On certain sums concerning the gcd's and lcm's of k positive integers

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Abstract

We use elementary arguments to prove results on the order of magnitude of certain sums concerning the gcd's and lcm's of k positive integers, where $k \geq 2$ is fixed. We refine and generalize an asymptotic formula of Bordellès (2007), and extend certain related results of Hilberdink and Tóth (2016). We also formulate some conjectures and open problems.

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1 Introduction

Consider the gcd-sum function

$$G(n) := \sum_{k=1}^{n} (k, n) = \sum_{d \mid n} d\varphi(n/d) \qquad (n \in \mathbb{N}),$$

where $\varphi(n)$ is Euler's totient function. The function $G(n)$ is multiplicative and the asymptotic formula

$$\sum_{n \leq x} G(n) = \frac{x^2}{2\zeta(2)} \left(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)} \right) + \mathcal{O}(x^{1+\theta+\varepsilon}), \tag{1.1}$$

holds for every $\varepsilon > 0$, where γ is Euler's constant, and θ is the exponent appearing in Dirichlet's divisor problem. See the survey paper [8] by the third author.

The function

$$G^{(-1)}(n) := \sum_{k=1}^{n} \frac{1}{(k, n)} = \sum_{d \mid n} \frac{\varphi(n/d)}{d} \qquad (n \in \mathbb{N}),$$

is also multiplicative. Bordellès [1, Th. 5.1] deduced that

$$\sum_{n \leq x} G^{(-1)}(n) = \frac{\zeta(3)}{2\zeta(2)} x^2 + O\left(x (\log x)^{2/3} (\log \log x)^{4/3} \right). \tag{1.2}$$

The error term of estimate (1.2) comes from the classical result of Walfisz [9, Satz 1, p. 144],

$$R(x) := \sum_{n \le x} \varphi(n) - \frac{1}{2\zeta(2)} x^2 = O\left(x(\log x)^{2/3}(\log\log x)^{4/3}\right). \tag{1.3}$$

We remark that recently (1.3) was improved by Liu [4] into

$$R(x) = O\left(x(\log x)^{2/3}(\log\log x)^{1/3}\right), \tag{1.4}$$

therefore, this serves as the remainder of (1.2). Also see the preprint by Suzuki [7].

The lcm-sum function

$$L(n) := \sum_{k=1}^{n} [k, n] = \frac{n}{2}\left(1 + \sum_{d \mid n} d\varphi(d)\right) \qquad (n \in \mathbb{N}).$$

was investigated by Bordellès [1], Ikeda and Matsuoka [3], and others. The function $L(n)$ is not multiplicative and one has, see [1, Th. 6.3],

$$\sum_{n \le x} L(n) = \frac{\zeta(3)}{8\zeta(2)} x^4 + O\left(x^3(\log x)^{2/3}(\log\log x)^{4/3}\right). \tag{1.5}$$

By using (1.4), the exponent of the $\log\log x$ factor in the error of (1.5) can be improved into 1/3.

Now let

$$L^{(-1)}(n) := \sum_{k=1}^{n} \frac{1}{[k, n]} \qquad (n \in \mathbb{N}).$$

Bordellès [1, Th. 7.1] proved that

$$\sum_{n \le x} L^{(-1)}(n) = \frac{1}{\pi^2}(\log x)^3 + A(\log x)^2 + O(\log x), \tag{1.6}$$

with an explicitly given constant A.

By the general identity

$$\sum_{m,n \le x} \psi(m, n) = 2 \sum_{n \le x} \sum_{m=1}^{n} \psi(m, n) - \sum_{n \le x} \psi(n, n),$$

valid for any function $\psi : \mathbb{N}^2 \to \mathbb{C}$, which is symmetric in the variables, (1.1), (1.2), (1.5) and (1.6), together with the remark on (1.4) lead to the asymptotic formulas

$$\sum_{m,n \le x} (m, n) = \frac{x^2}{\zeta(2)}\left(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta(2)}{2} - \frac{\zeta'(2)}{\zeta(2)}\right) + O\left(x^{1+\theta+\varepsilon}\right), \tag{1.7}$$

$$\sum_{m,n \le x} \frac{1}{(m, n)} = \frac{\zeta(3)}{\zeta(2)} x^2 + O\left(x(\log x)^{2/3}(\log\log x)^{1/3}\right), \tag{1.8}$$

$$\sum_{m,n\le x}[m,n]=\frac{\zeta(3)}{4\zeta(2)}x^4+O\left(x^3(\log x)^{2/3}(\log\log x)^{1/3}\right), \tag{1.9}$$

and

$$\sum_{m,n\le x}\frac{1}{[m,n]}=\frac{2}{\pi^2}(\log x)^3+A_1(\log x)^2+O(\log x), \tag{1.10}$$

respectively, where $A_1=2A$.

It is easy to generalize (1.7) and (1.8) for sums with k variables by using the general identity

$$\sum_{n_1,\ldots,n_k\le x}f((n_1,\ldots,n_k))=\sum_{d\le x}(\mu*f)(d)\lfloor x/d\rfloor^k,$$

where f is an arbitrary arithmetic function, μ is the Möbius function and $*$ stands for the Dirichlet convolution of arithmetic functions. For example, we have the next result: For any $k\ge 3$,

$$\sum_{n_1,\ldots,n_k\le x}\frac{1}{(n_1,\ldots,n_k)}=\frac{\zeta(k+1)}{\zeta(k)}x^k+O\left(x^{k-1}\right).$$

However, it is more difficult to derive asymptotic formulas for similar sums involving the lcm $[n_1,\ldots,n_k]$. As corollaries of more general results concerning a large class of functions f, the first and third authors [2, Cor 1] proved that for any $k\ge 3$ and any real number $r>-1$,

$$\sum_{n_1,\ldots,n_k\le x}[n_1,\ldots,n_k]^r=A_{r,k}x^{k(r+1)}+O\left(x^{k(r+1)-\frac{1}{2}\min(r+1,1)+\varepsilon}\right) \tag{1.11}$$

and

$$\sum_{n_1,\ldots,n_k\le x}\left(\frac{[n_1,\ldots,n_k]}{n_1\cdots n_k}\right)^r=A_{r,k}x^k+O\left(x^{k-\frac{1}{2}\min(r+1,1)+\varepsilon}\right),$$

where $A_{k,r}$ are explicitly given constants. Here, (1.11) is the k dimensional generalization of (1.9). Furthermore, [2, Cor 2] shows that for any $k\ge 3$ and any real number $r>0$,

$$\sum_{n_1,\ldots,n_k\le x}\left(\frac{[n_1,\ldots,n_k]}{(n_1,\ldots,n_k)}\right)^r=B_{r,k}x^{k(r+1)}+O\left(x^{k(r+1)-\frac{1}{2}+\varepsilon}\right),$$

with explicitly given constants $B_{k,r}$. The proofs use the fact that $(n_1,\ldots,n_k)$ and $[n_1,\ldots,n_k]$ are multiplicative functions of k variables and the associated multiple Dirichlet series factor over the primes into Euler products. The proofs given in [2] cannot be applied in the case $r=-1$.

It is the goal of the present paper to investigate the order of magnitude of the sums

$$S_k(x):=\sum_{n_1,\ldots,n_k\le x}\frac{1}{[n_1,\ldots,n_k]}, \tag{1.12}$$

$$T_k(x):=\sum_{n_1,\ldots,n_k\le x}\frac{(n_1,\ldots,n_k)}{[n_1,\ldots,n_k]}, \tag{1.13}$$

$$U_k(x) := \sum_{\substack{n_1,\ldots,n_k\le x\\ (n_1,\ldots,n_k)=1}} \frac{1}{[n_1,\ldots,n_k]}, \tag{1.14}$$

$$V_k(x) := \sum_{n_1,\ldots,n_k\le x} \frac{n_1\cdots n_k}{[n_1,\ldots,n_k]}, \tag{1.15}$$

where $k\ge 2$ is fixed, by using elementary arguments. Theorem 2.1, concerning the sum $S_2(x)$, refines formulas (1.6) and (1.10) of Bordellès [1]. Theorems 2.3 and 3.1 give the exact order of magnitude of the sums $S_k(x)$ and $U_k(x)$, respectively, for $k\ge 3$. Theorem 4.1 concerns the sums $V_k(x)$, while Theorem 5.2 provides an asymptotic formula with remainder term for $T_k(x)$, for any fixed $k\ge 2$. Some conjectures and open problems are formulated as well.

2 The sums $S_k(x)$

First consider the sums $S_k(x)$ defined by (1.12). In the case $k=2$ we use Dirichlet's hyperbola method to prove the next result, which improves formulas (1.6) and (1.10).

Theorem 2.1.

$$\sum_{n\le x} L^{(-1)}(n) = \frac{1}{\pi^2}(\log x)^3 + A(\log x)^2 + B\log x + C + O\left(x^{-1/2}(\log x)^2\right), \tag{2.1}$$

that is,

$$\sum_{m,n\le x} \frac{1}{[m,n]} = \frac{2}{\pi^2}(\log x)^3 + A_1(\log x)^2 + B_1\log x + C_1 + O\left(x^{-1/2}(\log x)^2\right),$$

where the constants A,B,C can be explicitly computed, and $A_1=2A$, $B_1=2B-1$, $C_1=C-\gamma$.

Proof. We have

$$L^{(-1)}(n) = \sum_{k=1}^{n} \frac{(k,n)}{kn} = \frac{1}{n}\sum_{d\mid n} d \sum_{\substack{k=1\\ (k,n)=d}}^{n} \frac{1}{k} = \frac{1}{n}\sum_{d\mid n} \sum_{\substack{t=1\\ (t,n/d)=1}}^{n/d} \frac{1}{t} = \frac{1}{n}\sum_{d\mid n} h(d), \tag{2.2}$$

where

$$h(n) := \sum_{\substack{m=1\\ (m,n)=1}}^{n} \frac{1}{m} = \sum_{m=1}^{n} \frac{1}{m} \sum_{d\mid (m,n)} \mu(d) = \sum_{d\mid n} \frac{\mu(d)}{d} \sum_{j=1}^{n/d} \frac{1}{j}$$

$$= \sum_{d\mid n} \frac{\mu(d)}{d}\left(\log\frac{n}{d} + \gamma + O\left(\frac{d}{n}\right)\right) = \sum_{d\mid n} \frac{\mu(d)}{d}\log\frac{n}{d} + \gamma\frac{\varphi(n)}{n} + O\left(\frac{2^{\omega(n)}}{n}\right).$$

Hence,

$$H(x) := \sum_{n\le x} h(n) = \sum_{d\le x} \frac{\mu(d)}{d} \sum_{m\le x/d} \log m + \gamma \sum_{n\le x} \frac{\varphi(n)}{n} + O\left(\sum_{n\le x} \frac{2^{\omega(n)}}{n}\right).$$

By using the known estimates

$$\sum_{n\le x} \log n = x\log x - x + O(\log x),$$
$$\sum_{n\le x} \frac{\varphi(n)}{n} = \frac{6}{\pi^2}x + O(\log x),$$
$$\sum_{n\le x} \frac{2^{\omega(n)}}{n} = O((\log x)^2),$$

we deduce that

$$H(x) = (x\log x - x)\sum_{d\le x}\frac{\mu(d)}{d^2} - x\sum_{d\le x}\frac{\mu(d)\log d}{d^2} + \frac{6}{\pi^2}\gamma x + O((\log x)^2)$$
$$= \frac{6}{\pi^2}(x\log x + cx) + O((\log x)^2), \tag{2.3}$$

with a certain constant c. Let $\mathbf{1}(n) = 1$ ($n \in \mathbb{N}$), and let $*$ denote the Dirichlet convolution. By Dirichlet's hyperbola method,

$$\sum_{n\le x}(\mathbf{1}*h)(n) = \sum_{n\le\sqrt{x}}\left(H(x/n) + h(n)\lfloor x/n\rfloor\right) - \lfloor\sqrt{x}\rfloor H(\sqrt{x})$$
$$= \sum_{n\le\sqrt{x}} H(x/n) + x\sum_{n\le\sqrt{x}}\frac{h(n)}{n} - \sqrt{x}H(\sqrt{x}) + O(H(\sqrt{x})).$$

By partial summation,

$$x\sum_{n\le\sqrt{x}}\frac{h(n)}{n} = \sqrt{x}H(\sqrt{x}) + x\int_1^{\sqrt{x}}\frac{H(t)}{t^2}\,dt,$$

and using (2.3) we deduce

$$\sum_{n\le x}(\mathbf{1}*h)(n) = \frac{6}{\pi^2}\sum_{n\le\sqrt{x}}\left(\frac{x}{n}\log\left(\frac{x}{n}\right) + c\left(\frac{x}{n}\right)\right) + \frac{6x}{\pi^2}\int_1^{\sqrt{x}}\left(\frac{\log t}{t} + c\right)\frac{dt}{t} + O\left(\sqrt{x}(\log x)^2\right)$$
$$= x\left(\frac{3}{\pi^2}(\log x)^2 + a\log x + b\right) + O(\sqrt{x}(\log x)^2),$$

for some constants a, b, which can be explicitly calculated.

Here $(\mathbf{1}*h)(n) = nL^{(-1)}(n)$, according to (2.2), and we obtain (2.1) by partial summation. □

It is more difficult to handle the sums $S_k(x)$ in the case $k \ge 3$. We will apply the following general result proved by the second and third authors [5], using elementary arguments.

Theorem 2.2. ([5]) *Let k be a positive integer and let $f:\mathbb{N}\to\mathbb{C}$ be a multiplicative function satisfying the following properties:*

(i) $f(p)=k$ for every prime p,

(ii) $f(p^\nu)=\nu^{O(1)}$ for every prime p and every integer $\nu\geq 2$, where the constant implied by the O symbol is uniform in p.

Then

$$\sum_{n\leq x}\frac{f(n)}{n}=\frac{1}{k!}C_f(\log x)^k+D_f(\log x)^{k-1}+O\left((\log x)^{k-2}\right),$$

where C_f and D_f are constants,

$$C_f=\prod_p\left(1-\frac{1}{p}\right)^k\left(\sum_{\nu=0}^{\infty}\frac{f(p^\nu)}{p^\nu}\right).$$

We have the following result.

Theorem 2.3. *Let $k\geq 3$ be a fixed integer. Then*

$$S_k(x)\asymp(\log x)^{2^k-1}\qquad \text{as } x\to\infty.$$

Proof. Since $[n_1,\dots,n_k]\leq n_1\cdots n_k\leq x^k$, we can write

$$S_k(x)=\sum_{n\leq x^k}\frac{1}{n}\sum_{\substack{n_1,\dots,n_k\leq x\\ [n_1,\dots,n_k]=n}}1 \tag{2.4}$$

Let

$$a_k(n):=\sum_{\substack{n_1,\dots,n_k\in\mathbb{N}\\ [n_1,\dots,n_k]=n}}1.$$

Now if $n\leq x$, then the inner sum in (2.4) is just $a_k(n)$ (since $n\leq x$ forces $n_1,\dots,n_k\leq x$), while in any case it is at most $a_k(n)$. Thus

$$\sum_{n\leq x}\frac{a_k(n)}{n}\leq S_k(x)\leq\sum_{n\leq x^k}\frac{a_k(n)}{n}. \tag{2.5}$$

To see the properties of the function $a_k(n)$ write

$$\sum_{d|n}a_k(d)=\sum_{d|n}\sum_{[n_1,\dots,n_k]=d}1=\sum_{[n_1,\dots,n_k]|n}1=\sum_{n_1|n,\dots,n_k|n}1=\tau(n)^k.$$

Therefore, by Möbius inversion, we have $a_k=\mu*\tau^k$. This shows that $a_k(n)$ is multiplicative and its values at the prime powers p^ν are given by $a_k(p^\nu)=(\nu+1)^k-\nu^k$ ($\nu\geq 1$). In particular, $a_k(p)=2^k-1$.

Applying Theorem 2.2 for the function $f(n)=a_k(n)$, with 2^k-1 instead of k, we get that

$$\sum_{n\leq x}\frac{a_k(n)}{n}\sim\alpha_k(\log x)^{2^k-1}\qquad\text{as } x\to\infty, \tag{2.6}$$

for some constant α_k. Now, from (2.5) and (2.6) the result follows. □

Remark 2.4. It is natural to expect that $S_k(x) \sim c_k(\log x)^{2^k-1}$ as $x \to \infty$, with a certain constant c_k. In fact, in view of Theorem 2.1, the plausible conjecture is that

$$S_k(x) = P_{2^k-1}(\log x) + O(x^{-r}), \tag{2.7}$$

where $P_{2^k-1}(t)$ is a polynomial in t of degree $2^k - 1$ and r is a positive real number. We pose as an open problem to find the constants c_k and to prove (2.7).

3 The sums $U_k(x)$

Next consider the sums $U_k(x)$ defined by (1.14). In the case $k = 2$,

$$U_2(x) \sim \frac{6}{\pi^2}(\log x)^2 \qquad \text{as } x \to \infty,$$

and it is not difficult to deduce a more precise asymptotic formula.

We have the following general result.

Theorem 3.1. *Let $k \geq 3$ be a fixed integer. Then*

$$U_k(x) \asymp (\log x)^{2^k-2} \qquad \text{as } x \to \infty.$$

Proof. Similar to the proof of Theorem 2.3. We have

$$U_k(x) = \sum_{\substack{n_1,\ldots,n_k \leq x \\ (n_1,\ldots,n_k)=1}} \frac{1}{[n_1,\ldots,n_k]} = \sum_{n \leq x^k} \frac{1}{n} \sum_{\substack{n_1,\ldots,n_k \leq x \\ [n_1,\ldots,n_k]=n \\ (n_1,\ldots,n_k)=1}} 1. \tag{3.1}$$

Let

$$b_k(n) = \sum_{\substack{n_1,\ldots,n_k \in \mathbb{N} \\ [n_1,\ldots,n_k]=n \\ (n_1,\ldots,n_k)=1}} 1.$$

Now if $n \leq x$, then the inner sum in (3.1) is exactly $b_k(n)$, while in any case it is at most $b_k(n)$. Thus

$$\sum_{n \leq x} \frac{b_k(n)}{n} \leq U_k(x) \leq \sum_{n \leq x^k} \frac{b_k(n)}{n}. \tag{3.2}$$

Write

$$\sum_{d|n} b_k(d) = \sum_{d|n} \sum_{\substack{[n_1,\ldots,n_k]=d \\ (n_1,\ldots,n_k)=1}} 1 = \sum_{\substack{[n_1,\ldots,n_k]|n \\ (n_1,\ldots,n_k)=1}} 1$$

$$= \sum_{n_1|n,\ldots,n_k|n} \sum_{\delta|(n_1,\ldots,n_k)} \mu(\delta) = \sum_{\delta a_1 b_1 = n,\ldots,\delta a_k b_k = n} \mu(\delta)$$

$$= \sum_{\delta t = n} \mu(\delta) \sum_{a_1 b_1 = t} 1 \cdots \sum_{a_k b_k = t} 1 = \sum_{\delta t = n} \mu(\delta) \tau(t)^k.$$

Therefore, by Möbius inversion $b_k = \mu * \mu * \tau^k$. This shows that $b_k(n)$ is multiplicative and its values at the prime powers p^ν are given by $b_k(p^\nu) = (\nu+1)^k - 2\nu^k + (\nu-1)^k$ $(\nu \geq 1)$. In particular, $b_k(p) = 2^k - 2$.

Applying now Theorem 2.2 for the function $f(n) = b_k(n)$, with $2^k - 2$ instead of k, we deduce that

$$\sum_{n \leq x} \frac{b_k(n)}{n} \sim \alpha_k'(\log x)^{2^k-2} \qquad \text{as } x \to \infty \tag{3.3}$$

for some constant α_k'. Now, from (3.2) and (3.3) we have $U_k(x) \asymp (\log x)^{2^k-2}$. □

Remark 3.2. We conjecture that $U_k(x) \sim d_k(\log x)^{2^k-2}$ as $x \to \infty$, with a certain constant d_k. The sums $S_k(x)$ and $U_k(x)$ are strongly related. Namely, by grouping the terms according to the values $(n_1, \ldots, n_k) = d$ one obtains

$$S_k(x) = \sum_{d \leq x} \frac{1}{d} U_k(x/d), \tag{3.4}$$

and conversely,

$$U_k(x) = \sum_{d \leq x} \frac{\mu(d)}{d} S_k(x/d). \tag{3.5}$$

If $U_k(x) \sim d_k(\log x)^{2^k-2}$ holds, then by (3.4) it follows that $S_k(x) \sim \frac{d_k}{2^k-1}(\log x)^{2^k-1}$. Conversely, assume that the asymptotic formula (2.7) is true, where c_k is the leading coefficient of the polynomial $P_{2^k-1}(t)$. Then (3.5), together with the well known results

$$\sum_{n \leq x} \frac{\mu(n)}{n} = O((\log x)^{-1}), \qquad \sum_{n=1}^{\infty} \frac{\mu(n)\log n}{n} = -1,$$

and Shapiro's estimates [6, Th. 4.1]

$$\sum_{n \leq x} \frac{\mu(n)}{n} \left(\log\left(\frac{x}{n}\right)\right)^m = m(\log x)^{m-1} + \sum_{i=1}^{m-2} c_j^{(m)} (\log x)^j + O(1),$$

valid for any integer $m \geq 2$, where $c_i^{(m)}$ are constants, imply that

$$U_k(x) = (2^k-1)c_k(\log x)^{2^k-2} + b_{2^k-3}(\log x)^{2^k-3} + \cdots + b_1 \log x + O(1),$$

with some constants b_i.

4 The sums $V_k(x)$

The sums $V_k(x)$ defined by (1.15) are sums of integers. In the case $k = 2$ we have, according to (1.7),

$$V_2(x) = \sum_{m,n \leq x} (m,n) \sim \frac{6}{\pi^2} x^2 \log x. \tag{4.1}$$

Theorem 4.1. *Let $k \ge 3$ be a fixed integer. Then*

$$x^k \ll V_k(x) \ll x^k (\log x)^{2^k-2} \qquad \text{as } x \to \infty.$$

Proof. The lower bound is trivial by $n_1 \cdots n_k \ge [n_1, \ldots, n_k]$. Also, by grouping the terms according to the values $(n_1, \ldots, n_k) = d$, and by denoting $M = \max(m_1, \ldots, m_k)$ we have

$$V_k(x) = \sum_{\substack{dm_1,\ldots,dm_k \le x \\ (m_1,\ldots,m_k)=1}} \frac{dm_1 \cdots dm_k}{[dm_1, \ldots, dm_k]} = \sum_{\substack{m_1,\ldots,m_k \le x \\ (m_1,\ldots,m_k)=1}} \frac{m_1 \cdots m_k}{[m_1, \ldots, m_k]} \sum_{d \le x/M} d^{k-1}$$

$$\ll x^k \sum_{\substack{m_1,\ldots,m_k \le x \\ (m_1,\ldots,m_k)=1}} \frac{m_1 \cdots m_k}{[m_1, \ldots, m_k] M^k} \le x^k \sum_{\substack{m_1,\ldots,m_k \le x \\ (m_1,\ldots,m_k)=1}} \frac{1}{[m_1, \ldots, m_k]} = x^k U_k(x),$$

and the upper bound follows from Theorem 3.1. ☐

Remark 4.2. We conjecture that $V_k(x) \sim \lambda_k x^k (\log x)^{2^k-k-1}$ as $x \to \infty$, with a certain constant λ_k, in accordance with (4.1) for the case $k = 2$. We pose as another open problem to prove this and to find the constants λ_k.

5 The sums $T_k(x)$

Finally, we investigate the sums $T_k(x)$ defined by (1.13) and establish an asymptotic formula with remainder term for it. We give a short direct proof in the case $k = 2$. Then for any fixed $k \ge 2$ we use multiple Dirichlet series to get the result.

Let

$$F(n) := \sum_{k=1}^{n} \frac{(k, n)}{[k, n]} \qquad (n \in \mathbb{N}). \tag{5.1}$$

Theorem 5.1.

$$\sum_{n \le x} F(n) = 2x + O\left((\log x)^2\right), \tag{5.2}$$

that is,

$$\sum_{m,n \le x} \frac{(m, n)}{[m, n]} = 3x + O\left((\log x)^2\right).$$

Proof. Let $\phi_2(n) = \sum_{d|n} d^2 \mu(n/d)$ be the Jordan function of order 2. We have

$$F(n) = \sum_{k=1}^{n} \frac{(k, n)^2}{kn} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k} \sum_{d|(k,n)} \phi_2(d) = \frac{1}{n} \sum_{d|n} \phi_2(d) \sum_{\substack{k=1 \\ d|k}}^{n} \frac{1}{k}$$

$$= \frac{1}{n} \sum_{d|n} \frac{\phi_2(d)}{d} \sum_{j=1}^{n/d} \frac{1}{j} = \frac{1}{n} \sum_{d|n} \frac{\phi_2(d)}{d} H_{n/d},$$

where $H_m = \sum_{j=1}^m 1/j$ is the harmonic sum. Therefore, using that

$$\sum_{n\le x} \frac{\phi_2(n)}{n^2} = \frac{x}{\zeta(3)} + O(1),$$

we deduce

$$\sum_{n\le x} F(n) = \sum_{dm\le x} \frac{\phi_2(d)}{d^2 m} H_m = \sum_{m\le x} \frac{H_m}{m} \sum_{d\le x/m} \frac{\phi_2(d)}{d^2}$$

$$= \sum_{m\le x} \frac{H_m}{m} \left(\frac{x}{\zeta(3)m} + O(1) \right) = \frac{x}{\zeta(3)} \sum_{m\le x} \frac{H_m}{m^2} + O\left(\sum_{m\le x} \frac{H_m}{m} \right)$$

$$= \frac{x}{\zeta(3)} \sum_{m=1}^{\infty} \frac{H_m}{m^2} + O(x \sum_{m>x} \frac{H_m}{m^2}) + O\left(\sum_{m\le x} \frac{H_m}{m} \right)$$

$$= \frac{x}{\zeta(3)} \cdot 2\zeta(3) + O\left(x \sum_{m>x} \frac{\log m}{m^2} \right) + O\left(\sum_{m\le x} \frac{\log m}{m} \right) = 2x + O((\log x)^2),$$

by using that

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3), \tag{5.3}$$

which is Euler's result. □

Theorem 5.2. *If $k \ge 2$, then*

$$T_k(x) = \beta_k x + O\left((\log x)^{2^k - 2} \right),$$

where

$$\beta_k := \sum_{\substack{n_1,\ldots,n_k=1 \\ (n_1,\ldots,n_k)=1}}^{\infty} \frac{1}{[n_1,\ldots,n_k] \max(n_1,\ldots,n_k)} = \frac{1}{\zeta(2)} \sum_{n_1,\ldots,n_k=1}^{\infty} \frac{1}{[n_1,\ldots,n_k] \max(n_1,\ldots,n_k)}.$$

Proof. By grouping the terms according to $(n_1,\ldots,n_k) = d$, where $n_j = dm_j$ $(1 \le j \le k)$, $(m_1,\ldots,m_k) = 1$, we have

$$T_k(x) = \sum_{\substack{dm_1,\ldots,dm_k \le x \\ (m_1,\ldots,m_k)=1}} \frac{d}{[dm_1,\ldots,dm_k]} = \sum_{\substack{dm_1,\ldots,dm_k \le x \\ (m_1,\ldots,m_k)=1}} \frac{1}{[m_1,\ldots,m_k]}$$

$$= \sum_{\substack{m_1,\ldots,m_k \le x \\ (m_1,\ldots,m_k)=1}} \frac{1}{[m_1,\ldots,m_k]} \sum_{d\le x/M} 1 = \sum_{\substack{m_1,\ldots,m_k \le x \\ (m_1,\ldots,m_k)=1}} \frac{\lfloor x/M \rfloor}{[m_1,\ldots,m_k]},$$

where $M=\max(m_1,\ldots,m_k)$. Let

$$h(n_1,\ldots,n_k):=\begin{cases}\frac{1}{[n_1,\ldots,n_k]}, & \text{if } (n_1,\ldots,n_k)=1,\\ 0, & \text{otherwise.}\end{cases}$$

Hence,

$$T_k(x)=x\sum_{n_1,\ldots,n_k\le x}\frac{h(n_1,\ldots,n_k)}{\max(n_1,\ldots,n_k)}+O\left(\sum_{n_1,\ldots,n_k\le x}h(n_1,\ldots,n_k)\right) \tag{5.4}$$

and we estimate the right-hand sums in turn. Here $h(n_1,\ldots,n_k)$ is a symmetric and multiplicative function of k variables and for prime powers $p^{\nu_1},\ldots,p^{\nu_k}$ $(\nu_1,\ldots,\nu_k\ge 0)$ one has

$$h(p^{\nu_1},\ldots,p^{\nu_k})=\begin{cases}\frac{1}{p^{\max(\nu_1,\ldots,\nu_k)}}, & \text{if } \min(\nu_1,\ldots,\nu_k)=0,\\ 0, & \text{otherwise.}\end{cases}$$

Consider its Dirichlet series

$$H(s_1,\ldots,s_k):=\sum_{n_1,\ldots,n_k=1}^{\infty}\frac{h(n_1,\ldots,n_k)}{n_1^{s_1}\cdots n_k^{s_k}}=\prod_p\sum_{\substack{\nu_1,\ldots,\nu_k=0\\ \min(\nu_1,\ldots,\nu_k)=0}}^{\infty}\frac{1}{p^{\max(\nu_1,\ldots,\nu_k)+\nu_1 s_1+\cdots+\nu_k s_k}}.$$

By grouping the terms according to the values of $r=\max(\nu_1,\ldots,\nu_k)$ we deduce

$$H(s_1,\ldots,s_k)=\prod_p\frac{1}{p^r}\sum_{r=0}^{\infty}\sum_{\substack{\nu_1,\ldots,\nu_k=0\\ \max(\nu_1,\ldots,\nu_k)=r\\ \min(\nu_1,\ldots,\nu_k)=0}}^{\infty}\frac{1}{p^{\nu_1 s_1+\cdots+\nu_k s_k}},$$

which converges absolutely for $\Re s_j>0$ $(1\le j\le k)$.

We shall need an estimate for $H_k(\varepsilon,\ldots,\varepsilon)$ for $\varepsilon>0$ (small). We have

$$H(\varepsilon,\ldots,\varepsilon)=\prod_p\left(1+\frac{1}{p}\sum_{j=1}^{k-1}\binom{k}{j}\frac{1}{p^{j\varepsilon}}+O\left(\frac{1}{p^2}\right)\right).$$

Therefore,

$$\log H(\varepsilon,\ldots,\varepsilon)=\sum_p\frac{1}{p}\sum_{j=1}^{k-1}\binom{k}{j}\frac{1}{p^{j\varepsilon}}+O(1)=\sum_{j=1}^{k-1}\binom{k}{j}\sum_p\frac{1}{p^{1+j\varepsilon}}+O(1).$$

But $\sum_p p^{-1-\varepsilon}=\log\frac{1}{\varepsilon}+O(1)$ as $\varepsilon\to 0$. Thus,

$$H(\varepsilon,\ldots,\varepsilon)=\exp\left(\sum_{j=1}^{k-1}\binom{k}{j}\log\frac{1}{\varepsilon}+O(1)\right)\asymp\left(\frac{1}{\varepsilon}\right)^{2^k-2}. \tag{5.5}$$

Furthermore, for any $\varepsilon > 0$, we have

$$\sum_{n_1,\dots n_k\le x} h(n_1,\dots,n_k) = \sum_{n_1,\dots n_k\le x} \frac{h(n_1,\dots,n_k)}{(n_1\cdots n_k)^{\varepsilon/k}}(n_1\cdots n_k)^{\varepsilon/k}$$
$$\le x^{\varepsilon} \sum_{n_1,\dots n_k\le x} \frac{h(n_1,\dots,n_k)}{(n_1\cdots n_k)^{\varepsilon/k}} \le x^{\varepsilon} H(\varepsilon/k,\dots,\varepsilon/k). \tag{5.6}$$

Next, note that $\max(n_1,\dots,n_k) \ge (n_1\cdots n_k)^{1/k}$, so that

$$\sum_{n_1,\dots,n_k\le x} \frac{h(n_1,\dots,n_k)}{\max(n_1,\dots,n_k)} \le \sum_{n_1,\dots,n_k\le x} \frac{h(n_1,\dots,n_k)}{(n_1\cdots n_k)^{1/k}} \le H(\varepsilon/k,\dots,\varepsilon/k),$$

which converges. Hence,

$$\beta_k = \sum_{n_1,\dots,n_k=1}^{\infty} \frac{h(n_1,\dots,n_k)}{\max(n_1,\dots,n_k)}$$

is finite and $\beta_k \le H(\varepsilon/k,\dots,\varepsilon/k)$. Also,

$$\beta_k - \sum_{n_1,\dots,n_k\le x} \frac{h(n_1,\dots,n_k)}{\max(n_1,\dots,n_k)}$$
$$= \sum_{\substack{n_1,\dots,n_k\in\mathbb{N}\\ \text{some}\, n_i > x}} \frac{h(n_1,\dots,n_k)}{\max(n_1,\dots,n_k)}$$
$$\le k \sum_{\substack{n_1\ge n_2,\dots,n_k\\ n_1 > x}} \frac{h(n_1,\dots,n_k)}{n_1} \le k \sum_{\substack{n_1\ge n_2,\dots,n_k\\ n_1 > x}} \frac{h(n_1,\dots,n_k)}{n_1^{1-\varepsilon}(n_1 n_2\cdots n_k)^{\varepsilon/k}}$$
$$\le \frac{k}{x^{1-\varepsilon}} \sum_{n_1,\dots,n_k=1}^{\infty} \frac{h(n_1,\dots,n_k)}{(n_1\cdots n_k)^{\varepsilon/k}} = kx^{\varepsilon-1} H(\varepsilon/k,\dots,\varepsilon/k). \tag{5.7}$$

Hence, (5.4) and the estimates (5.6), (5.7) give

$$T_k(x) = \beta_k x + O\left(x^{\varepsilon} H(\varepsilon/k,\dots,\varepsilon/k)\right).$$

Now we choose $\varepsilon = 1/\log x$ and use the bound (5.5). The proof is complete. □

Remark 5.3. For $k = 2$, Theorem 5.2 recovers Theorem 5.1. Note that

$$\beta_2 = \frac{1}{\zeta(3)} \sum_{m,n=1}^{\infty} \frac{1}{mn\max(m,n)} = \frac{2}{\zeta(3)} \sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{n=1}^{m} \frac{1}{n} - 1 = 3,$$

by Euler's result (5.3). Is it possible to evaluate the constants β_k for any $k \ge 2$?

The sums $T_k(x)$ and $U_k(x)$ are related by the formulas

$$T_k(x) = \sum_{d\le x} U_k(x/d), \qquad U_k(x) = \sum_{d\le x} \mu(d) T_k(x/d).$$

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