrak{X}}^r \overline{\mathfrak{Q}}^{\mathbf{r}} \otimes \mathfrak{X}^r_{\mathbf{r}} \right)\Bigg|_{\mathbf{R}} \\
&= \sum_{\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}}} \left(\overline{\mathfrak{X}}^r \overline{\mathfrak{Q}}^{\mathbf{r}} \right)\Big|_{\mathbf{R} - \mathbf{r}} \otimes \mathfrak{X}^r_{\mathbf{r}}
\end{aligned} \tag{97b}$$

Finally, Equations (116c) and (116d) follow by Equations (116a) and (116b) together with the assumption of $\mathcal{Q}$ being one-loop divergent and the application of Corollary 3.8. ■

Proposition 4.3 (Coproduct identities for (divergent or restricted) combinatorial charges)**.** *Let $\mathcal{Q}$ be a one-loop divergent local QFT, $\mathcal{H}_{\mathcal{Q}}$ its associated renormalization Hopf algebra, $v \in \mathcal{R}_{\mathcal{Q}}^{[0]}$ a vertex residue and $\mathbf{R} \in \mathbb{Z}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}}$ a residue-grading multi-index. Then, we have the following coproduct identities for combinatorial charges:*

$$\Delta\left(\mathfrak{Q}^v\right) = \sum_{\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}}} \overline{\mathfrak{Q}}^v \overline{\mathfrak{Q}}^{\mathbf{r}} \otimes \mathfrak{Q}^v_{\mathbf{r}} \, , \tag{98a}$$

$$\Delta\left(\mathfrak{Q}^v_{\mathbf{R}}\right) = \sum_{\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}}} \left(\overline{\mathfrak{Q}}^v \overline{\mathfrak{Q}}^{\mathbf{r}} \right)\Big|_{\mathbf{R} - \mathbf{r}} \otimes \mathfrak{Q}^v_{\mathbf{r}} \, , \tag{98b}$$

$$\Delta\left(\overline{\mathfrak{Q}}^v\right) = \sum_{\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}}} \overline{\mathfrak{Q}}^v \overline{\mathfrak{Q}}^{\mathbf{r}} \otimes \overline{\mathfrak{Q}}^v_{\mathbf{r}} \tag{98c}$$

and

$$\Delta\left(\overline{\mathfrak{Q}}^v_{\mathbf{R}}\right) = \sum_{\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}}} \left.\left(\overline{\mathfrak{Q}}^v \overline{\mathfrak{Q}}^{\mathbf{r}}\right)\right|_{\mathbf{R}-\mathbf{r}} \otimes \overline{\mathfrak{Q}}^v_{\mathbf{r}} \tag{98d}$$

Proof. Equations (98a) and (98b) follow by the following computations, using the linearity and multiplicativity of the coproduct and Proposition 4.2:

$$\begin{aligned}
\Delta\left(\mathfrak{Q}^v\right) &= \frac{\Delta\left(\mathfrak{X}^v\right)}{\prod_{e \in E(v)} \sqrt{\Delta\left(\mathfrak{X}^e\right)}} \\
&= \frac{\sum_{\mathbf{r}_v \in \mathbb{Z}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}}} \overline{\mathfrak{X}}^v \overline{\mathfrak{Q}}^{\mathbf{r}_v} \otimes \mathfrak{X}^v_{\mathbf{r}_v}}{\prod_{e \in E(v)} \sqrt{\sum_{\mathbf{r}_e \in \mathbb{Z}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}}} \overline{\mathfrak{X}}^e \overline{\mathfrak{Q}}^{\mathbf{r}_e} \otimes \mathfrak{X}^e_{\mathbf{r}_e}}} \\
&= \frac{\left(\overline{\mathfrak{X}}^v \otimes \mathbb{I}\right)\left(\sum_{\mathbf{r}_v \in \mathbb{Z}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}}} \overline{\mathfrak{Q}}^{\mathbf{r}_v} \otimes \mathfrak{X}^v_{\mathbf{r}_v}\right)}{\prod_{e \in E(v)} \sqrt{\left(\overline{\mathfrak{X}}^e \otimes \mathbb{I}\right)\left(\sum_{\mathbf{r}_e \in \mathbb{Z}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}}} \overline{\mathfrak{Q}}^{\mathbf{r}_e} \otimes \mathfrak{X}^e_{\mathbf{r}_e}\right)}} \\
&= \left(\overline{\mathfrak{Q}}^v \otimes \mathbb{I}\right)\left(\frac{\left(\sum_{\mathbf{r}_v \in \mathbb{Z}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}}} \overline{\mathfrak{Q}}^{\mathbf{r}_v} \otimes \mathfrak{X}^v_{\mathbf{r}_v}\right)}{\prod_{e \in E(v)} \sqrt{\left(\sum_{\mathbf{r}_e \in \mathbb{Z}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}}} \overline{\mathfrak{Q}}^{\mathbf{r}_e} \otimes \mathfrak{X}^e_{\mathbf{r}_e}\right)}}\right) \\
&= \left(\overline{\mathfrak{Q}}^v \otimes \mathbb{I}\right)\left(\sum_{\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}}} \overline{\mathfrak{Q}}^{\mathbf{r}} \otimes \mathfrak{Q}^v_{\mathbf{r}}\right) \\
&= \sum_{\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}}} \overline{\mathfrak{Q}}^v \overline{\mathfrak{Q}}^{\mathbf{r}} \otimes \mathfrak{Q}^v_{\mathbf{r}}
\end{aligned} \tag{99a}$$

and

$$\begin{aligned}
\Delta\left(\mathfrak{Q}^v_{\mathbf{R}}\right) &= \left.\left(\Delta\left(\mathfrak{Q}^v\right)\right)\right|_{\mathbf{R}} \\
&= \left.\left(\sum_{\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}}} \overline{\mathfrak{Q}}^v \overline{\mathfrak{Q}}^{\mathbf{r}} \otimes \mathfrak{Q}^v_{\mathbf{r}}\right)\right|_{\mathbf{R}} \\
&= \sum_{\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}}} \left.\left(\overline{\mathfrak{Q}}^v \overline{\mathfrak{Q}}^{\mathbf{r}}\right)\right|_{\mathbf{R}-\mathbf{r}} \otimes \mathfrak{Q}^v_{\mathbf{r}} \, ,
\end{aligned} \tag{99b}$$

Finally, Equations (98c) and (98d) follow by Equations (98a) and (98b) together with the assumption of $\mathcal{Q}$ being one-loop divergent and the application of Corollary 3.8. ∎

Proposition 4.4 (Coproduct identities for powers of (divergent or restricted) combinatorial charges)**.** *Let $\mathcal{Q}$ be a one-loop divergent local QFT, $\mathcal{H}_\mathcal{Q}$ its associated renormalization Hopf algebra, $v \in \mathcal{R}_\mathcal{Q}^{[0]}$ a vertex residue, $\mathbf{R} \in \mathbb{Z}^{\# \mathcal{R}_\mathcal{Q}^{[0]}}$ a residue-grading multi-index. Then, we have the following coproduct identities for powers of combinatorial charges and $m \in \mathbb{Q}$ a rational number:*[19]

$$\Delta \left(\mathfrak{Q}^{mv} \right) = \sum_{\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}_\mathcal{Q}^{[0]}}} \overline{\mathfrak{Q}}^{mv} \overline{\mathfrak{Q}}^{\mathbf{r}} \otimes \mathfrak{Q}^{mv}_{\mathbf{r}} \, , \tag{101a}$$

$$\Delta \left(\mathfrak{Q}^{mv}_{\mathbf{R}} \right) = \sum_{\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}_\mathcal{Q}^{[0]}}} \left(\overline{\mathfrak{Q}}^{mv} \overline{\mathfrak{Q}}^{\mathbf{r}} \right) \Bigg|_{\mathbf{R}-\mathbf{r}} \otimes \mathfrak{Q}^{mv}_{\mathbf{r}} \, , \tag{101b}$$

$$\Delta \left(\overline{\mathfrak{Q}}^{mv} \right) = \sum_{\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}_\mathcal{Q}^{[0]}}} \overline{\mathfrak{Q}}^{mv} \overline{\mathfrak{Q}}^{\mathbf{r}} \otimes \overline{\mathfrak{Q}}^{mv}_{\mathbf{r}} \tag{101c}$$

and

$$\Delta \left(\overline{\mathfrak{Q}}^{mv}_{\mathbf{R}} \right) = \sum_{\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}_\mathcal{Q}^{[0]}}} \left(\overline{\mathfrak{Q}}^{mv} \overline{\mathfrak{Q}}^{\mathbf{r}} \right) \Bigg|_{\mathbf{R}-\mathbf{r}} \otimes \overline{\mathfrak{Q}}^{mv}_{\mathbf{r}} \tag{101d}$$

Proof. Equations (101a) and (101b) follow by the following computations, using the linearity and multiplicativity of the coproduct and Proposition 4.3:

$$\begin{aligned}
\Delta \left(\mathfrak{Q}^{mv} \right) &= \big(\Delta \left(\mathfrak{Q}^{v} \right) \big)^m \\
&= \left(\sum_{\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}_\mathcal{Q}^{[0]}}} \overline{\mathfrak{Q}}^{v} \overline{\mathfrak{Q}}^{\mathbf{r}} \otimes \mathfrak{Q}^{v}_{\mathbf{r}} \right)^m \\
&= \left(\left(\overline{\mathfrak{Q}}^{v} \otimes \mathbb{I} \right) \left(\sum_{\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}_\mathcal{Q}^{[0]}}} \overline{\mathfrak{Q}}^{\mathbf{r}} \otimes \mathfrak{Q}^{v}_{\mathbf{r}} \right) \right)^m \\
&= \left(\overline{\mathfrak{Q}}^{mv} \otimes \mathbb{I} \right) \left(\sum_{\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}_\mathcal{Q}^{[0]}}} \overline{\mathfrak{Q}}^{\mathbf{r}} \otimes \mathfrak{Q}^{v}_{\mathbf{r}} \right)^m \\
&= \sum_{\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}_\mathcal{Q}^{[0]}}} \overline{\mathfrak{Q}}^{mv} \overline{\mathfrak{Q}}^{\mathbf{r}} \otimes \mathfrak{Q}^{mv}_{\mathbf{r}}
\end{aligned} \tag{102a}$$

[19] The power of an element in the renormalization Hopf algebra $\mathfrak{G} \in \mathcal{H}_\mathcal{Q}$ via a non-integer number $m \in \big(\mathbb{Q} \setminus \mathbb{Z} \big)$ is understood via the formal binomial series, i.e.

$$\mathfrak{G}^m = \sum_{n=0}^{\infty} \binom{m}{n} \left(\mathfrak{G} - \mathbb{I} \right)^n \, . \tag{100}$$

More generally, if the renormalization Hopf algebra is considered over the field $\mathbb{k}$, then the following statements are true for $m \in \mathbb{k}$.

and

$$
\begin{aligned}
\Delta\left(\mathfrak{Q}^{mv}_{\mathbf{R}}\right) = \left.\left(\sum_{\mathbf{r}\in\mathbb{Z}^{\#\mathcal{R}^{[0]}_{\mathcal{Q}}}} \overline{\mathfrak{Q}}^{mv}\overline{\mathfrak{Q}}^{\mathbf{r}} \otimes \mathfrak{Q}^{mv}_{\mathbf{r}}\right)\right|_{\mathbf{R}} \\
\sum_{\mathbf{r}\in\mathbb{Z}^{\#\mathcal{R}^{[0]}_{\mathcal{Q}}}} \left.\left(\overline{\mathfrak{Q}}^{mv}\overline{\mathfrak{Q}}^{\mathbf{r}}\right)\right|_{\mathbf{R}-\mathbf{r}} \otimes \mathfrak{Q}^{mv}_{\mathbf{r}}
\end{aligned}
\tag{103}
$$

Finally, Equations (101c) and (101d) follow by Equations (101a) and (101b) together with the assumption of $\mathcal{Q}$ being one-loop divergent and the application of Corollary 3.8. ■

5 Quantum gauge symmetries and subdivergences

In this section, we give a precise definition of quantum gauge symmetries (QGS) and proove, that they correspond to Hopf ideals in the associated renormalization Hopf algebra in Theorem 5.4. This means, that it is in principle compatible with multiplicative renormalization to choose for a priori different coupling constants the same Z-factor (exponentiated via a natural number, if necessary).

Definition 5.1 (Quantum gauge symmetries)**.** Let $\mathcal{Q}$ be a local QFT, $\mathbf{Q}_{\mathcal{Q}}$ its set of combinatorial charges and $\mathbf{q}_{\mathcal{Q}}$ its set of physical charges. Suppose, that $\#\mathbf{Q}_{\mathcal{Q}} > \#\mathbf{q}_{\mathcal{Q}}$, then we define the equivalence relation, which we will refer to as quantum gauge symmetry (QGS), via

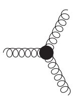
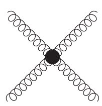
$$
\left(\overline{\mathfrak{Q}}^{v}\right)^{m} \sim \left(\overline{\mathfrak{Q}}^{w}\right)^{n} \quad \Longleftrightarrow \quad \operatorname{Cpl}\left(\mathfrak{Q}^{v}\right)^{m} \equiv \operatorname{Cpl}\left(\mathfrak{Q}^{w}\right)^{n} \tag{104}
$$

for all $v, w \in \mathcal{R}^{[0]}_{\mathcal{Q}}$ and $m, n \in \mathbb{N}_{>0}$. The set of all quantum gauge symmetries of $\mathcal{Q}$ is denoted via $\mathrm{QGS}_{\mathcal{Q}}$ and elements thereof are quadruples of the form $\{v, m; w, n\}$.

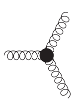
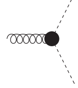
Example 5.2. Consider Quantum-Yang-Mills theory with Lagrange density

$$
\mathcal{L}_{\mathrm{QYM}} = -\frac{1}{4\mathrm{g}^2} F \wedge *F + \frac{1}{2\xi} \operatorname{Tr}\left((\delta A)^2\right) + \overline{c}\,(\delta \mathrm{d}c) + \mathrm{g}\overline{c} \operatorname{Tr}\left([\mathrm{d}c \wedge A]\right) . \tag{105}
$$

Then, we have

Cpl ((Q )^2) $=\mathrm{g}^2 = $ Cpl (Q ) (106)

and

Cpl (Q ) $=\mathrm{g} = $ Cpl (Q ) . (107)

Lemma 5.3 (Sums of Hopf ideals are Hopf ideals, quoted from [18])**.** *Let H be a Hopf algebra over a field $\Bbbk$ with characteristic zero and let $\{\mathfrak{i}_n\}_{n=1}^N$ be a set of N non-empty Hopf ideals, where $N \in \mathbb{N}_{\geq 1} \cup \infty$. Then the sum*

$$\mathfrak{i}_\Sigma := \sum_{n=1}^{N} \mathfrak{i}_n \, , \tag{108}$$

i.e. the ideal $\mathfrak{i}_\Sigma$ generated by $\Bbbk$-linear combinations of the generators of the Hopf ideals $\{\mathfrak{i}_n\}_{n=1}^N$, is also a Hopf ideal in H, i.e. $\mathfrak{i}_\Sigma$ satisfies:

1. $\Delta\left(\mathfrak{i}_\Sigma\right) \subseteq H \otimes \mathfrak{i}_\Sigma + \mathfrak{i}_\Sigma \otimes H$
2. $\hat{\mathbb{I}}\left(\mathfrak{i}_\Sigma\right) = 0$
3. $S\left(\mathfrak{i}_\Sigma\right) \subseteq \mathfrak{i}_\Sigma$

Proof. This follows directly by the linearity of the involved maps. ∎

Theorem 5.4 (Quantum gauge symmetries induce Hopf ideals)**.** *Given the situation of Definition 5.1 and a one-loop divergent local QFT $\mathcal{Q}$, then the ideals generated by quantum gauge symmetries*[20]

$$\mathfrak{i}_{\mathrm{QGS}_\mathcal{Q}}^{\{v,m;w,n\}} := \sum_{\mathbf{R} \in \mathbb{Z}^{\# \mathcal{R}_\mathcal{Q}^{[0]}}} \left\langle \overline{\mathfrak{X}}_{\mathbf{R}-mv}^{mv} - \overline{\mathfrak{X}}_{\mathbf{R}-nw}^{nw} \right\rangle_{\mathcal{H}_\mathcal{Q}} \tag{110}$$

are Hopf ideals in the (associated) renormalization Hopf algebra $\mathcal{H}_\mathcal{Q}$, as is their sum

$$\mathfrak{i}_{\mathrm{QGS}_\mathcal{Q}} := \sum_{\{v,m;w,n\} \in \mathrm{QGS}_\mathcal{Q}} \mathfrak{i}_{\mathrm{QGS}_\mathcal{Q}}^{\{v,m;w,n\}} \, , \tag{111}$$

i.e. the ideals $\mathfrak{i}_{\mathrm{QGS}_\mathcal{Q}}^{\{v,m;w,n\}}$ and $\mathfrak{i}_{\mathrm{QGS}_\mathcal{Q}}$ satisfy (where $\mathfrak{i}$ is used as a placeholder for both):

1. $\Delta\left(\mathfrak{i}\right) \subseteq \mathcal{H}_\mathcal{Q} \otimes \mathfrak{i} + \mathfrak{i} \otimes \mathcal{H}_\mathcal{Q}$
2. $\hat{\mathbb{I}}\left(\mathfrak{i}\right) = 0$
3. $S\left(\mathfrak{i}\right) \subseteq \mathfrak{i}$

Proof. We check conditions 1. to 3. for the ideals $\mathfrak{i}_{\mathrm{QGS}_\mathcal{Q}}^{\{v,m;w,n\}}$ on the level of generators and then conclude the second statement using Lemma 5.3:

[20] Actually, it is not necessary that $\{v,m;w,n\} \in \mathrm{QGS}_\mathcal{Q}$ is a quantum gauge symmetry — the statement is true for any $v, w \in \mathcal{R}_\mathcal{Q}^{[0]}$ and $m, n \in \mathbb{Z}$. Furthermore, we remark that the restrictions to the gradings are such, that

$$\operatorname{IntRes}\left(\overline{\mathfrak{X}}_{\mathbf{R}-mv}^{mv}\right) = \operatorname{IntRes}\left(\overline{\mathfrak{X}}_{\mathbf{R}-nw}^{nw}\right) . \tag{109}$$

1. This follows from Proposition 4.4:

$$\begin{aligned}\Delta\left(\overline{\mathfrak{Q}}^{mv}_{\mathbf{R}-mv}-\overline{\mathfrak{Q}}^{nw}_{\mathbf{R}-nw}\right) &= \sum_{\mathbf{r}\in\mathbb{Z}^{\#\mathcal{R}^{[0]}_{\mathcal{Q}}}}\left(\left.\left(\overline{\mathfrak{Q}}^{mv}\overline{\mathfrak{Q}}^{\mathbf{r}}\right)\right|_{\mathbf{R}-\mathbf{r}-mv}\otimes\overline{\mathfrak{Q}}^{mv}_{\mathbf{r}}\right.\\ &\qquad\left.-\left.\left(\overline{\mathfrak{Q}}^{nw}\overline{\mathfrak{Q}}^{\mathbf{r}}\right)\right|_{\mathbf{R}-\mathbf{r}-nw}\otimes\overline{\mathfrak{Q}}^{nw}_{\mathbf{r}}\right)\\ &= \sum_{\mathbf{r}\in\mathbb{Z}^{\#\mathcal{R}^{[0]}_{\mathcal{Q}}}}\left(\overline{\mathfrak{Q}}^{\mathbf{r}+mv}_{\mathbf{R}-\mathbf{r}-mv}\otimes\overline{\mathfrak{Q}}^{mv}_{\mathbf{r}}-\overline{\mathfrak{Q}}^{\mathbf{r}+nw}_{\mathbf{R}-\mathbf{r}-nw}\otimes\overline{\mathfrak{Q}}^{nw}_{\mathbf{r}}\right) \qquad (112)\\ &= \sum_{\mathbf{r}\in\mathbb{Z}^{\#\mathcal{R}^{[0]}_{\mathcal{Q}}}}\left(\overline{\mathfrak{Q}}^{\mathbf{r}}_{\mathbf{R}-\mathbf{r}}\otimes\overline{\mathfrak{Q}}^{mv}_{\mathbf{r}-mv}-\overline{\mathfrak{Q}}^{\mathbf{r}}_{\mathbf{R}-\mathbf{r}}\otimes\overline{\mathfrak{Q}}^{nw}_{\mathbf{r}-nw}\right)\\ &= \sum_{\mathbf{r}\in\mathbb{Z}^{\#\mathcal{R}^{[0]}_{\mathcal{Q}}}}\overline{\mathfrak{Q}}^{\mathbf{r}}_{\mathbf{R}-\mathbf{r}}\otimes\left(\overline{\mathfrak{Q}}^{mv}_{\mathbf{r}-mv}-\overline{\mathfrak{Q}}^{nw}_{\mathbf{r}-nw}\right)\end{aligned}$$

2. This follows immediately, as $\mathfrak{i}^{\{v,m;w,n\}}_{\mathrm{QGS}_{\mathcal{Q}}}\neq\emptyset$

3. This follows from Lemma 4.1 applied to 1.:

$$S\left(\overline{\mathfrak{Q}}^{mv}_{\mathbf{R}-mv}-\overline{\mathfrak{Q}}^{nw}_{\mathbf{R}-nw}\right)=-\sum_{\mathbf{r}\in\mathbb{Z}^{\#\mathcal{R}^{[0]}_{\mathcal{Q}}}}S\left(\overline{\mathfrak{Q}}^{\mathbf{r}}_{\mathbf{R}-\mathbf{r}}\right)\mathscr{P}\left(\overline{\mathfrak{Q}}^{mv}_{\mathbf{r}-mv}-\overline{\mathfrak{Q}}^{nw}_{\mathbf{r}-nw}\right) \qquad (113)$$

Finally, the ideal $\mathfrak{i}_{\mathrm{QGS}_{\mathcal{Q}}}$ is a Hopf ideal due to Lemma 5.3 applied to the previous argument. ■

Remark 5.5. Theorem 5.4 describes the most general situation. Slightly less general results in this direction can be found in [7, 8, 9, 10, 11, 28, 29, 31], some of them using the language of Hochschild cohomology.

Remark 5.6. The Hopf ideal $\mathfrak{i}_{\mathrm{QGS}}$ from Theorem 5.4 is defined such, that in the quotient $\mathcal{H}_{\mathcal{Q}}/\mathfrak{i}_{\mathrm{QGS}}$ the coproduct and antipode identities from Section 4, which are valid in the residue-grading, also hold for the coupling-grading. Thus it is possible to combine the Z-factors for the set $\mathbf{Q}_{\mathcal{Q}}$ to Z-factors for the set $\mathbf{q}_{\mathcal{Q}}$. However, we remark that in the quotient the residue-grading induces the coupling-grading, but they do not become equivalent, which would be the physical desirable situation. This will be studied in future work, as will be the relation between these symmetries and tree-identities.

6 Quantum gauge symmetries and renormalized Feynman rules

Having established, that quantum gauge symmetries (QGS) are compatible with the treatment of subdivergences and thus with multiplicative renormalization in Theorem 5.4, we now turn our attention to their relation with renormalized Feynman rules. We start this section with the definition of the gauge theory renormalization Hopf module, which implements the quantum gauge symmetries only on the left-hand side of the tensor product of the coproduct, i.e. only on the Feynman integrals contributing to the counterterm. More precisely, this corresponds to relations between the counterterms (i.e. Z-factors) of Greens functions related via quantum gauge symmetries. However, then we prove in Corollary 6.6, that under mild assumptions this already implies, that the renormalized Feynman rules coincide, i.e. instead of considering the gauge theory renormalization Hopf module, we can simply consider the renormalization Hopf algebra and take the quotient by the quantum gauge symmetry Hopf ideal, which coincides with recent calculations [32].

Definition 6.1 (Gauge theory renormalization Hopf module, [33])**.** Let $\mathcal{Q}$ be a local one-loop divergent QFT with quantum gauge symmetry, $\mathcal{H}_{\mathcal{Q}}$ its renormalization Hopf algebra and $\mathfrak{i}_{\text{QGS}}$ the corresponding Hopf ideal. Let

$$\pi : \mathcal{H}_{\mathcal{Q}} \twoheadrightarrow \mathcal{H}_{\mathcal{Q}} / \mathfrak{i}_{\text{QGS}} \tag{114}$$

denote the projection map. We consider $\mathcal{H}_{\mathcal{Q}} / \mathfrak{i}_{\text{QGS}}$ as a Hopf module over $\mathcal{H}_{\mathcal{Q}}$, with the usual multiplication restricted to the equivalence classes, the usual unit and counit as they are unaltered and the usual antipode restricted to the equivalence classes. The interesting map is the comodule map, defined as follows:

$$\delta := (\pi \otimes \operatorname{Id}) \circ \Delta \tag{115}$$

Then, we define the renormalized Feynman rules $\Phi_{\mathscr{R}}$ using the comodule map δ instead of the coproduct Δ, i.e. defining the counterterm map $S^{\Phi_{\mathscr{E}}}_{\mathscr{R}}$ on the quotient $\mathcal{H}_{\mathcal{Q}} / \mathfrak{i}_{\text{QGS}}$.

Corollary 6.2. *Given the situation of Definition 6.1, we have*

$$\delta \left(\mathfrak{X}^r \right) = \sum_{\mathbf{c} \in \mathbb{Z}^{\# \mathbf{Q}_{\mathcal{Q}}}} \overline{\mathfrak{X}}^r \overline{\mathfrak{Q}}^{\mathbf{c}} \otimes \mathfrak{X}^r_{\mathbf{c}} \, , \tag{116a}$$

$$\delta \left(\mathfrak{X}^r_{\mathbf{C}} \right) = \sum_{\mathbf{c} \in \mathbb{Z}^{\# \mathbf{Q}_{\mathcal{Q}}}} \left(\overline{\mathfrak{X}}^r \overline{\mathfrak{Q}}^{\mathbf{c}} \right) \Big|_{\mathbf{C} - \mathbf{c}} \otimes \mathfrak{X}^r_{\mathbf{c}} \, , \tag{116b}$$

$$\delta \left(\overline{\mathfrak{X}}^r \right) = \sum_{\mathbf{c} \in \mathbb{Z}^{\# \mathbf{Q}_{\mathcal{Q}}}} \overline{\mathfrak{X}}^r \overline{\mathfrak{Q}}^{\mathbf{c}} \otimes \overline{\mathfrak{X}}^r_{\mathbf{c}} \tag{116c}$$

and

$$\delta \left(\overline{\mathfrak{X}}^r_{\mathbf{C}} \right) = \sum_{\mathbf{c} \in \mathbb{Z}^{\# \mathbf{Q}_{\mathcal{Q}}}} \left(\overline{\mathfrak{X}}^r \overline{\mathfrak{Q}}^{\mathbf{c}} \right) \Big|_{\mathbf{C} - \mathbf{c}} \otimes \overline{\mathfrak{X}}^r_{\mathbf{c}} \, , \tag{116d}$$

where the left hand side of the tensor products are understood as representatives of the corresponding equivalence classes. Analogous results also hold in the cases of Propositions 4.3 and 4.4.

Proof. This follows from Proposition 4.2 and the fact, that on the quotient $\mathcal{H}_{\mathcal{Q}} / \mathfrak{i}_{\text{QGS}}$ the residue-grading induces the coupling-grading, c.f. Remark 5.6. This is the weakest requirement to define combinatorial charges for the equivalence classes coming from restricting the residue-grading to the coupling-grading. ■

Remark 6.3. Corollary 6.2 states, that the gauge theory renormalization Hopf module from Definition 6.1 possesses Hopf subalgebras in the sense of Definition 2.24 for coupling-grading, i.e. it allows multiplicative renormalization of local QFTs with quantum gauge symmetries if a compatible renormalization scheme $\mathscr{R}$ exists. Conversely, it is obvious by construction that this is the weakest requirement for multiplicative renormalization of quantum gauge symmetries.

Theorem 6.4 (Quantum gauge symmetries and renormalized Feynman rules)**.** *The gauge theory renormalization Hopf module from Definition 6.1 is compatible with renormalized Feynman rules,* *if the ideal* $\mathfrak{i}_{\mathrm{QGS}}$ *is in the kernel of the counterterm map, i.e. if for all* $\{v, m; w, n\} \in \mathrm{QGS}_{\mathcal{Q}}$ *we have*

$$S^{\Phi^\varepsilon_{\mathscr{E}}}_{\mathscr{R}}\left(\overline{\mathfrak{X}}^{mv}_{\mathbf{R}-mv}\right) = S^{\Phi^\varepsilon_{\mathscr{E}}}_{\mathscr{R}}\left(\overline{\mathfrak{X}}^{nw}_{\mathbf{R}-nw}\right) . \tag{117}$$

If the renormalization scheme $\mathscr{R}$ *is physical, c.f. Definition 2.28, then this is equivalent to*

$$\mathscr{R}\left[(\Phi \circ \mathscr{P})\left(\overline{\mathfrak{X}}^{mv}_{\mathbf{r}-mv}\right)\right] = \mathscr{R}\left[(\Phi \circ \mathscr{P})\left(\overline{\mathfrak{X}}^{nw}_{\mathbf{r}-nw}\right)\right] . \tag{118}$$

Proof. The first statement is equivalent to the well-definedness of the counterterm-map on the equivalence classes of the QGS-equivalence relation: Indeed, we have

$$\overline{\mathfrak{X}}^{mv}_{\mathbf{R}-mv} \simeq_{\mathrm{QGS}_{\mathcal{Q}}} \overline{\mathfrak{X}}^{nw}_{\mathbf{R}-nw} , \tag{119}$$

and thus Equation (117) ensures, that the counterterm-map can be uniquely defined on the corresponding equivalence classes. The second statement follows from the following argument: Using Theorem 5.4, we obtain

$$S^{\Phi^\varepsilon_{\mathscr{E}}}_{\mathscr{R}}\left(\overline{\mathfrak{X}}^{mv}_{\mathbf{R}-mv} - \overline{\mathfrak{X}}^{nw}_{\mathbf{R}-nw}\right) = - \sum_{\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}^{[0]}_{\mathcal{Q}}}} \mathscr{R}\left[S^{\Phi^\varepsilon_{\mathscr{E}}}_{\mathscr{R}}\left(\overline{\mathfrak{X}}^{\mathbf{r}}_{\mathbf{R}-\mathbf{r}}\right)(\Phi \circ \mathscr{P})\left(\overline{\mathfrak{X}}^{mv}_{\mathbf{r}-mv} - \overline{\mathfrak{X}}^{nw}_{\mathbf{r}-nw}\right)\right] , \tag{120}$$

which vanishes, if for all $\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}^{[0]}_{\mathcal{Q}}}$

$$\mathscr{R}\left[S^{\Phi^\varepsilon_{\mathscr{E}}}_{\mathscr{R}}\left(\overline{\mathfrak{X}}^{\mathbf{r}}_{\mathbf{R}-\mathbf{r}}\right)(\Phi \circ \mathscr{P})\left(\overline{\mathfrak{X}}^{mv}_{\mathbf{r}-mv} - \overline{\mathfrak{X}}^{nw}_{\mathbf{r}-nw}\right)\right] = 0 . \tag{121}$$

Since $\mathscr{R}$ is physical, its kernel consists only of convergent expressions. Therefore, Equation (121) vanishes, if

$$\mathscr{R}\left[(\Phi \circ \mathscr{P})\left(\overline{\mathfrak{X}}^{mv}_{\mathbf{r}-mv} - \overline{\mathfrak{X}}^{nw}_{\mathbf{r}-nw}\right)\right] = 0 , \tag{122}$$

since $S^{\Phi^\varepsilon_{\mathscr{E}}}_{\mathscr{R}}\left(\overline{\mathfrak{X}}^{\mathbf{r}}_{\mathbf{R}-\mathbf{r}}\right)$ is either 1, if $\overline{\mathfrak{X}}^{\mathbf{r}}_{\mathbf{R}-\mathbf{r}} = \mathbb{I}$, or it evaluates to a divergent expression, which is not in the kernel of $\mathscr{R}$. ■

Remark 6.5. Equations (117) and (118) from Theorem 6.4 are criteria for both, the unrenormalized Feynman rules Φ and the renormalization scheme $\mathscr{R}$, as they state, that the $\mathscr{R}$-divergent contributions of the Feynman rules Φ have to coincide.

Corollary 6.6 (Quantum gauge symmetries and renormalized Feynman rules)**.** *Given the situation of Theorem 6.4, the following two equations are equivalent:*

$$\Phi_{\mathscr{R}}\left(\overline{\mathfrak{X}}^{mv}_{\mathbf{r}-mv}\right) = \Phi_{\mathscr{R}}\left(\overline{\mathfrak{X}}^{nw}_{\mathbf{r}-nw}\right) \tag{123}$$

and

$$\Phi\left(\overline{\mathfrak{X}}^{mv}_{\mathbf{r}-mv}\right) = \Phi\left(\overline{\mathfrak{X}}^{nw}_{\mathbf{r}-nw}\right) . \tag{124}$$

Proof. Again, using Theorem 5.4 and the same reasoning as in the proof of Theorem 6.4, we obtain

$$\Phi_{\mathscr{R}}\left(\overline{\mathfrak{X}}^{mv}_{\mathbf{R}-mv} - \overline{\mathfrak{X}}^{nw}_{\mathbf{R}-nw}\right) = \sum_{\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}^{[0]}_{\mathcal{Q}}}} S^{\Phi^\varepsilon_{\mathscr{E}}}_{\mathscr{R}}\left(\overline{\mathfrak{X}}^{\mathbf{r}}_{\mathbf{R}-\mathbf{r}}\right) \Phi\left(\overline{\mathfrak{X}}^{mv}_{\mathbf{r}-mv} - \overline{\mathfrak{X}}^{nw}_{\mathbf{r}-nw}\right) , \tag{125}$$

which vanishes, if for all $\mathbf{r} \in \mathbb{Z}^{\# \mathcal{R}_{\mathcal{Q}}^{[0]}}$

$$S_{\mathscr{R}}^{\Phi_{\mathscr{E}}^{\varepsilon}} \left(\overline{\mathfrak{Q}}_{\mathbf{R}-\mathbf{r}}^{\mathbf{r}} \right) \Phi \left(\overline{\mathfrak{Q}}_{\mathbf{r}-mv}^{mv} - \overline{\mathfrak{Q}}_{\mathbf{r}-nw}^{nw} \right) = 0 \, , \tag{126}$$

which, as $S_{\mathscr{R}}^{\Phi_{\mathscr{E}}^{\varepsilon}} \left(\overline{\mathfrak{Q}}_{\mathbf{R}-\mathbf{r}}^{\mathbf{r}} \right) \neq 0$, is equivalent to

$$\Phi \left(\overline{\mathfrak{Q}}_{\mathbf{r}-mv}^{mv} \right) = \Phi \left(\overline{\mathfrak{Q}}_{\mathbf{r}-nw}^{nw} \right) \, . \tag{127}$$

■

Remark 6.7. Corollary 6.6 states, that if the Feynman rules Φ are compatible with the quantum gauge symmetries themselves, then this holds even for the renormalized Feynman rules $\Phi_{\mathscr{R}}$ independently of the chosen renormalization scheme $\mathscr{R}$. This statement shows the well-definedness of renormalized amplitudes for Yang-Mills theories obtained from scalar QFT with cubic interaction via the Corolla polynomial and differential, c.f. [12, 13, 14, 15, 16, 17]. However, a well-behaved renormalization scheme $\mathscr{R}$ (in the sense of Remark 6.5) is needed in order to obtain well-defined Feynman rules on the equivalence classes of the quotient $\mathcal{H}_{\mathcal{Q}} / i_{\text{QGS}}$.

7 Conclusion

We studied the effect of gauge symmetries of a classical gauge theory to the renormalization of its corresponding quantum gauge theory. The main results are Theorem 5.4 and Theorem 6.4, where the first states, that quantum gauge symmetries correspond to a Hopf ideal inside the associated renormalization Hopf algebra and the second provides a criterion for their validity on the level of Feynman rules. In future work, we will consider the relation between residue-grading and coupling-grading on the quotient $\mathcal{H}_{\mathcal{Q}} / i_{\text{QGS}}$ and investigate their relation with tree-identities.

Acknowledgments

It is my pleasure to thank Dirk Kreimer and Henry Kißler for illuminating and helpful discussions! This research is supported by the International Max Planck Research School for Mathematical and Physical Aspects of Gravitation, Cosmology and Quantum Field Theory.

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