

Updates and New Results in Models with Reduced Couplings

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Abstract

The idea of reduction of couplings consists in searching for renormalization group invariant relations between parameters of a renormalizable theory that hold to all orders of perturbation theory. Based on the principle of the reduction of couplings, one can construct Finite Unified Theories (FUTs) which are $N = 1$ supersymmetric Grand Unified Theories that can be made all-order finite. The prediction of the top quark mass well in advance of its experimental discovery and the prediction of the light Higgs boson mass in the range $\sim 121 - 126$ GeV much earlier than its experimental discovery are among the celebrated successes of such models. Here, after a brief review of the reduction of couplings method and the properties of the resulting finiteness in supersymmetric theories, we analyse four phenomenologically favoured models: a minimal version of the $N = 1$ $SU(5)$, a finite $N = 1$ $SU(5)$, a $N = 1$ finite $SU(3) \otimes SU(3) \otimes SU(3)$ model and a reduced version of the Minimal Supersymmetric Standard Model (MSSM). A relevant update in the phenomenological evaluation has been the improved light Higgs-boson mass prediction as provided by the latest version of `FeynHiggs`. All four models predict relatively heavy supersymmetric spectra that start just below or above the TeV scale, consistent with the non-observation LHC results. Depending on the model, the lighter regions of the spectra could be accessible at CLIC, while the FCC-hh will be able to test large parts of predicted spectrum of each model. The lightest supersymmetric particle (LSP), which is a neutralino, is considered as a cold dark matter candidate and put to test using the latest `MicrOMEGAs` code.

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1 Introduction

During the last years, a series of successes in developing frameworks such as String Theories and Noncommutativity have been presented, as a result of various theoretical endeavours that aim to describe the fundamental theory at the Planck scale. However, the essence of all theoretical efforts in Particle Physics is to understand the free parameters of the Standard Model (SM) in terms of fewer, fundamental ones. In other words, to achieve a *reduction of couplings* [1]. Unfortunately, the several recent successes in the above frameworks do not offer anything concerning the understanding of the SM free parameters. The problem of the large number of free parameters is deeply connected to the infinities that are present at the quantum level. The renormalization programme removes infinities by introducing counterterms, but it does so at the big cost of leaving the corresponding terms as free parameters.

Although the SM is successful in describing elementary particles and their interactions, it is a widespread belief that it should be the low-energy limit of a fundamental theory. The search for beyond the Standard Model (BSM) theories expands in various directions. One of the most efficient ways to reduce the plethora of free parameters of a theory (and thus render it more predictive) is the introduction of a symmetry. A very good example of such a procedure are the Grand Unified Theories (GUTs) [2–7]. In the early days, the minimal $SU(5)$ (because of an -approximate- gauge unification) reduced the gauge couplings of the SM, predicting one of them. It was the addition of an $N = 1$ global (softly broken) supersymmetry (SUSY) [8–10] that made the prediction viable. In the framework of GUTs, the Yukawa couplings can also be related among themselves. Again, $SU(5)$ demonstrated this by predicting the ratio of the tau to bottom mass [11] for the SM. Unfortunately, the requirement of additional gauge symmetry does not seem to help, since new complications due to new degrees of freedom arise.

One could relate seemingly independent parameters via the reduction of couplings method [12–14] (see also [15–17]). This technique reduces the number of free couplings by relating all or some of the couplings of the theory to a single parameter, the “primary coupling”. This method can identify previously hidden symmetries in a system, but it can also be applied in models with no apparent symmetry. It is necessary, though to make two assumptions: first, both the original and the reduced theory have to be renormalizable and second, the relations among parameters should be renormalization group invariant (RGI).

A natural continuation of the idea of Grand Unification is to achieve gauge-Yukawa Unification (GYU), that is to relate the gauge sector to the Yukawa sector. This is a feature of theories in which *reduction of couplings* has been applied. The original suggestion for the reduction of couplings in GUTs leads to the search for RGI relations that hold below the Planck scale, which are in turn preserved down to the unification scale. Impressively, this observation guarantees validity of such relations to all-orders in perturbation theory. This is done by studying their uniqueness at one-loop level. Even more remarkably, one can find such RGI relations that result in all-loop finiteness. In the sections to follow we will show cases in which these principles will be applied in $N = 1$ SUSY GUTs. The application of the GYU programme in the dimensionless couplings of such theories has been very successful, including the prediction of the mass of the top quark in the minimal [18] (see also Section 6 for the latest update) and in the finite $N = 1$ $SU(5)$ [19,20] before its experimental discovery [21].

In order to successfully apply the above-mentioned programme, SUSY appears to be essential. However, one has to understand its breaking as well. This naturally leads to the extension of this

search for RGI expressions to the SUSY breaking (SSB) sector of these models, which involves couplings of dimension one and two. There has been crucial progress on the renormalization properties of the SSB parameters based on the powerful supergraph method for studying SUSY theories, applied to the softly broken ones using the spurion external space-time independent superfields. In this method a softly broken supersymmetric theory is taken to be supersymmetric, where the various couplings and masses are considered external superfields. Then, relations among the soft term renormalization and that of an unbroken SUSY have been derived.

The application of *reduction of couplings* on $N = 1$ SUSY theories has led to many interesting phenomenological developments. In past work, the assumption of introducing a “universal” set of soft scalar masses that serve as one of the constraints preserving two-loop finiteness exhibited a number of problems due to its restrictive nature. Subsequently, this constraint was replaced by a more “relaxed” all-loop RGI sum rule that keeps the most attractive features of the universal case and overcomes the unpleasant phenomenological consequences. This arsenal of tools and results opened the way for the study of full finite models with few free parameters, with emphasis given on predictions for the SUSY spectrum and the light Higgs mass.

The Higgs mass prediction, that coincided with the LHC results (ATLAS [22, 23] and CMS [24, 25]) - combined with a predicted relatively heavy spectrum that is consistent with the as of yet non-observation of SUSY particles - was a success of the all-loop finite $N = 1$ SUSY $SU(5)$ model [26]. This case will be presented in Sect. 7, while the results for another finite theory, namely the $N = 1$ (two-loop) finite $SU(3) \otimes SU(3) \otimes SU(3)$, will be presented in Sect. 8. Furthermore, the above programme was also applied in the MSSM, with successful results concerning the top, bottom and Higgs masses, also featuring a relatively heavy SUSY spectrum that accommodates a dark matter candidate as well. These results will be presented in Sect. 9. The calculation of the lightest Higgs boson mass is done with the (new) `FeynHiggs` code [27–30].

Last but not least, it is a well known fact that the lightest neutralino, being the Lightest SUSY Particle (LSP), is an excellent candidate for Cold Dark Matter (CDM) [31]. Our analyses presented in Sect. 6, 7, 8 and 9 also include the calculation of the CDM relic density for each model, using the `MicrOMEGAs` code [32–34]. As will be discussed, none of the models satisfy the experimental bounds of the relic density exactly.

2 Theoretical Basis

2.1 Reduction of Dimensionless Parameters

We start by reviewing the basic *reduction of couplings* idea. The aim is, if possible, to express the parameters of a theory that are considered as free in terms of one basic parameter, which we call primary. The basic idea is to search for RGI relations among couplings and use them to reduce the seemingly independent parameters. Any RGI relation among parameters g_1, \dots, g_A of a given renormalizable theory can be expressed implicitly as $\Phi(g_1, \dots, g_A) = \text{const}$. This expression must satisfy the partial differential equation (PDE)

$$\mu \frac{d\Phi}{d\mu} = \vec{\nabla} \Phi \cdot \vec{\beta} = \sum_{a=1}^A \beta_a \frac{\partial \Phi}{\partial g_a} = 0, \quad (1)$$

with β_a the β -functions of g_a . Solving this PDE is equivalent to solving a set of ordinary differential equations (ODEs), the reduction equations (REs) [12–14],

$$\beta_g \frac{dg_a}{dg} = \beta_a, \quad a = 1, \dots, A-1, \quad (2)$$

Here, g and β_g are the primary coupling and its β -function, respectively. The Φ_a 's can impose up to $(A-1)$ independent RGI constraints in the A -dimensional parameter space. As a result, all couplings can be (in principle) expressed in terms of the primary coupling g .

This is not enough, as the number of integration constants of the general solutions of Eq. (2) matches the number of these equations, meaning that we just traded an integration constant for each ordinary renormalized coupling, and therefore these cannot be considered as reduced solutions.

The crucial requirement is the demand that the REs admit power series solutions,

$$g_a = \sum_n \rho_a^{(n)} g^{2n+1}, \quad (3)$$

that preserve perturbative renormalizability. This way, the integration constant corresponding to each RE is fixed and the RE is picked up as a special solution out of the set of the general ones. It is worth noting that a one-loop level examination is enough to decide for the uniqueness of these solutions [12–14]. As an illustration on the above, we assume β -functions of the form

$$\begin{aligned} \beta_a &= \frac{1}{16\pi^2} \left[\sum_{b,c,d \neq g} \beta_a^{(1)bcd} g_b g_c g_d + \sum_{b \neq g} \beta_a^{(1)b} g_b g^2 \right] + \dots, \\ \beta_g &= \frac{1}{16\pi^2} \beta_g^{(1)} g^3 + \dots, \end{aligned} \quad (4)$$

Here \dots stands for higher order terms and $\beta_a^{(1)bcd}$'s are symmetric in b, c and d . We assume that $\rho_a^{(n)}$ with $n \leq r$ are already determined uniquely. In order to obtain $\rho_a^{(r+1)}$, the power series (3) are inserted into the REs (2) and we collect terms of $\mathcal{O}(g^{2r+3})$. Thus, we find

$$\sum_{d \neq g} M(r)_a^d \rho_d^{(r+1)} = \text{lower order quantities},$$

where the right-hand side is known by assumption and

$$M(r)_a^d = 3 \sum_{b,c \neq g} \beta_a^{(1)bcd} \rho_b^{(1)} \rho_c^{(1)} + \beta_a^{(1)d} - (2r+1) \beta_g^{(1)} \delta_a^d, \quad (5)$$

$$0 = \sum_{b,c,d \neq g} \beta_a^{(1)bcd} \rho_b^{(1)} \rho_c^{(1)} \rho_d^{(1)} + \sum_{d \neq g} \beta_a^{(1)d} \rho_d^{(1)} - \beta_g^{(1)} \rho_a^{(1)}. \quad (6)$$

Therefore, the $\rho_a^{(n)}$ for all $n > 1$ for a given set of $\rho_a^{(1)}$ can be uniquely determined if $\det M(n)_a^d \neq 0$ for all $n \geq 0$. This is checked in all models that reductions of couplings is applied.

The search for power series solutions to the REs like (3) is more than justified in SUSY theories, where parameters often behave asymptotically in a similar way. This ‘‘completely reduced’’ theory

features only one independent parameter, rendering this unification very attractive. It is often unrealistic, however, and, usually, fewer RGI constraints are imposed, leading to a partial reduction [35, 36].

All the above give rise to hints towards an underlying connection among the requirement of reduction of couplings and SUSY.

As an example, we consider a $SU(N)$ gauge theory with $\phi^i(\mathbf{N})$ and $\hat{\phi}_i(\overline{\mathbf{N}})$ complex scalars, $\psi^i(\mathbf{N})$ and $\hat{\psi}_i(\overline{\mathbf{N}})$ left-handed Weyl spinors and $\lambda^a(a = 1, \dots, N^2 - 1)$ right-handed Weyl spinors in the adjoint representation of $SU(N)$.

The Lagrangian (kinetic terms are omitted) includes

$$\mathcal{L} \supset i\sqrt{2}\{ g_Y \bar{\psi} \lambda^a T^a \phi - \hat{g}_Y \bar{\hat{\psi}} \lambda^a T^a \hat{\phi} + \text{h.c.} \} - V(\phi, \bar{\phi}), \quad (7)$$

where

$$V(\phi, \bar{\phi}) = \frac{1}{4} \lambda_1 (\phi^i \phi_i^*)^2 + \frac{1}{4} \lambda_2 (\hat{\phi}_i \hat{\phi}^{*i})^2 + \lambda_3 (\phi^i \phi_i^*) (\hat{\phi}_j \hat{\phi}^{*j}) + \lambda_4 (\phi^i \phi_j^*) (\hat{\phi}_i \hat{\phi}^{*j}), \quad (8)$$

This is the most general renormalizable form in four dimensions. In search of a solution of the form of Eq. (3) for the REs, among other solutions, one finds in lowest order:

$$\begin{aligned} g_Y &= \hat{g}_Y = g, \\ \lambda_1 &= \lambda_2 = \frac{N-1}{N} g^2, \\ \lambda_3 &= \frac{1}{2N} g^2, \quad \lambda_4 = -\frac{1}{2} g^2, \end{aligned} \quad (9)$$

which corresponds to a $N = 1$ SUSY gauge theory. While these remarks do not provide an answer about the relation of reduction of couplings and SUSY, they certainly point to further study in that direction.

2.2 Reduction of Couplings in $N = 1$ SUSY Gauge Theories - Partial Reduction

Consider a chiral, anomaly free, $N = 1$ globally supersymmetric gauge theory that is based on a group G and has gauge coupling g . The superpotential of the theory is:

$$W = \frac{1}{2} m_{ij} \phi_i \phi_j + \frac{1}{6} C_{ijk} \phi_i \phi_j \phi_k, \quad (10)$$

m_{ij} and C_{ijk} are gauge invariant tensors and the chiral superfield ϕ_i belongs to the irreducible representation R_i of the gauge group. The renormalization constants associated with the superpotential, for preserved SUSY, are:

$$\phi_i^0 = \left(Z_i^j \right)^{(1/2)} \phi_j, \quad (11)$$

$$m_{ij}^0 = Z_{ij}^{i'j'} m_{i'j'}, \quad (12)$$

$$C_{ijk}^0 = Z_{ijk}^{i'j'k'} C_{i'j'k'}. \quad (13)$$

By virtue of the $N = 1$ non-renormalization theorem [37–40] there are no mass and cubic-interaction-term infinities. Therefore:

$$\begin{aligned} Z_{ij}^{i'j'} (Z_{i'}^{i''})^{(1/2)} (Z_{j'}^{j''})^{(1/2)} &= \delta_{(i}^{i''} \delta_{j)}^{j''} , \\ Z_{ijk}^{i'j'k'} (Z_{i'}^{i''})^{(1/2)} (Z_{j'}^{j''})^{(1/2)} (Z_{k'}^{k''})^{(1/2)} &= \delta_{(i}^{i''} \delta_{j)}^{j''} \delta_{k)}^{k''} . \end{aligned} \quad (14)$$

The only surviving infinities are the wave function renormalization constants Z_i^j , so just one infinity per field. The β -function of the gauge coupling g at the one-loop level is given by [41–45]

$$\beta_g^{(1)} = \frac{dg}{dt} = \frac{g^3}{16\pi^2} \left[\sum_i T(R_i) - 3 C_2(G) \right] , \quad (15)$$

where $C_2(G)$ is the quadratic Casimir operator of the adjoint representation of the gauge group G and $\text{Tr}[T^a T^b] = T(R) \delta^{ab}$, where T^a are the group generators in the appropriate representation. The β -functions of C_{ijk} are related to the anomalous dimension matrices γ_{ij} of the matter fields as:

$$\beta_{ijk} = \frac{dC_{ijk}}{dt} = C_{ijl} \gamma_k^l + C_{ikl} \gamma_j^l + C_{jkl} \gamma_i^l . \quad (16)$$

The one-loop γ_j^i is given by [41]:

$$\gamma_j^{(1)i} = \frac{1}{32\pi^2} [C^{ikl} C_{jkl} - 2g^2 C_2(R_i) \delta_j^i] , \quad (17)$$

where $C^{ijk} = C_{ijk}^*$. We take C_{ijk} to be real so that C_{ijk}^2 are always positive. The squares of the couplings are convenient to work with, and the C_{ijk} can be covered by a single index i ($i = 1, \dots, n$):

$$\alpha = \frac{g^2}{4\pi} , \quad \alpha_i = \frac{g_i^2}{4\pi} . \quad (18)$$

Then the evolution of α 's in perturbation theory will take the form

$$\begin{aligned} \frac{d\alpha}{dt} = \beta &= -\beta^{(1)} \alpha^2 + \dots , \\ \frac{d\alpha_i}{dt} = \beta_i &= -\beta_i^{(1)} \alpha_i \alpha + \sum_{j,k} \beta_{i,jk}^{(1)} \alpha_j \alpha_k + \dots , \end{aligned} \quad (19)$$

Here, \dots denotes higher-order contributions and $\beta_{i,jk}^{(1)} = \beta_{i,kj}^{(1)}$. For the evolution equations (19) we investigate the asymptotic properties. First, we define [12, 14, 16, 46, 47]

$$\tilde{\alpha}_i \equiv \frac{\alpha_i}{\alpha} , \quad i = 1, \dots, n , \quad (20)$$

and derive from Eq. (19)

$$\begin{aligned} \alpha \frac{d\tilde{\alpha}_i}{d\alpha} &= -\tilde{\alpha}_i + \frac{\beta_i}{\beta} = \left(-1 + \frac{\beta_i^{(1)}}{\beta^{(1)}} \right) \tilde{\alpha}_i \\ &\quad - \sum_{j,k} \frac{\beta_{i,jk}^{(1)}}{\beta^{(1)}} \tilde{\alpha}_j \tilde{\alpha}_k + \sum_{r=2} \left(\frac{\alpha}{\pi} \right)^{r-1} \tilde{\beta}_i^{(r)}(\tilde{\alpha}) , \end{aligned} \quad (21)$$

where $\tilde{\beta}_i^{(r)}(\tilde{\alpha})$ ($r = 2, \dots$) are power series of $\tilde{\alpha}$'s and can be computed from the r -th-loop β -functions. We then search for fixed points ρ_i of Eq. (20) at $\alpha = 0$. We have to solve the equation

$$\left(-1 + \frac{\beta_i^{(1)}}{\beta^{(1)}}\right) \rho_i - \sum_{j,k} \frac{\beta_{i,jk}^{(1)}}{\beta^{(1)}} \rho_j \rho_k = 0, \quad (22)$$

assuming fixed points of the form

$$\rho_i = 0 \text{ for } i = 1, \dots, n'; \quad \rho_i > 0 \text{ for } i = n' + 1, \dots, n. \quad (23)$$

Next, we treat $\tilde{\alpha}_i$ with $i \leq n'$ as small perturbations to the undisturbed system (defined by setting $\tilde{\alpha}_i$ with $i \leq n'$ equal to zero). It is possible to verify the existence of the unique power series solution of the reduction equations (21) to all orders already at one-loop level [12–14, 46]:

$$\tilde{\alpha}_i = \rho_i + \sum_{r=2} \rho_i^{(r)} \alpha^{r-1}, \quad i = n' + 1, \dots, n. \quad (24)$$

These are RGI relations among parameters, and preserve formally perturbative renormalizability. So, in the undisturbed system there is only one independent parameter, the primary coupling α .

The nonvanishing $\tilde{\alpha}_i$ with $i \leq n'$ cause small perturbations that enter in a way that the reduced couplings ($\tilde{\alpha}_i$ with $i > n'$) become functions both of α and $\tilde{\alpha}_i$ with $i \leq n'$. Investigating such systems with partial reduction is very convenient to work with the following PDEs:

$$\left\{ \tilde{\beta} \frac{\partial}{\partial \alpha} + \sum_{a=1}^{n'} \tilde{\beta}_a \frac{\partial}{\partial \tilde{\alpha}_a} \right\} \tilde{\alpha}_i(\alpha, \tilde{\alpha}) = \tilde{\beta}_i(\alpha, \tilde{\alpha}), \quad (25)$$

$$\tilde{\beta}_{i(a)} = \frac{\beta_{i(a)}}{\alpha^2} - \frac{\beta}{\alpha^2} \tilde{\alpha}_{i(a)}, \quad \tilde{\beta} \equiv \frac{\beta}{\alpha}.$$

These equations are equivalent to the REs (21), where, in order to avoid any confusion, we let a, b run from 1 to n' and i, j from $n' + 1$ to n . Then, we search for solutions of the form

$$\tilde{\alpha}_i = \rho_i + \sum_{r=2} \left(\frac{\alpha}{\pi}\right)^{r-1} f_i^{(r)}(\tilde{\alpha}_a), \quad i = n' + 1, \dots, n, \quad (26)$$

where $f_i^{(r)}(\tilde{\alpha}_a)$ are power series of $\tilde{\alpha}_a$. The requirement that in the limit of vanishing perturbations we obtain the undisturbed solutions (24) [36, 48] suggests this type of solutions. Once more, one can obtain the conditions for uniqueness of $f_i^{(r)}$ in terms of the lowest order coefficients.

2.3 Reduction of Dimension-1 and -2 Parameters

The extension of reduction of couplings to massive parameters is not straightforward, since the technique was originally aimed at massless theories on the basis of the Callan-Symanzik equation [12, 13]. Many requirements have to be met, such as the normalization conditions imposed on irreducible Green's functions [49], etc. Significant progress has been made towards this goal, starting from [50], where, as an assumption, a mass-independent renormalization scheme renders

all RG functions only trivially dependent on dimensional parameters. Mass parameters can then be introduced similarly to couplings.

This was justified later [51, 52], where it was demonstrated that, apart from dimensionless parameters, pole masses and gauge couplings, the model can also include couplings carrying a dimension and masses. To simplify the analysis, we follow Ref. [50] and use a mass-independent renormalization scheme as well.

Consider a renormalizable theory that contains $(N + 1)$ dimension-0 couplings, $(\hat{g}_0, \hat{g}_1, \dots, \hat{g}_N)$, L parameters with mass dimension-1, $(\hat{h}_1, \dots, \hat{h}_L)$, and M parameters with mass dimension-2, $(\hat{m}_1^2, \dots, \hat{m}_M^2)$. The renormalized irreducible vertex function Γ satisfies the RG equation

$$\mathcal{D}\Gamma \left[\Phi' s; \hat{g}_0, \hat{g}_1, \dots, \hat{g}_N; \hat{h}_1, \dots, \hat{h}_L; \hat{m}_1^2, \dots, \hat{m}_M^2; \mu \right] = 0, \quad (27)$$

with

$$\mathcal{D} = \mu \frac{\partial}{\partial \mu} + \sum_{i=0}^N \beta_i \frac{\partial}{\partial \hat{g}_i} + \sum_{a=1}^L \gamma_a^h \frac{\partial}{\partial \hat{h}_a} + \sum_{\alpha=1}^M \gamma_\alpha^{m^2} \frac{\partial}{\partial \hat{m}_\alpha^2} + \sum_J \Phi_I \gamma_J^{\phi I} \frac{\delta}{\delta \Phi_J}, \quad (28)$$

where β_i are the β -functions of the dimensionless couplings g_i and Φ_I are the matter fields. The mass, trilinear coupling and wave function anomalous dimensions, respectively, are denoted by $\gamma_\alpha^{m^2}$, γ_a^h and $\gamma_J^{\phi I}$ and μ denotes the energy scale. For a mass-independent renormalization scheme, the γ 's are given by

$$\begin{aligned} \gamma_a^h &= \sum_{b=1}^L \gamma_a^{h,b}(g_0, g_1, \dots, g_N) \hat{h}_b, \\ \gamma_\alpha^{m^2} &= \sum_{\beta=1}^M \gamma_\alpha^{m^2,\beta}(g_0, g_1, \dots, g_N) \hat{m}_\beta^2 + \sum_{a,b=1}^L \gamma_\alpha^{m^2,ab}(g_0, g_1, \dots, g_N) \hat{h}_a \hat{h}_b. \end{aligned} \quad (29)$$

The $\gamma_a^{h,b}$, $\gamma_\alpha^{m^2,\beta}$ and $\gamma_\alpha^{m^2,ab}$ are power series of the (dimensionless) g 's.

We search for a reduced theory where

$$g \equiv g_0, \quad h_a \equiv \hat{h}_a \quad \text{for } 1 \leq a \leq P, \quad m_\alpha^2 \equiv \hat{m}_\alpha^2 \quad \text{for } 1 \leq \alpha \leq Q$$

are independent parameters. The reduction of the rest of the parameters, namely

$$\begin{aligned} \hat{g}_i &= \hat{g}_i(g), \quad (i = 1, \dots, N), \\ \hat{h}_a &= \sum_{b=1}^P f_a^b(g) h_b, \quad (a = P + 1, \dots, L), \\ \hat{m}_\alpha^2 &= \sum_{\beta=1}^Q e_\alpha^\beta(g) m_\beta^2 + \sum_{a,b=1}^P k_\alpha^{ab}(g) h_a h_b, \quad (\alpha = Q + 1, \dots, M) \end{aligned} \quad (30)$$

is consistent with the RGEs (27,28). The following relations should be satisfied

$$\begin{aligned}
\beta_g \frac{\partial \hat{g}_i}{\partial g} &= \beta_i, & (i = 1, \dots, N), \\
\beta_g \frac{\partial \hat{h}_a}{\partial g} + \sum_{b=1}^P \gamma_b^h \frac{\partial \hat{h}_a}{\partial h_b} &= \gamma_a^h, & (a = P+1, \dots, L), \\
\beta_g \frac{\partial \hat{m}_\alpha^2}{\partial g} + \sum_{a=1}^P \gamma_a^h \frac{\partial \hat{m}_\alpha^2}{\partial h_a} + \sum_{\beta=1}^Q \gamma_\beta^{m^2} \frac{\partial \hat{m}_\alpha^2}{\partial m_\beta^2} &= \gamma_\alpha^{m^2}, & (\alpha = Q+1, \dots, M).
\end{aligned} \tag{31}$$

Using Eqs. (29) and (30), they reduce to

$$\begin{aligned}
\beta_g \frac{df_a^b}{dg} + \sum_{c=1}^P f_c^a \left[\gamma_c^{h,b} + \sum_{d=P+1}^L \gamma_c^{h,d} f_d^b \right] - \gamma_a^{h,b} - \sum_{d=P+1}^L \gamma_a^{h,d} f_d^b &= 0, \\
& (a = P+1, \dots, L; b = 1, \dots, P), \\
\beta_g \frac{de_\alpha^\beta}{dg} + \sum_{\gamma=1}^Q e_\alpha^\gamma \left[\gamma_\gamma^{m^2,\beta} + \sum_{\delta=Q+1}^M \gamma_\gamma^{m^2,\delta} e_\delta^\beta \right] - \gamma_\alpha^{m^2,\beta} - \sum_{\delta=Q+1}^M \gamma_\alpha^{m^2,d} e_\delta^\beta &= 0, \\
& (\alpha = Q+1, \dots, M; \beta = 1, \dots, Q), \\
\beta_g \frac{dk_\alpha^{ab}}{dg} + 2 \sum_{c=1}^P \left(\gamma_c^{h,a} + \sum_{d=P+1}^L \gamma_c^{h,d} f_d^a \right) k_\alpha^{cb} + \sum_{\beta=1}^Q e_\alpha^\beta \left[\gamma_\beta^{m^2,ab} + \sum_{c,d=P+1}^L \gamma_\beta^{m^2,cd} f_c^a f_d^b \right. \\
+ 2 \sum_{c=P+1}^L \gamma_\beta^{m^2,cb} f_c^a + \left. \sum_{\delta=Q+1}^M \gamma_\beta^{m^2,d} k_\delta^{ab} \right] - \left[\gamma_\alpha^{m^2,ab} + \sum_{c,d=P+1}^L \gamma_\alpha^{m^2,cd} f_c^a f_d^b \right. \\
+ 2 \sum_{c=P+1}^L \gamma_\alpha^{m^2,cb} f_c^a + \left. \sum_{\delta=Q+1}^M \gamma_\alpha^{m^2,\delta} k_\delta^{ab} \right] &= 0, \\
& (\alpha = Q+1, \dots, M; a, b = 1, \dots, P).
\end{aligned} \tag{32}$$

The above relations ensure that the irreducible vertex function of the reduced theory

$$\begin{aligned}
\Gamma_R [\Phi's; g; h_1, \dots, h_P; m_1^2, \dots, m_Q^2; \mu] &\equiv \\
\Gamma \left[\Phi's; g, \hat{g}_1(g), \dots, \hat{g}_N(g); h_1, \dots, h_P, \hat{h}_{P+1}(g, h), \dots, \hat{h}_L(g, h); \right. \\
& \left. m_1^2, \dots, m_Q^2, \hat{m}_{Q+1}^2(g, h, m^2), \dots, \hat{m}_M^2(g, h, m^2); \mu \right]
\end{aligned} \tag{33}$$

has the same renormalization group flow as the original one.

Assuming a perturbatively renormalizable reduced theory, the functions \hat{g}_i , f_a^b , e_α^β and k_α^{ab} are

expressed as power series in the primary coupling:

$$\begin{aligned}\hat{g}_i &= g \sum_{n=0}^{\infty} \rho_i^{(n)} g^n, & f_a^b &= g \sum_{n=0}^{\infty} \eta_a^{b(n)} g^n, \\ e_\alpha^\beta &= \sum_{n=0}^{\infty} \xi_\alpha^{\beta(n)} g^n, & k_\alpha^{ab} &= \sum_{n=0}^{\infty} \chi_\alpha^{ab(n)} g^n.\end{aligned}\tag{34}$$

These expansion coefficients are found by inserting the above power series into Eqs. (31), (32) and requiring the equations to be satisfied at each order of g . It is not trivial to have a unique power series solution; it depends both on the theory and the choice of independent couplings.

If there are no independent dimension-1 parameters (\hat{h}), their reduction becomes

$$\hat{h}_a = \sum_{b=1}^L f_a^b(g) M,$$

where M is a dimension-1 parameter (i.e. a gaugino mass, corresponding to the independent gauge coupling). If there are no independent dimension-2 parameters (\hat{m}^2), their reduction takes the form

$$\hat{m}_a^2 = \sum_{b=1}^M e_a^b(g) M^2.$$

2.4 Reduction of Couplings of Soft Breaking Terms in $N = 1$ SUSY Theories

The reduction of dimensionless couplings was extended [50,53] to the SSB dimensionful parameters of $N = 1$ supersymmetric theories. It was also found [54, 55] that soft scalar masses satisfy a universal sum rule.

We consider the superpotential (10)

$$W = \frac{1}{2} \mu^{ij} \Phi_i \Phi_j + \frac{1}{6} C^{ijk} \Phi_i \Phi_j \Phi_k ,\tag{35}$$

and the SSB Lagrangian

$$-\mathcal{L}_{\text{SSB}} = \frac{1}{6} h^{ijk} \phi_i \phi_j \phi_k + \frac{1}{2} b^{ij} \phi_i \phi_j + \frac{1}{2} (m^2)_i^j \phi^{*i} \phi_j + \frac{1}{2} M \lambda_i \lambda_i + \text{h.c.}\tag{36}$$

The ϕ_i 's are the scalar parts of chiral superfields Φ_i , λ are gauginos and M the unified gaugino mass.

The one-loop gauge β -function (15) is given by [41–45]

$$\beta_g^{(1)} = \frac{dg}{dt} = \frac{g^3}{16\pi^2} \left[\sum_i T(R_i) - 3C_2(G) \right] ,\tag{37}$$

whereas the one-loop C_{ijk} 's β -function (16) is given by

$$\beta_C^{ijk} = \frac{dC_{ijk}}{dt} = C_{ijl} \gamma_k^l + C_{ikl} \gamma_j^l + C_{jkl} \gamma_i^l ,\tag{38}$$

and the (one-loop) anomalous dimension $\gamma^{(1) i}_j$ of a chiral superfield (17) is

$$\gamma^{(1) i}_j = \frac{1}{32\pi^2} [C^{ikl} C_{jkl} - 2g^2 C_2(R_i)\delta_j^i] . \quad (39)$$

Then the $N = 1$ non-renormalization theorem [37, 38, 40] guarantees that the β -functions of C_{ijk} are expressed in terms of the anomalous dimensions.

We make the assumption that the REs admit power series solutions:

$$C^{ijk} = g \sum_{n=0} \rho_{(n)}^{ijk} g^{2n} . \quad (40)$$

Since we want to obtain higher-loop results instead of knowledge of explicit β -functions, we require relations among β -functions. The spurion technique [40, 56–59] gives all-loop relations among SSB β -functions [60–67]:

$$\beta_M = 2\mathcal{O} \left(\frac{\beta_g}{g} \right) , \quad (41)$$

$$\begin{aligned} \beta_h^{ijk} &= \gamma_i^i h^{ljk} + \gamma_l^j h^{ilk} + \gamma_l^k h^{ijl} \\ &\quad - 2(\gamma_1)_i^i C^{ljk} - 2(\gamma_1)_l^j C^{ilk} - 2(\gamma_1)_l^k C^{ijl} , \end{aligned} \quad (42)$$

$$(\beta_{m^2})_j^i = \left[\Delta + X \frac{\partial}{\partial g} \right] \gamma_j^i , \quad (43)$$

where

$$\mathcal{O} = \left(Mg^2 \frac{\partial}{\partial g^2} - h^{lmn} \frac{\partial}{\partial C^{lmn}} \right) , \quad (44)$$

$$\Delta = 2\mathcal{O}\mathcal{O}^* + 2|M|^2 g^2 \frac{\partial}{\partial g^2} + \tilde{C}_{lmn} \frac{\partial}{\partial C_{lmn}} + \tilde{C}^{lmn} \frac{\partial}{\partial C^{lmn}} , \quad (45)$$

$$(\gamma_1)_j^i = \mathcal{O}\gamma_j^i , \quad (46)$$

$$\tilde{C}^{ijk} = (m^2)_i^i C^{ljk} + (m^2)_l^j C^{ilk} + (m^2)_l^k C^{ijl} . \quad (47)$$

Assuming (following [64]) that the relation among couplings

$$h^{ijk} = -M(C^{ijk})' \equiv -M \frac{dC^{ijk}(g)}{d \ln g} , \quad (48)$$

is RGI and the use of the all-loop gauge β -function of [68–70]

$$\beta_g^{\text{NSVZ}} = \frac{g^3}{16\pi^2} \left[\frac{\sum_l T(R_l)(1 - \gamma_l/2) - 3C_2(G)}{1 - g^2 C_2(G)/8\pi^2} \right] , \quad (49)$$

we are led to an all-loop RGI sum rule [71] (assuming $(m^2)_j^i = m_j^2 \delta_j^i$),

$$\begin{aligned} m_i^2 + m_j^2 + m_k^2 &= |M|^2 \left\{ \frac{1}{1 - g^2 C_2(G)/(8\pi^2)} \frac{d \ln C^{ijk}}{d \ln g} + \frac{1}{2} \frac{d^2 \ln C^{ijk}}{d(\ln g)^2} \right\} \\ &\quad + \sum_l \frac{m_l^2 T(R_l)}{C_2(G) - 8\pi^2/g^2} \frac{d \ln C^{ijk}}{d \ln g} . \end{aligned} \quad (50)$$

It is worth noting that the all-loop result of Eq. (50) coincides with the superstring result for the finite case in a certain class of orbifold models [55, 72, 73] if $\frac{d \ln C^{ijk}}{d \ln g} = 1$ [20].

As mentioned above, the all-loop results on the SSB β -functions, Eqs.(41)-(47), lead to all-loop RGI relations. We assume:

(a) the existence of an RGI surface on which $C = C(g)$, or equivalently that the expression

$$\frac{dC^{ijk}}{dg} = \frac{\beta_C^{ijk}}{\beta_g} \quad (51)$$

holds (i.e. reduction of couplings is possible)

(b) the existence of a RGI surface on which

$$h^{ijk} = -M \frac{dC(g)^{ijk}}{d \ln g} \quad (52)$$

holds to all orders.

Then it can be proven [74, 75] that the relations that follow are all-loop RGI (note that in both assumptions we do not rely on specific solutions of these equations)

$$M = M_0 \frac{\beta_g}{g}, \quad (53)$$

$$h^{ijk} = -M_0 \beta_C^{ijk}, \quad (54)$$

$$b^{ij} = -M_0 \beta_\mu^{ij}, \quad (55)$$

$$(m^2)_j^i = \frac{1}{2} |M_0|^2 \mu \frac{d\gamma_j^i}{d\mu}, \quad (56)$$

where M_0 is an arbitrary reference mass scale to be specified shortly. Assuming

$$C_a \frac{\partial}{\partial C_a} = C_a^* \frac{\partial}{\partial C_a^*} \quad (57)$$

for an RGI surface $F(g, C^{ijk}, C^{*ijk})$ we are led to

$$\frac{d}{dg} = \left(\frac{\partial}{\partial g} + 2 \frac{\partial}{\partial C} \frac{dC}{dg} \right) = \left(\frac{\partial}{\partial g} + 2 \frac{\beta_C}{\beta_g} \frac{\partial}{\partial C} \right), \quad (58)$$

where Eq. (51) was used. Let us now consider the partial differential operator \mathcal{O} in Eq. (44) which (assuming Eq. (48)), becomes

$$\mathcal{O} = \frac{1}{2} M \frac{d}{d \ln g} \quad (59)$$

and β_M , given in Eq. (41), becomes

$$\beta_M = M \frac{d}{d \ln g} \left(\frac{\beta_g}{g} \right), \quad (60)$$

which by integration provides us [67, 74] with the generalized, i.e. including Yukawa couplings, all-loop RGI Hisano - Shifman relation [63]

$$M = \frac{\beta_g}{g} M_0 .$$

M_0 is the integration constant and can be associated to the unified gaugino mass M (of an assumed covering GUT), or to the gravitino mass $m_{3/2}$ in a supergravity framework. Therefore, Eq. (53) becomes the all-loop RGI Eq. (53). β_M , using Eqs.(60) and (53) can be written as follows:

$$\beta_M = M_0 \frac{d}{dt} (\beta_g/g) . \quad (61)$$

Similarly

$$(\gamma_1)_j^i = \mathcal{O} \gamma_j^i = \frac{1}{2} M_0 \frac{d\gamma_j^i}{dt} . \quad (62)$$

Next, from Eq.(48) and Eq.(53) we get

$$h^{ijk} = -M_0 \beta_C^{ijk} , \quad (63)$$

while β_h^{ijk} , using Eq.(62), becomes [74]

$$\beta_h^{ijk} = -M_0 \frac{d}{dt} \beta_C^{ijk} , \quad (64)$$

which shows that Eq. (63) is RGI to all loops. Eq. (55) can similarly be shown to be all-loop RGI as well.

Finally, it is important to note that, under the assumptions (a) and (b), the sum rule of Eq. (50) has been proven [71] to be RGI to all loops, which (using Eq. (53)) generalizes Eq. (56) for application in cases with non-universal soft scalar masses, a necessary ingredient in the models that will be examined in the next Sections. Another important point to note is the use of Eq. (53), which, in the case of product gauge groups (as in the MSSM), takes the form

$$M_i = \frac{\beta_{g_i}}{g_i} M_0 , \quad (65)$$

where $i = 1, 2, 3$ denotes each gauge group, and will be used in the Reduced MSSM case.

3 Finiteness in N=1 SUSY Gauge Theories

We start by considering a chiral, anomaly free, $N = 1$ globally supersymmetric gauge theory with gauge group G and g the theory's coupling constant. Again, the theory's superpotential is given by Eq. (10). Because of the $N = 1$ non-renormalization theorem, the one-loop β -function is given by Eq. (15), the β -function of C_{ijk} by Eq. (16) and the one-loop anomalous dimensions of the chiral superfields by Eq. (17).

It is obvious from Eqs. (15) and (17) that all one-loop β -functions of the theory vanish if $\beta_g^{(1)}$ and $\gamma^{(1)i}_j$ vanish:

$$\sum_i T(R_i) = 3C_2(G), \quad (66)$$

$$C^{ikl}C_{jkl} = 2\delta_j^i g^2 C_2(R_i). \quad (67)$$

In [76] one can find the finiteness conditions for $N = 1$ theories with $SU(N)$ gauge symmetry, while [77] discusses the requirements of anomaly-free and no-charge renormalization. Remarkably, the conditions (66,67) are necessary and sufficient for finiteness at the two-loop level as well [41–45].

In the case of soft SUSY breaking, requiring finiteness in the one-loop SSB sector imposes additional constraints among soft terms [78]. Again, the one-loop SSB finiteness conditions are enough to render the soft sector two-loop finite [79].

The above finiteness conditions impose considerable restrictions on the choice of irreducible representations (irreps) R_i for a given group G as well as the Yukawa couplings. These conditions cannot be applied to the MSSM, because the $U(1)$ gauge group is not compatible with condition (66), since $C_2[U(1)] = 0$. This points to the grand unified level, with the MSSM just being the low-energy theory.

Additionally, one(two)-loop finiteness causes SUSY to break only softly. Since gauge singlets are not acceptable, due to the condition given in Eq. (67) ($C_2(1) = 0$, i.e. singlets do not couple to the theory), F-type spontaneous symmetry breaking [80] terms are incompatible with finiteness, as well as D-type [81] spontaneous breaking which requires the existence of a $U(1)$ gauge group.

One can see that conditions (66,67) impose relations between the gauge and Yukawa sector. Imposing such relations, that make the parameters mutually dependent at a given renormalization point, is trivial. What is not trivial is to guarantee that relations leading to a reduction of the couplings hold at any renormalization point. As explained (see Eq. (51)), the necessary and sufficient condition is to require that such relations are solutions to the REs

$$\beta_g \frac{dC_{ijk}}{dg} = \beta_{ijk} \quad (68)$$

and hold at all orders. It is reminded that the existence of all-order power series solutions to (68) can be decided at one-loop level.

Concerning higher loop orders, a theorem [82, 83] exists that states the necessary and sufficient conditions to achieve all-loop finiteness for an $N = 1$ SUSY theory. It relies on the structure of the supercurrent in an $N = 1$ SUSY theory [84–86], and on the non-renormalization properties of $N = 1$ chiral anomalies [82, 83, 87–89]. Details and further discussion can be found in [82, 83, 87–91] Following [91] we briefly discuss the proof.

Consider an $N = 1$ SUSY gauge theory, with simple Lie group G . The content of this theory is given at the classical level by the matter supermultiplets S_i , which contain a scalar field ϕ_i and a Weyl spinor ψ_{ia} , and the vector supermultiplet V_a , which contains a gauge vector field A_μ^a and a gaugino Weyl spinor λ_α^a .

Let us first recall certain facts about the theory:

- (1) A massless $N = 1$ SUSY theory is invariant under a $U(1)$ chiral transformation R under which

the various fields transform as follows

$$\begin{aligned} A'_\mu &= A_\mu, \quad \lambda'_\alpha = \exp(-i\theta)\lambda_\alpha \\ \phi' &= \exp(-i\frac{2}{3}\theta)\phi, \quad \psi'_\alpha = \exp(-i\frac{1}{3}\theta)\psi_\alpha, \quad \dots \end{aligned} \quad (69)$$

The corresponding axial Noether current $J_R^\mu(x)$ is

$$J_R^\mu(x) = \bar{\lambda}\gamma^\mu\gamma^5\lambda + \dots \quad (70)$$

is conserved classically, while in the quantum case is violated by the axial anomaly

$$\partial_\mu J_R^\mu = r (\epsilon^{\mu\nu\sigma\rho} F_{\mu\nu} F_{\sigma\rho} + \dots). \quad (71)$$

From its known topological origin in ordinary gauge theories [92–94], one would expect the axial vector current J_R^μ to satisfy the Adler-Bardeen theorem and receive corrections only at the one-loop level. Indeed it has been shown that the same non-renormalization theorem holds also in SUSY theories [87–89]. Therefore

$$r = \hbar\beta_g^{(1)}. \quad (72)$$

(2) The massless theory we consider is scale invariant at the classical level and, in general, there is a scale anomaly due to radiative corrections. The scale anomaly appears in the trace of the energy momentum tensor $T_{\mu\nu}$, which is traceless classically. It has the form

$$T_\mu^\mu = \beta_g F^{\mu\nu} F_{\mu\nu} + \dots \quad (73)$$

(3) Massless, $N = 1$ SUSY gauge theories are classically invariant under the supersymmetric extension of the conformal group – the superconformal group. Examining the superconformal algebra, it can be seen that the subset of superconformal transformations consisting of translations, SUSY transformations, and axial R transformations is closed under SUSY, i.e. these transformations form a representation of SUSY. It follows that the conserved currents corresponding to these transformations make up a supermultiplet represented by an axial vector superfield called the supercurrent J ,

$$J \equiv \{J_R^\mu, Q_\alpha^\mu, T_\nu^\mu, \dots\}, \quad (74)$$

where J_R^μ is the current associated to R-invariance, Q_α^μ is the one associated to SUSY invariance, and T_ν^μ the one associated to translational invariance (energy-momentum tensor).

The anomalies of the R-current J_R^μ , the trace anomalies of the SUSY current, and the energy-momentum tensor, form also a second supermultiplet, called the supertrace anomaly

$$S = \{Re S, Im S, S_\alpha\} = \left\{ T_\mu^\mu, \partial_\mu J_R^\mu, \sigma_{\alpha\dot{\beta}}^\mu \bar{Q}_\mu^{\dot{\beta}} + \dots \right\}$$

where T_μ^μ is given in Eq. (73) and

$$\partial_\mu J_R^\mu = \beta_g \epsilon^{\mu\nu\sigma\rho} F_{\mu\nu} F_{\sigma\rho} + \dots \quad (75)$$

$$\sigma_{\alpha\dot{\beta}}^\mu \bar{Q}_\mu^{\dot{\beta}} = \beta_g \lambda^\beta \sigma_{\alpha\dot{\beta}}^{\mu\nu} F_{\mu\nu} + \dots \quad (76)$$

(4) It is important to note that the Noether current defined in (70) is not the same as the current associated to R-invariance that appears in the supercurrent J in (74), but they coincide in the tree approximation. So starting from a unique classical Noether current $J_{R(class)}^\mu$, the Noether current J_R^μ is defined as the quantum extension of $J_{R(class)}^\mu$ which allows for the validity of the non-renormalization theorem. On the other hand, J_R^μ , is defined to belong to the supercurrent J , together with the energy-momentum tensor. The two requirements cannot be fulfilled by a single current operator at the same time.

Although the Noether current J_R^μ which obeys (71) and the current J_R^μ belonging to the supercurrent multiplet J are not the same, there is a relation [82, 83] between quantities associated with them

$$r = \beta_g(1 + x_g) + \beta_{ijk}x^{ijk} - \gamma_A r^A, \quad (77)$$

where r is given in Eq. (72). The r^A are the non-renormalized coefficients of the anomalies of the Noether currents associated to the chiral invariances of the superpotential, and –like r – are strictly one-loop quantities. The γ_A 's are linear combinations of the anomalous dimensions of the matter fields, and x_g , and x^{ijk} are radiative correction quantities. The structure of Eq. (77) is independent of the renormalization scheme.

One-loop finiteness, i.e. vanishing of the β -functions at one loop, implies that the Yukawa couplings λ_{ijk} must be functions of the gauge coupling g . To find a similar condition to all orders it is necessary and sufficient for the Yukawa couplings to be a formal power series in g , which is solution of the REs (68).

We can now state the theorem for all-order vanishing β -functions [83].

Theorem:

Consider an $N = 1$ SUSY Yang-Mills theory, with simple gauge group. If the following conditions are satisfied

1. There is no gauge anomaly.
2. The gauge β -function vanishes at one loop

$$\beta_g^{(1)} = 0 = \sum_i T(R_i) - 3 C_2(G). \quad (78)$$

3. There exist solutions of the form

$$C_{ijk} = \rho_{ijk}g, \quad \rho_{ijk} \in \mathbb{C} \quad (79)$$

to the conditions of vanishing one-loop matter fields anomalous dimensions

$$\gamma^{(1)j}_i = 0 = \frac{1}{32\pi^2} [C^{ikl} C_{jkl} - 2 g^2 C_2(R) \delta_j^i]. \quad (80)$$

4. These solutions are isolated and non-degenerate when considered as solutions of vanishing one-loop Yukawa β -functions:

$$\beta_{ijk} = 0. \quad (81)$$

Then, each of the solutions (79) can be uniquely extended to a formal power series in g , and the associated super Yang-Mills models depend on the single coupling constant g with a β -function which vanishes at all orders.

Important note: The requirement of isolated and non-degenerate solutions guarantees the existence of a unique formal power series solution to the reduction equations. The vanishing of the gauge β -function at one loop, $\beta_g^{(1)}$, is equivalent to the vanishing of the R-current anomaly (71). The vanishing of the anomalous dimensions at one loop implies the vanishing of the Yukawa couplings β -functions at that order. It also implies the vanishing of the chiral anomaly coefficients r^A . This last property is a necessary condition for having β -functions vanishing at all orders.^a

Proof:

Insert β_{ijk} as given by the REs into the relationship (77). Since these chiral anomalies vanish, we get for β_g an homogeneous equation of the form

$$0 = \beta_g(1 + O(\hbar)). \tag{82}$$

The solution of this equation in the sense of a formal power series in \hbar is $\beta_g = 0$, order by order. Therefore, due to the REs (68), $\beta_{ijk} = 0$ too.

Thus we see that finiteness and reduction of couplings are intimately related. Since an equation like Eq. (77) is absent in non-SUSY theories, one cannot extend the validity of a similar theorem in such theories.

A very interesting development was done in [61]. Based on the all-loop relations among the β -functions of the soft SUSY breaking terms and those of the rigid supersymmetric theory with the help of the differential operators, discussed in Sect. 2.4, it was shown that certain RGI surfaces can be chosen, so as to reach all-loop finiteness of the full theory. More specifically, it was shown that on certain RGI surfaces the partial differential operators appearing in Eq. (41),(42) acting on the β - and γ -functions of the rigid theory can be transformed to total derivatives. Then the all-loop finiteness of the β and γ -functions of the rigid theory can be transferred to the β -functions of the SSB terms. Therefore, a totally all-loop finite $N = 1$ SUSY gauge theory can be constructed, including the soft SUSY breaking terms.

4 Phenomenologically Interesting Models with Reduced Couplings

In this section we review the basic properties of phenomenologically viable SUSY models that use the idea of reduction of couplings. Their predictions for quark masses, the light Higgs boson mass, the SUSY breaking scale (defined as the geometric mean of stops), M_S , the full SUSY spectrum and the Cold Dark Matter (CDM) relic density (in the case the lightest neutralino is considered a CDM candidate) are discussed in Sections 6-9. The set of experimental constraints employed can be found in Section 5. Note that in the examination of the various models we use the unified gaugino mass M instead of M_S , as a more indicative parameter of scale.

^aThere is an alternative way to find finite theories [95–97,99].

4.1 The Minimal $N = 1$ SUSY $SU(5)$

First, we present the partial reduction of couplings in the minimal $N = 1$ SUSY model based on the $SU(5)$ [18, 50]. $\Psi^I(\mathbf{10})$ and $\Phi^I(\mathbf{5})$ accommodate the three generations of quarks and leptons, I running over the three generations, an adjoint $\Sigma(\mathbf{24})$ breaks $SU(5)$ down to the MSSM gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$, and $H(\mathbf{5})$ and $\bar{H}(\mathbf{5})$ describe the two Higgs superfields of the electroweak symmetry breaking (ESB) [100, 101]. Only one set of $(\mathbf{5} + \bar{\mathbf{5}})$ is used to describe the Higgs superfields appropriate for ESB. This minimality renders the present version asymptotically free (negative β_g). Its superpotential is [100, 101]

$$W = \frac{g_t}{4} \epsilon^{\alpha\beta\gamma\delta\tau} \Psi_{\alpha\beta}^{(3)} \Psi_{\gamma\delta}^{(3)} H_\tau + \sqrt{2} g_b \Phi^{(3)\alpha} \Psi_{\alpha\beta}^{(3)} \bar{H}^\beta + \frac{g_\lambda}{3} \Sigma_\alpha^\beta \Sigma_\beta^\gamma \Sigma_\gamma^\alpha + g_f \bar{H}^\alpha \Sigma_\alpha^\beta H_\beta + \frac{\mu_\Sigma}{2} \Sigma_\alpha^\gamma \Sigma_\gamma^\alpha + \mu_H \bar{H}^\alpha H_\alpha . \quad (83)$$

where t, b and f are indices of the antisymmetric $\mathbf{10}$ and adjoint $\mathbf{24}$ tensors, α, β, \dots are $SU(5)$ indices, and the first two generations Yukawa couplings have been suppressed. The SSB Lagrangian is

$$\begin{aligned} -\mathcal{L}_{\text{soft}} = & m_{H_u}^2 \hat{H}^{*\alpha} \hat{H}_\alpha + m_{H_d}^2 \hat{\bar{H}}_\alpha^* \hat{H}^\alpha + m_\Sigma^2 \hat{\Sigma}_\beta^\dagger \hat{\Sigma}_\alpha^\beta + \sum_{I=1,2,3} [m_{\Phi_I}^2 \hat{\Phi}_\alpha^{*(I)} \hat{\Phi}^{(I)\alpha} \\ & + m_{\Psi_I}^2 \hat{\Psi}^\dagger{}^{(I)\alpha\beta} \hat{\Psi}_{\beta\alpha}^{(I)}] + \left\{ \frac{1}{2} M \lambda \lambda + B_H \hat{H}^\alpha \hat{H}_\alpha + B_\Sigma \hat{\Sigma}_\beta^\alpha \hat{\Sigma}_\alpha^\beta + h_f \hat{H}^\alpha \hat{\Sigma}_\alpha^\beta \hat{H}_\beta \right. \\ & \left. + \frac{h_\lambda}{3} \hat{\Sigma}_\alpha^\beta \hat{\Sigma}_\beta^\gamma \hat{\Sigma}_\gamma^\alpha + \frac{h_t}{4} \epsilon^{\alpha\beta\gamma\delta\tau} \hat{\Psi}_{\alpha\beta}^{(3)} \hat{\Psi}_{\gamma\delta}^{(3)} \hat{H}_\tau + \sqrt{2} h_b \hat{\Phi}^{(3)\alpha} \hat{\Psi}_{\alpha\beta}^{(3)} \hat{H}^\beta + \text{h.c.} \right\} , \quad (84) \end{aligned}$$

where the hat denotes the scalar components of the chiral superfields. The β - and γ -functions and a detailed presentation of the model can be found in [102] and in [98, 103].

The minimal number of SSB terms that do not violate perturbative renormalizability is required in the reduced theory. The perturbatively unified SSB parameters significantly differ from the universal ones. The gauge coupling g is assumed to be the primary coupling. We should note that the dimensionless sector admits reduction solutions that are independent of the dimensionful sector. Two sets of asymptotically free (AF) solutions can achieve a Gauge-Yukawa Unification in this model [102]:

$$\begin{aligned} a : g_t &= \sqrt{\frac{2533}{2605}} g + \mathcal{O}(g^3) , \quad g_b = \sqrt{\frac{1491}{2605}} g + \mathcal{O}(g^3) , \quad g_\lambda = 0 , \quad g_f = \sqrt{\frac{560}{521}} g + \mathcal{O}(g^3) , \\ b : g_t &= \sqrt{\frac{89}{65}} g + \mathcal{O}(g^3) , \quad g_b = \sqrt{\frac{63}{65}} g + \mathcal{O}(g^3) , \quad g_\lambda = 0 , \quad g_f = 0 . \end{aligned} \quad (85)$$

The higher order terms denote uniquely computable power series in g . These solutions describe the boundaries of an AF RGI surface in the parameter space, on which g_λ and g_f may differ from zero. This fact makes possible a partial reduction where g_λ and g_f are (non-vanishing) independent parameters without endangering AF. The proton-decay safe region of that surface favours solution a . Therefore, we choose to be exactly at the boundary defined by solution a ^b.

^b $g_\lambda = 0$ is inconsistent, but $g_\lambda < \sim 0.005$ is necessary in order for the proton decay constraint [104] to be satisfied. A small g_λ is expected not to affect the prediction of unification of SSB parameters.

The reduction of dimensionful couplings is performed as in Eq. (30). It is understood that μ_Σ , μ_H and M cannot be reduced in a desired form and they are treated as independent parameters. The lowest-order reduction solution is found to be:

$$B_H = \frac{1029}{521} \mu_H M, \quad B_\Sigma = -\frac{3100}{521} \mu_\Sigma M, \quad (86)$$

$$\begin{aligned} h_t &= -g_t M, \quad h_b = -g_b M, \quad h_f = -g_f M, \quad h_\lambda = 0, \\ m_{H_u}^2 &= -\frac{569}{521} M^2, \quad m_{H_d}^2 = -\frac{460}{521} M^2, \quad m_\Sigma^2 = \frac{1550}{521} M^2, \\ m_{\Phi^3}^2 &= \frac{436}{521} M^2, \quad m_{\Phi^{1,2}}^2 = \frac{8}{5} M^2, \quad m_{\Psi^3}^2 = \frac{545}{521} M^2, \quad m_{\Psi^{1,2}}^2 = \frac{12}{5} M^2. \end{aligned} \quad (87)$$

The gaugino mass M characterizes the scale of the SUSY breaking. It is noted that we may include B_H and B_Σ as independent parameters without changing the one-loop reduction solution (87). Also note that, although we have found specific relations among the soft scalar masses and the unified gaugino mass, the sum rule still holds.

4.2 The Finite $N = 1$ SUSY $SU(5)$

Next, we review an $SU(5)$ gauge theory which is finite (FUT) to all orders, with reduction of couplings applied to the third fermionic generation. This FUT was selected in the past due to agreement with experimental constraints at the time [26] and predicted the light Higgs mass between 121–126 GeV almost five years prior to the discovery. ^c The particle content consists of three $(\bar{\mathbf{5}} + \mathbf{10})$ supermultiplets, a pair for each generation of quarks and leptons, four $(\bar{\mathbf{5}} + \mathbf{5})$ and one $\mathbf{24}$ considered as Higgs supermultiplets. When the finite GUT group is broken, the theory is no longer finite, and we are left with the MSSM [18–20, 105–107].

A predictive all-order finite GYU $SU(5)$ model should also have the following properties:

1. One-loop anomalous dimensions are diagonal, i.e., $\gamma_i^{(1)j} \propto \delta_i^j$.
2. The fermions in the irreps $\bar{\mathbf{5}}_i, \mathbf{10}_i$ ($i = 1, 2, 3$) do not couple to the adjoint $\mathbf{24}$.
3. The two Higgs doublets of the MSSM are mostly made out of a pair of Higgs quintet and anti-quintet, which couple to the third generation.

Reduction of couplings enhances the symmetry, and the superpotential is then given by [55, 108]:

$$\begin{aligned} W &= \sum_{i=1}^3 \left[\frac{1}{2} g_i^u \mathbf{10}_i \mathbf{10}_i H_i + g_i^d \mathbf{10}_i \bar{\mathbf{5}}_i \bar{H}_i \right] + g_{23}^u \mathbf{10}_2 \mathbf{10}_3 H_4 \\ &+ g_{23}^d \mathbf{10}_2 \bar{\mathbf{5}}_3 \bar{H}_4 + g_{32}^d \mathbf{10}_3 \bar{\mathbf{5}}_2 \bar{H}_4 + g_2^f H_2 \mathbf{24} \bar{H}_2 + g_3^f H_3 \mathbf{24} \bar{H}_3 + \frac{g^\lambda}{3} (\mathbf{24})^3. \end{aligned} \quad (88)$$

A more detailed description of the model and its properties can be found in [18–20]. The non-degenerate and isolated solutions to $\gamma_i^{(1)} = 0$ give:

$$(g_1^u)^2 = \frac{8}{5} g^2, \quad (g_1^d)^2 = \frac{6}{5} g^2, \quad (g_2^u)^2 = (g_3^u)^2 = \frac{4}{5} g^2, \quad (89)$$

^cImproved Higgs mass calculations would yield a different interval, still compatible with current experimental data (see below).

$$(g_2^d)^2 = (g_3^d)^2 = \frac{3}{5} g^2, \quad (g_{23}^u)^2 = \frac{4}{5} g^2, \quad (g_{23}^d)^2 = (g_{32}^d)^2 = \frac{3}{5} g^2, \\ (g^\lambda)^2 = \frac{15}{7} g^2, \quad (g_2^f)^2 = (g_3^f)^2 = \frac{1}{2} g^2, \quad (g_1^f)^2 = 0, \quad (g_4^f)^2 = 0.$$

Furthermore, we have the $h = -MC$ relation, while from the sum rule (see Subsection 2.4) we obtain:

$$m_{H_u}^2 + 2m_{\mathbf{10}}^2 = M^2, \quad m_{H_d}^2 - 2m_{\mathbf{10}}^2 = -\frac{M^2}{3}, \quad m_{\frac{5}{3}}^2 + 3m_{\mathbf{10}}^2 = \frac{4M^2}{3}. \quad (90)$$

This shows that we have only two free parameters $m_{\mathbf{10}}$ and M for the dimensionful sector.

The GUT symmetry breaks to the MSSM, where we want only two Higgs doublets. This is achieved with the introduction of appropriate mass terms that allow a rotation in the Higgs sector [19, 20, 109–111], that permits only one pair of Higgs doublets (which couple mostly to the third family) to remain light and acquire vacuum expectation values. The usual fine tuning to achieve doublet-triplet splitting helps the model to avoid fast proton decay (but this mechanism has differences compared to the one used in the minimal $SU(5)$ because of the extended Higgs sector of the finite case).

Thus, below the GUT scale we have the MSSM with the first two generations unrestricted, while the third is given by the finiteness conditions.

4.3 Finite $SU(N)^3$ Unification

One can consider the construction of FUTs that have a product gauge group. Let us consider an $N = 1$ theory with a $SU(N)_1 \times SU(N)_2 \times \dots \times SU(N)_k$ and n_f copies (number of families) of the supermultiplets $(N, N^*, 1, \dots, 1) + (1, N, N^*, \dots, 1) + \dots + (N^*, 1, 1, \dots, N)$. Then, the one-loop β -function coefficient of the RGE of each $SU(N)$ gauge coupling is

$$b = \left(-\frac{11}{3} + \frac{2}{3}\right)N + n_f \left(\frac{2}{3} + \frac{1}{3}\right) \left(\frac{1}{2}\right) 2N = -3N + n_f N. \quad (91)$$

The necessary condition for finiteness is $b = 0$, which occurs only for the choice $n_f = 3$. Thus, it is natural to consider three families of quarks and leptons.

From a phenomenological point of view, the choice is the $SU(3)_C \times SU(3)_L \times SU(3)_R$ model, which is discussed in detail in Ref. [112]. The discussion of the general well-known example can be found in [113–116]. The quarks and the leptons of the model transform as follows:

$$q = \begin{pmatrix} d & u & h \\ d & u & h \\ d & u & h \end{pmatrix} \sim (3, 3^*, 1), \quad q^c = \begin{pmatrix} d^c & d^c & d^c \\ u^c & u^c & u^c \\ h^c & h^c & h^c \end{pmatrix} \sim (3^*, 1, 3), \quad (92)$$

$$\lambda = \begin{pmatrix} N & E^c & \nu \\ E & N^c & e \\ \nu^c & e^c & S \end{pmatrix} \sim (1, 3, 3^*), \quad (93)$$

where h are down-type quarks that acquire masses close to M_{GUT} . We have to impose a cyclic Z_3 symmetry in order to have equal gauge couplings at the GUT scale, i.e.

$$q \rightarrow \lambda \rightarrow q^c \rightarrow q, \quad (94)$$

where q and q^c are given in Eq. (92) and λ in Eq. (93). Then the vanishing of the one-loop gauge β -function, which is the first finiteness condition (66), is satisfied. This leads us to the second condition, namely the vanishing of the anomalous dimensions of all superfields Eq. (67). Let us write down the superpotential first. For one family we have just two trilinear invariants that can be used in the superpotential as follows:

$$f \text{Tr}(\lambda q^c q) + \frac{1}{6} f' \epsilon_{ijk} \epsilon_{abc} (\lambda_{ia} \lambda_{jb} \lambda_{kc} + q_{ia}^c q_{jb}^c q_{kc}^c + q_{ia} q_{jb} q_{kc}), \quad (95)$$

where f and f' are the Yukawa couplings associated to each invariant. The quark and leptons obtain masses when the scalar parts of the superfields (\tilde{N}, \tilde{N}^c) obtain vacuum expectation values (vevs),

$$m_d = f \langle \tilde{N} \rangle, \quad m_u = f \langle \tilde{N}^c \rangle, \quad m_e = f' \langle \tilde{N} \rangle, \quad m_\nu = f' \langle \tilde{N}^c \rangle. \quad (96)$$

For three families, the most general superpotential has 11 f couplings and 10 f' couplings. Since anomalous dimensions of each superfield vanish, 9 conditions are imposed on these couplings:

$$\sum_{j,k} f_{ijk} (f_{ljk})^* + \frac{2}{3} \sum_{j,k} f'_{ijk} (f'_{ljk})^* = \frac{16}{9} g^2 \delta_{il}, \quad (97)$$

where

$$f_{ijk} = f_{jki} = f_{kij}, \quad (98)$$

$$f'_{ijk} = f'_{jki} = f'_{kij} = f'_{ikj} = f'_{kji} = f'_{jik}. \quad (99)$$

Quarks and leptons receive masses when the scalar part of the superfields $\tilde{N}_{1,2,3}$ and $\tilde{N}_{1,2,3}^c$ obtain vevs:

$$(\mathcal{M}_d)_{ij} = \sum_k f_{kij} \langle \tilde{N}_k \rangle, \quad (\mathcal{M}_u)_{ij} = \sum_k f_{kij} \langle \tilde{N}_k^c \rangle, \quad (100)$$

$$(\mathcal{M}_e)_{ij} = \sum_k f'_{kij} \langle \tilde{N}_k \rangle, \quad (\mathcal{M}_\nu)_{ij} = \sum_k f'_{kij} \langle \tilde{N}_k^c \rangle. \quad (101)$$

When the FUT breaks at M_{GUT} , we are left with the MSSM^d, where both Higgs doublets couple maximally to the third generation. These doublets are the linear combinations $\tilde{N}^c = \sum_i a_i \tilde{N}_i^c$ and $\tilde{N} = \sum_i b_i \tilde{N}_i$. For the choice of the particular combinations we can use the appropriate masses in the superpotential [109], since they are not constrained by the finiteness conditions. The FUT breaking leaves remnants in the form of the boundary conditions on the gauge and Yukawa couplings, i.e. Eq. (97), the $h = -Mf$ relation and the soft scalar mass sum rule at M_{GUT} . The latter takes the following form in this model:

$$m_{H_u}^2 + m_{\tilde{t}^c}^2 + m_{\tilde{q}}^2 = M^2 = m_{H_d}^2 + m_{\tilde{b}^c}^2 + m_{\tilde{q}}^2. \quad (102)$$

If the solution of Eq. (97) is both unique and *isolated*, the model is finite in all orders. This leads f' to vanish and we are left with the relations

$$f^2 = f_{111}^2 = f_{222}^2 = f_{333}^2 = \frac{16}{9} g^2. \quad (103)$$

^d [117, 118] and refs therein discuss in detail the spontaneous breaking of $SU(3)^3$.

Since all f' parameters are zero in one-loop level, the lepton masses are zero. They cannot appear radiatively (as one would expect) due to the finiteness conditions, and remain as a problem for further study.

If the solution is just unique (but not isolated, i.e. parametric) we can keep non-vanishing f' and achieve two-loop finiteness, in which case lepton masses are not fixed to zero. Then we have a slightly different set of conditions that restrict the Yukawa couplings:

$$f^2 = r \left(\frac{16}{9} \right) g^2, \quad f'^2 = (1-r) \left(\frac{8}{3} \right) g^2, \quad (104)$$

where r is free and parametrizes the different solutions to the finiteness conditions. It is important to note that we use the sum rule as boundary condition to the soft scalars.

4.4 Reduction of Couplings in the MSSM

Finally, we present a version of the MSSM with reduced couplings. All work is carried out in the framework of the MSSM, but with the assumption of a covering GUT. The original partial reduction in this model was done and analysed in [119,120] and is once more restricted to the third fermionic generation. The superpotential is given by

$$W = Y_t H_2 Q t^c + Y_b H_1 Q b^c + Y_\tau H_1 L \tau^c + \mu H_1 H_2, \quad (105)$$

and the SSB Lagrangian is

$$\begin{aligned} -\mathcal{L}_{\text{SSB}} = & \sum_{\phi} m_{\phi}^2 \hat{\phi}^* \hat{\phi} + \left[m_3^2 \hat{H}_1 \hat{H}_2 + \sum_{i=1}^3 \frac{1}{2} M_i \lambda_i \lambda_i + \text{h.c.} \right] \\ & + \left[h_t \hat{H}_2 \hat{Q} \hat{t}^c + h_b \hat{H}_1 \hat{Q} \hat{b}^c + h_\tau \hat{H}_1 \hat{L} \hat{\tau}^c + \text{h.c.} \right], \end{aligned} \quad (106)$$

The Yukawa $Y_{t,b,\tau}$ and the trilinear $h_{t,b,\tau}$ couplings correspond only to the third family.

Starting with the dimensionless sector we consider the top and bottom Yukawa couplings, which will be expressed in terms of the strong coupling. The other gauge couplings, as well as the tau Yukawa coupling are treated as corrections. The REs give

$$\frac{Y_i^2}{4\pi} \equiv \alpha_i = G_i^2 \alpha_3, \quad i = t, b,$$

and, using the Yukawa RGE,

$$G_i^2 = \frac{1}{3}, \quad i = t, b.$$

Furthermore, the above reduction is dictated by the different running behaviour of the couplings of $SU(2)$ and $U(1)$ compared to the strong one [35], as well as the incompatibility of including the tau Yukawa, since its G^2 coefficient turns negative [121]. Adding all three couplings as corrections, one obtains

$$G_t^2 = \frac{1}{3} + \frac{71}{525} \rho_1 + \frac{3}{7} \rho_2 + \frac{1}{35} \rho_\tau, \quad G_b^2 = \frac{1}{3} + \frac{29}{525} \rho_1 + \frac{3}{7} \rho_2 - \frac{6}{35} \rho_\tau \quad (107)$$

where

$$\rho_{1,2} = \frac{g_{1,2}^2}{g_3^2} = \frac{\alpha_{1,2}}{\alpha_3}, \quad \rho_\tau = \frac{g_\tau^2}{g_3^2} = \frac{Y_\tau^2}{\alpha_3} \quad (108)$$

Corrections in Eq. (107) are calculated at the M_{GUT} and assuming

$$\frac{d}{dg_3} \left(\frac{Y_{t,b}^2}{g_3^2} \right) = 0.$$

This assumption practically states that, even including these corrections, at M_{GUT} the ratio of the top (or bottom) coupling over the strong coupling is constant, thus they have negligible scale dependence. This requirement sets the boundary condition at M_{GUT} , given in Eq. (107).

At two-loop level, we assume the corrections to be of the form

$$\alpha_i = G_i^2 \alpha_3 + J_i^2 \alpha_3^2, \quad i = t, b,$$

where the J_i 's are

$$J_i^2 = \frac{1}{4\pi} \frac{17}{24}, \quad i = t, b$$

when only top, bottom and strong gauge couplings are active. If we switch on the rest of the above-mentioned couplings as corrections, we have

$$J_t^2 = \frac{1}{4\pi} \frac{N_t}{D}, \quad J_b^2 = \frac{1}{4\pi} \frac{N_b}{5D},$$

where D , N_t and N_b are known quantities given in [122].

Let us now move to the dimensionful couplings of the SSB sector of the Lagrangian, namely the trilinear couplings $h_{t,b,\tau}$ given in Eq. (106). Following the same pattern as in the dimensionless case, we first reduce $h_{t,b}$, while h_τ is treated as a correction.

$$h_i = c_i Y_i M_3 = c_i G_i M_3 g_3, \quad i = t, b,$$

with M_3 the gluino mass. The use of the h_t and h_b RGEs gives

$$c_t = c_b = -1,$$

where we used the 1-loop relation between the gluino mass and the gauge couplings RGE

$$2M_i \frac{dg_i}{dt} = g_i \frac{dM_i}{dt}, \quad i = 1, 2, 3.$$

Switching on the other gauge couplings and h_τ as corrections, we have

$$c_t = -\frac{A_A A_{bb} + A_{tb} B_B}{A_{bt} A_{tb} - A_{bb} A_{tt}}, \quad c_b = -\frac{A_A A_{bt} + A_{tt} B_B}{A_{bt} A_{tb} - A_{bb} A_{tt}}.$$

Again, A_{tt} , A_{bb} and A_{tb} are given in [122].

Finally, we turn our attention to the soft scalar masses m_ϕ^2 of the SSB Lagrangian. Their reduction (see Sect. 2.3) takes the form

$$m_i^2 = c_i M_3^2, \quad i = Q, u, d, H_u, H_d. \quad (109)$$

Then, the soft scalar masses RGEs at one loop reduce to the following (the corrections from the tau Yukawa, h_τ and the two gauge couplings are included)

$$\begin{aligned} c_Q &= -\frac{c_{Q\text{Num}}}{D_m}, & c_u &= -\frac{1}{3} \frac{c_{u\text{Num}}}{D_m}, & c_d &= -\frac{c_{d\text{Num}}}{D_m}, \\ c_{H_u} &= -\frac{2}{3} \frac{c_{H_u\text{Num}}}{D_m}, & c_{H_d} &= -\frac{c_{H_d\text{Num}}}{D_m}, \end{aligned}$$

where D_m , $c_{Q\text{Num}}$, $c_{u\text{Num}}$, $c_{d\text{Num}}$, $c_{H_u\text{Num}}$, $c_{H_d\text{Num}}$ and the complete analysis are again given in [122].

For the completely reduced system, i.e. g_3, Y_t, Y_b, h_t, h_b , the coefficients of the soft scalar masses become

$$c_Q = c_u = c_d = \frac{2}{3}, \quad c_{H_u} = c_{H_d} = -1/3,$$

obeying the sum rules

$$\frac{m_Q^2 + m_u^2 + m_{H_u}^2}{M_3^2} = c_Q + c_u + c_{H_u} = 1, \quad \frac{m_Q^2 + m_d^2 + m_{H_d}^2}{M_3^2} = c_Q + c_d + c_{H_d} = 1.$$

Concerning the gaugino masses, the Hisano-Shiftman relation (Eq. (53)) is applied to each gaugino mass as a boundary condition at the GUT scale, where the gauge couplings are considered unified. Thus, at one-loop level, each gaugino mass is only dependent on the b-coefficients of the gauge β -functions and the arbitrary M_0 :

$$M_i = b_i M_0. \quad (110)$$

This means that we can make a choice of M_0 such that the gluino mass equals the unified gaugino mass, and the other two gaugino masses are equal to the gluino mass times the ratio of the appropriate b-coefficients.

In Sect. 9 we begin with the selection of the free parameters. This discussion is intimately connected to the fermion masses predictions.

5 Phenomenological Constraints

In our phenomenological analysis we apply several experimental constraints, which we will briefly review in this section.

Starting from the quark masses, we calculate the top quark pole mass, while the bottom quark mass is evaluated at M_Z , in order not to encounter uncertainties inherent to its pole mass. Their experimental values are [123],

$$m_b(M_Z) = 2.83 \pm 0.10 \text{ GeV}. \quad (111)$$

and

$$m_t^{\text{exp}} = (173.1 \pm 0.9) \text{ GeV} . \quad (112)$$

The discovery of a Higgs-like particle at ATLAS and CMS in July 2012 [22, 124] can be interpreted as the discovery of the light \mathcal{CP} -even Higgs boson of the MSSM Higgs spectrum [125–127]. The experimental average for the (SM) Higgs boson mass is [123]^e

$$M_H^{\text{exp}} = 125.10 \pm 0.14 \text{ GeV} . \quad (113)$$

The theoretical accuracy [27, 28, 30], however, for the prediction of M_h in the MSSM, dominates the uncertainty. In our following analysis of each of the models described, we use the new **FeynHiggs** code [27–30] (Version 2.16.0) to predict the Higgs mass. **FeynHiggs** evaluates the Higgs masses using a combination of fixed order diagrammatic calculations and resummation of the (sub)leading logarithmic contributions at all orders, and thus provides a reliable evaluation of M_h even for large SUSY scales. The refinements in this combination (w.r.t. previous versions [29]) result in a downward shift of M_h of order $\mathcal{O}(2 \text{ GeV})$ for large SUSY masses. This version of **FeynHiggs** computes the uncertainty of the Higgs boson mass point by point. This theoretical uncertainty is added linearly to the experimental error in Eq. (113).

We also consider four types of flavour constraints, in which SUSY has non-negligible impact, namely the flavour observables $\text{BR}(b \rightarrow s\gamma)$, $\text{BR}(B_s \rightarrow \mu^+\mu^-)$, $\text{BR}(B_u \rightarrow \tau\nu)$ and ΔM_{B_s} . Although we do not use the latest experimental values, no major effect would be expected.

- For the branching ratio $\text{BR}(b \rightarrow s\gamma)$ we take a value from the Heavy Flavor Averaging Group (HFAG) [128, 129]:

$$\frac{\text{BR}(b \rightarrow s\gamma)^{\text{exp}}}{\text{BR}(b \rightarrow s\gamma)^{\text{SM}}} = 1.089 \pm 0.27 . \quad (114)$$

- For the branching ratio $\text{BR}(B_s \rightarrow \mu^+\mu^-)$ we use a combination of CMS and LHCb data [130–134]:

$$\text{BR}(B_s \rightarrow \mu^+\mu^-) = (2.9 \pm 1.4) \times 10^{-9} . \quad (115)$$

- For the B_u decay to $\tau\nu$ we use the limit [129, 135, 136]:

$$\frac{\text{BR}(B_u \rightarrow \tau\nu)^{\text{exp}}}{\text{BR}(B_u \rightarrow \tau\nu)^{\text{SM}}} = 1.39 \pm 0.69 . \quad (116)$$

- For ΔM_{B_s} we use [137, 138]:

$$\frac{\Delta M_{B_s}^{\text{exp}}}{\Delta M_{B_s}^{\text{SM}}} = 0.97 \pm 0.2 . \quad (117)$$

We finally consider Cold Dark Matter (CDM) constraints. Since the lightest neutralino, being the Lightest SUSY Particle (LSP), is a very promising candidate for CDM [31], we demand that our LSP is indeed the lightest neutralino and we discard parameters leading to different LSPs. The current bound on the CDM relic density at 2σ level is given by [139, 140]^f

$$\Omega_{\text{CDM}} h^2 = 0.1120 \pm 0.0112 . \quad (118)$$

^eThis is the latest available LHC combination. More recent measurements confirm this value.

^fWhile this is not the latest value, updates would have no visible effect on our analysis.

For the calculation of the relic density of each model we use the `MicrOMEGAs` code [32–34] (Version 5.0). The calculation of annihilation and coannihilation channels is also included. It should be noted that other CDM constraints do not affect our models significantly, and thus were not included in our analysis.

6 Numerical Analysis of the Minimal $N = 1$ $SU(5)$

Here, we analyse the particle spectrum predicted by the Minimal $N = 1$ SUSY $SU(5)$ as discussed in Subsection 4.1 for $\mu < 0$. Below M_{GUT} all couplings and masses of the theory run according to the RGEs of the MSSM. Thus we examine the evolution of these parameters according to their RGEs up to two-loops for dimensionless parameters and at one-loop for dimensionful ones imposing the corresponding boundary conditions. In Fig. 1, we show the predictions for $m_b(M_Z)$ and m_t as a function of the unified gaugino mass M . The green points include the B-physics constraints. The ΔM_{B_s} channel is responsible for the gap at the B-physics allowed points. One can see that, once more, the model (mostly) prefers the higher energy region of the spectrum (especially with the admission of B-physics constraints). The orange (blue) lines denote the 2σ (3σ) experimental uncertainties, while the black dashed lines in the left plot add a ~ 6 MeV theory uncertainty to that. The uncertainty for the boundary conditions of the Yukawa couplings is taken to be 7%, which is included in the spread of the points shown. In the evaluation of the bottom mass we have included the corrections coming from bottom squark-gluino loops and top squark-chargino loops [141]. One can see in the left plot of Fig. 1 that only by taking all uncertainties to their limit, some points at very high M are within these bounds. I.e. confronting the Minimal $N = 1$ SUSY $SU(5)$ with the quark mass measurements “nearly” excludes this model, and only a very heavy spectrum might be in agreement with the experimental data.

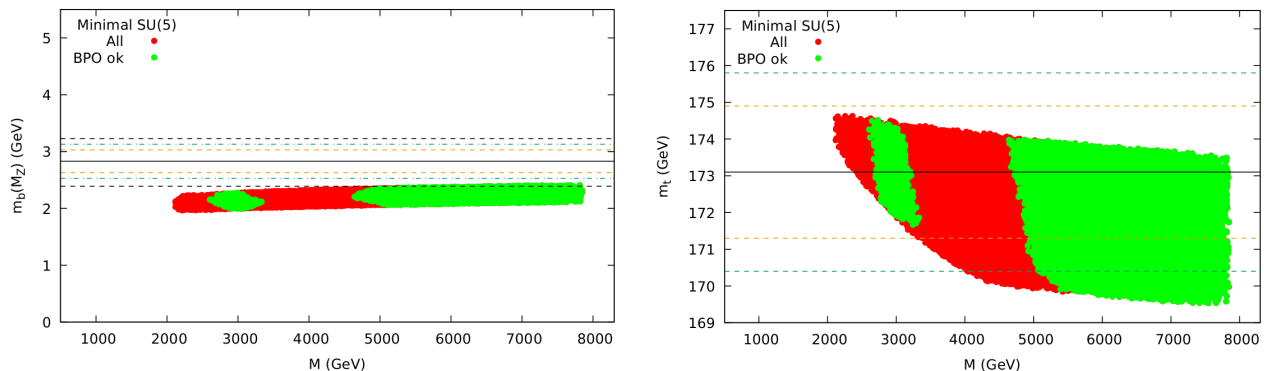


Figure 1: The bottom quark mass at the Z boson scale (left) and top quark pole mass (right) are shown as a function of M for the Minimal $N = 1$ $SU(5)$. The green points are the ones that satisfy the B-physics constraints. The orange (blue) dashed lines denote the 2σ (3σ) experimental uncertainties, while the black dashed lines in the left plot add a ~ 6 MeV theory uncertainty to that.

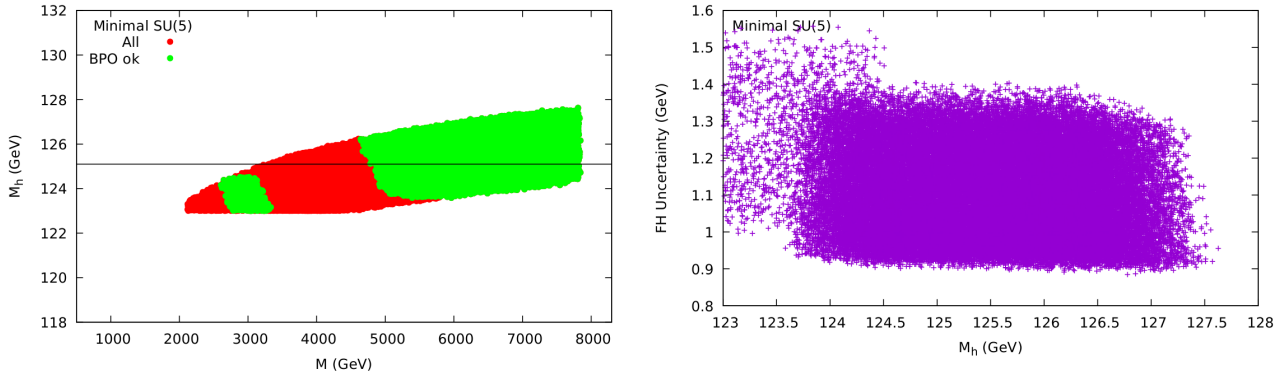


Figure 2: Left: The lightest Higgs mass, M_h , as a function of M for the Minimal $N = 1$ $SU(5)$ model. The B -physics constraints allow (mostly) higher scale points (with green colour). Right: The lightest Higgs mass theoretical uncertainty [30].

The prediction for M_h with $\mu < 0$ is given in Fig. 2 (left), for a unified gaugino mass between 2 TeV and 8 TeV, where again the green points satisfy B -physics constraints. Fig. 2 (right) gives the theoretical uncertainty of the Higgs mass for each point, calculated with `FeynHiggs 2.16.0` [30]. There is substantial improvement to the Higgs mass uncertainty compared to past analyses, since it has dropped by more than 1 GeV.

The full particle spectrum of the model (third generation of fermions only) that complies with quark mass and B -physics constraints as well as with the Higgs-boson mass constraint is shown in Fig. 3. Here the points used have a Higgs mass within the bounds $125.1 \pm \text{unc}$, where “unc” denotes the uncertainty shown in Fig. 2 (right). Correspondingly, in Tab. 1 we present an example spectrum, that is in agreement with all the constraints. The table shows the lightest and the heaviest spectrum (based on $m_{\tilde{\chi}_1^0}$). The Higgs boson masses are denoted as M_h , M_H , M_A and M_{H^\pm} . $m_{\tilde{t}_{1,2}}$, $m_{\tilde{b}_{1,2}}$, $m_{\tilde{g}}$ and $m_{\tilde{\tau}_{1,2}}$, are the scalar top, bottom, gluino and tau masses, respectively. $m_{\tilde{\chi}_{1,2}^\pm}$ and $m_{\tilde{\chi}_{1,2,3,4}^0}$ stand for chargino and neutralino masses, respectively. As expected from the quark mass discussion, one can observe that the allowed spectrum is extremely heavy. Depending on the details, the FCC-hh might be able to observe some parts of the (colored) spectrum [155]. On the other hand, improved predictions for the bottom-quark mass may rule out this model, independent of further experimental data.

Furthermore, no point fulfills the strict bound of Eq. (118), since the relic abundance turns out to be too high. Thus, our model needs a mechanism that can reduce the CDM abundance in the early universe. This issue could be related to the problem of neutrino masses. These masses cannot be generated naturally in this particular model, although a non-zero value for neutrino masses has been established [136]. However, the model could be, in principle, extended by introducing bilinear R -parity violating terms and introduce neutrino masses [142, 143]. R -parity violation [144] would have a small impact on the above collider phenomenology (apart from the fact that supersymmetry search strategies could not rely on a ‘missing energy’ signature), but remove the CDM bound of Eq. (118) completely. Other mechanisms, not involving R -parity violation and keeping the ‘missing energy’ signature, that could be invoked if the amount of CDM appears to be

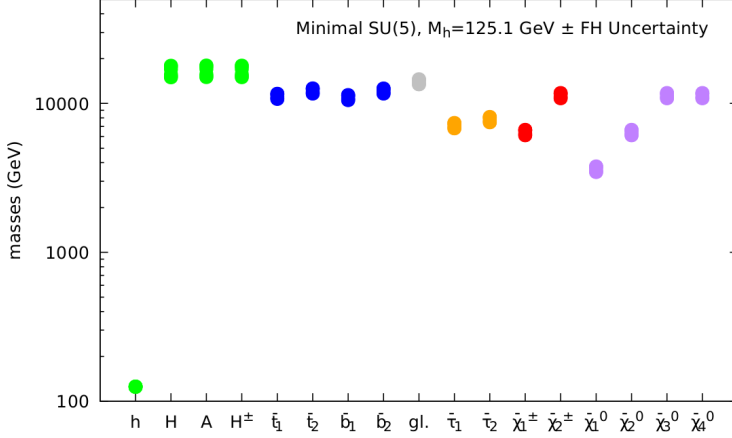


Figure 3: The plot shows the spectrum of the Minimal $N = 1$ $SU(5)$ model for points with Higgs mass within its calculated uncertainty. The green points are the various Higgs boson masses; the blue points are the two scalar top and bottom masses; the gray ones are the gluino masses; then come the scalar tau masses in orange; the red points are the two chargino masses; followed by the purple points indicating the neutralino masses.

	M_h	M_H	M_A	M_{H^\pm}	$m_{\tilde{t}_1}$	$m_{\tilde{t}_2}$	$m_{\tilde{b}_1}$	$m_{\tilde{b}_2}$	$m_{\tilde{g}}$
lightest	124.6	15163	15163	15163	10755	11683	10551	11683	13477
heaviest	125.3	17920	17920	17920	11609	12609	11390	12615	14532
	$m_{\tilde{\tau}_1}$	$m_{\tilde{\tau}_2}$	$m_{\tilde{\chi}_1^\pm}$	$m_{\tilde{\chi}_2^\pm}$	$m_{\tilde{\chi}_1^0}$	$m_{\tilde{\chi}_2^0}$	$m_{\tilde{\chi}_3^0}$	$m_{\tilde{\chi}_4^0}$	$\tan \beta$
lightest	6819	7486	6125	10873	3468	6126	10870	10873	49
heaviest	7396	8107	6645	11745	3772	6647	11747	111752	49.5

Table 1: Example spectrum of the Minimal $N = 1$ $SU(5)$. Masses are in GeV and rounded to 1 (0.1) GeV (for the light Higgs mass).

too large, concern the cosmology of the early universe. For example, “thermal inflation” [145] or “late time entropy injection” [146] can bring the CDM density into agreement with WMAP measurements.

7 Numerical Analysis of the Finite $N = 1$ $SU(5)$

In this section we discuss the full particle spectrum predicted in the Finite $N = 1$ SUSY $SU(5)$ model, as discussed in Subsection 4.2. The gauge symmetry breaks spontaneously below the GUT

scale, so conditions set by finiteness do not restrict the renormalization properties at low energies. We are left with boundary conditions on the gauge and Yukawa couplings (89), the $h = -MC$ relation and the soft scalar-mass sum rule at M_{GUT} . Again, the uncertainty for the boundary conditions of the Yukawa couplings is at 7%, which again is included in the spread of the points.

In Fig. 4, $m_b(M_Z)$ and m_t are shown as functions of the unified gaugino mass M , where the green points satisfy the B-physics constraints with the same color coding as in Fig. 1. Here we omitted the additional theoretical uncertainty of ~ 6 MeV. The only phenomenologically viable option is to consider $\mu < 0$, as is shown in earlier work [147–149]. The experimental values are indicated by the horizontal lines with the uncertainties at the 2σ and 3σ level. The value of the bottom mass is lower than in past analyses, sending the allowed energy scale higher. Also the top-quark mass turns out slightly lower than in previous analyses.

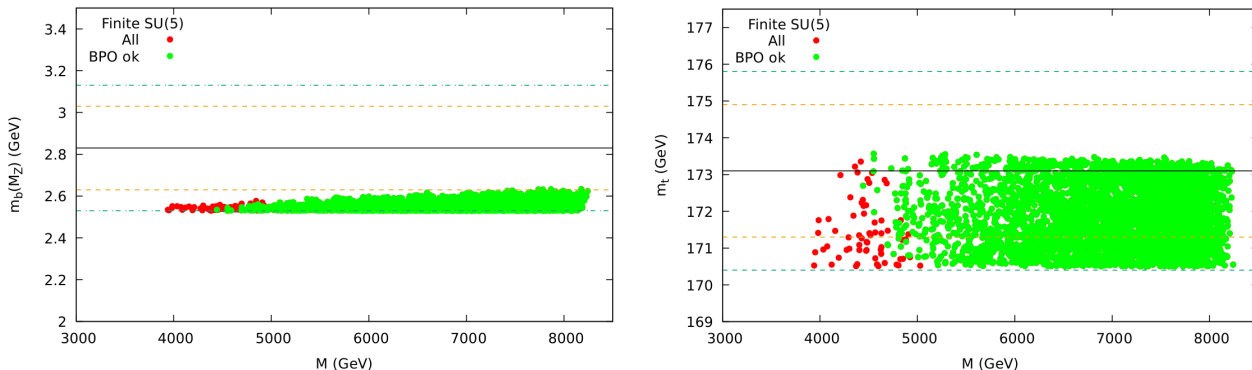


Figure 4: $m_b(M_Z)$ (left) and m_t (right) as a function of M for the Finite $N = 1$ $SU(5)$, with the color coding as in Fig. 1.

The light Higgs boson mass is given in Fig. 5 (left) as a function of the unified gaugino mass. Like in the previous section, these predictions are subject to a theory uncertainty [30] that is given in Fig. 5 (right). This point-by-point uncertainty (calculated with `FeynHiggs`) drops significantly from the flat estimate of 2 and 3 GeV of past analyses to the much improved $0.65 - 0.70$ GeV. The B -physics constraints (green points) and the smaller Higgs uncertainty drive the energy scale above ~ 4.5 TeV. Older analyses, including in particular less refined evaluations of the light Higgs mass, are given in Refs. [147–149]. It should be noted that, w.r.t. previous analyses the top-quark mass turns out to be slightly lower. Consequently, higher scalar top masses have to be reached in order to yield the Higgs-boson mass around its central value of Eq. (113), resulting in a correspondingly heavier spectrum.

In Fig. 6 we show the full particle spectrum (for the third fermionic generation), where we only keep points that fulfill all the experimental constraints (see above). Correspondingly, in Tab. 2 we give an example spectrum which shows the mass range of the parameter space that complies with all the above-mentioned experimental constraints. Compared to our previous analyses [147–149, 156–158], the improved evaluation of M_h and its uncertainty, together with a lower prediction of the top-quark mass prefers a heavier Higgs and SUSY spectrum. In particular, very heavy coloured SUSY particles are favoured (nearly independent of the M_h uncertainty), in agreement with the non-

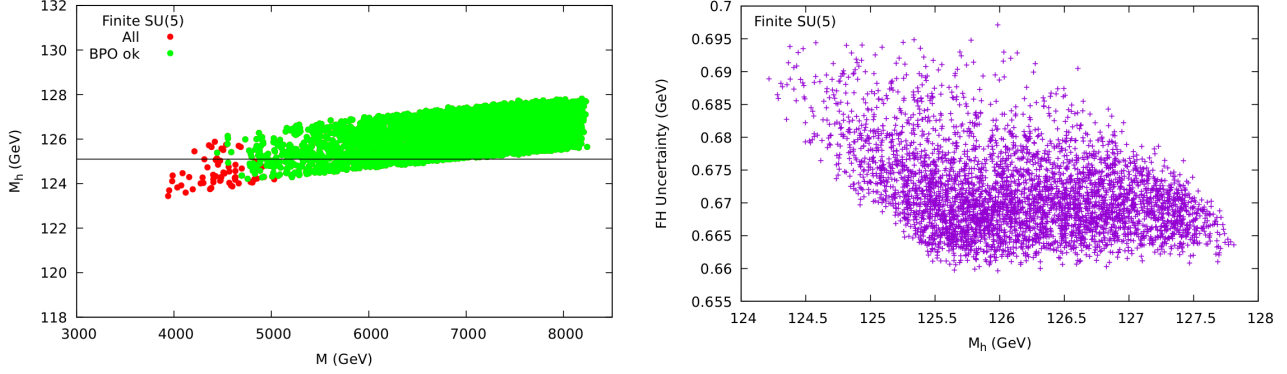


Figure 5: Left: M_h as a function of M . Green points comply with B -physics constraints. Right: The lightest Higgs mass theoretical uncertainty calculated with FeynHiggs 2.16.0 [30].

observation of those particles at the LHC [154]. Overall, the allowed coloured SUSY masses would remain unobservable at the (HL-)LHC, the ILC or CLIC. However, the coloured spectrum would be accessible at the FCC-hh [155], as could the lower part of the heavy Higgs-boson spectrum.

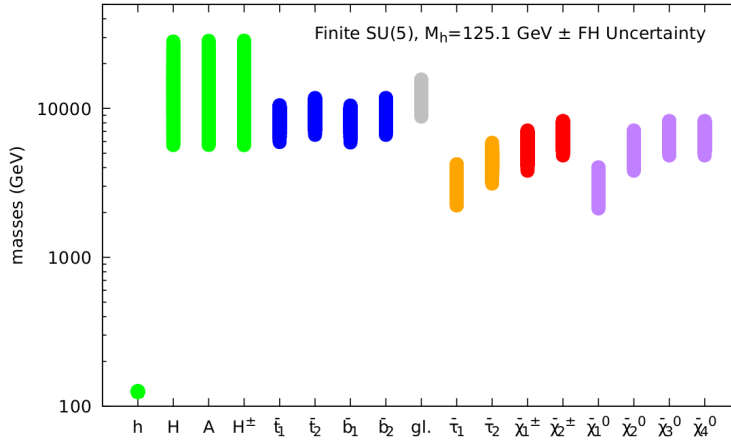


Figure 6: The plot shows the spectrum of the Finite $N = 1$ $SU(5)$ model for points in agreement with all experimental constraints (see text). The color coding is as in Fig. 3.

	M_h	M_H	M_A	M_{H^\pm}	$m_{\tilde{t}_1}$	$m_{\tilde{t}_2}$	$m_{\tilde{b}_1}$	$m_{\tilde{b}_2}$	$m_{\tilde{g}}$
lightest	124.4	5513	5513	5510	5940	6617	5888	6617	8819
heaviest	125.8	28121	28121	28120	10486	11699	10318	11686	15509
	$m_{\tilde{\tau}_1}$	$m_{\tilde{\tau}_2}$	$m_{\tilde{\chi}_1^\pm}$	$m_{\tilde{\chi}_2^\pm}$	$m_{\tilde{\chi}_1^0}$	$m_{\tilde{\chi}_2^0}$	$m_{\tilde{\chi}_3^0}$	$m_{\tilde{\chi}_4^0}$	$\tan \beta$
lightest	2225	3123	3819	4801	2120	3811	4820	4811	50
heaviest	4215	5788	7108	8200	4019	7108	8227	8227	51

Table 2: Example spectrum of the Finite $N = 1$ $SU(5)$. Masses are in GeV and rounded to 1 (0.1) GeV (for the light Higgs mass).

Concerning DM, the model exhibits a high relic abundance for CDM. The CDM alternatives proposed for the Minimal $SU(5)$ model can also be applied here. It should be noted that the bilinear R-parity violating terms proposed in the previous section preserve finiteness, as well.

8 Numerical Analysis of the Two-Loop Finite $N = 1$ $SU(3) \otimes SU(3) \otimes SU(3)$

We continue our analysis with the two-loop finite $N = 1$ SUSY $SU(3) \otimes SU(3) \otimes SU(3)$ model, as described in the Subsection 4.3. Again, below M_{GUT} we get the MSSM. We further assume a unique SUSY breaking scale M_{SUSY} and below that scale the effective theory is just the SM. The boundary condition uncertainty is at 5% for the Yukawa couplings and at 1% for the strong gauge coupling and the soft parameters.

We take into account two new thresholds for the masses of the new particles h 's and E 's (of the third family in particular) at $\sim 10^{13}$ GeV and $\sim 10^{14}$ GeV. This results in a wider phenomenologically viable parameter space [159]. Specifically, one of the down-like exotic particles decouples at 10^{14} GeV, while the rest decouple at 10^{13} GeV.

We compare our predictions with the experimental value of m_t^{exp} , while in the case of the bottom quark we take again the value evaluated at M_Z , see Eq. (111). We single out the $\mu < 0$ case as the most promising model. With the inclusion of thresholds for the decoupling of the exotic particles, the parameter space allowed predicts a top quark mass in agreement with experimental bounds (see Eq. (112)), which is an important improvement from past versions of the model [112, 160–162]. Looking for the values of the parameter r (see Subsection 4.3) which comply with the experimental limits (see Section 5) for $m_b(M_Z)$ and m_t , we find, as shown in Fig. 7, that both masses are in the experimental range for the same value of r between 0.65 and 0.80. It is important to note that the two masses are simultaneously within two sigmas of the experimental bounds.

In Fig. 8 (left) the light Higgs boson mass is shown as a function of the unified gaugino mass, while with the point-by-point calculated theoretical uncertainty drops below 1 GeV [30] (Fig. 8 (right)). As in the previous models examined, the B -physics constraints (green points in Fig. 8 (left) satisfy them) and the new, more restrictive Higgs mass uncertainty exclude most of the low

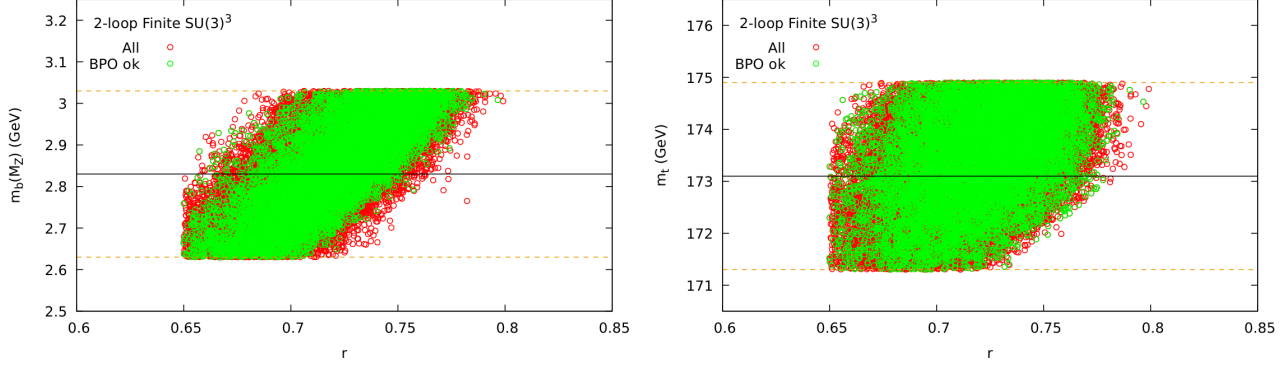


Figure 7: Bottom and top quark masses for the Finite $N = 1$ $SU(3) \otimes SU(3) \otimes SU(3)$ model, with $\mu < 0$, as functions of r . The color coding is as in Fig. 1.

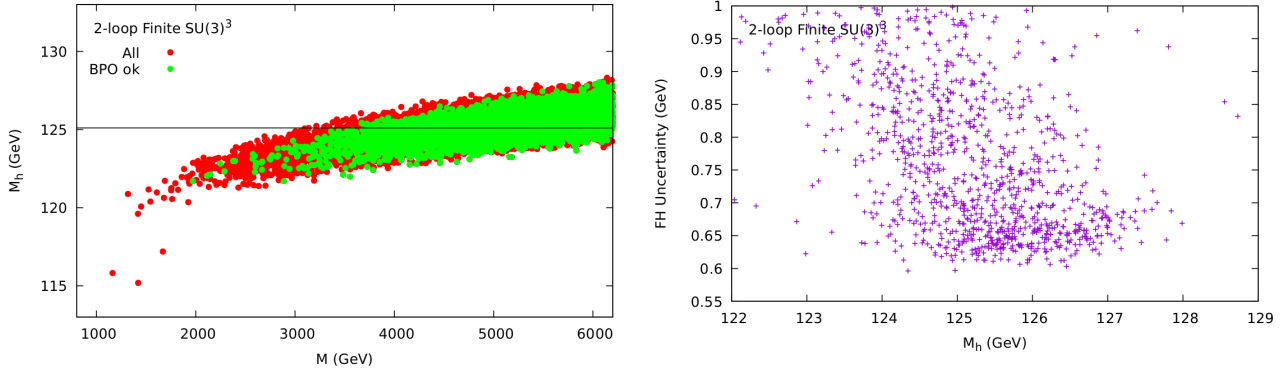


Figure 8: Left: M_h as a function of M for the Finite $N = 1$ $SU(3) \otimes SU(3) \otimes SU(3)$. Right: The Higgs mass theoretical uncertainty [30].

range of M , pushing the particle spectrum to higher values. This is obvious in Fig. 9 where the full SUSY spectrum is shown. As before, an example spectrum of Tab. 3 gives the lightest and heaviest values for each value of the spectrum. In fact, all constraints regarding quark masses, the light Higgs boson mass and B-physics are satisfied, rendering the model very successful. The only observable that fails to comply with the experimental bounds is the CDM relic density (see Eq. (118)). The lightest neutralino is the LSP and considered as a CDM candidate, but its relic density does not go below 0.15, since it is strongly Bino-like and would require a lower scale of the particle spectrum. It should be noted that if the B-physics constraints allowed for a unified gaugino mass ~ 0.5 TeV lower, then agreement with the CDM bounds as well could be achieved.

The SUSY and Higgs spectrum corresponding to the experimentally allowed points turns out to be too heavy for current or most future experiments. The FCC-hh will be able to test most of the

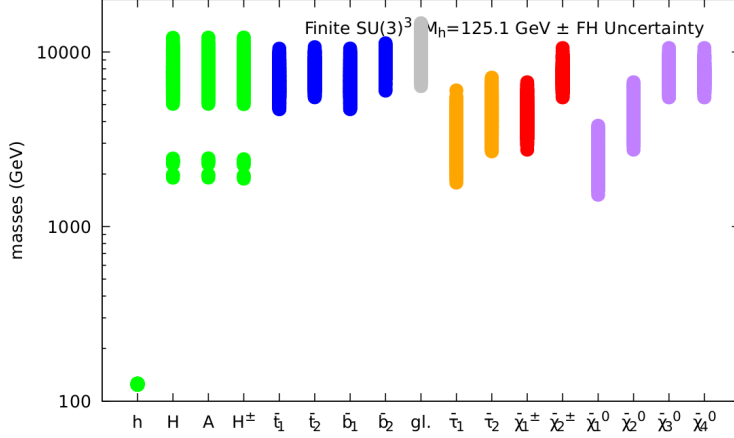


Figure 9: The spectrum of the Finite $N = 1$ $SU(3) \otimes SU(3) \otimes SU(3)$ model for points with light Higgs mass that satisfies its calculated theoretical uncertainty. The color coding is as in Fig. 3.

	M_h	M_H	M_A	M_{H^\pm}	$m_{\tilde{t}_1}$	$m_{\tilde{t}_2}$	$m_{\tilde{b}_1}$	$m_{\tilde{b}_2}$	$m_{\tilde{g}}$
lightest	124.2	1918	1918	1917	4703	5480	4671	6013	6329
heaviest	125.9	12053	12053	12050	10426	10631	10426	11193	14550
	$m_{\tilde{\tau}_1}$	$m_{\tilde{\tau}_2}$	$m_{\tilde{\chi}_1^\pm}$	$m_{\tilde{\chi}_2^\pm}$	$m_{\tilde{\chi}_1^0}$	$m_{\tilde{\chi}_2^0}$	$m_{\tilde{\chi}_3^0}$	$m_{\tilde{\chi}_4^0}$	$\tan \beta$
lightest	1774	2694	2736	5469	1517	2736	5480	5481	44
heaviest	5999	7113	6713	10522	3767	6703	10522	10523	53

Table 3: Example spectrum of the Finite $N = 1$ $SU(3) \otimes SU(3) \otimes SU(3)$. Masses are in GeV and rounded to 1 (0.1) GeV (for the light Higgs mass).

spectrum, in particular for colored particles. However, also here the highest parts of the allowed parameter space might be inaccessible even to this collider.

9 Numerical Analysis of the Reduced MSSM

The relations among reduced parameters in terms of the fundamental ones derived in Sect. 4.4 have an RGI part and a part that originates from the corrections, and thus scale dependent. In the present analysis we choose the unification scale to apply the corrections to all these RGI relations. As was noted earlier, the Hisano-Shiftman relation sets a hierarchy among the gaugino masses, rendering Wino the lightest of them. As such, we have a Wino-like lightest neutralino (which is

the LSP).

In the dimensionless sector of the theory, since Y_τ is not reduced in favour of the fundamental parameter α_3 , the tau lepton mass is an input parameter and, consequently, ρ_τ is an independent parameter, too. At low energies we fix ρ_τ and $\tan\beta$ using the mass of the tau lepton $m_\tau(M_Z) = 1.7462$ GeV. Then, we determine the top and bottom masses using the value found for $\tan\beta$ together with $G_{t,b}$, as obtained from the REs and their corrections.

Correspondingly, concerning the dimensionful sector, h_τ cannot be expressed in terms of the unified gaugino mass scale, leaving ρ_{h_τ} a free parameter. μ is a free parameter as well, as it cannot be reduced in favour of M_3 as discussed above. On the other hand, m_3^2 could be reduced, but here we choose to leave it free. However, μ and m_3^2 are restricted from the requirement of EWSB, and only μ is taken as an independent parameter. Finally, the other parameter in the Higgs-boson sector, the \mathcal{CP} -odd Higgs-boson mass M_A is evaluated from μ , as well as from $m_{H_u}^2$ and $m_{H_d}^2$, which are obtained from the REs. In total, we vary the parameters ρ_τ , ρ_{h_τ} , M and μ .

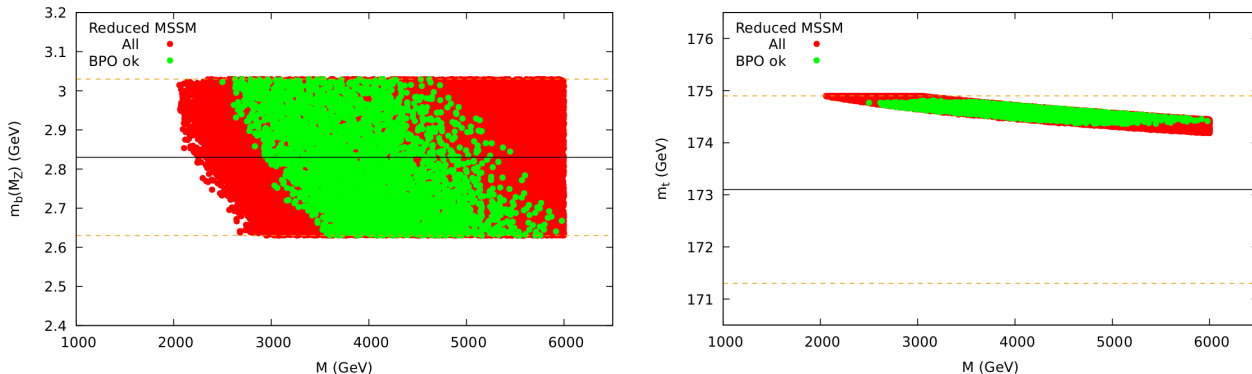


Figure 10: The left (right) plot shows the bottom (top) quark mass for the Reduced MSSM, with the color coding as in Fig. 1.

As we have already mentioned, the variation of ρ_τ gives the running bottom quark mass at the Z boson mass scale and the top pole mass, where points not within 2σ of the experimental data are neglected, as it is shown in Fig. 10. The experimental values (see Sect. 5) are denoted by the horizontal lines with the uncertainties at the 2σ level. The green dots satisfy the flavour constraints. One can see that the scan yields many parameter points that are in very good agreement with the experimental data and give restrictions in the allowed range of M (the common gaugino mass at the unification scale).

The prediction for M_h is shown in Fig. 11 (left). Once again, one should keep in mind that the theory uncertainty given in Fig. 11 (right) has dropped below 1 GeV [30]. The Higgs mass predicted by the model is in the range measured at the LHC, favoring this time relatively small values of M . This in turn sets a limit on the low-energy SUSY masses, rendering the Reduced MSSM highly predictive and testable. In Fig. 12 we show its full spectrum (again, third generation of sfermions only), which complies with the B -physics and the Higgs mass uncertainty (with the color coding as in Fig. 3). Correspondingly, in Tab. 4 we show an example spectrum of the lightest and heaviest value of each parameter of the SUSY spectrum of the Reduced MSSM, in agreement

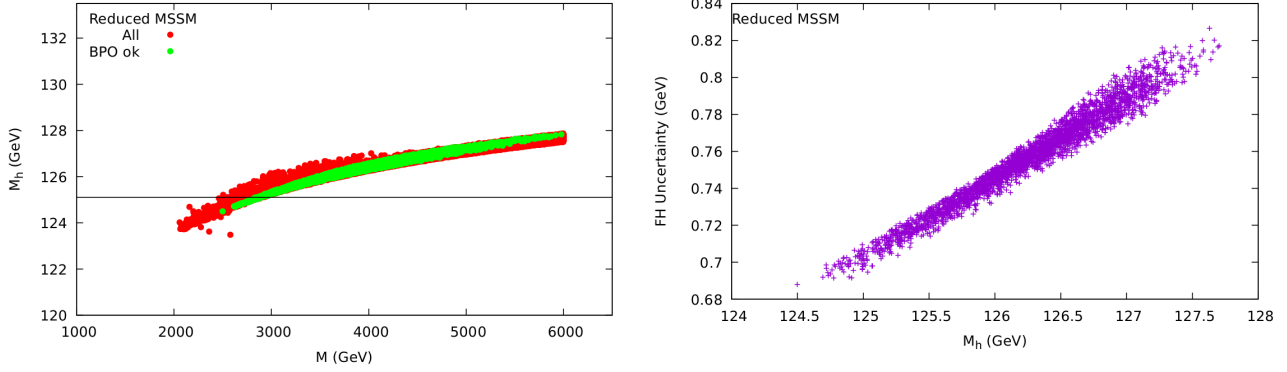


Figure 11: Left: The lightest Higgs boson mass, M_h in the Reduced MSSM. The green points is the full model prediction. Right: the lightest Higgs mass theoretical uncertainty [30].

with the Higgs-boson mass measurement and its calculated theoretical uncertainty, as well as with the B -physics constraints.

From the spectra shown in Fig. 12 and Tab. 4 it can be concluded that already the HL-LHC [163] will be able to test the full Higgs spectrum. The lighter SUSY particles, which are given by the electroweak spectrum, will mostly remain unobservable at the LHC and at future e^+e^- colliders such as the ILC or CLIC. An exception are the lightest neutralino and chargino masses, which could be covered by CLIC3TeV. The coloured mass spectrum will remain unobservable at the (HL-)LHC, but could be accessible at the FCC-hh [155], which could either confirm the SUSY spectrum of the Reduced MSSM or rule it out.

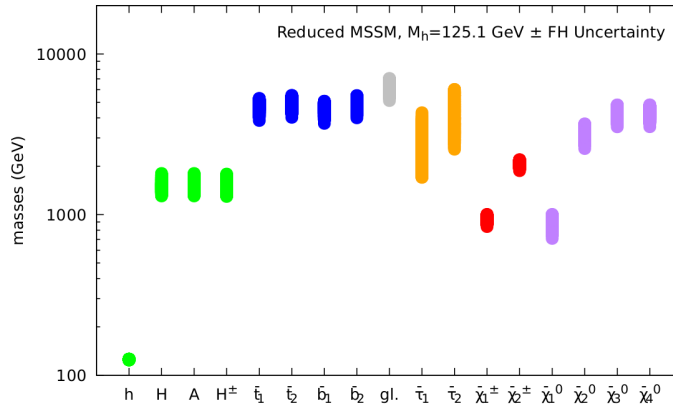


Figure 12: The full spectrum of the Reduced MSSM after keeping only the points with Higgs mass that complies with its theoretical uncertainty. The colours denoting each mass are as described in Sect. 6.

	M_h	M_H	M_A	M_{H^\pm}	$m_{\tilde{t}_1}$	$m_{\tilde{t}_2}$	$m_{\tilde{b}_1}$	$m_{\tilde{b}_2}$	$m_{\tilde{g}}$
lightest	124.5	1305	1305	1297	3851	4029	3699	4007	5126
heaviest	125.8	1801	1801	1780	5275	5564	5076	5502	7017
	$m_{\tilde{\tau}_1}$	$m_{\tilde{\tau}_2}$	$m_{\tilde{\chi}_1^\pm}$	$m_{\tilde{\chi}_2^\pm}$	$m_{\tilde{\chi}_1^0}$	$m_{\tilde{\chi}_2^0}$	$m_{\tilde{\chi}_3^0}$	$m_{\tilde{\chi}_4^0}$	$\tan \beta$
lightest	1705	2536	843	1875	711	2579	3516	3517	40
heaviest	4288	6008	1004	2195	1001	3666	4814	4815	45

Table 4: Example spectrum of the Reduced MSSM. All masses are in GeV and rounded to 1 (0.1) GeV (for the light Higgs mass).

Concerning the DM predictions, it should be noted that the Hisano-Shiftman relation imposes a Wino-like LSP, which unfortunately lowers the CDM relic density below the boundaries of Eq. (118). This renders this model viable if Eq. (118) is applied only as an upper limit and additional sources of CDM are allowed. This is in contrast to the other three models discussed previously.

10 Conclusions

In this review we have briefly discussed the ideas concerning the reduction of couplings of renormalizable theories and the theoretical tools which have been developed to confront the problem. Updates and new results were given for four specific models, in which the reduction of parameters has been theoretically explored and tested against the experimental data. Important updates w.r.t. previous analyses are the improved Higgs-boson mass predictions as provided by the latest version of `FeynHiggs` (version 2.16.0), including in particular the improved uncertainty evaluation. Furthermore, we have evaluated the CDM predictions of each model with `MicrOMEGAs`(version 5.0). From a phenomenological point of view, the reduction of couplings method described in the article provides selection rules that single out realistic GUTs. It is also possible to work with the reduction of couplings method directly in the MSSM. In this case, the number of free parameters is decreased substantially and the model becomes more predictive [119, 120, 122, 150, 158].

We focused our analysis in four models, namely the Minimal $N = 1$ $SU(5)$, the Finite $N = 1$ $SU(5)$, the Two-Loop Finite $N = 1$ $SU(3) \otimes SU(3) \otimes SU(3)$ and the Reduced MSSM, which are presented in Sect. 6-9 respectively and share similar features. The Minimal $N = 1$ $SU(5)$ model predicts the top quark mass and the light Higgs boson mass in agreement with LHC measurements, as well as the full SUSY spectrum of the MSSM. However, concerning the bottom-quark mass predictions, relatively small values are obtained, and agreement with the experimental data can be found at the 3σ level only if additionally a ~ 6 MeV theory uncertainty is included, favoring an extremely heavy SUSY spectrum. The Finite $N = 1$ $SU(5)$ model, the Finite $N = 1$ $SU(3) \otimes SU(3) \otimes SU(3)$ model and the Reduced MSSM are in natural agreement with all LHC measurements and searches. Concerning the DM predictions, the three former models have an excess of CDM w.r.t. the experimental measurements, while the latter has a lower relic density than required by

experimental searches. This renders this model viable if the experimental value is applied only as an upper limit and additional sources of CDM are allowed. This is in contrast to the other three models discussed previously.

All models predict relatively heavy spectra, the heavy parts of which evade detection in present and near-future colliders, with the exception of the lighter part of the Reduced MSSM spectrum. The Higgs sector of that model can be fully tested already at the HL-LHC, and the lighter electroweak spectrum could be covered by CLIC3TeV. On the other hand, the FCC-hh will have the capacity to test large parts of the predicted parameter spaces of all four models. From this point of view, the Reduced MSSM is the model with the best prospect, since it allows the lightest spectrum out of the four models. On the theoretical side, the long-term challenge is in the development of a framework in which the above successes of the field theory models are combined with gravity.

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