Quantum approach to a Bianchi I singularity

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Abstract. The approach of a quantum state to a cosmological singularity is studied through the evolution of its moments in a simple version of a Bianchi I model. In spite of the simplicity, the model exhibits several instructive and unexpected features of the moments. They are investigated here both analytically and numerically in an approximation in which anisotropy is assumed to vary slowly, while numerical methods are also used to analyze the case of a rapidly evolving anisotropy. Physical conclusions are drawn mainly regarding two questions. First, quantum uncertainty of anisotropies does not necessarily eliminate the existence of isotropic solutions, with potential implications for the interpretation of minisuperspace truncations as well as structure-formation scenarios in the early universe. Secondly, back-reaction of moments on basic expectation values is found to delay the approach to the classical singularity.

1 Introduction

The dynamics of anisotropic cosmological models are believed to give a reliable description of the approach to a space-like singularity in general relativity, based on the Belinskii–Khalatnikov–Lifshitz (BKL) [1] scenario. It is therefore of interest to analyze in detail the behavior of quantized anisotropic models in order to determine whether a singularity may persist in quantum gravity. The most generic dynamics, given by the Bianchi IX model, can be rather complicated classically [2], but even in this case it consists of long stretches of time during which the dynamics resembles that of the simpler Bianchi I model.

The main goal of this paper is to analyze how the presence of anisotropies may affect the behavior of a quantum state, parametrized by its quantum fluctuations and higher-order moments. These parameters can be considered coordinates of a quantum phase space that extends the classical phase space of the volume and anisotropy degrees of freedom, parametrized here in a Misner-like fashion. The quantum parameters, as opposed to a wave function, preserve the geometrical nature of the classical phase-space problem and are therefore appropriate for a quantum understanding of the BKL scenario.

Misner variables [3,4] describe a homogeneous geometry not directly through the coefficients in a line element but rather through the volume and two anisotropy (or shape) parameters. We will further restrict the dynamics by assuming that only one of the anisotropy parameters is non-zero. As geometrical variables, we will therefore have the volume, one anisotropy parameter, the moments of each of these variables and their momenta, and cross-moments between volume and anisotropy. Even in the restricted setting of a single anisotropy parameter and a quantum dynamics truncated to some fixed moment order, the parameter space is therefore rather large, making the analysis non-trivial and instructive.

For generic anisotropy, the system of dynamical equations for moments is highly coupled and hard to solve analytically. We will therefore introduce an approximation in which anisotropy varies much more slowly than the volume, in which case several analytical expressions can be obtained. Numerical results for moments up to fifth order are shown for generic anisotropy.

In addition to computational questions in the analysis of our system, this paper highlights two kinds of physical interpretations of the technical results. First, isotropic models within anisotropic ones can be used as test systems of the minisuperspace truncation, in which the relation between a symmetric quantum model and a less-symmetric one is an important open question; see for instance [5]. Our equations will allow us to determine conditions on the

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moments of a state in the anisotropic model such that it follows the behavior of the isotropic model. A general argument against minisuperspace truncations is that quantum uncertainty relations prevent anisotropic degrees of freedom from being completely absent, questioning the validity of a quantum model in which those degrees of freedom have been neglected. We will find that in our model, on the contrary, it is possible to find states that follow exactly isotropic behavior.

While this result may be considered supportive of minisuperspace truncations at least in the types of models studied here, it also strengthens questions that have been raised about quantum scenarios of structure formation [6, 7]: In early-universe cosmology, inhomogeneity is supposed to be generated out of quantum fluctuations of an initially homogeneous state, but if the dynamics is translation invariant, it should preserve the homogeneity of any initial state. In our case, similarly, the isotropy of an initial state is preserved by quantum evolution, but only under additional conditions on higher-order moments.

Our second application is about the behavior of a quantum state approaching an anisotropic singularity. We find that different kinds of moments play different roles. We therefore determine which moments can be used as indicators of singular behavior. Such results are useful for establishing the genericness of various proposals to avoid singularities by quantum effects. Often, such proposals are analyzed by using a specific class of initial or evolving states. While we also fix our initial states, making the common Gaussian choice, we are able to track the moments that grow most strongly and might therefore have a dominant effect on the quantum behavior near a singularity. We also draw lessons about possible modifications of the approach to a singularity, which seems to be slowed down by back-reaction at least with respect to the time variable chosen here, given by deparametrization with respect to a scalar field.

2   Canonical description of the classical model

The metric of an anisotropic Bianchi I universe is given by,

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \sum_{k=1}^{3} a_k^2 \mathrm{d}x_k^2\,,$$

where a_k are the scale factors in the different spatial directions, and N is the lapse function. In the variables introduced by Misner [3, 4], this line element takes the form

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + e^{\alpha} \sum_{k=1}^{3} e^{\beta_k} \mathrm{d}x_k^2\,. \tag{1}$$

The spatial volume is described by the variable α, defined by $e^{\alpha} := (a_1 a_2 a_3)^{1/3}$, whereas the three shape-parameters $\beta_k = \ln(a_k/e^{\alpha})$ measure the degree of anisotropy of each spatial direction. These three variables are not independent but satisfy the constraint $\beta_1 + \beta_2 + \beta_3 = 0$. Therefore, for convenience, we construct two independent shape-parameters defined as

$$\beta_+ := -\beta_3 = -\ln(a_3/(a_1 a_2 a_3)^{1/3}), \qquad \beta_- := \frac{1}{\sqrt{3}}(\beta_1 - \beta_2) = \frac{1}{\sqrt{3}}\ln(a_1/a_2)\,. \tag{2}$$

We will consider a free, massless scalar field ϕ as the matter source, with conjugate momentum p_ϕ. The Hamiltonian constraint is then given by

$$\mathcal{C} := e^{-3\alpha}(-p_\alpha^2 + p_+^2 + p_-^2) + \frac{1}{2}e^{-3\alpha}p_\phi^2 = 0, \tag{3}$$

where we have absorbed constant factors (such as Newton's constant) in p_ϕ. Using this constraint, the conjugate momenta of the configuration variables $(\alpha, \beta_\pm, \phi)$ are obtained in terms of their derivatives with respect to coordinate time t,

$$p_\alpha = -\frac{e^{3\alpha}}{2N}\alpha_{,t}, \qquad p_\pm = \frac{e^{3\alpha}}{2N}\beta_{\pm,t}, \qquad p_\phi = \frac{e^{3\alpha}}{N}\phi_{,t}\,. \tag{4}$$

In order to simplify the effects of the anisotropy on the system, we will consider only one anisotropic direction by choosing a vanishing β_-. In this way, the directions x_1 and x_2 will be isotropic, as their corresponding scale

factors are equal, $a_1 = a_2$, whereas the direction x_3 will generically be anisotropic. Therefore, we have just one shape-parameter, $\beta := \beta_+$, which measures the ratio of the scale factor a_3 with respect to the geometric mean of the three scale factors. Alternatively, we could eliminate the matter content and use β_- as internal time, such that $\frac{1}{2}p_\phi^2$ in the following expressions would play the role of p_-^2. Our results therefore apply to a matter model with restricted anisotropy, or to a vacuum model with full anisotropy.

Instead of choosing the logarithm of the scale factor α as our basic variable, we will use the spatial volume $v = e^{3\alpha}$. Its conjugate momentum is proportional to the Hubble parameter, describing the isotropic rate of expansion of the universe:

$$p_v = \frac{1}{3}e^{-3\alpha}p_\alpha = -\frac{2v_{,t}}{Nv}\,. \tag{5}$$

This variable is preferred for numerical purposes because it places the singularity at a finite value of the geometric variable, $v = 0$. Moreover, even though the constraint

$$\mathcal{C} = \frac{1}{v}\left(-v^2p_v^2 + p_\beta^2 + \frac{1}{2}p_\phi^2\right) = 0 \tag{6}$$

in the volume parameter may appear more complicated than the original (3), it will be straightforward to interpret the quantum dynamics of moments of the volume, as opposed to moments of its logarithm.

Since neither ϕ nor β appear explicitly in the constraint, their conjugate momenta p_ϕ and p_β are conserved quantitities. The expression of the Hamiltonian constraint $\mathcal{C}$ is also conserved through evolution, thus one can infer that the combination vp_v is another constant of motion, which will appear throughout this paper. As can be seen in the definition (5), this combination is proportional to the momentum p_α, but we will refer to it as $\gamma := vp_v$ because we will not consider it a basic canonical variable.

In fact, after performing the deparametrization with respect to ϕ, γ will represent the unconstrained Hamiltonian $H_{\mathrm{iso}} = -p_\phi|_{p_\beta=0}$ of the reference isotropic model (27). Including anisotropy, the deparametrized dynamics is generated by the Hamiltonian

$$H := -p_\phi = (\gamma^2 - p_\beta^2)^{1/2}\,, \tag{7}$$

implying the classical equations of motion

$$\dot{v} = \frac{\gamma}{H}v, \tag{8}$$

$$\dot{p}_v = -\frac{\gamma}{H}p_v, \tag{9}$$

$$\dot{\beta} = -\frac{p_\beta}{H}, \tag{10}$$

$$\dot{p}_\beta = 0, \tag{11}$$

where the dot represents a derivative with respect to the scalar field ϕ. Since the equations are symmetric under the transformation $p_v \to -p_v$, $p_\beta \to -p_\beta$, and $\phi \to -\phi$ and we are interested in an expanding universe with a singularity in the past, towards decreasing ϕ, we will without loss of generality choose a positive sign for both p_β and p_v (and, thus, also for γ). In this way, and with the choice of sign taken for p_ϕ when solving the constraint (6), the universe expands as ϕ increases and the singularity is located at $\phi \to -\infty$.

The equations of motion can easily be solved:

$$v(\phi) = v(0)\exp\left(\frac{\gamma}{H}\phi\right), \qquad p_v(\phi) = p_v(0)\exp\left(-\frac{\gamma}{H}\phi\right), \qquad \beta(\phi) = \beta(0) - \frac{p_\beta}{H}\phi, \tag{12}$$

while p_β is a constant of motion, as already seen. Here, $v(0), p_v(0)$ and $\beta(0)$ are initial values of the different variables at $\phi = 0$. As the volume v tends to zero, approaching the singularity located at $\phi \to -\infty$, its conjugate momentum, p_v, diverges exponentially, keeping their product γ constant. The ratio γ/H parametrizes the rate of collapse of the volume towards the singularity. On the other hand, the shape-parameter β increases as a linear function of ϕ with a velocity controlled by the constant of motion p_β, making the universe more and more anisotropic as it approaches the singularity. The variable β tends to (plus) infinity for $\phi \to -\infty$, producing a singularity as the scale factor a_3 tends to zero. The other two scale factors, a_1 and a_2, which are equal in our restricted model, may be non-zero, but such that $v = a_1a_2a_3 \to 0$ for $\phi \to -\infty$. Using the defining relationships of our variables and the solutions (12), we can write

$$a_1(\phi) = v(\phi)^{1/3}\exp(\beta(\phi)/2) = v(0)^{1/3}\exp(\beta(0)/2)\exp((\gamma/3 - p_\beta/2)\phi/H)\,. \tag{13}$$

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Therefore, a_1 approaches zero or $+\infty$ at the singularity, depending on the sign of $(\gamma/3 - p_\beta/2)$. In the vacuum model, we would have $p_\phi = 0$ and therefore $\gamma = p_\beta$, such that $a_1 \propto \exp(-p_\beta\phi/(6H)) \to \infty$ at the usual Kasner singularity. With scalar matter, however, p_β is a free parameter restricted only by $p_\beta \leq \gamma$. This condition does not fix the sign of $\gamma/3 - p_\beta/2$, and a_1 may approach zero or $+\infty$ depending on the initial conditions.

3 Quantum dynamics

Having the classical dynamics of the system under control, we proceed to analyze its quantum dynamics following a formalism based on a moment decomposition of the wave function developed for quantum cosmology in [8,9]. The quantum dynamics of this model is ruled by a Hamiltonian $\hat{H}$, that depends on the basic operators $\hat{v}$, $\hat{p}_v$, $\hat{\beta}$, and $\hat{p}_\beta$. In order to analyze the quantum evolution produced by this Hamiltonian, we will define the following moments, which encode the complete information of the quantum state,

$$G^{ijkl} := \langle (\hat{v} - v)^i (\hat{p}_v - p_v)^j (\hat{\beta} - \beta)^k (\hat{p}_\beta - p_\beta)^l \rangle_{\text{Weyl}}, \tag{14}$$

where the subscript "Weyl" indicates totally symmetric ordering of the operators, and the expectation values $v := \langle \hat{v} \rangle$, $p_v := \langle \hat{p}_v \rangle$, $\beta := \langle \hat{\beta} \rangle$ and $p_\beta := \langle \hat{p}_\beta \rangle$ have been defined. We will refer to the sum of the indices of a given moment $i + j + k + l$ as its *order*. This definition will be relevant later on when we consider truncations of the system.

Unlike the basic expectation values, moments of a state are not completely arbitrary but restricted by (generalized) uncertainty relations which follow from the positivity condition of an algebraic state (the derivation of such generalized inequalities for the case of one degree of freedom is studied in [18]). These restrictions will play an important role in some of our discussions, but in specific cases we will mainly refer to the well-known second-order version, which is nothing but Heisenberg's uncertainty relation. Provided these general conditions are obeyed by a given set of moments, a state with these moments does exist. However, it is not guaranteed to be a pure state, demonstrating the general nature of states included in the parametrization by moments.

3.1 Effective Hamiltonian and equations of motion

The dynamics of these variables is given by the following effective Hamiltonian, defined as the expectation value of the quantum Hamiltonian operator, which is assumed to be Weyl-ordered:

$$H_Q := \langle \hat{H}(\hat{v}, \hat{p}_v, \hat{p}_\beta) \rangle = \sum_{i,j,k} \frac{1}{i!j!k!} \frac{\partial^{i+j+k} H(v, p_v, p_\beta)}{\partial v^i \partial p_v^j \partial p_\beta^k} G^{ij0k}, \tag{15}$$

where $H(v, p_v, \beta)$ is the classical Hamiltonian (7) and the sum runs over all non-negative integer values of i, j and k. (If a Hamiltonian operator with a different ordering is preferred, the effective Hamiltonian would contain terms explicitly depending on $\hbar$ that result from re-ordering operations.) In particular, if $i = j = k = 0$, then $G^{ij0k} = 1$ because the state is normalized. The corresponding term in the sum therefore produces the classical Hamiltonian, $H(v, p_v, p_\beta)$, evaluated in the basic expectation values. For instance, the second-order Hamiltonian is

$$H_Q = H - \frac{v^2 p_v^2}{2H^3} G^{0002} + \frac{v p_v}{H^3} \left(H^2 - p_\beta^2 \right) G^{1100} + \frac{p_\beta}{H^3} v p_v^2 G^{1001} - \frac{v^2 p_\beta^2}{2H^3} G^{0200} - \frac{p_v^2 p_\beta^2}{2H^3} G^{2000} + \frac{p_\beta}{H^3} v^2 p_v G^{0101}.$$

Since β does not appear in the classical Hamiltonian, only moments unrelated to β (and thus of the form G^{ij0k}) appear in the expression of the effective Hamiltonian. This fact implies that p_β, as well as all its pure fluctuations (moments of the form G^{000i}), are constants of motion for the quantum dynamics. Nonetheless, $\gamma = v p_v$ will not be a constant of motion at the quantum level because the full quantum Hamiltonian H_Q (15), but not the classical Hamiltonian H (7), is conserved by quantum evolution. Here, we define γ as the product of expectation values of $\hat{v}$ and $\hat{p}_v$. An alternative definition, using

$$\tilde{\gamma} = \frac{1}{2}\langle \hat{v}\hat{p}_v + \hat{p}_v\hat{v} \rangle = v p_v + G^{1100}, \tag{16}$$

is less convenient for our purposes. If one were to use $\hat{\gamma}$ as a basic operator, as done in affine quantum cosmology [10–14], $\langle \hat{\gamma} \rangle$ would be conserved as a consequence of $[\hat{\gamma}, \hat{H}] = 0$. However, neither γ nor $\tilde{\gamma}$ need be conserved in our

system because the assumed Weyl ordering in $\hat{v}$ and $\hat{p}_v$ as basic operators implies that an operator quantizing a classical expression, that depends on v and p_v only through γ, is not required to depend on $\hat{v}$ and $\hat{p}_v$ only through $\frac{1}{2}\langle \hat{v}\hat{p}_v + \hat{p}_v\hat{v}\rangle$. We will see explicit solutions in which, indeed, neither γ nor $\tilde{\gamma}$ are conserved.

The equations of motion for the different variables are obtained by computing Poisson brackets with the effective Hamiltonian. The Poisson brackets between two expectation values are related to the expectation value of their commutator by the relation

$$\{\langle \hat{X}\rangle, \langle \hat{Y}\rangle\} = -i\langle [\hat{X}, \hat{Y}]\rangle, \tag{17}$$

extended to products of expectation values by the Leibniz rule. This expression is standard for basic expectation values, while it defines an extension of the classical bracket for moments. A general expression for the brackets of moments is known in closed form [8,9], but it is rather lengthy and will not be displayed here. (See also [15,16] for the structure of the underlying Poisson manifold.) These brackets are not canonical and they contain linear and quadratic terms in moments. In particular, the origin of the linear terms lies in the reordering of operators and, therefore, they appear multiplied by certain power of $\hbar$. Each of this $\hbar$ factors is considered as increasing the total moment-order by two. Our main arguments will use the schematic form of the bracket,

$$\{G^{ijkl}, G^{mnop}\} \sim GG^{qrst}, \tag{18}$$

where "GG" on the right-hand side represents a finite sum of terms quadratic in moments (or a moment multiplied by certain power of $\hbar$) of a total order $(qrst)$ such that $q + r + s + t = i + j + k + l + m + n + o + p - 2$. This general statement about orders follows from an application of (17), in which the commutator always reduces the total moment order by two.

In this way one can, for instance, obtain the equation of motion for the volume:

$$\dot{v} = \{v, H_Q\} = \frac{\gamma}{H}v + \sum_{i,j,k} \frac{1}{i!j!k!} \frac{\partial^{i+j+k+1} H(v, p_v, p_\beta)}{\partial v^i \partial p_v^{j+1} \partial p_\beta^k} G^{ij0k}, \tag{19}$$

where we have used the fact [8] that all the expectation values (in particular v) Poisson commute with the moments (14). One can then proceed in this way to find the equations of motion for all the variables.

In general, the equations of motion for the moments and expectation values form a highly coupled infinite system of equations. Therefore, one usually needs to implement a truncation in order to solve them. The main assumption is that for semiclassical states peaked around a classical trajectory, there is a hierarchy of moments ruled by their order. For such states, higher-order moments are then less relevant than lower-order moments. In particular, for the numerical solutions that will be performed later on, we will consider the system of equations up to fifth order in moments. The equations of motion up to such a high order are much too lengthy to be displayed here. Hence, in order to give a grasp of the system we are dealing with, all the equations up to second order in moments are displayed in Appendix A. In addition, in Appendix B the evolution equation for the volume is given, truncated at fifth order.

For the specific Hamiltonian (7) under consideration, the equations are not completely coupled. In particular, since only moments G^{ij0k} unrelated to the shape-parameter appear in the Hamiltonian, the set of equations of motion for the variables $\{v, p_v, \beta, p_\beta, G^{ij0k}\}$ forms an independent subsystem of equations that can be solved on its own. This is due to the fact that the Poisson bracket $\{G^{ij0l}, G^{kl0m}\}$, which must be computed to obtain the evolution equation for G^{ij0l}, does not generate any moment of the form G^{abcd}, with $c \neq 0$. In fact, the equation of motion for β depends on $\{v, p_v, p_\beta, G^{ij0k}\}$ but not the other way around, and one can thus conclude that the system $\{v, p_v, p_\beta, G^{ij0k}\}$ is independent of the rest. One can even remove the dependence on the volume from this system by performing a further change of variables, as shown below.

For the main analysis of this paper, instead of using the absolute moments G^{ijkl}, we will use the relative moments

$$K^{ijkl} := \frac{G^{ijkl}}{v^i p_v^j} = \gamma^{-j} v^{j-i} G^{ijkl}. \tag{20}$$

Furthermore, the momentum of the volume, p_v, will be replaced by the isotropic Hamiltonian $\gamma := vp_v$. As will be explained in Sec. 4, a convenient property of this new set of variables $\{v, \gamma, \beta, p_\beta, K^{ijkl}\}$ is that all but the volume v are constants of motion in the limit of a slowly-evolving anisotropy ($p_\beta \ll \gamma$). More importantly, with this new set of variables the couplings between different equations of motion simplify considerably as the equations of motion for the variables $\{\gamma, p_\beta, K^{ij0l}\}$ decouple from the equations for v, β and the rest of the moments, K^{ijkl}

with $k \neq 0$. In fact, the volume v and the shape-parameter β only appear explicitly in their own equations of motion as a time derivative (or a logarithmic derivative in the case of the volume). Schematically one can write the equations of motion as

$$\frac{\dot{v}}{v} = F_1(\gamma, p_\beta, K^{ij0l}), \tag{21}$$

$$\dot{\beta} = F_2(\gamma, p_\beta, K^{ij0l}), \tag{22}$$

$$\dot{\gamma} = F_3(\gamma, p_\beta, K^{ij0l}), \tag{23}$$

$$\dot{p}_\beta = 0, \tag{24}$$

$$\dot{K}^{ab0d} = F_{ab0d}(\gamma, p_\beta, K^{ij0l}) \tag{25}$$

where the right-hand sides are given in terms of the constant p_β, the Hamiltonian of the reference isotropic model γ, and moments K^{ij0l} unrelated to the shape-parameter, but are independent of the volume v and β. Therefore, in order to obtain the quantum back-reaction effects on the classical trajectories, it is enough to consider this subsystem of equations. Similarly, the equation of motion for a generic moment has the form

$$\dot{K}^{abcd} = F_{abcd}(\gamma, p_\beta, K^{ijkl}). \tag{26}$$

Hence, the dynamics of the moments is only affected by the expectation values γ and p_β, but not by v and β. The explicit form of this system of equations, truncated at second order in moments, is shown in Appendix C.

Finally, as with the classical system, the equations are symmetric under the transformation $p_v \to -p_v$, $\gamma \to -\gamma$, $p_\beta \to -p_\beta$, and $\phi \to -\phi$, provided that the moments are also transformed as $G^{ijkl} \to (-1)^{j+l}G^{ijkl}$ or, equivalently, $K^{ijkl} \to (-1)^l K^{ijkl}$. Therefore, as already commented above, a positive sign for γ and p_β will be considered throughout the paper.

3.2 The isotropic (harmonic) case

Using the basic variables defined here, the isotropic case is formally recovered by choosing β and p_β to vanish, along with all the moments with some contribution from the anisotropic sector (that is, G^{ijkl} with $k + l \neq 0$). Restricting all moments of this form is not consistent with uncertainty relations in the anisotropy sector. The restriction therefore amounts to a minisuperspace truncation of isotropic geometries within anisotropic (but still homogeneous) ones. In principle, therefore, the reduction is not expected to define a subset of quantum solutions in the anisotropic model. A detailed analysis of solutions will nevertheless show that isotropic solutions do exist within the anisotropic quantum model.

The classical Hamiltonian (7) is then simplified to be a linear function of v and p_v,

$$H^{\text{iso}} = \gamma = vp_v. \tag{27}$$

From the perspective of the quantum dynamics, this case is very special, as the Hamiltonian turns out to be *harmonic*. (It is quadratic in basic variables. A linear canonical transformation maps it to an inverted harmonic oscillator.) The most important property of this kind of Hamiltonians is that different orders in moments are not coupled to one another. Furthermore, the classical equations of motion do not get corrections by quantum moments; there is no quantum back-reaction. Therefore, the expectation values v and p_v follow exactly their classical trajectories (12),

$$v = v(0)e^\phi, \qquad p_v = p_v(0)e^{-\phi}. \tag{28}$$

In addition, it is easy to obtain and solve the equations of motion for the quantum moments. Note that the infinite sum that defines the effective Hamiltonian (15) is reduced to a finite sum, as only second-order derivatives are nonvanishing. Therefore, for this harmonic case one obtains the following quantum Hamiltonian,

$$H_Q^{\text{iso}} = vp_v + G^{1100} \tag{29}$$

without any truncation. The equations of motion for the different moments,

$$\dot{G}^{ij00} = (i - j)G^{ij00}, \tag{30}$$

indeed shows that there is no coupling between the equations of motion for different orders. Solving this equation, one obtains an exponential evolution for the moments,

$$G^{ij00}(\phi) = G^{ij00}(0)e^{(i-j)\phi}. \tag{31}$$

In summary, as one approaches the singularity at $v = 0$, moments G^{ij00} with $i > j$ decrease exponentially, whereas moments G^{ij00} with $i < j$ follow an exponentially increasing behavior. Finally, moments of the form G^{ii00} are constants of motion. Taking into account the time dependence for the expectation values (28), we note also that all relative moments K^{ij00} (20) are constant throughout evolution since the time dependence of the absolute moments G^{ij00} is compensated for by that of the expectation values.

4 Slowly-evolving anisotropy ($p_\beta \ll \gamma$)

As noted after equation (12), p_β is the velocity of the shape-parameter β and therefore measures the rate of (an)isotropization of the universe, while γ is a measure of the velocity of expansion of the isotropic reference model. Therefore, the system dynamics should be close to the isotropic dynamics whenever $p_\beta \ll \gamma$ is obeyed so that $H \approx H^{\text{iso}}$. This condition means that the evolution of the anisotropy is slow compared with the rate of expansion of the volume. These are statements about the rates of change rather than the size of the homogeneous region. The approximation may therefore be used in the late universe (where the homogeneous volume may be assumed macroscopic) or in the early universe close to a spacelike singularity (where the BKL scenario suggests the existence of microscopic homogeneous patches).

In this section we will consider an expansion of the system of equations for large values of the parameter γ. In particular, the equations of motion for the different moments take the form

$$\dot{G}^{ijkl} = (i-j)G^{ijkl} + \mathcal{O}(\gamma^{-1}). \tag{32}$$

We therefore recover similar equations as in the harmonic case above (30), but for all the moments and not only for moments G^{ij00} of the isotropic sector. It is easy to see that the generator of these equations is the effective Hamiltonian $H = vp_v + \mathcal{O}(\gamma^{-1})$. These equations can be solved right away,

$$G^{ijkl}(\phi) = G^{ijkl}(0)e^{(i-j)\phi}. \tag{33}$$

As one would expect, in the case of slowly-evolving anisotropy the dynamics is dominated by the isotropic sector. In particular, the increasing or decreasing behavior of a corresponding moment G^{ijkl} is completely determined by the difference between its isotropic indices i and j. Furthermore, at this level of approximation, the equations for the expectation values do not get any quantum back-reaction effects from the moments, and thus expectation values follow the classical trajectory. In this way, the relative moments K^{ijkl} (20) are conserved quantitities,

$$K^{ijkl} = K^{ijkl}(0). \tag{34}$$

Let us now analyze the behavior of the system at next order in $1/\gamma$. We will consider an expansion for large γ, keeping the volume and the relative moments K^{ijkl} constant. The classical Hamiltonian takes the form

$$H = \gamma\left(1 - \frac{p_\beta^2}{2\gamma^2}\right) + \mathcal{O}(p_\beta^4/\gamma^3). \tag{35}$$

It can be expanded in order to get the quantum Hamiltonian,

$$H_Q = \gamma(1 + K^{1100}) - \frac{1}{2\gamma}\sum_{i,j}(-1)^{i+j}\left(p_\beta^2 K^{ij00} + 2p_\beta K^{ij01} + K^{ij02}\right) + \mathcal{O}(3), \tag{36}$$

where the sum runs over all non-negative integer values of i and j. In this expansion, $\mathcal{O}(3)$ stands for terms of the form $K^{ij0k}p_\beta^{n-k}/\gamma^{n-1}$ for $n \geq 3$ and $0 \geq k \geq n$. Therefore, the present approximation should be valid as long as all those terms are small. Here, we first expanded the classical Hamiltonian and then derived its effective expression. It is easy to see that the order can be reversed without changing the result, for instance using the second-order example (16).

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In the classical Hamiltonian one can define the dimensionless anharmonicity parameter p_β/γ, which is a constant of motion and measures the departure of the system from the harmonic behavior. In the quantum system, however, there are infinitely many more parameters that produce an anharmonic behavior, such as the moments K^{ij02} that appear explicitly in the Hamiltonian above (36) and might generate an anisotropy even if $p_\beta = 0$ and $\beta = 0$ at some initial time. In fact, this case will be analyzed in detail in the next section.

The equations of motion for the expectation values, generated by the approximate Hamiltonian (36), are

$$\dot{v} = v, \tag{37}$$

$$\dot{\gamma} = -\frac{1}{2\gamma}\sum_{i,j}(-1)^{i+j}(i-j)\left(p_\beta^2 K^{ij00} + 2p_\beta K^{ij01} + K^{ij02}\right), \tag{38}$$

$$\dot{\beta} = -\frac{1}{\gamma}\sum_{i,j}(-1)^{i+j}\left(p_\beta K^{ij00} + K^{ij01}\right). \tag{39}$$

At this level of approximation, all moments of the form K^{ij0k} are constants of motion since the Poisson brackets $\{K^{ij0k}, K^{lm0n}\}/\gamma$ are of the form KK^{ab0d}/γ^2 with $a = i + j - 1$, $b = l + m - 1$ and $d = k + n$, while "KK" is interpreted as explained for "GG" in (18). Any such term turns out to be of order $\mathcal{O}(3)$ and, thus, should be neglected. In particular, this includes all *purely isotropic moments* of the form K^{ij00}, which only involve the isotropic variables. Furthermore, all the moments that appear in the Hamiltonian (36) and in the equations for the expectation values above (37)–(39), are also included in this category. Therefore, these last equations can be easily integrated to obtain the evolution of the expectation values,

$$v = v_0 e^\phi, \tag{40}$$

$$\gamma = \sqrt{\gamma_0^2 + r\phi}, \tag{41}$$

$$\beta = \beta_0 - \frac{2\widetilde{p}_\beta}{r}\sqrt{\gamma_0^2 + r\phi}, \tag{42}$$

where the following constants have been defined:

$$r := -\sum_{i,j}(-1)^{i+j}(i-j)\left(p_\beta^2 K^{ij00} + 2p_\beta K^{ij01} + K^{ij02}\right), \tag{43}$$

$$\widetilde{p}_\beta := \sum_{i,j}(-1)^{i+j}\left(p_\beta K^{ij00} + K^{ij01}\right) = p_\beta + \sum_{i\geq 1, j\geq 1}\left(p_\beta K^{ij00} + K^{ij01}\right). \tag{44}$$

(For these generic solutions, we have assumed that $r \neq 0$; see below.) Note that, in general, neither $\gamma = vp_v$ nor $\tilde{\gamma} = vp_v + G^{1100}$ are constant, in contrast to the classical solution. The dynamics of γ is instead governed by the constant r, which is purely quantum and vanishes in the classical limit. Since we are assuming a large value of γ_0, the solutions (41)–(42) can be approximated by linear functions:

$$\gamma \approx \gamma_0 + \frac{r}{2\gamma_0}\phi, \tag{45}$$

$$\beta \approx \beta_0 - \frac{\widetilde{p}_\beta}{\gamma_0}\phi. \tag{46}$$

The solution (42) for the shape-parameter is valid only for the generic case $r \neq 0$. For special states in the quantum case, $r = 0$ is compatible with uncertainty relations. For instance, at second order, while r depends on the moments K^{2000} and K^{0200}, which cannot both be zero, it does so in an antisymmetric way because of the factor of $(i-j)$ in (43). Provided $K^{2000} = K^{0200}$, these moments therefore cancel out. Moreover, while the general expression for r depends on K^{ij02}, the second-order moment K^{0002} does not contribute because of the same factor of $(i-j)$. In the special case of $r = 0$, then, $\gamma = \gamma_0$ is constant while β changes linearly with time as in the approximate solution (46),

$$\beta = \beta_0 - \frac{\widetilde{p}_\beta}{\gamma_0}\phi. \tag{47}$$

Therefore, as in the classical limit, β is a linear function of ϕ, but with a regularized value of p_β which takes into account quantum effects.

The moments, except for the constant K^{ij0k} which still follow their isotropic behavior, do feel the effects of anisotropy and are no longer constant. One class of moments — those that imply only one factor in the shape-parameter K^{ij1k} — have simple equations of motion since they only contain constant moments of the form K^{nm0l}. Schematically, the equations for such moments are given as

$$\dot{K}^{ij1k} = \frac{c_{ijk}}{\gamma}, \tag{48}$$

with certain constants c_{ijk} that depend on p_β and moments of the form K^{nm0l}. This equation can be integrated, which gives rise to

$$K^{ij1k} = d_{ijk} + \frac{2c_{ijk}\sqrt{\gamma_0^2 + r\phi}}{r}, \tag{49}$$

with integration constants d_{ijk} for $r \neq 0$, and

$$K^{ij1k} = d_{ijk} + \frac{c_{ijk}}{\gamma_0}\phi, \tag{50}$$

for $r = 0$.

This pattern continues, allowing us to iteratively solve for the behavior of all moments. In the next step, using (18), $\dot{K}^{ij2k}$ is given by a sum of terms of the form $K^{ij1(k-1)}/\gamma$, each of which has a time dependence $d_{ij(k-1)}(\gamma_0^2 + r\phi)^{-1/2} + 2c_{ij(k-1)}/r$ for $r \neq 0$. Integrating, we have

$$K^{ij2k} = \frac{c_{ijk}^{(2)}\phi}{r} + d_{ijk}^{(2)}\frac{\sqrt{\gamma_0^2 + r\phi}}{r} + e_{ijk}^{(2)} \tag{51}$$

with new constants $c_{ijk}^{(2)}$, $d_{ijk}^{(2)}$ and $e_{ijk}^{(2)}$. The dominant behavior is linear in ϕ. Finally, it is possible to obtain that, at this level of approximation, a general moment K^{ijnk} has the dominant behavior

$$K^{ijnk} \sim \frac{(\gamma_0^2 + r\phi)^{n/2}}{r^n}. \tag{52}$$

The demonstration follows by induction. Note that $\dot{K}^{ij(n+1)k}$ is a sum of terms of the form $KK^{ijn(k-1)}/\gamma$ (again, see (18) for the meaning of "KK"), which all have the dominant behavior

$$\dot{K}^{ij(n+1)k} \sim \frac{(\gamma_0^2 + r\phi)^{(n-1)/2}}{r^n}, \tag{53}$$

according to (52). Integrating this expression, it is then straightforward to obtain the form (52) for $n + 1$. For third-order moments, this result is confirmed in Appendix D. For the particular case $r = 0$, in which (52) no longer applies, the evolution of the moments is faster and a moment of the form K^{ijkl} is given by a polynomial of order k in ϕ.

In summary, for this quasiharmonic case, we have found that up to order $\mathcal{O}(2)$ the volume follows its classical trajectory, whereas γ is not constant anymore but is a linear function in ϕ. The shape-parameter β is a linear function of ϕ, but with a quantum-corrected slope. Finally, depending on whether the constant r is vanishing or not, relative moments K^{ijkl} go either as ϕ^k or as $\phi^{k/2}$. Therefore, their index on β governs their evolution rate.

5 On the quantum generation of anisotropy

Before analyzing numerically generic values of p_β, let us look at the particular case of $\beta = 0$ and $p_\beta = 0$. Classicaly there is then no initial anisotropy and the spacetime will remain isotropic throughout evolution. In a quantized model, however, one would expect that some anisotropy is generated by quantum fluctuations (or certain higher moments) which are constrained by uncertainty relations to be non-zero. This expectation is a common criticism of minisuperspace quantizations, which start with symmetry reductions at the classical level and therefore ignore fluctuations of non-symmetric variables. While symmetry reduction leads to special solutions of the classical theory, it is not clear whether their minisuperspace quantizations can be considered approximations of solutions of some full theory of quantum gravity.

In a more specific context, it would be interesting if non-symmetric degrees of freedom could, in fact, be generated by quantum effects. This possibility, as a physical scenario, is usually considered for inhomogeneity rather than anisotropy in order to explain structure formation in the early universe. In this context, it would be desirable to excite non-symmetric degrees of freedom even if the initial state is symmetric (such as the homogeneous vacuum). Our model can be used as a test system in which inhomogeneity is replaced by more tractable anisotropy.

We will therefore be interested in initial states with vanishing p_β. Nevertheless, the discussion in the present section goes beyond what we found in the preceding section because we will assume that some anharmonicity parameters of the form $K^{ij0k} p_\beta^{n-k}/\gamma^{n-1}$ are not negligible for certain n, such that $0 \geq k \geq n$.

In order to make the appearance of anisotropy transparent, we begin by analyzing the equation of motion for the shape-parameter, β. If $p_\beta = 0$, it takes the particularly simple form

$$\dot{\beta} = \sum_{i,j,k} \frac{1}{\gamma^k} f(i,j,k) K^{ij0(2k+1)}, \tag{54}$$

where the sum runs over all non-negative integer values of i, j and k, and f is a function that only depends on the indices i, j and k. Therefore, the moments that produce a nonvanishing derivative for the shape-parameter are precisely those that are unrelated to this variable and, moreover, have odd order in p_β. For any such moment, uncertainty relations do not imply any lower bound. Therefore, it is consistent to assume that all $K^{ij0(2k+1)}$ are zero in a certain class of states. Specific examples can easily be constructed using products of Gaussian wave functions, such that $K^{ij0(2k+1)} = K^{ij00} K^{000(2k+1)} = 0$ because all odd-order moments vanish for a Gaussian.

It is therefore possible to choose an initial state for which the right-hand side of equation (54) is zero. As time goes on, the dynamics might activate some of the relevant moments, $K^{ij0(2k+1)}$, in which case the time derivative of the shape-parameter would become non-zero and an anisotropy would be generated. However, using the detailed dynamics at least up to fifth order in moments, we have analytically confirmed that the time derivative of any $\dot{K}^{ij0(2k+1)} = 0$ is identically zero provided $p_\beta = 0$ and all $K^{ij0(2k+1)} = 0$ at an initial time. The right-hand side of (54) is then vanishing at all times, and no anisotropy is generated.

This result can be understood based on the general behavior of moment equations. Since the effective Hamiltonian does not depend on β or its moments, a non-zero $\{K^{ij0(2k+1)}, H_Q\}$ can be obtained only via the (v, p_v)-part of the moments. Moreover, H is a function of p_β^2, such that only even-order moments of the form $K^{mn0(2l)}$ contribute to H_Q when $p_\beta = 0$. The first property implies that p_β-orders of moments add up in $\{K^{ij0(2k+1)}, H_Q\}$, which then contains only moments of odd order in p_β based on the second property. Therefore, if all $K^{ij0(2k+1)}$ are zero initially, they remain zero if $p_\beta = 0$.

Although this result follows directly from properties of the moment brackets, it is somewhat unexpected based on general arguments about limitations of minisuperspace quantization. Our result, however, relies on the specific dynamics of the moments and not just on general expectations on implications of uncertainty relations. It is also consistent with detailed studies made in the case of inhomogeneity [6,7], where an expectation opposite to the usual criticism of minisuperspace quantization has been formulated. Nevertheless, since our result relies on the detailed dynamics, it may well change if other models are considered, for instance those with a non-vanishing anisotropy potential. In our model, the coupling between the different degrees of freedom is not strong. It could therefore be possible that our result comes about because the Hamiltonian (7) is a function (a square root) of the harmonic (free) Hamiltonian $(v^2 p_v^2 - p_\beta^2)$, where the different sectors (v, p_v) and (β, p_β) are completely decoupled.

At high orders in moments, the nonlinearities and strong couplings make it difficult to obtain analytical solutions. We have, however, been able to obtain the general solution of the system up to third order in moments. At this order, the evolution of the expectation value of the volume v and γ are given by,

$$v = v_0 \left| \gamma^2 + r\phi \right|^{-(K^{0102} + \frac{1}{2} K^{1002} - \frac{1}{2} K^{0002})/r} e^\phi \qquad \gamma = \sqrt{\gamma_0^2 + r\phi}, \tag{55}$$

where $r = K^{1002} - K^{0102}$ (here assumed to be non-zero) is the truncation of (43) to third order. The shape-parameter has the following form,

$$\beta = \beta_0 + \frac{K^{0003}}{r\gamma} - \frac{2\widetilde{p}_\beta}{r}\gamma, \tag{56}$$

in terms of γ from (55), where

$$\widetilde{p}_\beta = (K^{1001} + K^{0101} + K^{1101} + K^{2001} + K^{0201}), \tag{57}$$

is the truncation of (44) to third order.

All moments that appear in our solutions for the expectation values are constants of motion. In general a moment of the form K^{ijkl} is a polynomial of order k in γ and therefore changes like $\phi^{k/2}$ for large ϕ. Moreover, the second-order moments that involve either v or p_v depend on ϕ through an expression logarithmic in γ. There is no such logarithmic term in third-order moments, except for K^{0120} and K^{1020}, in which case this term is multiplied by $\hbar^2$. Therefore, in addition to the momentum p_β and its pure fluctuations, K^{000i}, almost all the moments K^{ij0l} unrelated to the shape-parameter β are constants of motion. The only three nonconstant fluctuations of the isotropic sector increase as logarithmic functions of γ,

$$K^{0200} \;\; = \;\; b_1 + \left(1 + \frac{K^{0102} + K^{1002}}{K^{0102} - K^{1002}}\right) \ln\gamma, \tag{58}$$

$$K^{1100} \;\; = \;\; b_2 - \ln\gamma, \tag{59}$$

$$K^{2000} \;\; = \;\; b_3 + \left(1 - \frac{K^{0102} + K^{1002}}{K^{0102} - K^{1002}}\right) \ln\gamma\ , \tag{60}$$

where b_1, b_2, b_3 are real constants. The explicit form for the rest of the moments is given in Appendix E, including fluctuations of the anisotropic sector and different correlations between the two sectors.

The generic solution presented above is not valid in the particular case in which the moments K^{0102} and K^{1002} are equal, such that $r = 0$. If $r = 0$, γ is constant and the volume depends on ϕ by

$$v = v_0 \exp\left[\phi + \left(2K^{0002} - 3(K^{0102} + K^{1002})\right)\frac{\phi}{4\gamma^2}\right], \tag{61}$$

while the shape-parameter increases as a linear function in ϕ, as in the classical case,

$$\beta = \beta_0 - \frac{2\gamma^2 \widetilde{p}_\beta + K^{0003}}{2\gamma^3}\phi\ . \tag{62}$$

Therefore, even if β_0 is vanishing, the quantum moments will produce an anisotropy by acting as an effective p_β, unless K^{0003} and those that appear in (57) vanish. Nonetheless, as commented above, this is allowed by uncertainty relations and one can indeed choose initial states that will never generate an anisotropy.

The moments that were constant in the previous generic solution are also constant in the case of $r = 0$, as well as the isotropic correlation $K^{1100} = \tilde{b}_2$. Therefore, in the isotropic sector, only pure fluctuations of v and p_v are dynamical, and they increase faster (as linear functions of ϕ) than in the previous case:

$$K^{0200} = \tilde{b}_1 - \frac{K^{0102} + K^{1002}}{2\gamma^2}\phi \qquad K^{2000} = \frac{K^{0102} + K^{1002}}{2\gamma^2}\phi + \tilde{b}_3. \tag{63}$$

The rest of the moments are explicitly given in Appendix E.

Our model therefore suggests some middle ground between the pessimistic expectations formulated in the two distinct contexts of minisuperspace quantization on one hand, and structure formation on the other. General criticism of the minisuperspace quantization argues that none of the solutions of a minisuperspace model are relevant for the full dynamics because non-symmetric degrees of freedom will always get excited, while concerns about structure formation are based on the statement that a symmetric initial state cannot evolve into a structured state in which the symmetry is broken. In our model, we find that anisotropy is, generically, generated, but there are also states that retain an initial isotropic form. The latter is not prohibited by uncertainty relations.

6 Numerical analysis of the model

The complicated structure of equations of motion at high orders in moments is illustrated by the equations collected in the appendices. It implies that analytical investigations are possible only in certain particular cases, as shown here for exact isotropy or slowly varying anisotropy. Going beyond these regimes requires a numerical implementation to solve the equations of motion and interpret the dynamics. For this numerical study, a truncation of the system to fifth order in moments has been considered by neglecting sixth and higher-order moments. (For the consistency of such truncations, see [17].)

The full phase space of our system has coordinates $\{v, \gamma, \beta, p_\beta, K^{ijkl}\}$. As shown by the schematic equations (21)–(26), the equations of motion for $\{\gamma, p_\beta, K^{ij0k}\}$ form an independent subsystem that is decoupled from the remaining equations. Therefore, one can solve this subsystem without considering the whole set of equations of motion. Once the evolution of $\{\gamma, p_\beta, K^{ij0k}\}$ has been obtained, the equations of motion for the rest of the moments, K^{ijkl} with $k \neq 0$, and for the expectation values v and β can be solved.

For specific numerical solutions, we will assume an initial quantum state given by a product of two Gaussians, one in the volume v and the other one in the shape-parameter β, centered at initial expectation values v_0 and β_0 respectively. The moments in such a state are

$$
K^{ijkl} = \begin{cases} \gamma_0^{-j} v_0^{j-i} \, 2^{-(i+j+k+l)} \hbar^{j+l} \sigma_v^{i-j} \sigma_\beta^{k-l} \frac{i!\,j!}{(i/2)!(j/2)!} \frac{k!\,l!}{(k/2)!(l/2)!}, & \text{if all indices are even,} \\[2ex] 0, & \text{otherwise,} \end{cases}
\tag{64}
$$

where σ_v and σ_β are the Gaussian widths in the volume and in the shape-parameter, respectively, and γ_0 is the initial value of γ.

In order to construct a semiclassical state peaked on a classical trajectory, we will impose small initial relative fluctuations. In particular, for the isotropic sector, both K^{2000} and K^{0200} have to be small:

$$
K^{2000} = \frac{\sigma_v^2}{2v_0^2} \ll 1, \qquad K^{0200} = \frac{v_0^2 \hbar^2}{2\sigma_v^2 \gamma_0^2} \ll 1 \,.
\tag{65}
$$

Therefore, the Gaussian width needs to be chosen as $\hbar v_0/\gamma_0 \ll \sigma_v \ll v_0$. In addition, if one requires the state to be unsqueezed, with equal absolute fluctuations for both conjugate variables, $G^{0200} = G^{2000}$, then one gets the specific value $\sigma_v = \sqrt{\hbar}$ for the width. This is the value we will consider for both the Gaussian width in the volume σ_v and in the shape-parameter $\sigma_\beta = \sqrt{\hbar}$. From this point on, and for all the numerical simulations, $\hbar$ will be set equal to one. In these units, the Gaussian widths will then be chosen as $\sigma_v = \sigma_\beta = 1$, and the requirement of an initial peaked state is summarized by the condition $1 \ll v_0 \ll \gamma_0$. However, the values of the Gaussian widths have also been altered by several orders of magnitudes to check that this choice does not qualitatively affect our main results.

Regarding initial conditions for the expectation values $\{v_0, \gamma_0, \beta_0, p_\beta\}$, following the discussion of peaked states, v_0 and γ_0 have been chosen very large, obeying the constraint $v_0 \ll \gamma_0$. In the anisotropic sector, β does not appear in the equations of motion for the moments and is therefore less relevant than the other variables, while we would like to analyze the behavior of the system for different values of p_β. For convenience, we will choose a small initial value for β_0 (around unity), so that we begin our simulations with a nearly isotropic universe. Due to the form of the Hamiltonian (7), the maximum allowed value for p_β is $p_\beta = \gamma$. Therefore, we will explore the behavior of the system for different values of p_β between a very small value (corresponding to slowly varying anisotropy) and the fixed γ_0. Since we allow for small values of both β_0 and p_β, we do not use a sharply peaked state in these variables because relative moments in the anisotropy sector may be large if the basic expectation values are small.

Based on the possible values of p_β, this section is divided into two parts. We will first consider slowly varying anisotropy, that is $p_\beta \ll \gamma$, in order to test the analytical results obtained in Sections 3.2 and 4. In the second part we will consider solutions in which the shape-parameter is evolving more rapidly, leaving the previous quasi-harmonic regime. In this case, p_β will be of the same order of magnitude as γ. Since we observe different behaviors of the system for different values of γ_0, we will further subdivide the second part into three parts. In the first and second parts we will analyze the evolution of the moments for small and big values of back-reaction on the evolution of the expectation values v and β will be studied.

6.1 Slowly varying anisotropy ($p_\beta \ll \gamma$)

In Section 4, we have found approximate analytical solutions for slowly-evolving anisotropy. At zeroth order in the anharmonicity parameters, all moments K^{ijkl} are constants of motion, whereas at next order, including terms of order $1/\gamma$, their evolution is determined either by a polynomial of order k in internal time ϕ ($r = 0$) or by a dependence of the form $\phi^{k/2}$ ($r \neq 0$).

Numerically we have observed that moments of the form K^{ij00} follow the same qualitative behavior as the isotropic ones: They are constant throughout the whole evolution and do not feel the presence of an anisotropy as long as p_β is small. Moments of the anisotropy sector, K^{00ij}, do not have an isotropic counterpart. According to
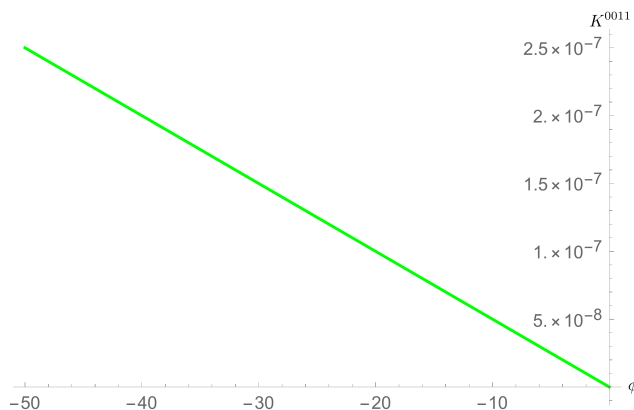
Figure 1: This plot shows an example of the evolution of a purely anisotropic moment, more precisely the correlation between the shape-parameter and its conjugate momentum K^{0011}, for the case of a slowly-varying anisotropy. For this plot $\gamma_0 = 10^8$ and $p_\beta = 10^{-1}$ have been chosen.

our analytical results, pure moments of p_β, K^{000n}, are exactly conserved during evolution, which is easily confirmed numerically. Perhaps surprisingly, we find out that pure fluctuations of β, K^{00n0}, are also conserved up to a high degree of precision. Based on the approximate analytical solution, by contrast, one would expect an evolution of the form ϕ^n or $\phi^{n/2}$. This discrepancy seems to be a consequence of the specific initial state, in particular the uncorrelated nature of the anisotropic state for β and p_β. Therefore, most correlations of the form K^{ijnm}, with $n \neq 0$ and $m \neq 0$, are zero; and they are the only moments that contribute to $\dot{K}^{00n0}$. The correlations are not conserved and may therefore build up during evolution, but the rate is suppressed by a factor of $1/H$ compared with the evolution of expectation values.

While generic correlations of the form K^{00nm}, with $n \neq 0$ and $m \neq 0$, are not conserved in numerical solutions but rather evolve as linear functions in time ϕ, they do not follow the analytical behavior (52) unless $n = 2$.

For general moments G^{ijkl} mix both sectors, the evolution slightly differs from the approximate one. In general the behavior of a specific moment G^{ijkl} is qualitatively the same as its isotropic counterpart, either increasing or decreasing depending on the sign of the difference $(i-j)$, but some of them are slightly accelerated or decelerated. The different behavior does not seem to follow any specific rule based on the values of the indices. Some examples of such corrections are shown in the plots depicted in Figs. 1–3, where the evolution of some relative moments K^{ijkl} are shown. The study clearly shows how the presence of anisotropy affects the evolution of the moments. In all the cases we observe that, instead of a polynomial of order k, the moments follow a linear dependence in time, that is, $K^{ijkl} = c^{ijkl}\phi$ with constants c^{ijkl}.

There might be several reasons for such a disagreement. On the one hand, the approximate analytical solution could be invalid because one or several anharmonicity parameters might not be negligible. On the other hand, it might well happen that closer to the singularity, where this formalism ceases to be valid, one recovers the commented polynomial behavior. Finally, the choice of peaked states could in principle play a relevant role in the behavior of the system, but we have tested that this is not the case. If one allows for a squeezed state, either by increasing or decreasing the value of the Gaussian widths, σ_v and σ_β, moments depart a little bit more from their corresponding isotropic behavior. But in all cases, deviations from their isotropic counterparts stay small during the whole evolution.

In summary, for uncorrelated Gaussian initial states we have found that pure fluctuations of β (K^{00n0}) and p_β (K^{00n0}) are conserved quantities, while the remaining moments K^{ijkl} are not stabilized to any specific constant values as they approach the singularity. In fact they diverge linearly in internal time ϕ, either to plus or minus infinity.
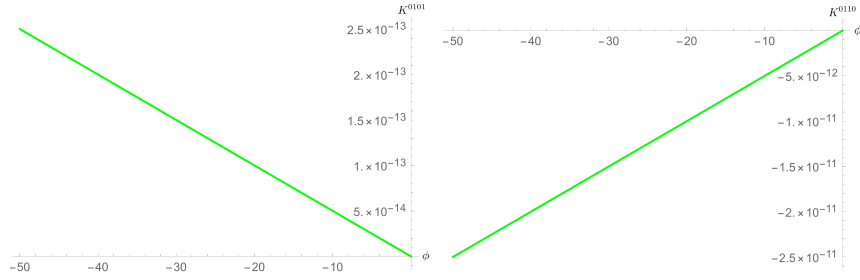
13

Figure 2: Here we show the linear evolution of two moments, K^{0101} and K^{0110}, for the slowly varying anisotropy case. Even if both have a common isotropic counterpart, as their v and p_v indices are the same, they exhibit two opposite behaviors: K^{0101} is increasing, whereas K^{0110} is decreasing. But, in module, both go to infinity and thus they slightly accelerate the exponential diverging of the corresponding absolute moments, G^{0101} and G^{0110}. In any case, this variation is very small as compared with the dominant isotropic behavior. As in the previous figure, here we have chosen $\gamma_0 = 10^8$ and $p_\beta = 10^{-1}$.
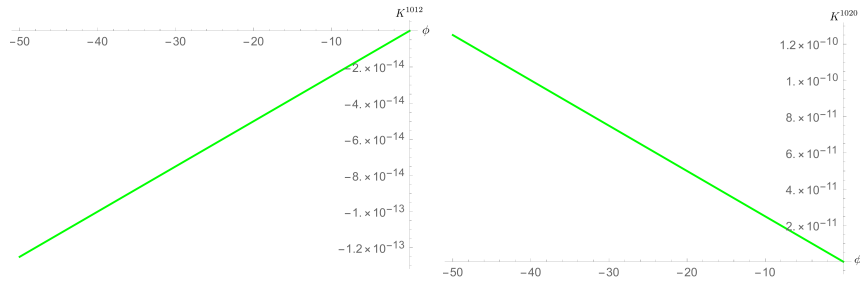

Figure 3: The evolution of the moments K^{1012} and K^{1020} for $\gamma_0 = 10^8$ and $p_\beta = 10^{-1}$. As opossed to the case shown in the previous figure, in this case, the isotropic counterpart G^{10} exponentially approaches zero towards the singularity. Therefore, as can be seen in the plots, the anisotropy decelerates this approach to zero. The only difference between both moments, G^{1012} and G^{1020}, is that the former will approach zero from negative values, whereas the latter will tend to zero from positive values.
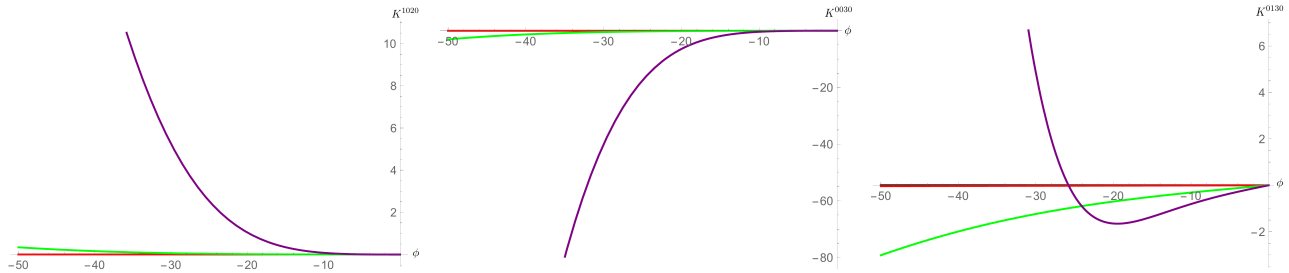
Figure 4: Examples of evolution of dominant moments of the form K^{00m0}, K^{n0m0}, and K^{0lm0} for a relatively small value of $\gamma_0 \sim 10^{20}$. Different colors corresponds to different value of the ratio p_β/γ_0. In particular, purple (0.3), green (10^{-2}), red (10^{-8}) and black (10^{-16}). In fact, the black line is visible only in the last plot. The first two plots show the increasing and decreasing behavior of moments with an even or odd β-index, respectively. The third plot shows the only moment of this type, K^{0130}, that follows the generic rule only for certain values of p_β.

6.2 General anisotropy

We now present our extension of the numerical study to the case of a general anisotropy. We have systematically studied different ranges of values for all the parameters involved in the evolution in order to understand the global behavior. In particular, we have found a qualitative change in the evolution of the moments, and in the approach of the system to the singularity, depending on the value of γ_0. Apart from this, another important variable is p_β, which measures the departure from the harmonic behavior. As always, for the Hamiltonian (7) to be real, the relation $p_\beta < \gamma_0$ must hold, indicating a relationship between these two scales. The relation between p_β and p_{v_0} or v_0, by contrast, appears to be irrelevant for the qualitative physical behavior of the system.

Accordingly, this subsection is divided into three parts. In the first two we analyze the behavior of the moments for different ranges of values of γ_0, whereas in the last one, the effects of the quantum back-reaction on classical trajectories are studied.

6.2.1 Evolution of the moments for small values of γ_0

We first turn to the behavior of the model for $\gamma_0 \lesssim 10^{20}$. In this regime, we have analyzed the increase of p_β approaching its upper bound given by γ_0. As commented above, in order to construct the formalism under consideration, we have introduced a truncation, assuming that sixth- and higher-order moments are negligible. It is expected that this approximation holds only as long as the state is sufficiently peaked around the classical trajectory. As one departs from the harmonic case the numerical solutions might eventually break down, signaling the limited validity of the approximation. Due to such limitations, in this case, it was not possible for us to consider values of p_β greater than $0.3\gamma_0$.

While we depart from the limiting case of slowly varying approximation, more and more moments begin to deviate from their harmonic behavior. None of the relative moments K^{ijkl} stabilize their behavior; they either increase or decrease continually towards the singularity. It is interesting to note that when we depart from the previous regime, we see a certain dilation in the evolution of moments for a given amount of scalar-field time. That is, the evolution of a given moment for a large value of p_β during a short period of time corresponds exactly to the whole evolution of the same moment for a small value of p_β but a longer period of time. This result indicates some scaling in time, parametrized by p_β. Therefore, this variable drives the velocity of evolution of the moments, in much the same way that it controls the velocity of the anisotropy, even though it is not canonically conjugate to the moments. This effect can be seen in the three plots depicted in Fig. 4.

Concerning the moments, there is a subset that dominate the dynamics in the sense that they evolve faster than the other ones, diverging exponentially towards the singularity, and get a larger absolute value than the rest of the moments. In particular the most dominant moments are the pure fluctuations of β (K^{00m0}). The other relevant moments are the correlations between β and v (K^{n0m0}) and between β and p_v (K^{0lm0}). All the mentioned moments (with the particular exception of K^{0130} for certain values of p_β, see Fig. 4) are increasing for even values of the index m, corresponding to β, and decreasing for odd values. But this rule does not apply to other generic
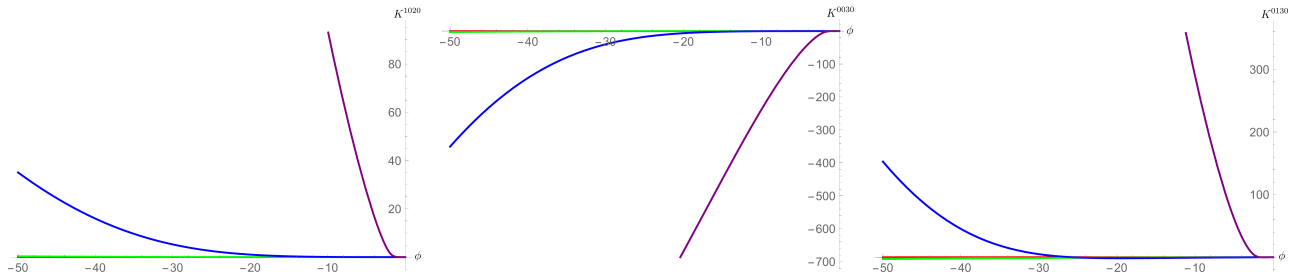
Figure 5: Evolution of the same moments as in Fig. (4) for a larger value of $\gamma_0 \sim 10^{25}$. In this case we have chosen the following p_β/γ_0: 0.7 (purple line), 0.3 (blue line), 10^{-2} (green line), 10^{-8} (red line) and 10^{-16} (black line). The first two plots show that the increasing and decreasing behavior in terms of the parity of β is the same as in the case of small γ_0. The third plot shows the anomalous behavior of K^{0130}: even if it has an odd-order in β it increases towards plus infinity.

moments K^{ijkl}. In fact, some of them, depending on the value of p_β, diverge to minus or plus infinity, as the commented K^{0130}. Nonetheless, in the following section we will see that this rule generalizes to all the moments when the value of γ_0 is larger than the one considered here.

On the contrary, pure fluctuations of p_v (K^{0n00}) evolve very slowly, and they keep a small value along the whole evolution. Therefore, these are the least affected moments by the presence of the anisotropy.

6.2.2 Evolution of the moments for large values of γ_0

For large values of γ_0, approximately in the range 10^{21}–10^{25}, the isotropic dynamics completely dominates the behavior of different moments. Even for large values of p_β, up to $0.4\gamma_0$, the evolution of the moments follow exactly the isotropic one and all K^{ijkl} are constant. This is also the case if one squeezes the state by modifying the relation $\sigma_v = \sqrt{\hbar}$ by several orders of magnitude. The regime of large γ_0 might also be interpreted as the classical limit of the model since it implies a very large value of the classical part of the Hamiltonian, which then dominates over all moment terms.

Contrary to the previous case where the maximum value we could consider for p_β was found to be around $0.3\gamma_0$, here we can choose values as large as $0.7\gamma_0$. This result is consistent with the smallness of moment terms relative to the classical contribution to the Hamiltonian, such that truncation effects should be negligible. Only for values of p_β larger than $0.4\gamma_0$ do the moments depart from their corresponding slowly varying anisotropic behavior. Classical trajectories are then modified by quantum back-reaction, as will be shown in the next subsection.

For this case, we observe that more moments depart from their harmonic behavior than in the previous (small γ_0) case. For such moments, we have been able to find a general rule that characterizes their divergence when approaching the singularity. The key parameter is the moment index that refers to the shape-parameter β. More precisely, as we approach the singularity, moments $K^{nm(2l)m}$ with an even index in β are increasing functions, diverging to positive infinity, whereas moments $K^{nm(2l+1)m}$ with an odd index in β are decreasing functions. This rule agrees with what we found in the previous section for a certain subset of moments, but here it is obeyed by all activated moments, with an only exception: the moment K^{0130}, as can be seen in Fig. 5. Note that this is a fourth-order moment and, thus, the origin of this anomalous behavior might be the truncation we have introduced. It might well happen that, considering higher-order moments, K^{0130} would obey too the commented general rule.

The absolute value the moments reach at the end of their evolution is much greater in this case than in the previous one. Furthermore, moments, which in the previous case were approximately constant, are now evolving. This outcome is not related to the fact that here we have been able to get closer to the limit of $p_\beta \approx \gamma_0$: Even here, the moments are completely constant if we use the maximum value of the ratio p_β/γ_0 considered before (around 0.3), while for larger ratios the moments start increasing much faster than in the previous case.

Finally, for extremely large values of $\gamma_0 \gtrsim 10^{25}$, the previously mentioned effects are absent and all the K^{ijkl} follow an almost constant evolution even when we reach the limit $p_\beta \approx \gamma_0$. This limiting case is relevant because it shows that the dilation effects in evolution cannot be explained simply by a p_ϕ enlarged by moment terms, which
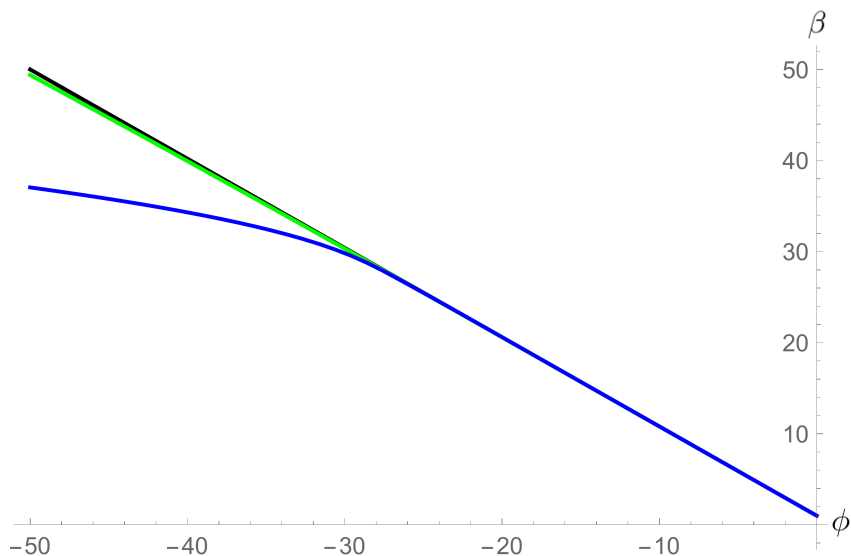
Figure 6: In this plot one can see how the quantum corrections produce a decrease in the growth of the anisotropy close to the singularity for $10^{21} \lesssim \gamma_0 \lesssim 10^{25}$. Different colors correspond to different truncations in moments: black (classical solution), red (second-order), green (third-order) and blue (fourth-order).

would rescale any ϕ-derivative in the equations of motion. If this were the reason for dilation effects, it should occur even in the case of very large γ_0, in particular for $p_\beta \approx \gamma_0$ which implies that p_ϕ is tiny, leading to large rescaling factors of $1/p_\phi$.

6.2.3 Quantum modifications of the classical trajectories

The evolution of the expectation values β and v is generically modified by quantum back-reaction effects. Nonetheless, in the approximation of slowly varying anisotropy we have analytically found, and numerically checked, that these parameters follow their classical trajectories up to a high degree of precision. But in the case of rapidly varying anisotropy, $p_\beta \approx \gamma_0$, we do observe a departure from the classical behavior.

In the classical setting, the evolution of the shape-parameter β is a linearly increasing function of time, viewed towards the singularity. However, as seen in Fig. 6, for large values of γ_0, quantum effects give rise to a decrease of the anisotropization toward the singularity. That is, quantum modifications decelerate the divergent behavior of the shape-parameter towards the singularity.

In the evolution of the volume v, which classically follows an exponentially decreasing behavior towards the singularity, we observe three qualitatively different behaviors in the presence of quantum effects: For relatively small values of γ_0, $\gamma_0 \lesssim 10^{20}$, the volume collapses faster at the beginning of the evolution, and then follows the standard exponential behavior, as seen in Fig. 7. Thus, it approaches the singularity faster in the presence of quantum corrections. This early quantum modification, appearing as a "jump" in the volume, is due to the fact that the chosen initial state is not a coherent state of the model. Therefore, initially vanishing moments are turned on, reaching their "natural" value adapted to this particular dynamics. This transition, which has also been seen at late times in other models [9], produces the fast but short initial collapse. Once the moments settle down to a more coherent behavior, we observe the usual exponential collapse of the volume. In this case, the small value of γ_0 is not strong enough to hide these quantum back-reaction effects at an early epoch of the evolution.

Correspondingly, for greater values of γ_0, $10^{21} \lesssim \gamma_0 \lesssim 10^{25}$, we do not observe the initial "jump" in the volume. In that case, quantum corrections lead to a slow-down of the rate of collapse of the volume towards the singularity, as seen in Fig. 8. This result is consistent with the softening in the increase of shape-parameter towards the
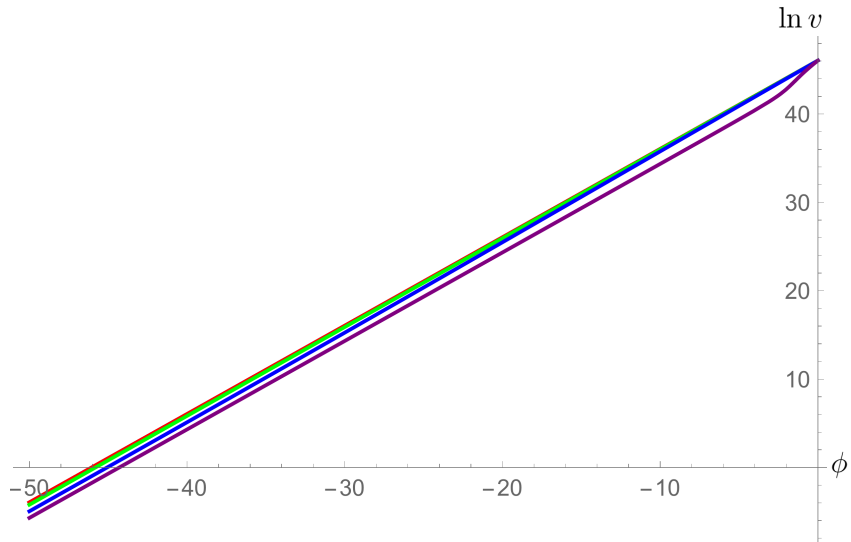
Figure 7: The approach of the volume to the singularity in a logarithmic plot for $\gamma_0 \lesssim 10^{20}$. Different colors represent different order of truncations: black (classical solution), red (second-order), green (third-order), blue (fourth-order), and purple (fifth-order).

singularity, and points to a smoothening of the singularity by quantum effects. [4] (We would like to note that for this range of γ_0 the fifth-order truncation of the system has shown strong numerical instabilities and, therefore, the commented result has been derived from the system truncated up to fourth order.)

Finally, as explained previously, very large values of γ_0, $\gamma_0 \gtrsim 10^{25}$, correspond to the classical limit of the model. For this case, quantum effects are not visible in our numerical implementation, and expectation values follow exactly their classical trajectories. In fact, due to the large value of the classical Hamiltonian, one would expect that quantum modifications of these trajectories will appear only much closer to the singularity.

7   Conclusions

We have considered a Bianchi I model written in terms of Misner-type variables. For the sake of simplicity, but without loss of generality, we have taken only one anisotropic direction to perform our analysis. The classical model contains a singularity located at a vanishing value of the volume $v = 0$. In this paper we have studied the quantum evolution of the system when approaching this singularity. With additional approximations the model is simple enough to allow analytical statements about generic moments, yet complex enough to show non-trivial behavior.

For such a purpose, we have made use of a formalism based on a decomposition of a quantum state into its infinite set of moments. Furthermore, we have considered a set of variables that allow us to decouple the quantum evolution for some of the relevant variables, simplifying both the analysis and interpretation. Because of this choice, we have been able to find analytical solutions for all moments when the anisotropy evolves slowly (that is, when the conjugate momentum of the shape-parameter p_β is much smaller than the Hamiltonian γ of the reference isotropic model) by performing an expansion in p_β/γ. In this limiting case we can compare with the isotropic solution ($p_\beta = 0$), showing that the evolution of the volume is not affected. But moments that in the isotropic case were constant feel the anisotropy and evolve with a polynomial dependence on time. We remark that the moments with vanishing index on β play a special role in the evolution, being constants of motion.

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[4]We would like to point out that a previous study in the context of quantum geometrodynamics also suggests an avoidance of the classical singularity [19].
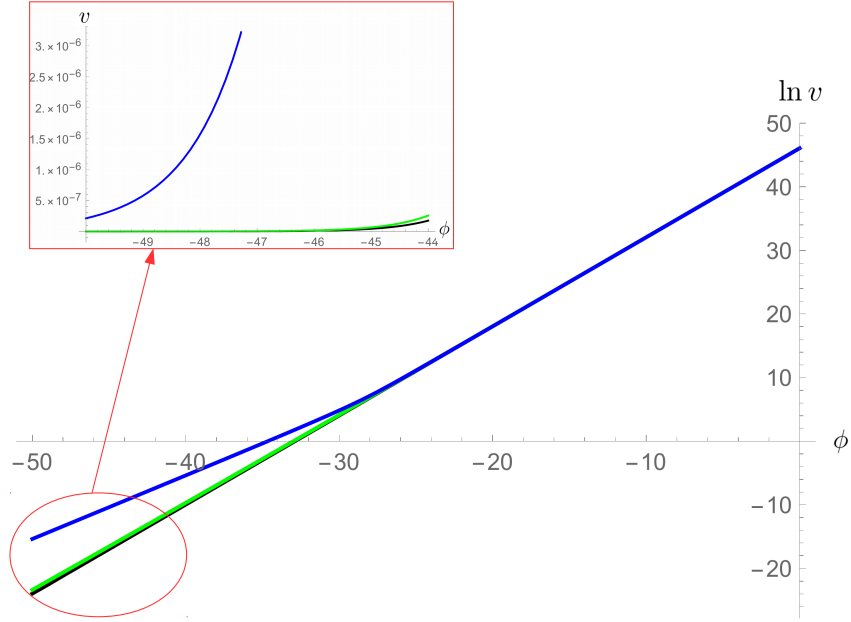
Figure 8: The logarithm of the volume approaching the singularity for $10^{21} \lesssim \gamma_0 \lesssim 10^{25}$. The zoom-in shows (in a linear plot) the last stages of the evolution of the volume towards the singularity. Different colors correspond to different truncations in moments: black (classical solution), red (second-order), green (third-order) and blue (fourth-order). The collapse of the volume is slowed down by quantum effects.

Going beyond the previous approximation by including terms of arbitrary powers in γ^{-1}, we have analyzed the particular scenario when, at an initial time, the shape-parameter β and its momentum p_β are zero. In this case the universe is initially isotropic, but one would expect the generation of an anisotropy during quantum evolution, owing to the performed classical reduction of symmetry that is not consistent in the quantum realm. (Note that, based on the uncertainty principle, not all moments can vanish.) Nonetheless, we have found an analytic solution and shown that anisotropy is not necessarily generated during quantum evolution. This interesting new result could be a consequence of the completely decoupled nature of degrees of freedom in the classical Hamiltonian, which may allow a classical symmetry to be unaltered by quantization. Nevertheless, it suggests an interesting middle ground between having no symmetry-preserving solutions at all (a common criticism of minisuperspace quantization) and not generating any non-symmetric degrees of freedom (a concern sometimes voiced about cosmological structure formation).

In order to go beyond the limiting case, we have performed a numerical simulation of the complete system up to fifth order in moments. We have used the limiting case to check our method and study the evolution of the moments towards the singularity. After that, we have considered a general case and analyzed the evolution for the different ranges of p_β and the initial value of γ. We have shown how some moments are activated during evolution, departing from their isotropic behavior and diverging as they approach the singularity. The greater the value of p_β with respect to γ_0 the more dominant the moments, mainly those that represent pure fluctuations in β. In addition, they are not only more dominant for that range, but also allow us to see a longer stretch of their evolution.

Concerning the expectation values, we have analyzed the quantum evolution of the volume v and the shape-parameter β, which are determined by previous variables and moments. We have shown that quantum back-reaction has a more relevant effect when the initial value of γ is bigger. It then appreciably acts only in a region near the singularity, where we expect quantum effects even if the initial state is classical. In that case, the shape-parameter, which classically increases towards the singularity, decelerates this divergent behavior, softening the anisotropization close to the singularity. In a similar sense, the exponential collapse of volume towards the singularity has been smoothed by quantum back-reaction effects, decreasing the rate of collapse.

Finally, we would like to note that several statements about the generic moment orders relied on the β-

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independence of the classical Hamiltonian, which would no longer be the case in models with an anisotropy potential. In such models, it would be more difficult to obtain corresponding statements, even if a similar behavior might still be realized. One would then have to rely on numerical investigations based on the methods developed here. Such results, however, would always be state dependent and require a careful analysis of implications of how one chooses an initial state.

Acknowledgments

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A Second-order equations of motion for absolute moments

In order to give an idea about the system of equations, we present the second-order truncation of equations of motion, used up to fifth order in the numerics. The equations of motion for the variables $\{v, p_v, \beta, p_\beta\}$ are

$$
\begin{aligned}
\dot{v} &= \frac{1}{H^5}\bigg[H^4 v^2 p_v + \frac{3}{2} v^4 p_v p_\beta^2 G^{0200} - v^2 p_\beta \left(p_\beta^2 + 2 p_v^2 v^2 \right) G^{0101} - v p_v p_\beta \left(2 p_\beta^2 + p_v^2 v^2 \right) G^{1001}, \\
&+ v^2 p_v \left(\frac{p_v^2 v^2}{2} + p_\beta^2 \right) G^{0002} + v p_\beta^2 \left(2 p_\beta^2 + p_v^2 v^2 \right) G^{1100} + p_v p_\beta^2 \left(p_\beta^2 + \frac{1}{2} p_v^2 v^2 \right) G^{2000} \bigg],
\end{aligned}
$$

$$
\begin{aligned}
\dot{p}_v &= \frac{1}{H^5}\bigg[- H^4 v p_v^2 - \frac{3}{2} v p_v^4 p_\beta^2 G^{2000} - \frac{1}{2} v p_v^2 \left(2 p_\beta^2 + p_v^2 v^2 \right) G^{0002} + p_\beta p_v^2 \left(p_\beta^2 + 2 p_v^2 v^2 \right) G^{1001}, \\
&- p_\beta^2 p_v \left(2 p_\beta^2 + p_v^2 v^2 \right) G^{1100} + v p_v p_\beta \left(2 p_\beta^2 + v^2 p_v^2 \right) G^{0101} - \frac{1}{2} v p_\beta^2 \left(2 p_\beta^2 + v^2 p_v^2 \right) G^{0200} \bigg],
\end{aligned}
$$

$$
\begin{aligned}
\dot{\beta} &= \frac{1}{H^5}\bigg[- H^4 p_\beta - \frac{3}{2} v^2 p_v^2 p_\beta G^{0002} - \frac{1}{2} p_v^2 p_\beta \left(p_\beta^2 + 2 v^2 p_v^2 \right) G^{2000} - v p_v p_\beta \left(2 p_\beta^2 + v^2 p_v^2 \right) G^{1100}, \\
&+ v^2 p_v \left(v^2 p_v^2 + 2 p_\beta^2 \right) G^{0101} - \frac{1}{2} v^2 p_\beta \left(p_\beta^2 + 2 v^2 p_v^2 \right) G^{0200} + v p_v^2 \left(v^2 p_v^2 + 2 p_\beta^2 \right) G^{1001} \bigg],
\end{aligned}
$$

$$
\dot{p}_\beta = 0.
$$

The equations for second-order moments are

$$
\dot{G}^{0002} = 0,
$$

$$
\dot{G}^{0011} = \frac{v p_v}{H^3}\bigg[p_v p_\beta G^{1001} + v p_\beta G^{0101} - v p_v G^{0002} \bigg],
$$

$$
\dot{G}^{0020} = \frac{v p_v}{H^3}\bigg[2 p_v p_\beta G^{1010} + 2 v p_\beta G^{0110} - 2 v p_v G^{0011} \bigg],
$$

$$
\dot{G}^{0101} = \frac{p_v}{H^3}\bigg[- v p_v p_\beta G^{0002} + p_v p_\beta^2 G^{1001} + v \left(2 p_\beta^2 - v^2 p_v^2 \right) G^{0101} \bigg],
$$

$$
\dot{G}^{0110} = \frac{p_v}{H^3}\bigg[- v p_v p_\beta G^{0011} - v^2 p_v G^{0101} + p_v p_\beta^2 G^{1010} + v p_v p_\beta G^{1100} + v^2 p_\beta G^{0200} + v \left(2 p_\beta^2 - v^2 p_v^2 \right) G^{0110} \bigg],
$$

$$
\dot{G}^{0200} = \frac{p_v}{H^3}\bigg[- 2 v p_v p_\beta G^{0101} + 2 p_v p_\beta^2 G^{1100} + 2 v \left(2 p_\beta^2 - v^2 p_v^2 \right) G^{0200} \bigg],
$$

$$
\dot{G}^{1001} = \frac{v}{H^3}\bigg[v p_v p_\beta G^{0002} - v p_\beta^2 G^{0101} + p_v \left(v^2 p_v^2 - 2 p_\beta^2 \right) G^{1001} \bigg],
$$

$$
\dot{G}^{1010} = \frac{v}{H^3}\bigg[p_v^2 p_\beta G^{2000} - v p_v^2 G^{1001} + v p_v p_\beta G^{0011} + v p_v p_\beta G^{1100} - v p_\beta^2 G^{0110} + p_v \left(v^2 p_v^2 - 2 p_\beta^2 \right) G^{1010} \bigg],
$$

$$
\dot{G}^{1100} = \frac{1}{H^3}\bigg[p_v^2 p_\beta^2 G^{2000} - v p_v^2 p_\beta G^{1001} + v^2 p_v p_\beta G^{0101} - v^2 p_\beta^2 G^{0200} \bigg],
$$

$$
\dot{G}^{2000} = \frac{2v}{H^3}\bigg[v p_v p_\beta G^{1001} - v p_\beta^2 G^{1100} + p_v \left(v^2 p_v^2 - 2 p_\beta^2 \right) G^{2000} \bigg].
$$

B The equation for the volume at fifth order in moments

$$
\begin{aligned}
\dot{v} =\ & -\frac{3p_\beta^2\left(p_\beta^4+12\gamma^2 p_\beta^2+8\gamma^4\right)G^{0500}v^6}{8H^{11}}+\frac{5\left(3\gamma p_\beta^4+4\gamma^3 p_\beta^2\right)G^{0400}v^5}{8H^9}+\frac{5p_\beta\gamma\left(15p_\beta^4+40\gamma^2 p_\beta^2+8\gamma^4\right)G^{0401}v^5}{8H^{11}}\\
& -\frac{\left(p_\beta^4+4\gamma^2 p_\beta^2\right)G^{0300}v^4}{2H^7}-\frac{p_\beta\left(3p_\beta^4+24\gamma^2 p_\beta^2+8\gamma^4\right)G^{0301}v^4}{2H^9}-\frac{\left(12p_\beta^6+159\gamma^2 p_\beta^4+136\gamma^4 p_\beta^2+8\gamma^6\right)G^{0302}v^4}{4H^{11}}\\
& -\frac{5p_\beta^2\gamma\left(18p_\beta^4+41\gamma^2 p_\beta^2+4\gamma^4\right)G^{1400}v^4}{8H^{11}}+\frac{3p_\beta^2\gamma G^{0200}v^3}{2H^5}+\frac{3\left(3\gamma p_\beta^3+2\gamma^3 p_\beta\right)G^{0201}v^3}{2H^7}\\
& +\frac{3\left(2\gamma^5+21p_\beta^2\gamma^3+12p_\beta^4\gamma\right)G^{0202}v^3}{4H^9}+\frac{15p_\beta\gamma\left(4p_\beta^4+13\gamma^2 p_\beta^2+4\gamma^4\right)G^{0203}v^3}{4H^{11}}\\
& +\frac{p_\beta^2\left(4p_\beta^4+27\gamma^2 p_\beta^2+4\gamma^4\right)G^{1300}v^3}{2H^9}+\frac{p_\beta\left(12p_\beta^6+159\gamma^2 p_\beta^4+136\gamma^4 p_\beta^2+8\gamma^6\right)G^{1301}v^3}{2H^{11}}\\
& -\frac{p_\beta\left(p_\beta^2+2\gamma^2\right)G^{0101}v^2}{H^5}-\frac{\left(2p_\beta^4+11\gamma^2 p_\beta^2+2\gamma^4\right)G^{0102}v^2}{2H^7}-\frac{p_\beta\left(2p_\beta^4+21\gamma^2 p_\beta^2+12\gamma^4\right)G^{0103}v^2}{2H^9}\\
& -\frac{\left(8p_\beta^6+136\gamma^2 p_\beta^4+159\gamma^4 p_\beta^2+12\gamma^6\right)G^{0104}v^2}{8H^{11}}-\frac{3p_\beta^2\gamma\left(4p_\beta^2+\gamma^2\right)G^{1200}v^2}{2H^7}\\
& -\frac{3p_\beta\gamma\left(12p_\beta^4+21\gamma^2 p_\beta^2+2\gamma^4\right)G^{1201}v^2}{2H^9}-\frac{3\gamma\left(48p_\beta^6+186\gamma^2 p_\beta^4+79\gamma^4 p_\beta^2+2\gamma^6\right)G^{1202}v^2}{4H^{11}}\\
& -\frac{p_\beta^2\left(12p_\beta^6+159\gamma^2 p_\beta^4+136\gamma^4 p_\beta^2+8\gamma^6\right)G^{2300}v^2}{4H^{11}}+\frac{\gamma v}{H}+\frac{\left(\gamma^3+2p_\beta^2\gamma\right)G^{0002}v}{2H^5}+\frac{\left(2\gamma p_\beta^3+3\gamma^3 p_\beta\right)G^{0003}v}{2H^7}\\
& +\frac{\gamma\left(8p_\beta^4+24\gamma^2 p_\beta^2+3\gamma^4\right)G^{0004}v}{8H^9}+\frac{p_\beta\gamma\left(8p_\beta^4+40\gamma^2 p_\beta^2+15\gamma^4\right)G^{0005}v}{8H^{11}}+\frac{p_\beta^2\left(2p_\beta^2+\gamma^2\right)G^{1100}v}{H^5}\\
& +\frac{p_\beta\left(2p_\beta^4+11\gamma^2 p_\beta^2+2\gamma^4\right)G^{1101}v}{H^7}+\frac{\left(4p_\beta^6+54\gamma^2 p_\beta^4+45\gamma^4 p_\beta^2+2\gamma^6\right)G^{1102}v}{2H^9}\\
& +\frac{p_\beta\left(4p_\beta^6+98\gamma^2 p_\beta^4+177\gamma^4 p_\beta^2+36\gamma^6\right)G^{1103}v}{2H^{11}}+\frac{3\left(12\gamma p_\beta^6+21\gamma^3 p_\beta^4+2\gamma^5 p_\beta^2\right)G^{2200}v}{4H^9}\\
& +\frac{3p_\beta\gamma\left(36p_\beta^6+177\gamma^2 p_\beta^4+98\gamma^4 p_\beta^2+4\gamma^6\right)G^{2201}v}{4H^{11}}-\frac{p_\beta\gamma\left(2p_\beta^2+\gamma^2\right)G^{1001}}{H^5}-\frac{\left(\gamma^5+10p_\beta^2\gamma^3+4p_\beta^4\gamma\right)G^{1002}}{2H^7}\\
& -\frac{\left(4\gamma p_\beta^5+22\gamma^3 p_\beta^3+9\gamma^5 p_\beta\right)G^{1003}}{2H^9}-\frac{\left(9\gamma^7+138p_\beta^2\gamma^5+152p_\beta^4\gamma^3+16p_\beta^6\gamma\right)G^{1004}}{8H^{11}}\\
& -\frac{\left(2p_\beta^6+11\gamma^2 p_\beta^4+2\gamma^4 p_\beta^2\right)G^{2100}}{2H^7}-\frac{\left(2p_\beta^7+45\gamma^2 p_\beta^5+54\gamma^4 p_\beta^3+4\gamma^6 p_\beta\right)G^{2101}}{2H^9}\\
& -\frac{\left(4p_\beta^8+194\gamma^2 p_\beta^6+549\gamma^4 p_\beta^4+194\gamma^6 p_\beta^2+4\gamma^8\right)G^{2102}}{4H^{11}}-\frac{3\left(8\gamma p_\beta^8+56\gamma^3 p_\beta^6+39\gamma^5 p_\beta^4+2\gamma^7 p_\beta^2\right)G^{3200}}{4H^{11}}\\
& +\frac{p_\beta^2\gamma\left(2p_\beta^2+\gamma^2\right)G^{2000}}{2H^5 v}+\frac{\left(2\gamma p_\beta^5+11\gamma^3 p_\beta^3+2\gamma^5 p_\beta\right)G^{2001}}{2H^7 v}+\frac{\left(2\gamma^7+45p_\beta^2\gamma^5+54p_\beta^4\gamma^3+4p_\beta^6\gamma\right)G^{2002}}{4H^9 v}\\
& +\frac{\left(4\gamma p_\beta^7+98\gamma^3 p_\beta^5+177\gamma^5 p_\beta^3+36\gamma^7 p_\beta\right)G^{2003}}{4H^{11}v}+\frac{\left(12\gamma^2 p_\beta^6+21\gamma^4 p_\beta^4+2\gamma^6 p_\beta^2\right)G^{3100}}{2H^9 v}\\
& +\frac{\left(4p_\beta\gamma^8+98p_\beta^3\gamma^6+177p_\beta^5\gamma^4+36p_\beta^7\gamma^2\right)G^{3101}}{2H^{11}v}-\frac{p_\beta^2\gamma^3\left(4p_\beta^2+\gamma^2\right)G^{3000}}{2H^7 v^2}\\
& -\frac{\left(2p_\beta\gamma^7+21p_\beta^3\gamma^5+12p_\beta^5\gamma^3\right)G^{3001}}{2H^9 v^2}-\frac{\left(2\gamma^9+79p_\beta^2\gamma^7+186p_\beta^4\gamma^5+48p_\beta^6\gamma^3\right)G^{3002}}{4H^{11}v^2}\\
& -\frac{\left(12\gamma^2 p_\beta^8+159\gamma^4 p_\beta^6+136\gamma^6 p_\beta^4+8\gamma^8 p_\beta^2\right)G^{4100}}{8H^{11}v^2}+\frac{\left(4p_\beta^2\gamma^7+27p_\beta^4\gamma^5+4p_\beta^6\gamma^3\right)G^{4000}}{8H^9 v^3}\\
& +\frac{\left(8p_\beta\gamma^9+136p_\beta^3\gamma^7+159p_\beta^5\gamma^5+12p_\beta^7\gamma^3\right)G^{4001}}{8H^{11}v^3}-\frac{\left(4p_\beta^2\gamma^9+41p_\beta^4\gamma^7+18p_\beta^6\gamma^5\right)G^{5000}}{8H^{11}v^4}.
\end{aligned}
$$

C   Second-order equations of motion for relative moments

Performing the change of variables $\{v, p_v, \beta, p_\beta, G^{ijkl}\}$ to $\{v, \gamma, \beta, p_\beta, K^{ijkl}\}$, the equations of motion for $\{\gamma, p_\beta, K^{ij0l}\}$ decouple from the rest, p_β and all its fluctuations K^{000n} are constants of motion.

$$
\begin{aligned}
\dot{\gamma} &= \frac{\gamma^2 p_\beta}{H^3}\left(p_\beta(K^{0200} - K^{2000}) - K^{0101} + K^{1001}\right), \\
\dot{p_\beta} &= 0, \\
\dot{K}^{0002} &= 0, \\
\dot{K}^{0101} &= \frac{\gamma p_\beta}{H^3}\left(p_\beta(K^{0101} + K^{1001}) - K^{0002}\right), \\
\dot{K}^{0200} &= \frac{2\gamma p_\beta}{H^3}\left(p_\beta(K^{0200} + K^{1100}) - K^{0101}\right), \\
\dot{K}^{1001} &= -\frac{\gamma p_\beta}{H^3}\left(p_\beta(K^{0101} + K^{1001}) - K^{0002}\right), \\
\dot{K}^{1100} &= -\frac{\gamma p_\beta}{H^3}\left(p_\beta(K^{0200} - K^{2000}) - K^{0101} + K^{1001}\right), \\
\dot{K}^{2000} &= -\frac{2\gamma p_\beta}{H^3}\left(p_\beta(K^{1100} + K^{2000}) - K^{1001}\right).
\end{aligned}
$$

The moments K^{ijkl} related to β are given by

$$
\begin{aligned}
\dot{K}^{0011} &= \frac{\gamma^2}{H^3}\left(p_\beta(K^{0101} + K^{1001}) - K^{0002}\right), \\
\dot{K}^{0020} &= \frac{2\gamma^2}{H^3}\left(p_\beta(K^{0110} + K^{1010}) - K^{0011}\right), \\
\dot{K}^{0110} &= \frac{\gamma}{H^3}\left(-\gamma K^{0101} + p_\beta(\gamma(K^{0200} + K^{1100}) - K^{0011}) + p_\beta^2(K^{0110} + K^{1010})\right), \\
\dot{K}^{1010} &= -\frac{\gamma}{H^3}\left(\gamma K^{1001} - p_\beta(\gamma(K^{1100} + K^{2000}) + K^{0011}) + p_\beta^2(K^{0110} + K^{1010})\right).
\end{aligned}
$$

Just as these equations, the equation for β is also independent of the volume:

$$
\begin{aligned}
\dot{\beta} = -\frac{1}{2H^5}\bigg(&-2\gamma^4(K^{0101} + K^{1001}) + \gamma^2 p_\beta^3(K^{0200} + 4K^{1100} + K^{2000} - 4)) \\
&-4\gamma^2 p_\beta^2(K^{0101} + K^{1001} + p_\beta\left(2\gamma^4(K^{0200} + K^{1100} + K^{2000} + 1) + 3\gamma^2 K^{0002}\right) + 2p_\beta^5\bigg).
\end{aligned}
$$

Finally, the equation of motion for the volume is given by a logarithmic derivative,

$$
\begin{aligned}
\frac{\dot{v}}{v} = \frac{\gamma}{2H^5}\bigg(&2\gamma^4 - 4\gamma^2 p_\beta^2 + 2p_\beta^4 + \gamma^2 K^{0002} + p_\beta^2\left(\gamma^2(3K^{0200} + 2K^{1100} + K^{2000}) + 2K^{0002}\right) \\
&- 2\gamma^2 p_\beta(2K^{0101} + K^{1001}) + 2p_\beta^4(2K^{1100} + K^{2000}) - 2p_\beta^3(K^{0101} + 2K^{1001})\bigg).
\end{aligned}
$$

D   The solution for the quasiharmonic case truncated at third order in moments

In this appendix we present the solution for the quasiharmonic case up to order $\mathcal{O}(2)$ and truncated at third order in moments. For the generic case, with $r \neq 0$, moments K^{ijkl} go as $\phi^{k/2}$ and, in particular moments of the form K^{ij0l} are constants of motion. The other moments are

$$K^{0011} = c_1 + \frac{2\sqrt{\gamma_0^2 + r\phi}}{r}(-K^{0002} + K^{0102} + K^{1002} + p_\beta(K^{0101} - K^{0201} + K^{1001} - K^{1101} - K^{2001})),$$

$$K^{0012} = c_2 + \frac{2\sqrt{\gamma_0^2 + r\phi}}{r}(p_\beta(K^{0102} + K^{1002}) - K^{0003}),$$

$$K^{0110} = c_3 + \frac{2\sqrt{\gamma_0^2 + r\phi}}{r}(-K^{0101} + K^{0201} + K^{1101} + p_\beta(K^{0200} - K^{0300} + K^{1100} - K^{1200} - K^{2100})),$$

$$K^{0111} = c_4 + \frac{2\sqrt{\gamma_0^2 + r\phi}}{r}(p_\beta(K^{0201} + K^{1101}) - K^{0102}),$$

$$K^{0210} = c_5 + \frac{2\sqrt{\gamma_0^2 + r\phi}}{r}(p_\beta(K^{0300} + K^{1200}) - K^{0201}),$$

$$K^{1010} = c_6 + \frac{2\sqrt{\gamma_0^2 + r\phi}}{r}(-K^{1001} + K^{1101} + K^{2001} + p_\beta(K^{1100} - K^{1200} + K^{2000} - K^{2100} - K^{3000})),$$

$$K^{1011} = c_7 + \frac{2\sqrt{\gamma_0^2 + r\phi}}{r}\left(p_\beta(K^{1101} + K^{2001}) - K^{1002}\right),$$

$$K^{0021} = c_8 - \frac{4\sqrt{\gamma_0^2 + r\phi}}{r}(c_2 - p_\beta c_4 - p_\beta c_7)$$
$$+ \frac{4\phi}{r}\left(K^{0003} + p_\beta(-2K^{0102} - 2K^{1002} + p_\beta(K^{0201} + 2K^{1101} + K^{2001}))\right),$$

$$K^{1110} = c_9 + \frac{2\sqrt{\gamma_0^2 + r\phi}}{r}(p_\beta(K^{1200} + K^{2100}) - K^{1101}),$$

$$K^{0120} = c_{10} - \frac{4\sqrt{\gamma_0^2 + r\phi}}{r}(c_4 - p_\beta c_5 - p_\beta c_9)$$
$$+ \frac{4\phi}{r}\left(K^{0102} + p_\beta(-2K^{0201} - 2K^{1101} + p_\beta(K^{0300} + 2K^{1200} + K^{2100}))\right),$$

$$K^{2010} = c_{11} + \frac{2\sqrt{\gamma_0^2 + r\phi}}{r}\left(p_\beta(K^{2100} + K^{3000}) - K^{2001}\right),$$

$$K^{0020} = c_{12} - \frac{4\sqrt{\gamma_0^2 + r\phi}}{r}(c_1 - p_\beta c_3 - c_4 + p_\beta c_5 - p_\beta c_6 - c_7 + p_\beta c_9 + p_\beta c_{11})$$
$$+ \frac{4\phi}{r}(K^{0200} - 2K^{0300} + 2K^{1100} - 4K^{1200} + K^{2000} - 4K^{2100} - 2K^{3000})p_\beta^2$$
$$\frac{-8\phi}{r}\left((K^{0101} - 2K^{0201} + K^{1001} - 3K^{1101} - 2K^{2001})p_\beta + K^{0002} - 2(K^{0102} + K^{1002})\right),$$

$$K^{1020} = c_{13} - \frac{4\sqrt{\gamma_0^2 + r\phi}}{r}(c_7 - p_\beta c_9 - p_\beta c_{11})$$
$$+ \frac{4\phi}{r}\left(K^{1002} + p_\beta(-2K^{1101} - 2K^{2001} + p_\beta(K^{1200} + 2K^{2100} + K^{3000}))\right),$$

and finally

$$
\begin{aligned}
K^{0030} \;=\; & c_{14} + \frac{6}{r}\Bigg(12\phi\big(c_2 + p_\beta(-2c_4 - 2c_7 + p_\beta(c_5 + 2c_9 + c_{11}))\big) + \sqrt{\gamma_0^2 + r\phi}\,(p_\beta(c_{10} + c_{13}) - c_8) \Bigg) \\
& + \frac{12\gamma_0^2}{r^2}\Bigg(c_2 + p_\beta(-2c_4 - 2c_7 + p_\beta(c_5 + 2c_9 + c_{11})) \Bigg) \\
& - \frac{24\gamma_0^2\sqrt{\gamma_0^2 + r\phi}}{r^2}\,p_\beta^3(K^{0300} + 3K^{1200} + 3K^{2100} + K^{3000}) \\
& - \frac{24\gamma_0^2\sqrt{\gamma_0^2 + r\phi}}{r^2}\,p_\beta^2(K^{0201} + 2K^{1101} + K^{2001}) \\
& + \frac{24\gamma_0^2\sqrt{\gamma_0^2 + r\phi}}{r^3}\,p_\beta(K^{0102} + K^{1002}) \\
& + \frac{24\gamma_0^2\sqrt{\gamma_0^2 + r\phi}}{r^3}\,K^{0003} \\
& + \frac{8(\gamma_0^2 + r\phi)^{3/2}}{r^3}\,p_\beta^3(K^{0300} + 3K^{1200} + 3K^{2100} + K^{3000}) \\
& - \frac{24(\gamma_0^2 + r\phi)^{3/2}}{r^3}\,p_\beta^2(K^{0201} + 2K^{1101} + K^{2001}) \\
& + \frac{24(\gamma_0^2 + r\phi)^{3/2}}{r^2}\,p_\beta(K^{0102} + K^{1002}) \\
& - \frac{8(\gamma_0^2 + r\phi)^{3/2}}{r^2}\,K^{0003} \,,
\end{aligned}
$$

with integration constants c_i.

For the particular case with $r = 0$, a moment of the form K^{ijkl} is given by a polynomial of order k in ϕ:

$$
\begin{aligned}
K^{0011} \;=\; & c_1 + \frac{\phi}{\gamma_0}\Bigg(-K^{0002} + K^{0102} + K^{1002} + p_\beta(K^{0101} - K^{0201} + K^{1001} - K^{1101} - K^{2001}) \Bigg), \\
K^{0012} \;=\; & \frac{1}{\gamma_0}\Bigg(\gamma_0 c_2 - \phi K^{0003} + p_\beta\phi(K^{0102} + K^{1002}) \Bigg), \\
K^{0020} \;=\; & c_3 + \frac{p_\beta^2\phi^2}{\gamma_0^2}\Bigg(K^{0200} - 2K^{0300} + 2K^{1100} - 4K^{1200} + K^{2000} - 4K^{2100} - 2K^{3000} \Bigg) \\
& - \frac{\phi^2}{\gamma_0^2}\Bigg(2(K^{0101} - 2K^{0201} + K^{1001} - 3K^{1101} - 2K^{2001})p_\beta + K^{0002} - 2(K^{0102} + K^{1002}) \Bigg) \\
& - \frac{2\phi}{\gamma_0}\Bigg(c_1 - c_7 - c_{11} + p_\beta(-c_6 + c_9 - c_{10} + c_{13} + c_{14}) \Bigg), \\
K^{0021} \;=\; & c_4 + \frac{\phi^2}{\gamma_0^2}\Bigg(K^{0003} + p_\beta(-2K^{0102} - 2K^{1002} + p_\beta(K^{0201} + 2K^{1101} + K^{2001})) \Bigg) \\
& + \frac{2\phi}{\gamma_0}\Bigg(p_\beta(c_7 + c_{11}) - c_2 \Bigg), \\
K^{0030} \;=\; & c_5 + \frac{p_\beta\phi^3}{\gamma_0^3}\Bigg((K^{0300} + 3K^{1200} + 3K^{2100} + K^{3000})p_\beta^2 - 3(K^{0201} + 2K^{1101} + K^{2001})p_\beta \Bigg) \\
& + \frac{\phi^3}{\gamma_0^3}\Bigg(p_\beta\left(+3K^{0102} + 3K^{1002}\right) - K^{0003} \Bigg) + \frac{3\phi^2}{\gamma_0^2}\Bigg(c_2 + p_\beta(-2c_7 - 2c_{11} + p_\beta(c_9 + 2c_{13} + c_{14})) \Bigg) \\
& + \frac{3\phi}{\gamma_0}\Bigg(p_\beta(c_8 + c_{12}) - c_4 \Bigg),
\end{aligned}
$$

for pure anisotropy moments, and

$$K^{0110} = c_6 + \frac{\phi}{\gamma_0}\bigg(-K^{0101} + K^{0201} + K^{1101} + p_\beta(K^{0200} - K^{0300} + K^{1100} - K^{1200} - K^{2100})\bigg),$$

$$K^{0111} = \frac{1}{\gamma_0}\bigg(\gamma_0 c_7 - \phi K^{0102} + p_\beta\phi(K^{0201} + K^{1101})\bigg),$$

$$K^{0120} = c_8 + \frac{\phi^2}{\gamma_0^2}\bigg(K^{0102} + p_\beta(-2K^{0201} - 2K^{1101} + p_\beta(K^{0300} + 2K^{1200} + K^{2100}))\bigg)$$
$$+ \frac{2\phi}{\gamma_0}\bigg(p_\beta(c_9 + c_{13}) - c_7\bigg),$$

$$K^{0210} = \frac{1}{\gamma_0}\bigg(\gamma_0 c_9 - \phi K^{0201} + p_\beta\phi(K^{0300} + K^{1200})\bigg),$$

$$K^{1010} = c_{10} + \frac{\phi}{\gamma_0}\bigg(-K^{1001} + K^{1101} + K^{2001} + p_\beta(K^{1100} - K^{1200} + K^{2000} - K^{2100} - K^{3000})\bigg),$$

$$K^{1011} = \frac{1}{\gamma_0}\bigg(\gamma_0 c_{11} - \phi K^{1002} + p_\beta\phi(K^{1101} + K^{2001})\bigg),$$

$$K^{1020} = c_{12} + \frac{\phi^2}{\gamma_0^2}\bigg(K^{1002} + p_\beta(-2K^{1101} - 2K^{2001} + p_\beta(K^{1200} + 2K^{2100} + K^{3000}))\bigg)$$
$$+ \frac{2\phi}{\gamma_0}\bigg(p_\beta(c_{13} + c_{14}) - c_{11}\bigg),$$

$$K^{1110} = \frac{1}{\gamma_0}\bigg(\gamma_0 c_{13} - \phi K^{1101} + p_\beta\phi(K^{1200} + K^{2100})\bigg),$$

$$K^{2010} = \frac{1}{\gamma_0}\bigg(\gamma_0 c_{14} - \phi K^{2001} + p_\beta\phi(K^{2100} + K^{3000})\bigg),$$

for moments with volume-anisotropy correlations.

E Third-order solution for the $p_\beta = 0$ case

We present the solution for the system truncated at third order in moments for the particular case $p_\beta = 0$. For the case with $r \neq 0$ the fluctuations of the anisotropic sector are

$$K^{0011} = c_1 + \frac{2(K^{0002} - (K^{0102} + K^{1002}))}{K^{0102} - K^{1002}}\gamma, \qquad K^{0012} = c_2 + \frac{2K^{0003}}{K^{0102} - K^{1002}}\gamma,$$

$$K^{0020} = c_3 + \frac{4\gamma}{(K^{0102} - K^{1002})^2}\bigg[(K^{0002} - 2(K^{0102} + K^{1002}))\gamma + (K^{0102} - K^{1002})(c_1 + c_4 + c_5)\bigg],$$

$$K^{0021} = \frac{1}{(K^{0102} - K^{1002})^2}\bigg[-8\gamma_0^2 K^{0003} - (K^{0102} - K^{1002})^2 c_6 +$$
$$4(K^{0102} - K^{1002})c_2\gamma + 4K^{0003}\gamma^2\bigg],$$

$$K^{0030} = \frac{1}{(K^{0102} - K^{1002})^3}\bigg[(K^{0102} - K^{1002})^3 c_7 - 24(K^{0102} - K^{1002})\gamma_0^2 c_2$$
$$-6\left(8\gamma_0^2 K^{0003} + (K^{0102} - K^{1002})^2 c_6\right)\gamma + 12(K^{0102} - K^{1002})c_2\gamma^2 + 8K^{0003}\gamma^3\bigg],$$

where c_i with $i = 1, ..., 6$ are real constants. The second-order correlations between the two sectors, with their logarithmic behavior read as:

$$K^{0101} = d_1 + \frac{K^{0003}}{(K^{0102} - K^{1002})} \ln \gamma,$$

$$K^{0110} = d_2 - \frac{2}{K^{0102} - K^{1002}} \left(K^{0201} - d_1 + K^{1101} \right) \gamma + \frac{((K^{0102} - K^{1002})c_2 + 2\gamma K^{0003})}{(K^{0102} - K^{1002})^2} \ln \gamma,$$

$$K^{1001} = d_3 - \frac{K^{0003}}{(K^{0102} - K^{1002})} \ln \gamma,$$

$$K^{1010} = d_4 - \frac{2(-d_3 + K^{1101} + K^{2001})}{K^{0102} - K^{1002}} \gamma - \frac{((K^{0102} - K^{1002})c_2 + 2\gamma K^{0003})}{(K^{0102} - K^{1002})^2} \ln \gamma,$$

where d_i with $i = 1, ..., 5$ are real constants. And, finally, the third-order correlations between the two sectors:

$$K^{0111} = -c_5 + \left(1 + \frac{K^{0102} + K^{1002}}{K^{0102} - K^{1002}} \right) \gamma,$$

$$K^{0120} = f_1 + \frac{4K^{0102}}{(K^{0102} - K^{1002})^2} \gamma^2 - \frac{4c_5}{K^{0102} - K^{1002}} \gamma - \frac{\hbar^2}{2(K^{0102} - K^{1002})} \ln \gamma,$$

$$K^{0210} = f_2 + \frac{2K^{0201}}{K^{0102} - K^{1002}} \gamma,$$

$$K^{1011} = -c_4 + \left(\frac{K^{0102} + K^{1002}}{K^{0102} - K^{1002}} - 1 \right) \gamma,$$

$$K^{1020} = f_3 + \frac{4K^{1002}}{(K^{0102} - K^{1002})^2} \gamma^2 - \frac{4c_4}{K^{0102} - K^{1002}} \gamma + \frac{\hbar^2}{2(K^{0102} - K^{1002})} \ln \gamma,$$

$$K^{1110} = f_4 + \frac{2K^{1101}}{K^{0102} - K^{1002}} \gamma,$$

$$K^{2010} = f_5 + \frac{2K^{2001}}{K^{0102} - K^{1002}} \gamma,$$

where f_i with $i = 1, ..., 5$ are real constants.

For the case with $r = 0$, the solution is slightly different and the fluctuations of the anisotropic sector take the form

$$K^{0011} = \tilde{c}_1 - \frac{(K^{0002} - K^{0102} - K^{1002})}{\gamma} \phi, \qquad K^{0012} = \tilde{c}_2 - \frac{K^{0003}}{\gamma} \phi,$$

$$K^{0020} = \tilde{c}_3 - \frac{2}{\gamma} (\tilde{c}_1 + \tilde{c}_4 + \tilde{c}_5) \phi + \frac{(K^{0002} - 2K^{0102} - 2K^{1002})\phi^2}{\gamma^2},$$

$$K^{0021} = -\tilde{c}_6 - \frac{2\tilde{c}_2}{\gamma} \phi + \frac{K^{0003}}{\gamma^2} \phi^2, \qquad K^{0030} = \tilde{c}_7 + \frac{3\tilde{c}_6}{\gamma} \phi + \frac{3\tilde{c}_2}{\gamma^2} \phi^2 - \frac{K^{0003}}{\gamma^3} \phi^3.$$

The second-order correlations between the two sectors are given as

$$K^{0101} = \tilde{d}_1 - \frac{K^{0003}}{2\gamma^2} \phi, \qquad K^{0110} = -\tilde{d}_2 - \frac{(\tilde{c}_2 - 2\gamma(K^{0201} + K^{1101} - \tilde{d}_1))}{2\gamma^2} \phi + \frac{K^{0003}}{2\gamma^3} \phi^2,$$

$$K^{1001} = \tilde{d}_3 + \frac{K^{0003}}{2\gamma^2} \phi, \qquad K^{1010} = \tilde{d}_4 + \frac{(2\gamma(K^{1101} + K^{2001} - \tilde{d}_3) + \tilde{c}_2)}{2\gamma^2} \phi - \frac{K^{0003}}{2\gamma^3} \phi^2.$$

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And finally, the third-order correlations between the two sectors take the form,

$$K^{0111} = -\tilde{c}_5 - \frac{K^{0102} + K^{1002}}{2\gamma}\phi, \qquad K^{0120} = \tilde{f}_1 + \frac{\left(8\gamma\tilde{c}_5 + \hbar^2\right)}{4\gamma^2}\phi + \frac{K^{0102} + K^{1002}}{2\gamma^2}\phi^2,$$

$$K^{0210} = \tilde{f}_2 - \frac{K^{0201}}{\gamma}\phi,$$

$$K^{1011} = -\tilde{c}_4 - \frac{K^{0102} + K^{1002}}{2\gamma}\phi, \qquad K^{1020} = \tilde{f}_3 - \frac{\left(\hbar^2 - 8\gamma\tilde{c}_4\right)}{4\gamma^2}\phi + \frac{K^{0102} + K^{1002}}{2\gamma^2}\phi^2,$$

$$K^{1110} = \tilde{f}_4 - \frac{K^{1101}}{\gamma}\phi,$$

$$K^{2010} = \tilde{f}_5 - \frac{K^{2001}}{\gamma}\phi.$$