

# Optimal utility and probability functions for agents with finite computational precision

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## Abstract

When making economic choices, such as those between goods or gambles, humans act as if their internal representation of the value and probability of a prospect is distorted away from its true value. These distortions give rise to decisions which apparently fail to maximise reward, and preferences that reverse without reason. Why would humans have evolved to encode value and probability in a distorted fashion, in the face of selective pressure for reward-maximising choices? Here, we show that under the simple assumption that humans make decisions with finite computational precision – in other words, that decisions are irreducibly corrupted by noise – the distortions of value and probability displayed by humans are approximately optimal in that they maximise reward and minimise uncertainty. In two empirical studies, we manipulate factors that change the reward-maximising form of distortion, and find that in each case, humans adapt optimally to the manipulation. This work suggests an answer to the longstanding question of why humans make “irrational” economic choices.

## Introduction

Utility theories describe how economic choices are made under risk (1). The expected utility of a risky prospect is a function of its value  $x$  and probability  $p$  (2). Utility functions characterise the potentially nonlinear transformations that  $x$  and  $p$  undergo when humans make economic choices, such as deciding among monetary gambles. For example, the subjective expected utility  $U_i$  of gamble  $i$  offering monetary amount  $x_i$  with probability  $p_i$  might be described by the function

$$U_i = v(x_i) \cdot w(p_i) \quad [1]$$

where  $v(x)$  and  $w(p)$  are psychometric transduction functions. Faced with uncertain prospects, human participants will often make choices that fail to maximise expected value (2, 3). For example, many people will prefer \$3000 with certainty over an 80% chance of winning \$4000, even though the uncertain sum has higher expected value. Human preferences also tend to reverse irrationally with irrelevant factors. For example, an agent who prefers the certain sum above might well opt for a 0.2 chance of \$4000 over a 0.25 chance of \$3000 – even though this gamble is identical except for a rescaling of the probabilities by a factor of 1/4. Utility theories typically propose variants of  $v(x)$  and  $w(p)$  that can capture violations of rationality such as this, allowing researchers to build predictive models of human economic choice (4). However, the resulting models *describe* rather than *explain* the policies that humans adopt when making risky decisions. Here, instead, we seek a normative account of the idiosyncratic forms of the empirically measured functions  $v(x)$  and  $w(p)$ , under the assumption that human decisions are corrupted by irreducible noise in neural computation (5–7).

While the precise form of  $v(x)$  and  $w(p)$  that best capture human choices remains controversial, there is consensus over several points. Firstly,  $v(x)$  is a compressive nonlinearity, such as a (sign-conserving) power law function, that inflects around zero, or the status quo wealth. For example, it is often assumed that

$$v(x) = x^\kappa \quad [2]$$

in the domain of gains. A number of different forms of  $w(p)$  have been proposed, which mostly assert that the probability function approximates an inverse s-shape, i.e. is largely convex but with an initial concavity, often up to a fixed point around  $p \sim 0.3$ . For example, one popular model that conforms to these requirements proposes that

$$w(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}} \quad [3]$$

where typically  $\gamma$  is empirically found to be on the order of 0.6–0.8 (8–10).

Our goal here is not to debate the empirical form of  $v(x)$  and  $w(p)$ . Instead, we ask a different question, which pertains to the processing limits of cognitive function, and consequently to the intrinsic variability of human choices. Assuming Gaussian noise in the internal representation of decision information, the probability of choosing gamble 1 over 2 is

$$q(g_1) = \Phi[U_1 - U_2] \quad [4]$$

where  $\Phi[\cdot]$  is a logistic function with inverse slope  $\sigma$ . The parameter  $\sigma$  scales with the inverse computational precision of human choices: decisions made under higher values of  $\sigma$  are more prone to error. In standard econometric accounts,  $\sigma$  is treated as a nuisance; it simply allows for random normal variability in decisions, consistent with the ubiquitously observed ogival form of psychometric functions. We similarly assume here that  $\sigma$  is an inevitable feature of our psychological apparatus, defined as an irreducible “late” noise term or bound on the precision of information processing. We then ask a simple question: given a fixed processing capacity which constrains  $\sigma$ , what is the reward-maximising form of  $v(x)$  and  $w(p)$ ? We address this optimisation problem, and in doing so, attempt to derive the minimal assumptions it is necessary to make about the quantity that humans are seeking to optimise during economic choices, in order for the canonically described form(s) of the functions  $v(x)$  and  $w(p)$  to be optimal, i.e. reward maximising (and/or uncertainty minimising) for the agent.

## Results

Consider an agent choosing among a pair of lotteries. Each lottery  $[x_i, p_i]$  involves an opportunity to obtain a monetary sum  $x_i$  with probability  $p_i$ , or otherwise nothing. We denote the binary choice between two lotteries as  $[x_1, p_1; x_2, p_2]$ . Next, we define a quantity  $Y_i$  that indicates the quality of each lottery with respect to an externally defined objective function, i.e. according to an assumption about what is “optimal” for humans. In the simplest case, we assume that this objective corresponds to the expected value of lottery  $i$  so that  $Y_i = EV_i = x_i \cdot p_i$  and thus humans are simply wealth maximising. We highlight the difference between  $U$  and  $Y$ : in our formulation  $U$  is an internal psychological quantity that is calculated from distorted probability and magnitude of each lottery, whereas  $Y$  is defined externally and depends on our assumptions about what is optimal.

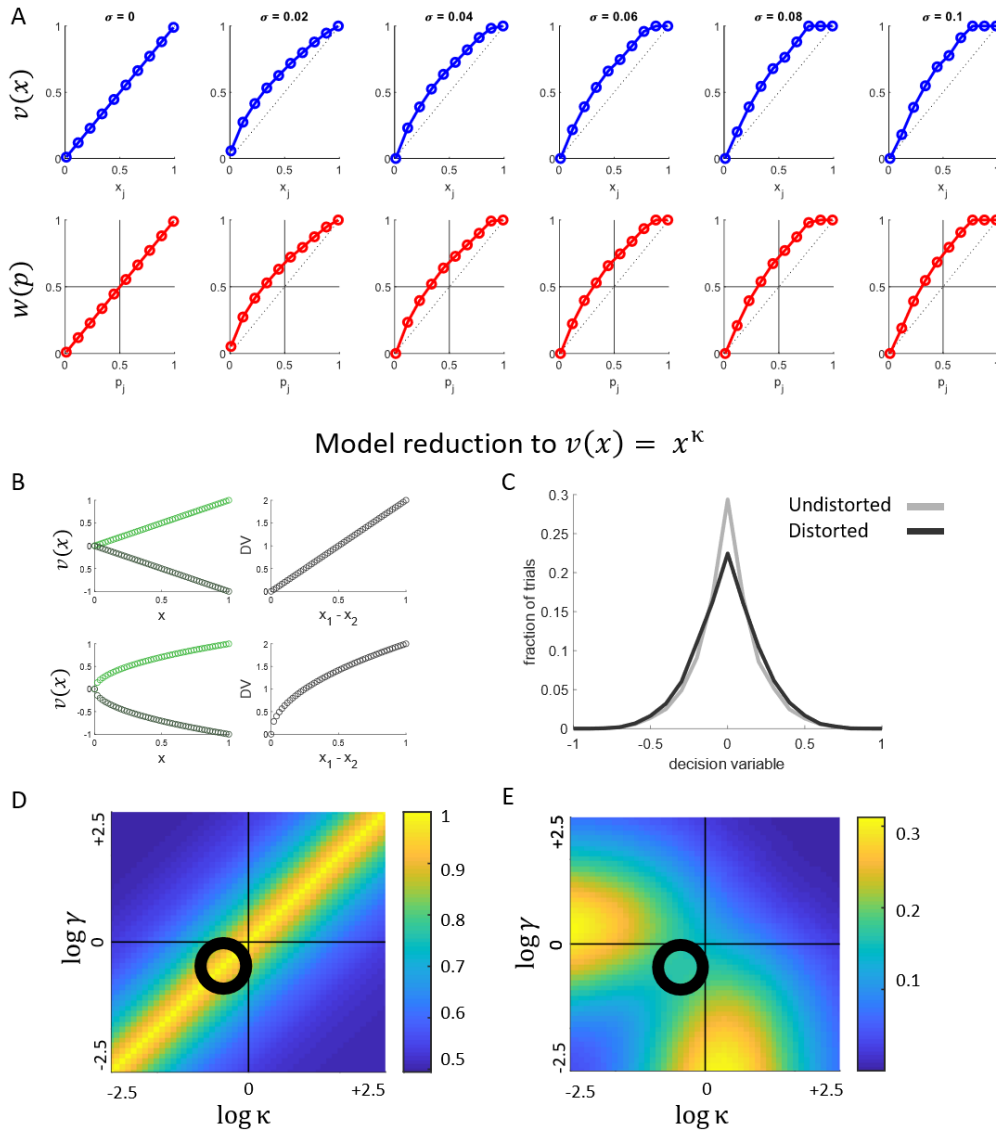
The loss function over  $n$  choices is computed as follows, where  $q(g_{i,n})$  is the probability of choosing lottery  $i$  on trial  $n$ :

$$L = -\frac{1}{n} \sum_1^n q(g_{1,n}) \cdot Y_{1,n} + q(g_{2,n}) \cdot Y_{2,n} \quad [5]$$

Critically, we also assume that agents make decisions with finite precision; in other words, decisions are corrupted by irreducible Gaussian noise at the level of evaluation or choice.

To begin with, we relax the assumptions that typify classical utility theories and treat this as an unconstrained optimisation problem. We assume a set of choices for which magnitudes  $x_i$  (e.g.

gamble outcomes in currency units) are drawn uniformly in the range  $[0,1]$  and probabilities  $p_i$  span the full range  $[0,1]$ . We define each of  $j$  intervals into which  $x$  and  $p$  may fall, and estimate freely decision coefficients  $c_j^x$  and  $c_j^p$  for each interval (here,  $j = 10$ ) that minimise the loss term in eq. 4. We solve the optimisation problem separately for different values of  $\sigma \in [0:0.02:0.1]$  (see methods). The resulting coefficients  $c_j^x$  (upper panels) and  $c_j^p$  (lower panels) for each (fixed) value of late noise  $\sigma$  (columns) are plotted in Fig. 1a.



**Fig. 1. A.** Optimal (i.e. reward-maximising) values for  $v(x)$  and  $w(p)$  as derived from eq. 4 under the assumption that  $Y_i = x_i \cdot p_i$  (y-axis) plotted against their untransformed counterparts (x-axis) under variable levels of decision noise (columns). **B.** Illustration of the effect of compression (in terms of power law distortion with exponent kappa ( $\kappa$ )) on decisions. Top panels: under  $\kappa = 1$ , the decision variable (DV) reflecting the relative utility of the lotteries is a linear function of differences in  $x$ . Lower panels: under  $\kappa = 0.5$ , smaller values of the decision variable are inflated away from zero, rendering these choices more robust to noise. **C.** Correspondingly, the distribution of decision values for the reward-maximising parameterisation under the double exponent model (black line) is broader than under no distortion (grey line). **D.** Probability that  $\text{sign}(U_1 - U_2) = \text{sign}(Y_1 - Y_2)$  for different values of  $\log(\kappa)$  and  $\log(\gamma)$  under the double-exponent model. **E.** Absolute decision values, i.e. relative expected value  $|U_1 - U_2|$  for different values of  $\log(\kappa)$  and  $\log(\gamma)$  under the double-exponent model. In B and C, the black circle indicates the reward-maximising values of  $\log(\kappa)$  and  $\log(\gamma)$ .

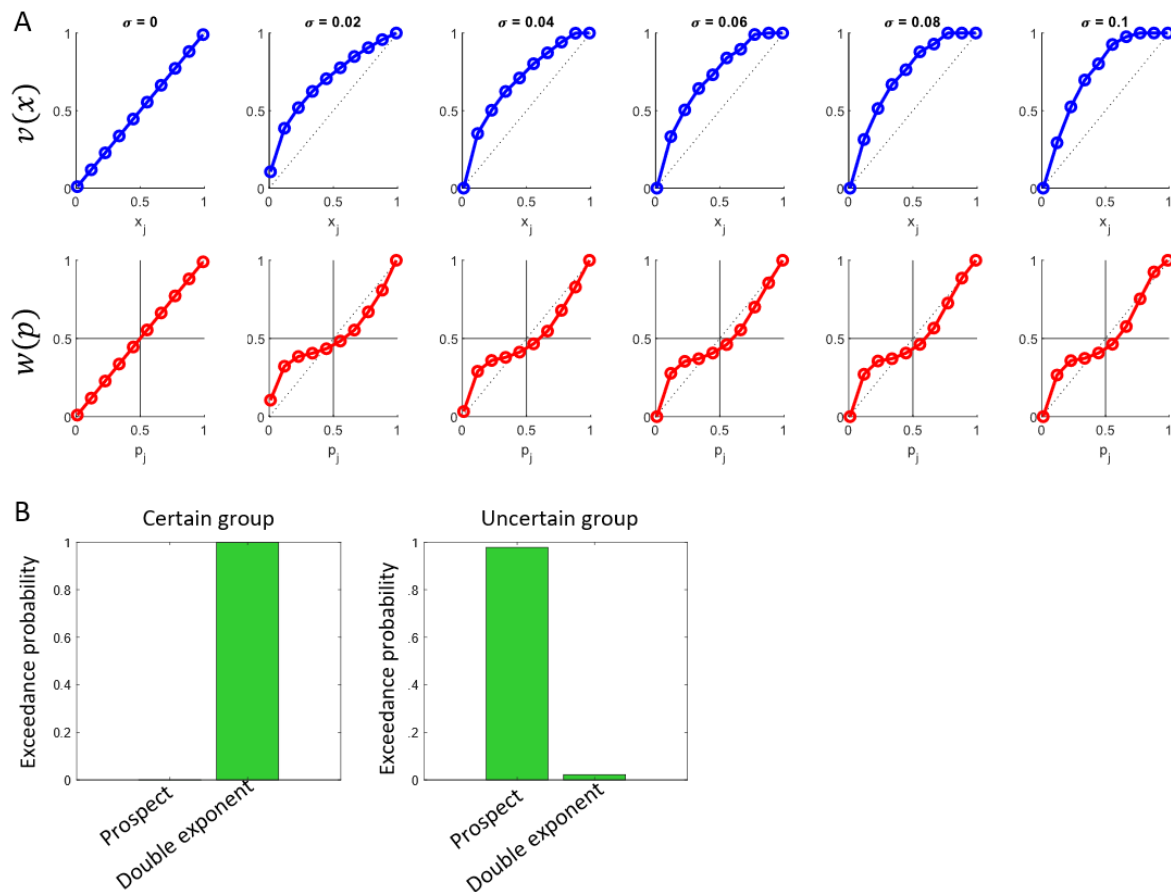
We highlight several features of these data. Firstly, in the case where  $\sigma = 0$  (leftmost panel of **Fig. 1a**) the reward-maximising policy is to recover parameters  $c_j^x$  and  $c_j^p$  that are identical to their untransformed counterparts  $x_j$  and  $p_j$ . This simply means that in the absence of decision noise, expected value can trivially be maximised by multiplying  $x$  and  $p$ . More interestingly, however, when we assume finite computational precision ( $\sigma > 0$ ) then the policy that maximises expected value is distorted away from the identity line. In fact, both the reward-maximising value function  $v(x)$  and weighting function  $w(p)$  take the form of a compressive nonlinearity (akin to a power function with exponent  $\kappa < 1$ ), such that an optimal observer will magnify differences between lower magnitudes and between lower probabilities, relative to their higher counterparts. Thus, a compressive non-linearity of the form presented in eq. 2 provides a compact description of the optimal decision weights. To corroborate this claim, we repeated our simulations under the assumption of power-law transducer functions (eq. 2), again finding a compressive non-linearity that mimics the optimal agent behaviour identified in free-fitting.

This might seem counterintuitive, but it follows naturally from a consideration of where a limited resource (e.g. selective attention, or neuronal firing rates) can best be allocated to maximise rewards. We illustrate in **Fig. 1b-e**. The decision variable on which a choice is based is jointly determined by the difference in the utility of the gambles, i.e. will depend on  $+U_1$  and  $-U_2$ . A compressive value or probability function that distorts  $U$  away from  $Y$  will increase the probability that the lottery with lower objective value will be mistakenly chosen, i.e. distortion increases the number of “sign-flipped” gambles where  $p[\text{sign}(U_1 - U_2) \neq \text{sign}(Y_1 - Y_2)]$ . Thus, in the absence of decision noise, linear transducers maximise reward. However, an auxiliary effect of the compressive function is to increase the spread of the decision variable  $U_1 - U_2$  around zero, including for the majority of cases where the decision sign is not flipped. This ensures that small decision values are inflated away from the indifference point, rendering them more robust to decision noise and less likely to result in suboptimal choices. Similar phenomena, including the compressive form of the reward-maximising transducer under “late” decision noise (11), have been reported elsewhere (7, 12).

Whilst the imposition of a compressive nonlinearity has been a hallmark of the utility function  $v(x)$  since Bernoulli (13), the compressive form of the optimal probability weighting function shown in **Fig.1** (lower panels) is rarely observed (although see the proposal in (14), and the empirical data recorded from rodents in (15) for exceptions). Instead, a more typical form for  $w(p)$  is an inverse s-shape with an inflection below the midpoint. However, the curves above were derived under the assumption that humans simply wish to maximise expected value. Whilst this is desirable, it is also sensible to minimise the uncertainty associated with a given choice. For example, by taking actions whose outcome is predictable in advance, it is possible to formulate plans for the future, and potentially to avoid deleterious negative outcomes that accrue from overly risky choices (16). We thus considered a scenario in which the observer evaluates gambles according to a different objective value  $Y_i = x_i \cdot p_i \cdot (1 - H_i)$  where the new term  $H_i$  is the Shannon entropy of the lottery (with base  $e$ ):

$$H_i = -p_i \cdot \log(p_i) - (1 - p_i) \cdot \log(1 - p_i)$$

Next, thus, we assume that human choices seek to maximise reward and minimise outcome uncertainty. Using this new approach, we recomputed the optimal functions in an unconstrained fashion. The results are shown in Fig.2a.

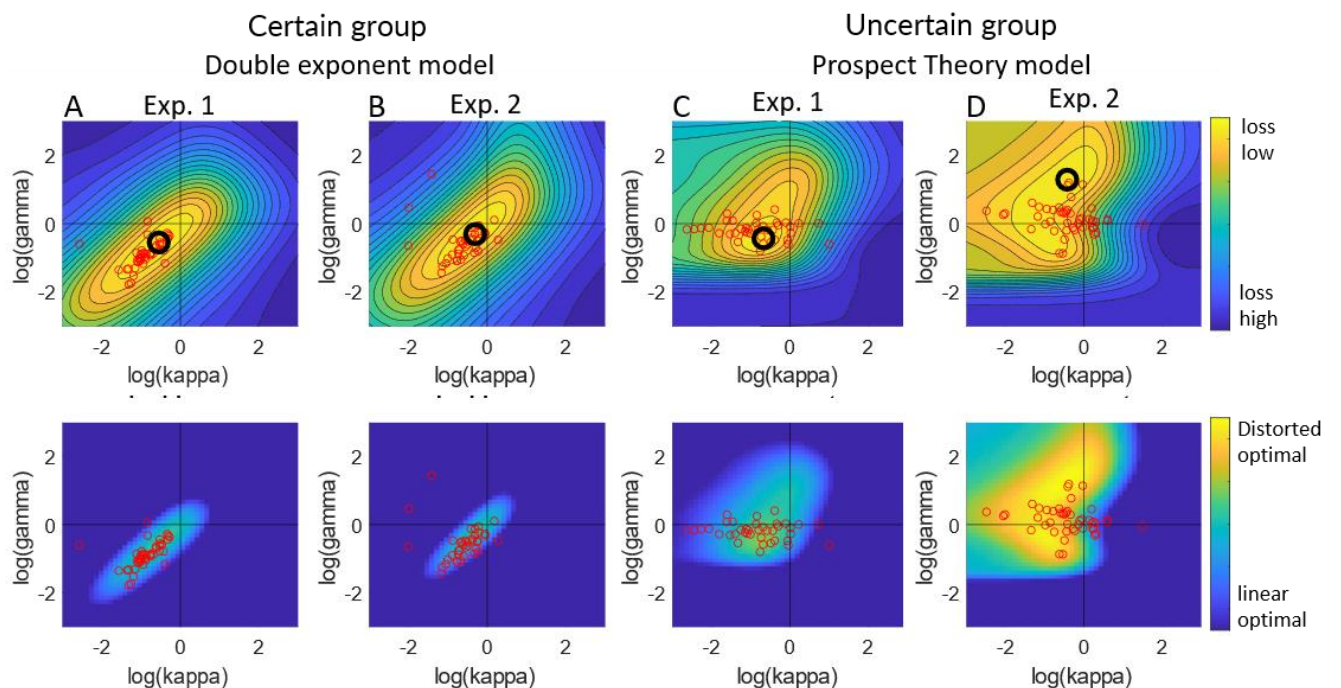


**Fig. 2. A.** Optimal (i.e. reward-maximising) values for  $v(x)$  and  $w(p)$  as derived from eq. 4 under the assumption that  $Y_i = x_i \cdot p_i \cdot (1 - H_i)$  (y-axis) plotted against their untransformed counterparts (x-axis) under variable levels of decision noise (columns). **B.** Exceedance probabilities for the Prospect Theory model and double-exponent model in the certain outcome condition (left panel) and uncertain outcome condition (right panel).

As can be seen, under these assumptions we can recover (i) the compressive form of the value function for the domain of gains, (ii) the inverse-s shape of the probability weighting function, and (iii) the inflection around  $p \sim 0.3$ . These functions clearly resemble the canonical utility functions  $v(x)$  and  $w(p)$  (8, 17). To verify this contention, we used a model mimicry approach, fitting several classical utility models to these freely derived optimal functions. Under the reward-maximising objective  $Y_i = x_i \cdot p_i$  both  $w(p)$  and  $v(x)$  were well captured by a power law function with an exponent of less than one; whereas when risk minimisation was an additional concern, the functions were best described by classical forms of  $w(p)$ , such as that shown in eq.3 or that proposed by Prelec (18). Together, these simulations thus imply that (i) when decisions are made with finite computational precision, distorted utility functions are optimal; and (ii) if we assume that human participants seek to maximise reward and minimise entropy, then a distortion of the form predicted by classical utility theory is optimal.

However, these considerations also allow us to make new predictions about human behaviour. Near-optimal probability weighting functions differ according to whether the agent is maximising reward only, or whether she is additionally minimizing uncertainty. Thus, when choices carry no risk, we should expect human behaviour to more closely resemble a model where both value and probability are compressive. In the following we will term this the “double exponent model”. By contrast, when choices carry risk, we should expect behaviour to resemble that of classical models, where  $w(p)$  follows an inverse s-shape – we term this the “Prospect Theory” model. Thus, we should expect the form of the probability weighting function to vary according to whether outcomes are risky or not, whereas the value function should not be affected by this manipulation.

We tested this prediction in two cohorts of human participants ( $n = 200$  total) who made incentive-compatible decisions about financial lotteries (in the domain of gains) of the form  $[x_1, p_1; x_2, p_2]$ . In each of two experiments, one group received the chosen gamble value  $x_c$  with probability  $p_c$  and zero with probability  $1 - p_c$  (uncertain outcome condition,  $n = 99$ ) the other group always received the expected value of the chosen gamble  $x_c \cdot p_c$  (certain outcome condition,  $n = 101$ ). Experiments 1 and 2 were identical except that on each trial in Exp.1,  $x$  and  $p$  were drawn randomly for each gamble, whereas in Exp.2 the probability of one of the gambles (e.g.  $p_2$  but not  $p_1$ ) was set to certainty but the expected values were otherwise matched to Exp.1 (see methods). Our model predicts that  $w(p)$  and  $v(x)$  will both be captured by power law functions (the “double-exponent model”) in the certain outcome condition, but by classical utility theory in the uncertain outcome condition, even though the expected value of every gamble is matched between certain and uncertain outcome groups. This is exactly what we found (Fig. 2b). Using Bayesian model selection (19) on cross-validated model fits to human data, we found that the former fit the data best in the certain outcome condition (exceedance probability  $[xP] > 0.99$ ) and the latter fit better in the uncertain outcome condition ( $xP > 0.95$ ).

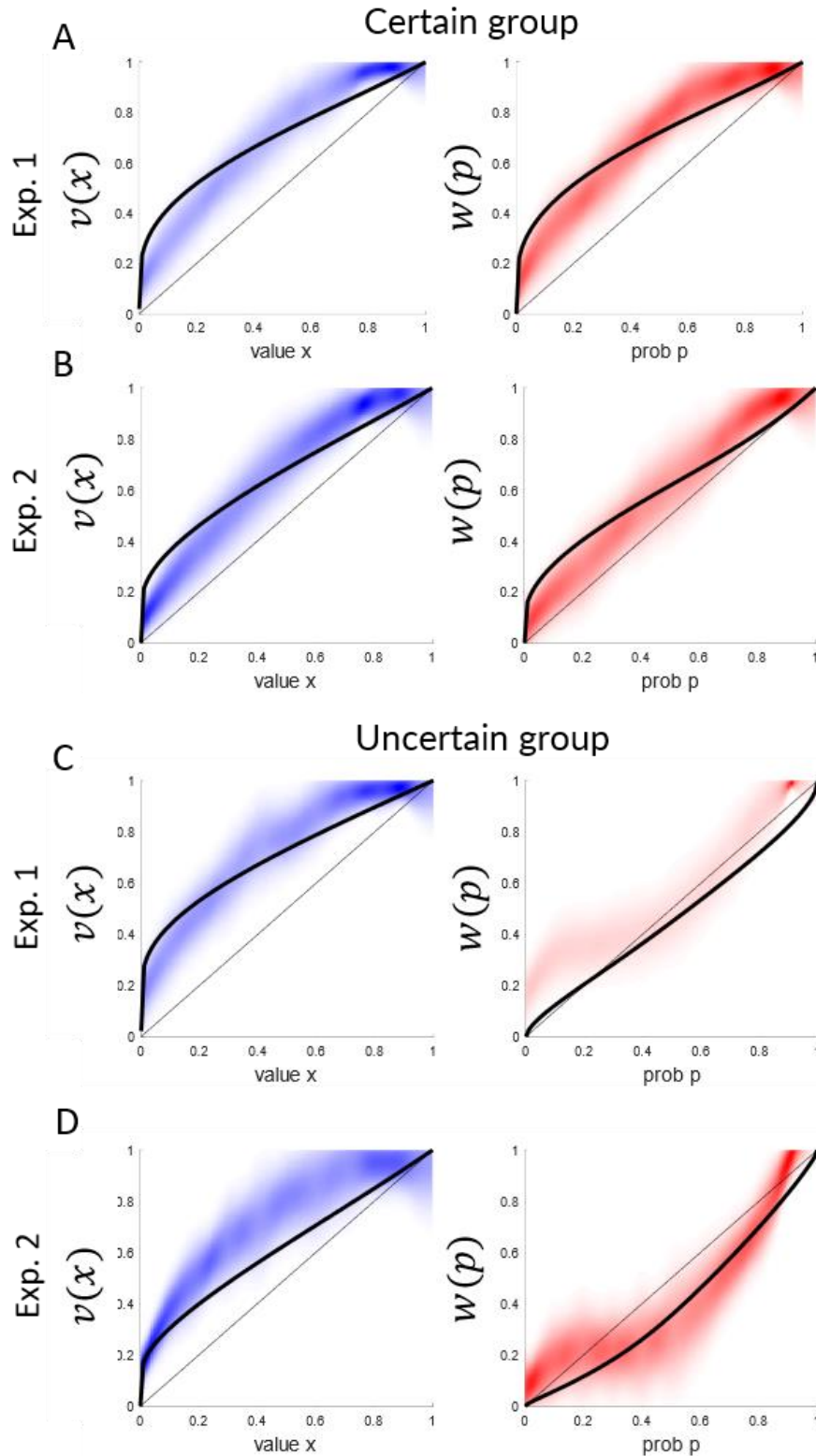




**Fig. 3.** Upper panels: Loss landscapes showing parameterisations of the winning model in each group that maximise reward (for the certain outcome condition; panels A [Exp.1] and B [Exp.2]) and that maximise reward/minimise risk (for the uncertain outcome condition; panels C [Exp.1] and D [Exp.2]). Warmer colours signal parameterisations that are closer to optimal, and the black circle shows the maximum. Red circles show parameter estimates for individual human participants in each experiment (panel A-B: double exponent model; panel C-D: PT model). Note how humans cluster in the same quadrant as the maximum; see statistics below. Lower panels: the difference in loss between the distorted model and a linear model, i.e. a model with the same estimated noise but with  $\log(\kappa) = 0$  and  $\log(\gamma) = 0$ . Red circles are individual participants. Warmer colours show regions where the distortion increases return over the linear model.

We conducted our next analysis under the assumption that the double-exponent and Prospect theory (PT) models provided a good reduction of the human policy in the certain outcome condition and uncertain outcome conditions respectively. Both models assume that  $v(x) = x^\kappa$  (in the domain of gains explored here) but make different assumptions about  $w(p)$ : the double exponent model assumes that  $w(p) = p^\gamma$  whereas PT assumes the probability weighting function shown in eq.2. This allowed us to plot the (negative) loss landscape  $-\mathcal{L}$  under each parameterisation (defined by  $\kappa$  and  $\gamma$ ) for a theoretical agent exhibiting the mean level of decision noise estimated across the cohort (**Fig. 3**) and to compare it to the best-fitting parameters for human participants. The surfaces for the certain outcome condition (**Fig. 3a-b**) confirm that the reward-maximising policy for an agent with the same average noise as our participants is to transduce  $x$  and  $p$  with compressive functions, so that the highest return is obtained for  $\log(\kappa) < 0$  and  $\log(\gamma) < 0$  (warm colour shading; black circle is the maximum given human levels of noise). For the uncertain outcome conditions (**Fig 3c-d**), the optimal value of  $\log(\kappa)$  was always negative but the optimal value of  $\gamma$  varied with experiment: for Exp.1, rewards are maximised with  $\log(\gamma) < 0$  but for Exp.2,  $\log(\gamma) > 0$  is optimal (note that in the case of the uncertain outcome condition we use the term “optimal” to refer to those solutions that jointly maximise reward and minimise risk, as proposed above). This pattern of predictions closely resembles what was observed from human participants. In each case, the human parameters (red circles in Fig.3) fall in the area that minimises the relevant objective function. Moreover, as can be seen in the lower panels, the human parameters were those for which the expected return from distorted transducers exceeds the expected return from a linear transducer to the greatest extent.

These intuitions were confirmed by statistical comparison. Parameters  $\log(\kappa)$  fell reliably below zero in both conditions of both experiments (Exp 1 Certain Outcome:  $t = 12.5$ ; Exp 2 Certain Outcome:  $t = 8.41$ ; Exp 1 Uncertain Outcome:  $t = 7.17$ ; Exp 2 Uncertain Outcome:  $t = 3.87$ ; all p-values  $< 0.001$ ). Parameters  $\log(\gamma)$  fell reliably below zero in both Certain Outcome conditions (Exp 1:  $t = 11.6$ ; Exp 2:  $t = 6.67$ ; both p-values  $< 0.001$ ). However, in the Uncertain Outcome condition  $\log(\gamma)$  was significantly below zero in Exp 1 ( $t = 4.2$ ,  $p < 0.001$ ) and above zero in Exp2 ( $t = 2.46$ ,  $p < 0.02$ ). These results are consistent with model predictions in each of the cases tested.



**Fig. 4.** Each panel shows the reward-maximising form of the functions  $v(x)$  (blue shading) and  $w(p)$  (red shading). Each function is expressed as a density over optimal estimates derived from each participant in the certain outcome condition (A [Exp.1] and B [Exp.2]) and the uncertain outcome condition (C [Exp.1] and D [Exp.2]). Optimal estimates vary from participant to participant because of distinct noise levels and variation in lottery sampling. Superimposed on each reward-maximising function is the form of the distortion that best fit human choices (estimated from median parameters, shown on each plot). This was estimated from the double-exponent model for the certain outcome condition (A and B) and from Prospect Theory for the uncertain outcome condition (C and D).

A further comparison of the human and reward-maximising value and probability functions is shown in **Fig.4**. Here, we use the unconstrained model to freely estimate the reward-maximising coefficients for each participant individually. The optimal coefficients vary across the cohort as individuals differ in their levels of estimated internal noise, and in the sample of

lotteries they viewed. We plot the distribution of coefficients obtained across the cohort for  $v(x)$  [blue shading] and  $w(p)$  [red shading] and superimpose on top the best-fitting value and probability functions from the double exponent model (certain outcome condition) and Prospect Theory (uncertain outcome condition). As can be seen, human distortions resemble the optimal solution qualitatively in each case tested, although they are slightly weaker than optimal in the uncertain outcome condition.

The thesis advanced here is that distortions in the subjective representation of value and probability are reward-maximising. Our theory thus makes predictions about the relationships among observed parameters themselves, and how they relate to performance. Firstly, because the theory predicts that distortions have a common origin in reward-maximisation (at least in the certain outcome condition), the parameters  $\kappa$  and  $\gamma$  should be correlated: those participants with greater distortion for value should also have greater distortion for probability. Empirically, this is what we found in the certain outcome condition (both r-values  $> 0.3$ , both p-values  $< 0.001$ ) but not the uncertain outcome condition (both p-values  $> 0.1$ ). This follows from the assumption that in the uncertain outcome condition  $\kappa$  controls compression of value in the service of reward maximisation, whereas  $\gamma$  largely reflects risk sensitivity by controlling the inverse s-shape of the probability weighting function in eq. 3.

## Discussion

The nature of the internal representations that guide economic choices has been a question of longstanding interest for psychologists, economists and neuroscientists. Here, we shed new light on this question by taking a normative rather than a descriptive perspective. We asked why humans behave as if they distorted estimates of probability and magnitude when making economic decisions. We suggest that they do so because in the presence of decision noise, distorted functions yield higher reward (and sometimes lower risk) than undistorted functions. Moreover, we show that the precise form of the distortions commonly observed in economic choice tasks (e.g. where humans make binary choices among lotteries) can be explained under three simple and uncontroversial assumptions: (i) human decisions are made with finite computational precision; (ii) humans wish to maximise expected value; (iii) humans wish to minimise risk (or outcome variance). Our empirical work shows that this account explains the effects of switching between otherwise equivalent safe and risky outcomes and can even explain a reversal in the shape of the probability weighting function that occurs when one sum is offered with certainty (Exp 2) or not (Exp 1).

We note that our explanations for the form of the utility functions differ sharply from those proposed previously. A common view is that decisions are irrational because observers have a preference for “fast and frugal” computation, i.e. they are willing to sacrifice accuracy for speed when evaluating decision-relevant information (20). With regard to the value function, a compressive nonlinearity has typically been interpreted as implying an aversion to risk, because it accounts for the oft-observed preference for lottery  $[x, 1]$  over  $[\frac{x}{k}, k]$  where  $k < 1$ , i.e. the preference for certainty equivalents that are matched to the expected value of a risky gamble. We suggest, instead, that the compressive form of the transducer is simply a means to maximise return in the face of decision noise: by subjectively inflating smaller magnitudes and probabilities of reward, it renders otherwise uncertain choices less susceptible to decision noise, as long as that noise is distributed in an approximately Gaussian fashion.

Our assumptions about the “canonical” form of the human utility functions draws principally on Prospect Theory. This is not meant to imply a commitment to the specific form of  $w(p)$  proposed by eq.2. In fact, very similar results are obtained with the two-parameter forms of the probability weighting function described elsewhere (21). However, we found that the extra flexibility permitted by these functions did not improve the cross-validated model fit to human data, and so we chose to focus on the canonical function proposed by Prospect Theory. Moreover, the focus of our paper is on distortions of probability and value in the domain of gains. Whilst an identical argument would explain the mirror-symmetric compressive form of the value function in the domain of losses, the theory proposed here does not explain why losses “loom larger” than gains, having disproportionate influence on choices and further contributing to participants’ failure to maximise expected value (although see recent review articles that question the extent or replicability of loss aversion (22, 23)). Nor does our theory consider the most distinctive contribution of Prospect Theory – the intuition that computation of value is reference-dependent, with all utilities evaluated relative to a status quo given by the current context. The normative properties of such reference-dependence, for example in the context of efficient coding and efficient computation, have been discussed elsewhere (24).

### **Author Contributions**

BS conceived of the project idea, JB, BS, and CS conceived of the simulation and experiment idea, JB and KJ collected the data, all authors carried out the stimulations, analysed the experimental data, discussed and interpreted the results, and edited the manuscript.

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## Methods

### Simulations

We obtained the optimal shape of the value and probability weighting functions under decision noise using a model-free approach. We first created all possible combinations of sets of two lotteries of the form  $[v,p; 0, 1-p]$  where  $p$  and  $v$  were drawn from one of  $j = 10$  bins of equal size and spacing within the range  $[0.01,0.99]$ . These  $N = 10,000$  samples were randomly split into training and test sets, after discarding trivial pairs where one lottery had both greater probability and value than its competitor. Next, we asked what decision weight an optimal agent should apply to each bin of value and probability in order to minimize one of two loss functions described below in the training set. Optimization was carried out via gradient descent using Matlab's `fmincon()` function, with parameters for each bin initialised to  $x_j$  and  $p_j$ . We varied the amount of noise in equation (4) by adjusting the parameter  $\sigma$  between 0 and 0.1 in intervals of 0.02, where higher values of  $\sigma$  indicate stronger noise, and repeated the simulation for each level of  $\sigma$ . Having obtained the best fitting estimates we evaluated cross-validated fit on the held-out test set. For convenience and to allow direct comparison, values were scaled to fall within the interval  $(0, 1)$ . Note, however, that the results of our simulations hold regardless of value scale. Where we used the entropic loss function (eq.6), the logarithm was base  $e$ .

### Human Experiments

Ethical approval.

All participants gave informed consent to participate in the study and were free to withdraw at any point. The study was approved by the ethics board of the Medical Sciences Division, University of Oxford (R50750/RE001)

Participants

We recruited  $n = 100$  participants online for each of the two experiments on Amazon Mechanical Turk (MTurk). In experiment 1, 59 participants were male and 41 were female. In experiment 2, 62 were male, and 38 were female. Age was only assessed to decade resolution, with most participants falling within the 21 to 30 years range (74% in both experiments). Participants were remunerated for their time with \$5 plus a bonus equivalent to the value of their chosen lottery of one randomly selected trial (scaled to \$0-\$10). We included only those participants whose performance differed significantly ( $p < 0.001$ ) from random as determined by a binomial test ( $n = 167$ ).

Task

Participants performed the task online in their browser and were required run it in full screen mode for the duration of the experiment. On each of the 250 trials, participants saw two lotteries to the left and right of the centre of the screen and were asked to indicate which of two lotteries they preferred. Participants were only instructed about the visual components of the task and that one lottery would be chosen at random at the end as their bonus, but received no further instructions about how to choose between the lotteries. During the response window of 20s, both lotteries remained on the screen until participants indicated

their response using the left or right arrow key. A dial at the top of the screen indicated the time remaining within the trial. After participants pressed a button, the chosen lottery remained on screen for 1s and was then replaced with the outcome feedback for a further 2s. The next trial began immediately after this. Throughout the experiment, the time elapsed and trials completed were displayed at the top of the screen. Participants took on average 2.38s in experiment 1 and 2.03s in experiment 2 to respond, leading to a total average time for task completion of 23.6 minutes in experiment 1 and 21.5 minutes in experiment 2. There were no breaks during the experiment.

### Stimuli

Lotteries consisted of one probability cue expressed as a percentage and one value cue expressed as dollar amount. This indicated that participants had the chance to win \$X with Y% or \$0 otherwise. For convenience, the shared \$0 outcome across lotteries was not displayed on screen. Whether probabilities were displayed above values or vice versa was assigned randomly for each participant. Values and probabilities were sampled from a uniform distribution between \$1-\$99 and 1%-99, excluding trivial samples where one lottery was better on both value and probability than its competitor (these trivial gambles were also excluded from our simulations). Experiment 2 changed this to always include one certain lottery with 100% probability, with the remaining probability and values sampled from the same uniform distribution and subject to the same constraint. In all experiments, participants completed 250 trials and had to indicate their choice by pressing the left or right arrow key.

### Feedback

Our main manipulation between groups pertained to how feedback was given to participants. The “certain outcome” condition received as feedback the product of value and probability, in other words the lottery’s expected value, whereas the “uncertain outcome” group received feedback depending on the lottery’s probability: either its dollar value (with probability  $p$ ) or \$0 (with probability  $1-p$ ). Note that the optimal strategy in the absence of noise is always to multiply the value and probability regardless of feedback. Hence, a noiseless ideal observer would not differ in its behaviour between conditions.

### Model fitting

Similar to our simulations, we first assessed participants’ weighting functions using a model-free approach. For this purpose, we again binned values and probabilities into  $j = 10$  bins and fit one decision weight to each bin using the genetic algorithm. This allowed us to compute the density of decision weight estimates as illustrated in Figure 4, without having to assume a specific parameterized functional form. We assessed how well economic decision-making models accounted for participants’ choices using maximum likelihood fitting. The two models we used were:

First, a standard model from the Prospect Theory (PT) family of models of the form given in equations 2-3 of the main text.

Second, we assumed that participants in the certain outcome group were better fit by a double exponent model as our simulations indicated no difference between the probability and value weighting functions for the return-maximizing agent. Thus, the double exponent model assumed both the  $v(x)$  and  $w(p)$  weighting to be of the form described in equation (2), albeit

with different parameters:  $\kappa$  for  $v(x)$  and  $\gamma$  for  $w(p)$ , employing the same naming convention as in equations (2-3).

Models were fit to the data using a hierarchical model and optimized using the Expectation-Maximization algorithm following software provided in (25). The model first draws a number of parameter samples from a group prior distribution and assesses fit for each sample. The group prior is then updated to better reflect those samples that more accurately predicted behaviour. These two processes proceed iteratively until convergence (i.e. no further improvement to fit is observed by adjusting the group prior or increasing the number of samples). We used the standard normal distribution as group priors. Data were fit separately for each group and experiment.

#### Model comparison

We compared models using the Variational Bayesian Analysis toolbox (26). We employed a random effect analysis, using the models' log likelihoods to compute the exceedance probability that a given (crossvalidated) model fit participants' data better than all other models. This procedure calculates the likelihood that a given model is more frequently the best model (across participants) compared to all others within the set. This produces a more nuanced model comparison metric than comparisons based on overall, fixed effects model fits (e.g. Bayesian Information Criterion).

#### Loss landscapes

Having established which of the two models best fit participants' choices in each group and experiment, we asked whether the best fitting parameters fell within the range of optimal parameters for a given model. For this purpose, we plotted the loss landscape of each model over a range of parameter values, evaluating performance on the exact gambles given to participants. We fixed the noise in the simulation to the mean across participants and derived for each parameter pair  $[\kappa, \gamma]$  how the model fared relative to a linear agent with parameters  $[1, 1]$ . As both models used power functions, we explored the parameters in log-space (with base  $e$ ), where a value of 0 indicates a linear mapping from objective to subjective magnitude. We linearly sampled  $n = 50$  points within the range of  $[-3, +3]$  in logspace and plotted the value of the loss function (either return-maximizing or entropic) at that point, **Fig. 3**.

#### Data and code availability

All data, and code to reproduce all figures and to run the simulations will be made freely available at the point of publication on our GitHub repository.

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