

Casimir stresses in active nematic films

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Abstract

We calculate the Casimir stresses in a thin layer of active fluid with nematic order. By using a stochastic hydrodynamic approach for an active fluid layer of finite thickness L , we generalize the Casimir stress for nematic liquid crystals in thermal equilibrium to active systems. We show that the active Casimir stress differs significantly from its equilibrium counterpart. For contractile activity, the active Casimir stress, although attractive like its equilibrium counterpart, diverges logarithmically as L approaches a threshold of the spontaneous flow instability from below. In contrast, for small extensile activity, it is repulsive, has no divergence at any L and has a scaling with L different from its equilibrium counterpart.

INTRODUCTION

It is well-known that although the zero-point energy of the electromagnetic field inside a cavity bounded by conducting walls is formally diverging, its variation upon displacements of the boundaries remains finite. It corresponds to a weak but measurable attractive force, known as the Casimir force [1]. For example, in the case of two parallel conducting plates at a distance L , the attractive Casimir force per unit area, or the Casimir stress is given by $C_F = -\frac{\pi^2}{240} \frac{h^2 \pi c}{L^4}$ [1]. It is of purely quantum origin.

Subsequently, thermal analogs of the Casimir stress associated to various fluctuating fields at a finite temperature T have been studied. In nematic liquid crystals confined between two parallel plates, the thermal fluctuations of the director field that describes the nematic order, play the role of the electromagnetic fluctuations in the electromagnetic Casimir effect. In all such classical systems, the boundary conditions on the relevant fields (e.g., the director field for nematic liquid crystals) constrain their thermal fluctuations and lead to a thermal analog of the Casimir stress. For instance, for a nematic liquid crystal between parallel confining plates separated by a distance L with the director field rigidly anchored to them, one again obtains an attractive Casimir stress that varies with the thickness L of the liquid crystal film as $1/L^3$ [3].

Studies on non-equilibrium analogs of thermal Casimir stresses are relatively new. In Ref. [4], Casimir stresses between two parallel plates due to non-thermal noises are calculated. Further, embedding objects or inclusions in a correlated fluid are shown to generate effective Casimir-like stresses between the inclusions [5]. There are direct biologically relevant examples as well: more recently, Ref. [6] elucidated the dependence of this Casimir-like forces on inclusions in a fluctuating active fluids on active noises and hydrodynamic interaction of the inclusion with the boundaries. Subsequently, Ref. [7] studied the role of active Casimir effects on the deformation dynamics of the cell nucleus and showed the appearance of a fluctuation maximum at a critical level of activity, a result in agreement with recent experiments [8]. The active fluid models considered by Refs. [6, 7] are effectively one-dimensional and hence do not include any soft orientational fluctuations.

In this article, we calculate the Casimir stress between two parallel plates confining a layer of an active nematic fluid with a uniform macroscopic orientation [9–11]. The active fluid is driven out of equilibrium by a locally constant supply of energy. Our work directly

generalizes thermal Casimir stresses in equilibrium nematics [3] into the nonequilibrium domain.

The hydrodynamic active fluid model [9, 10] has been proposed as a generic coarse-grained model for a driven orientable fluid with nematic or polar symmetry. The main feature of the active fluid is the existence of an *active stress* of non-equilibrium origin that describes the constant consumption of energy by the system, which drives the system away from equilibrium. Due to its very general nature, the active fluid model is able to describe a broad range of phenomena, observed in very different physical systems and at very different length scales [9–11]. Notable examples include the dynamics of actin filaments in the cortex of eukaryotic cells or bird flocks and bacterial biofilms. In particular, in the case of actin filament dynamics, the active stress results from the release of free energy due to the chemical conversion of Adenosine-Triphosphate (ATP) to Adenosine-Diphosphate (ADP).

In this article, we study Casimir forces using a stochastically driven coarse-grained hydrodynamic approach for active fluids [9–11], with a nematic order, described by a unit vector polarization field p_α , $\alpha = x, y, z$. The film is infinite along the x, y plane, but has a finite thickness L in the z -direction. A typical example of ordered active nematic where our results may apply is the cortical actin layer in a cell where the orientation of the actin filaments can have a component parallel to the cell membrane. It has been recently shown that a liquid contractile active film of thickness L with polarization either parallel or perpendicular to its surface has a spontaneous flow instability, above a critical value of the activity [12, 13]. This is the nonequilibrium analog of the “Frederiks transition” in equilibrium classical nematic liquid crystals. It is driven by the coupling between the polarization orientation and the active stress. We here calculate C , the *active* analog of the thermal equilibrium Casimir stress, that we formally define below.

ACTIVE CASIMIR STRESS

We consider a thin film of active fluid with a fixed thickness L along the z -direction confined between the planes $z = 0$ and $z = L$. In the passive case, i.e., without any activity, the Casimir stress C_{eq} is defined as[3]

$$C_{eq} = \langle \sigma_{zz}^{eq} \rangle|_{z=L} - \langle \sigma_{zz}^{eq} \rangle|_{z=\infty}. \quad (1)$$

Here, σ_{zz}^{eq} is the normal component of the equilibrium stress that diverges for all z (or, all L); C_{eq} however is finite for any non-zero L [3]. Here, $\langle \dots \rangle$ implies averages of thermal noise ensembles (see below). In an active system, we define the Casimir stress C as

$$C = \langle \sigma_{zz}^{tot} \rangle|_{z=L} - \langle \sigma_{zz}^{tot} \rangle|_{\Delta\mu=0, z=L} - \left[\langle \sigma_{zz}^{tot} \rangle|_{z=L} - \langle \sigma_{zz}^{tot} \rangle|_{\Delta\mu=0, z=L} \right]_{K \rightarrow \infty}, \quad (2)$$

where K is the Frank elastic constant of the nematics (assuming a one Frank constant description). Here, σ_{zz}^{tot} is the normal component of the total stress in an active fluid and $\Delta\mu$ is the activity parameter that parametrizes the free energy release in the chemical conversion of ATP to ADP. Here, $\sigma_{zz}^{tot}|_{\Delta\mu=0} = \sigma_{zz}^{eq}$, the normal component of the equilibrium stress. Note that the last term in (2) in the limit $K \rightarrow \infty$ represents the stresses $\langle \sigma_{zz}^{tot} \rangle$ and $\langle \sigma_{zz}^{eq} \rangle$ in the absence of any orientation fluctuations which are independent of layer thickness L . By using stochastic hydrodynamic descriptions for orientationally ordered active fluids, we show below that (2) reduces to

$$C = -\frac{K}{2} \langle (\partial_z p_i)^2 \rangle|_{z=L} + \frac{K}{2} \langle (\partial_z p_i)^2 \rangle|_{\Delta\mu=0, z=L}. \quad (3)$$

The quantity C is difficult to measure directly. However changes of C due to changes in L can in principle be measured.

When the thickness L of a contractile active fluid layer approaches the critical thickness L_c for the spontaneous flow instability from below [12], we show that C remains attractive, scales with L in a way same as its equilibrium counterpart, but diverges logarithmically as L approaches L_c from below. We also calculate C for extensile activity, and contrast it with the active Casimir stress for the contractile case: in this case, C is found to be repulsive, has no divergence at any finite L , and scales with L *differently* from the equilibrium result.

STEAD STATE STRESSES IN A FLUCTUATING ACTIVE FLUID

We consider an incompressible viscous active fluid film with nematic order. Our analysis below closely follows the physical discussion of Ref. [15], where the diffusion coefficient of a test particle immersed in an active fluid with nematic order is calculated. The force balance in an incompressible active fluid is given by

$$\partial_\beta (\tilde{\sigma}_{\alpha\beta} + \sigma_{\alpha\beta}^a - P\delta_{\alpha\beta} + \sigma_{\alpha\beta}^e) = 0, \quad (4)$$

where fluid inertia is neglected [9, 14]. Here, $\tilde{\sigma}_{\alpha\beta}$ denotes the traceless part of the symmetric deviatoric stress and the antisymmetric deviatoric stress is given by

$$\sigma_{\alpha\beta}^a = \frac{1}{2}(p_\alpha h_\beta - p_\beta h_\alpha). \quad (5)$$

Here $h_\alpha = -\delta F/\delta p_\alpha$ is the orientational field conjugate to the nematic director p_α , where $F = \int d^3r f$ denotes the nematic director free energy with a free energy density f . Furthermore, P denotes the hydrostatic pressure. Note that in a nematic system the equilibrium stress can have anisotropies described by the Ericksen stress

$$\sigma_{\alpha\beta}^e = -\frac{\partial f}{\partial(\partial_\beta p_\gamma)} \partial_\alpha p_\gamma. \quad (6)$$

Here, $\alpha, \beta = x, y, z$. The total normal stress is thus given by

$$\sigma_{\alpha\beta}^{tot} = \tilde{\sigma}_{\alpha\beta} + \sigma_{\alpha\beta}^a - P\delta_{\alpha\beta} + \sigma_{\alpha\beta}^e; \quad (7)$$

see Eq. (4) above.

In the following, we impose for simplicity a constant amplitude of the nematic director $p_\gamma p_\gamma = 1$. The constitutive equations of a single-component active fluid then read [14]

$$\left\{ \tilde{\sigma}_{\alpha\beta} + \zeta \Delta \mu q_{\alpha\beta} + \frac{\nu_1}{2}(p_\alpha h_\beta + p_\beta h_\alpha - \frac{2}{3} p_\gamma h_\gamma \delta_{\alpha\beta}) \right\} = 2\eta v_{\alpha\beta} + \xi_{\alpha\beta}^\sigma, \quad (8)$$

$$\frac{D}{Dt} p_\alpha = \frac{1}{\gamma_1} h_\alpha - \nu_1 p_\beta \tilde{v}_{\alpha\beta} + \xi_{\perp\alpha} \quad (9)$$

where $q_{\alpha\beta} = (p_\alpha p_\beta - \frac{1}{3} \delta_{\alpha\beta})$ is the nematic tensor. The symmetric velocity gradient tensor is $\tilde{v}_{\alpha\beta} = (\partial_\alpha v_\beta + \partial_\beta v_\alpha)/2$, where v_α is the three-dimensional velocity field of the active fluid ($\alpha = x, y, z$). The shear viscosity is denoted by η , γ_1 is the rotational viscosity and ν_1 the flow alignment parameter which is a number of order one. Functions $\xi_{\alpha\beta}^\sigma$ and $\xi_{\perp\alpha}$ are stochastic noises, which we assume to be thermal noises of zero-mean and variances given by

$$\langle \xi_{\alpha\beta}^\sigma(t, \mathbf{x}) \xi_{\gamma\delta}^\sigma(t', \mathbf{x}') \rangle = 2k_B T \eta \left[(\delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma} - \frac{2}{3} \delta_{\alpha\beta} \delta_{\gamma\delta}) \right] \delta(t - t') \delta(\mathbf{x} - \mathbf{x}'), \quad (10)$$

$$\langle \xi_{\perp\alpha}(t, \mathbf{x}) \xi_{\perp\beta}(t', \mathbf{x}') \rangle = 2 \frac{k_B T}{\gamma_1} [\delta_{\alpha\beta} - p_\alpha p_\beta] \delta(t - t') \delta(\mathbf{x} - \mathbf{x}'). \quad (11)$$

where k_B is Boltzmann constant and T denotes temperature. Notice that the noises $\xi_{\perp\alpha}(t, \mathbf{x})$ are *multiplicative* in nature (see noise variance (11)). However, since we are interested in a linearized description about uniform ordered states (see below), the multiplicative nature of

these noises do not affect us. Furthermore, we do not consider any athermal or active noises for simplicity. We consider an incompressible system imposed by the constraint $\partial_\alpha v_\alpha = 0$.

The pressure P plays the role of a Lagrange multiplier used to impose the incompressibility constraint $\partial_\alpha v_\alpha = 0$. The incompressibility leads to the following equation for P :

$$\nabla^2 P = -\frac{\nu_1}{2} \partial_\alpha \partial_\beta (p_\alpha h_\beta + p_\beta h_\alpha - \frac{2}{3} \mathbf{p} \cdot \mathbf{h} \delta_{\alpha\beta}) - \zeta \Delta \mu \partial_\alpha \partial_\beta (p_\alpha p_\beta) + \partial_\alpha \partial_\beta \sigma_{\alpha\beta}^e + \partial_\alpha \partial_\beta \xi_{\alpha\beta}^\sigma. \quad (12)$$

We consider a film of the active fluid with a fixed thickness L along the z direction, confined between the planes $z = 0$ and $z = L$. We consider a non-flowing reference state together with $p_z = 1$, which is a steady state solution of (4) and (9). We study small fluctuations $\delta \mathbf{p} = (p_x, p_y, 0)$ around this state; $\delta p = |\delta \mathbf{p}|$. We impose boundary conditions $(p_x, p_y) = 0$ and vanishing shear stress at $z = 0$ and $z = L$. The total normal stress on the surface at $z = L$, $\langle \sigma_{zz}^{tot} \rangle_{z=L}$ should depend on L and also contains a constant piece independent of L [3]. From the definition of σ_{zz}^{tot}

$$\begin{aligned} \langle \sigma_{zz}^{tot} \rangle_{z=L} &= \eta \left\langle \frac{\partial v_z}{\partial z} \right\rangle_{z=L} - \zeta \Delta \mu \langle p_z^2 \rangle_{z=L} - \frac{\nu_1}{3} \langle p_i h_i \rangle_{z=L} - \frac{\nu_1}{3} \langle p_z h_z \rangle_{z=L} \\ &+ \langle \sigma_{zz}^e \rangle_{z=L} - \langle P \rangle_{z=L}. \end{aligned} \quad (13)$$

Here, $i, j = x, y$ are the coordinates along the film surface. Using, for simplicity and analytical convenience, a single Frank elastic constant K for the nematic liquid crystals, the Frank free energy density is given by $f = K(\nabla_\alpha p_\beta)^2/2$. Below we evaluate the pressure P which obeys Eqs (12). The remaining terms in (13) are also to be evaluated using the relevant equations of motion and then averaging over the various noise terms. The contributions to the stress that are linear in small fluctuations $\delta \mathbf{p}$ vanish upon averaging; therefore, a non-vanishing Casimir stress is obtained from contributions to the stress quadratic in $\delta \mathbf{p}$ in (13). It is instructive to analyze the different contributions in (13) to C term by term. This will allow us to considerably simplify (13) as we will see below.

We first consider the contribution $\eta \langle \frac{\partial v_z}{\partial z} \rangle_{z=L}$ in (13). Using the condition of incompressibility $\nabla \cdot \mathbf{v} = 0$, this may be written as

$$\eta \left\langle \frac{\partial v_z}{\partial z} \right\rangle_{z=L} = -\eta \langle \nabla_\perp \cdot \mathbf{v}_\perp \rangle_{z=L} = -\eta \nabla_\perp \cdot \langle \mathbf{v}_\perp \rangle_{z=L} = 0, \quad (14)$$

since, there is no flow on an average. Here, $\nabla_\perp = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ is the two-dimensional gradient operator and $\mathbf{v}_\perp = (v_x, v_y)$ is the in-plane component of the three-dimensional velocity \mathbf{v} .

Secondly, $\langle p_i h_i \rangle_{z=L} = 0$ since $p_i = 0$ at $z = L$. Further, $\langle p_z h_z \rangle_{z=L} = \langle h_\parallel \rangle$, since $p_z = 1$ at $z = L$. Here, h_\parallel is a Lagrange multiplier, which must be introduced to impose $p^2 = 1$,

or to the leading order $p_z = 1$ in the geometry that we consider. Using $p_z = 1$ in Eq. (9) and linearizing around $p_z = 1$, we obtain $h_{\parallel} \sim \frac{\partial v_z}{\partial z}$ at all z [14]. Using the incompressibility condition, $\partial v_z / \partial z = -\nabla_{\perp} \cdot \mathbf{v}_{\perp}$. This then gives $\langle h_z \rangle = 0$ to the leading order in fluctuations.

In order to evaluate the form of the pressure P , we consider the equation for the velocity field v_{α} that obeys the generalized Stokes equation

$$\eta \nabla^2 v_{\alpha} = \partial_{\alpha} P + \zeta \Delta \mu \partial_{\beta} (p_{\alpha} p_{\beta}) - \frac{\nu_1}{2} \partial_{\beta} (p_{\beta} h_{\alpha} + p_{\alpha} h_{\beta}) - \frac{1}{2} \partial_{\beta} (p_{\alpha} h_{\beta} - p_{\beta} h_{\alpha}) - \partial_{\beta} \sigma_{\alpha\beta}^e - \partial_{\beta} \xi_{\alpha\beta}^{\sigma}. \quad (15)$$

We focus on the in-plane velocity v_i , $\alpha = i = x, y$ in (15) above. Now consider the different terms in (15) with $\alpha = i$ at $z = L$ and note that (i) $p_i = 0$ at $z = L$, (ii) in the absence of any mean flow and consistent with the in-plane rotational invariance, velocity fluctuations $+v_i$ and $-v_i$ should be equally likely in the statistical steady state, i.e., the steady state average of any function odd in v_i should be zero. This implies that $\langle v_i \rangle = 0 = \langle \partial_z^2 v_i \rangle$ in steady states. Similarly, in an oriented state having nematic order with $p_z = 1$, fluctuations $+p_i$ and $-p_i$ should be equally likely in the steady state, i.e., any function odd in p_i must have a vanishing average in the steady state. Furthermore, since h_i is odd in p_i , we must have $\langle h_i \rangle = 0$ in the steady states. Similarly,

$$\zeta \Delta \mu \langle \partial_{\beta} (p_i p_{\beta}) \rangle|_{z=L} = \zeta \Delta \mu \langle \partial_j (p_i p_j) \rangle|_{z=L} + \zeta \Delta \mu \langle \partial_z (p_i p_z) \rangle|_{z=L} = \zeta \Delta \mu \langle \partial_z (p_i p_z) \rangle|_{z=L} \quad (16)$$

vanishes in the steady state due to the inversion symmetry of p_i . Lastly, we note that

$$\partial_{\beta} \sigma_{i\beta}^e|_{z=L} = -\partial_j (\partial_i p_{\gamma} \partial_j p_{\gamma})|_{z=L} - \partial_z (\partial_i p_{\gamma} \partial_z p_{\gamma})|_{z=L} = -\frac{1}{2} \partial_i (\partial_z p_j)^2|_{z=L}, \quad (17)$$

where we have used $[(\partial_i p_{\gamma})(\partial_z^2 p_{\gamma})]_{z=L} = [(\partial_i p_j)(\partial_z^2 p_j)]_{z=L} + [(\partial_i p_z)(\partial_z^2 p_z)]_{z=L} = 0$, since $p_j = 0$ and $p_z = 1$ exactly at $z = L$. Putting together everything and averaging in the steady states, we then obtain at $z = L$

$$\partial_i P = \partial_{\beta} \sigma_{i\beta}^e = -\frac{K}{2} \partial_i (\partial_z p_j)^2, \quad (18)$$

giving

$$P = \frac{K}{2} (\partial_z p_j)^2 + a_0 \quad (19)$$

at $z = L$, where a_0 is a constant of integration. Then substituting P in (13)

$$\langle \sigma_{zz}^{tot} \rangle_{z=L} = -\frac{K}{2} \langle (\partial_z p_i)^2 \rangle|_{z=L} - \zeta \Delta \mu + a_0 = -\frac{K}{2} \langle (\partial_z p_i)^2 \rangle|_{z=L} + \tilde{a}_0, \quad (20)$$

where \tilde{a}_0 is another constant.

Notice that the constant \tilde{a}_0 , which in general can depend upon $\Delta\mu$ is actually $\langle\sigma_{zz}^{tot}\rangle$ evaluated in the limit $K \rightarrow \infty$ (i.e., with all the orientation fluctuations suppressed): $\tilde{a}_0 = \langle\sigma_{zz}^{tot}\rangle|_{K \rightarrow \infty}$ and is independent of L . Similarly in the passive case [3]

$$\langle\sigma_{zz}^{tot}\rangle|_{\Delta\mu=0, z=L} = \langle\sigma_{zz}^{eq}\rangle_{z=L} = -\frac{K}{2}\langle(\partial_z p_i)^2\rangle|_{z=L, \Delta\mu=0} + a_0^{eq}, \quad (21)$$

where a_0^{eq} is a constant that is given by $\langle\sigma_{zz}^{eq}\rangle|_{K \rightarrow \infty}$ and is independent of L . We are now in a position to formally define active Casimir stress C as

$$\begin{aligned} C &= \langle\sigma_{zz}^{tot}\rangle|_{z=L} - \langle\sigma_{zz}^{tot}\rangle|_{\Delta\mu=0, z=L} - \tilde{a}_0 + a_0^{eq} \\ &= -\frac{K}{2}\langle(\partial_z p_i)^2\rangle|_{z=L} + \frac{K}{2}\langle(\partial_z p_i)^2\rangle|_{\Delta\mu=0, z=L}. \end{aligned} \quad (22)$$

We show below that C in an ordered active nematic layer is fundamentally different from its equilibrium counterpart, primarily because the dynamics of orientation fluctuations here is very different from its equilibrium counterpart.

We calculate C for small fluctuations around the chosen reference state by using the dynamical equations (4) and (9). Since $\langle\sigma_{zz}^{tot}\rangle \sim \delta p_\alpha^2$, it suffices to study the dynamics after linearizing about the reference state. Considering a contractile active fluid, i.e., $\Delta\mu < 0$, we find that as thickness L approaches L_c from below, where L_c is the critical thickness for the spontaneous flow instability (see Ref. [12]; see also below), akin to the Frederiks transition in equilibrium nematics [2], the Casimir stress C diverges logarithmically. We find that

$$C = k_B T \frac{-\pi^2}{2L_c^3} \frac{\Gamma\gamma_1}{8\eta + \gamma_1(\nu_1 - 1)^2} \ln \left| \frac{[2/\gamma_1 + (\nu_1 - 1)^2/4\eta]\gamma_1}{2\Gamma\delta} \right|, \quad (23)$$

Here, $\Gamma = 2\eta/\gamma_1 + (\nu_1 - 1)^2/4$ is a positive dimensionless number. The critical thickness L_c is determined by the relation [14]

$$\frac{K}{\gamma_1} \frac{\pi^2}{L_c^2} + \frac{(\nu_1 - 1)^2}{4\eta} K \frac{\pi^2}{L_c^2} = -\xi \Delta\mu \frac{\nu_1 - 1}{2\eta}. \quad (24)$$

Clearly, L_c diverges as $\Delta\mu \rightarrow 0$, consistent with the fact that there are no instabilities at any thickness in equilibrium. Further we have used, $L = L_c(1 - \delta)$, where $0 < \delta \ll 1$ is a small, dimensionless number parameterizing the thickness L approaching the critical thickness L_c from below. Clearly, C vanishes for $L_c \rightarrow \infty$ for fixed L (equivalently for $\Delta\mu = 0$ for a fixed L), as expected. Compare this with the corresponding equilibrium result

$$C_{eq} = -\frac{1}{8\pi} \frac{K_B T}{L^3} \zeta_R(3), \quad (25)$$

where $\zeta_R(3)$ is the Riemann-Zeta function [3]. Clearly, C_{eq} has no divergence at any finite L , in contrast to C in (23). It follows from (23) and (25) that both C and C_{eq} are negative. This implies that the surfaces at $z = 0$ and $z = L$ are attracted towards each other. This feature is similar to the equilibrium problem [3]. Although both contributions scale as $1/L^3$, the active contribution clearly dominates the corresponding equilibrium contribution for a sufficiently small δ . In contrast, for an extensile active system, C scales as $\zeta\Delta\gamma_1\mu/(\eta L)$ for small activity, and is repulsive in nature.

We now argue that $C + C_{eq} = C_{tot}$ indeed has the interpretation of the *total* Casimir stress on the system for $L < L_c$. For instance, in the contractile case consider the differences $\Delta\sigma$ in the total normal stresses for two different thicknesses $L_1 = L_c(1 - \delta_1)$ and $L_2 = L_c(1 - \delta_2)$ with $0 < \delta_1, \delta_2 \ll 1$. We note that

$$\Delta\sigma = \sigma_{zz}^{tot}|_{z=L_1} - \sigma_{zz}^{tot}|_{z=L_2} = C_{tot}|_{z=L_1} - C_{tot}|_{z=L_2} = C(\delta_1) + C_{eq}(L_1) - C(\delta_2) - C_{eq}(L_2). \quad (26)$$

Since $\Delta\sigma$ is a measure of the change in the force per unit area on the wall as the thickness changes from L_1 to L_2 , we can conclude that C_{tot} can indeed be interpreted as the total Casimir stress on the system. Similar arguments can be made in the extensile case as well, with $C_{tot} = C + C_{eq}$ as the total Casimir stress.

In order to better understand the result given by Eqs. (23) and (25), we first present arguments at the scaling level using a simplified analysis of the problem that highlights the general features of the active contributions in (23). This is similar to the scaling analysis of Ref. [15]. We provide the results of the full fluctuating hydrodynamic equations in appendix that confirm the scaling analysis and yield (23).

We consider a small perturbation to the non-flowing steady state with $\mathbf{p} = \hat{e}_z$ along the z -axis. In a simplified picture, we describe the tilt of the polarity with respect to the z -axis normal to the film surface by a single small angle θ . The rate of variation of the angle θ is due to the elastic nematic torque with a Frank elastic constant K and according to Eq. (9) to a coupling to the strain rate u ,

$$\frac{\partial\theta}{\partial t} = \frac{K}{\gamma_1}\nabla^2\theta - \nu_1u + \tilde{\xi}_\perp(\mathbf{x}, t). \quad (27)$$

We have added in this equation the thermal noise of the orientation fluctuations $\tilde{\xi}_\perp(\mathbf{r}, t)$ introduced above. Noise $\tilde{\xi}_\perp$ is a simplified form of $\xi_{\perp\alpha}(t, \mathbf{x})$ in Eq. (9). It is Gaussian-

distributed with zero mean and variance given by

$$\langle \tilde{\xi}_\perp(\mathbf{x}, t) \tilde{\xi}_\perp(0, 0) \rangle = 2 \frac{K_B T}{\gamma_1} \delta(\mathbf{x}) \delta(t), \quad (28)$$

in analogy with (11). We ignore here for simplicity the tensorial character of the strain rate and represent it by a scalar u which represents one of its typical components.

If the polarization angle θ does not vanish, the active stress is finite and it is compensated by the viscous stress in the film

$$\eta u \simeq \zeta \Delta \mu \theta \quad , \quad (29)$$

where we have for simplicity ignored the noise in the stress. Including this noise does not qualitatively change the final result. The two equations (27) and (29) can be solved by Fourier expansion both in space and time, writing the polarization angle as

$$\theta(\mathbf{x}, t) = \sum_n \sin(n\pi z/L) \int d\omega \int \frac{d\mathbf{q}}{(2\pi)^2} \exp i(\mathbf{q} \cdot \mathbf{r} - \omega t) \tilde{\theta}(n, \omega, \mathbf{q}). \quad (30)$$

Here, the position vector is $\mathbf{x} = (\mathbf{r}, z)$ where \mathbf{r} denotes the position in the plane parallel to the film, and the wave vector is $\mathbf{k} = (\mathbf{q}, n\pi/L)$ where \mathbf{q} denotes the wave vector parallel to the plane, while n describes the discrete Fourier mode along the z direction. The Fourier transform of the orientation angle satisfies the equation

$$-i\omega \tilde{\theta}(n, \omega, \mathbf{q}) = \frac{\nu_1}{\eta} \left((\zeta \Delta \mu - \zeta \Delta \mu_c(n)) - \frac{\eta K q^2}{\nu_1 \gamma_1} \right) \tilde{\theta} + \tilde{\xi}_\perp(n, \omega, \mathbf{q}). \quad (31)$$

Here, $\zeta \Delta \mu < 0$ for a contractile active fluid, where as $\zeta \Delta \mu > 0$ for an extensile active fluid. Equation (31) defines the relaxation time $\tau_n(q)$ of $\tilde{\theta}$:

$$\tau_n(q)^{-1} = -\frac{\nu_1}{\eta} \left[\zeta \Delta \mu - \zeta \Delta \mu_c(n) - \frac{\eta K q^2}{\nu_1 \gamma_1} \right]. \quad (32)$$

Clearly, the system gets unstable for $|\zeta \Delta \mu| > \zeta \Delta \mu_c$. We have defined here $\zeta \Delta \mu_c(n) = \eta K n^2 / (\nu_1 \gamma_1 \pi^2 L^2)$. We further note that in the equilibrium limit, $u = 0$ in our simplified description and hence the equilibrium relaxation time $\tau_{qe,n}$ is given by

$$\tau_{qe,n}^{-1} = \frac{K}{\gamma_1} \left(q^2 + \frac{n^2 \pi^2}{L^2} \right). \quad (33)$$

The orientation angle correlation function can be directly calculated from Eq. (31) leading to

$$\langle \tilde{\theta}_n(\mathbf{q}, \omega) \tilde{\theta}_n(\mathbf{q}', \omega') \rangle = \frac{k_B T \gamma_1}{\omega^2 + \tau_q^{-2}} (2\pi)^3 \delta(\mathbf{q} + \mathbf{q}') \delta(\omega + \omega'). \quad (34)$$

Using Eq. (22), this then yields

$$C = -\frac{\pi^3 k_B T}{L^3} \sum_n \int d^2 q \left[\frac{1}{\tau_n(q)^{-1}} - \frac{1}{\tau_{qe,n}^{-1}} \right]. \quad (35)$$

Equation (35) applies to both contractile and extensile active fluids. We now consider contractile and extensile active fluids separately below.

For a contractile active fluid with $\zeta \Delta \mu < 0$; clearly the system can get unstable for sufficiently large $\zeta \Delta \mu$ for a given L , or equivalently, for sufficiently large L for a fixed $\zeta \Delta \mu$. The nature of C depends sensitively on whether $|\zeta \Delta \mu| \rightarrow \zeta \Delta \mu_c$ from below (near the threshold for spontaneous flow instability), or $|\zeta \Delta \mu| \ll \zeta \Delta \mu_c$ (far away from the instability threshold). Concentrating first on the near threshold behavior of C , we focus only on the $n = 1$ mode that is dominant near the instability threshold, which gets unstable first as L approaches L_c from below. For ease of notations, we denote $\Delta \mu_c(n = 1) = \Delta \mu_c$, and $\tau_{n=1}(q)^{-1} = \tau(q)^{-1}$, $\tau_{qe,n=1} = \tau_{qe}$ with

$$\tau(q)^{-1} = -\frac{\nu_1}{\eta} [\zeta \Delta \mu - \zeta \Delta \mu_c] - \frac{\eta K q^2}{\nu_1 \gamma_1}, \quad (36)$$

$$\tau_{qe}^{-1} = \frac{K}{\gamma_1} \left(q^2 + \frac{\pi^2}{L^2} \right). \quad (37)$$

If the active stress $|\zeta \Delta \mu|$ is larger than this threshold, the non-flowing steady state is unstable and the film spontaneously flows. Retaining only the $n = 1$ mode, C for an orientationally ordered contractile active fluid is given by

$$C = -\frac{\pi^3}{L^3} K_B T \int d^2 q \left(\frac{1}{q^2 + q_c^2} - \frac{1}{q^2 + \frac{\pi^2}{L^2}} \right), \quad (38)$$

valid for all $|\zeta \Delta \mu| < \zeta \Delta \mu_c$. Here, we have defined the wave vector q_c such that $q_c^2 = (\gamma_1 \nu_1 / \eta)(\zeta \Delta \mu_c - |\zeta \Delta \mu|) / K$ and a is a small length-scale cut off. Now, in the vicinity of the spontaneous flow instability, $|\zeta \Delta \mu| \rightarrow \zeta \Delta \mu_c$ from below. Then,

$$C = -\frac{\pi^2 k_B T}{L^3} \ln \left| \frac{1 + a^2 q_c^2}{a^2 q_c^2} \right| \sim -\frac{k_B T}{L^3} \ln |a q_c|, \quad (39)$$

retaining only the divergent contribution to C as $q_c^2 \rightarrow 0$, or equivalently, $\zeta \Delta \mu \rightarrow \zeta \Delta \mu_c$ from below or $L \rightarrow L_c = \eta \kappa / (\nu_1 \gamma_1 \zeta \Delta \mu_c)$ from below. We find that the Casimir stress (39) diverges logarithmically as $q_c \rightarrow 0$ near the instability threshold, i.e., $\zeta \Delta \mu \rightarrow \zeta \Delta \mu_c$. The Casimir stress (39) is clearly attractive. Comparing this with (23) above we note that our simplified analysis does capture the correct sign and the logarithmic divergence near the

instability threshold. Compare this with the corresponding equilibrium result given in (25); clearly has the same scaling with L as C , but has no divergence at any finite L .

We now consider the scaling of C far away from the threshold ($|\zeta\Delta\mu| \ll \zeta\Delta\mu_c$) as well. Assuming small $\zeta\Delta\mu$ (i.e., small q_c), we expand the denominator of (35) up to the linear order in $|\zeta\Delta\mu|$. We obtain for

$$C \sim -K_B T \frac{\zeta\Delta\mu\gamma_1}{\eta L} \quad (40)$$

to the leading order in $\zeta\Delta\mu$, valid for $L \ll L_c$. Thus far away from the threshold, the leading active contribution to C scales as $1/L$ with L that is different from both its form near the instability threshold as well as the equilibrium contribution to C . It remains attractive, however. We thus conclude that C remains attractive for all $L < L_c$ for a nematically ordered active fluid.

We now discuss the extensile case, i.e., $\zeta\Delta\mu > 0$ for which there are no instabilities at any L . The active Casimir stress C is still formally defined by Eq. (22), which yields (35) with the sign of $\zeta\Delta\mu$ reversed. The time-scale τ_q is now given by

$$\tau_n(q)^{-1} = \frac{\nu_1}{\eta} \sum_n [(\zeta\Delta\mu + \zeta\Delta\mu_c(n)) + \frac{\eta K q^2}{\nu_1 \gamma_1}] \quad (41)$$

which is positive definite implying stability. The active Casimir stress in this case now reads

$$C = -\frac{\pi^3 k_B T}{2L^3} \frac{\eta}{\gamma_1} \sum_n \int d^2q \left[\frac{1}{\zeta\Delta\mu + \zeta\Delta\mu_c(n) + \eta K q^2 / (\nu_1 \gamma_1)} - \frac{1}{\tau_{qe,n}^{-1}} \right]. \quad (42)$$

We expand in $\Delta\mu$, assuming small activity, and extract the leading order active contribution to C as

$$C \sim \frac{k_B T}{L} \frac{\gamma_1}{\eta} \zeta\Delta\mu, \quad (43)$$

that vanishes with $\Delta\mu$, scales with L as $1/L$ and is *positive* in sign. This implies that C for an extensile active fluid with nematic order is repulsive to the leading order in $\zeta\Delta\mu$, in contrast to C for a contractile active fluid, or the corresponding equilibrium contribution C_E . Furthermore, it does not diverge for any finite L , unlike C for the contractile case.

So far, we have considered a macroscopically oriented state where the reference orientation is assumed to be perpendicular to the film. An alternative choice of boundary condition would be a polarization oriented parallel to the surface of the film: $p_x = 1$ as the reference state for orientation, and $p_z = 0 = p_y$ at $z = 0, L$. Similar arguments show that at the scaling level the Casimir stress C in these conditions is still given by Eq. (39). A third

choice for boundary conditions is $p_z = 0$ and p_y to be free at $z = 0, L$ with $p_x = 1$ as the ordered reference state. This is qualitatively different from what we have considered above, owing to the fact that p_y is a soft mode. Further, as discussed in Ref. [13], with this choice of the reference state there are no instabilities at any given thickness of the system. Thus, the Casimir stress will be significantly different from (39). We do not discuss this case here.

A potential biological system where the active Casimir stresses could be relevant is the thin cell cortex or the cell lamellipodium. Due to the active Casimir forces acting in the direction of the thickness of the actin layer, because of the overall incompressibility, the active layer tends to stretch along the in-plane directions. This causes the cell membrane to stretch and contributes to the *active tension* of the cell cortex. If the thickness of the system is close to the critical threshold of instability, the Casimir force contribution could become important.

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APPENDIX

Here, we discuss the full calculation of polarization fluctuations in a stochastically driven active fluid layer. The scheme of the calculations here is very similar to the detailed calculation for the diffusion coefficient of a test particle immersed in an active fluid layer, as given in Ref. [15] with full details. Nonetheless, we reproduce the basic outline here for the sake of completeness. We start from the relations (5)-(9) and determine the conjugate field to the polarity vector h_α from a Frank free energy which describes the energies of splay, bend and twist deformations by parameters K_1 , K_2 and K_3 . For simplicity we consider here the limit $K_1 \rightarrow \infty$ (i.e., the splay modes are suppressed, $\nabla \cdot \mathbf{p} = 0$). We furthermore introduce the constraints $\mathbf{p}^2 = 1$ and $\nabla \cdot \mathbf{v} = 0$, i.e. we ignore fluctuations of the magnitude of \mathbf{p} and we treat the fluid as incompressible. The two constraints $\nabla \cdot \mathbf{p} = 0$ and $p^2 = 1$ are imposed by two Lagrange multipliers h_\parallel and ϕ in the free energy functional

$$F = \frac{1}{2} \int d^3x [K_2(\nabla \times \mathbf{p})^2 + K_3(\partial_z \mathbf{p})^2 - h_\parallel p^2 + 2\phi \nabla \cdot \mathbf{p}], \quad (44)$$

where we have assumed that \mathbf{p} exhibits small fluctuations around a reference state $\mathbf{p}_0 = \hat{\mathbf{e}}_z$, the unit vector along the z -axis. The incompressibility constraint is imposed via the pressure

P as Lagrange multiplier. The active fluid is confined between two surfaces at $z = 0$ and $z = L$. We impose the following boundary conditions: no flow across the boundary surfaces $v_z(z = 0) = 0$ and $v_z(z = L) = 0$ and vanishing surface shear stress at the boundaries: $\partial v_\alpha / \partial z = 0$, at $z = 0$ and $z = L$ for $\alpha = x, y$. In addition we impose $\mathbf{p}(z = 0) = \hat{\mathbf{e}}_z$ and $\mathbf{p}(z = L) = \hat{\mathbf{e}}_z$. These boundary conditions are satisfied by the Fourier mode expansions

$$v_\alpha(\mathbf{x}, t) = \int \frac{d^2q}{(2\pi)^2} \frac{d\omega}{2\pi} \sum_n \tilde{v}_\alpha^n(\mathbf{q}, \omega) \exp[-i\omega t + i\mathbf{r} \cdot \mathbf{q}] \cos\left(\frac{n\pi z}{L}\right), \quad (45)$$

$$v_z(\mathbf{x}, t) = \int \frac{d^2q}{(2\pi)^2} \frac{d\omega}{2\pi} \sum_n \tilde{v}_z^n(\mathbf{q}, \omega) \exp[-i\omega t + i\mathbf{r} \cdot \mathbf{q}] \sin\left(\frac{n\pi z}{L}\right), \quad (46)$$

$$p_\alpha(\mathbf{x}, t) = \int \frac{d^2q}{(2\pi)^2} \frac{d\omega}{2\pi} \sum_n \tilde{p}_\alpha^n(\mathbf{q}, \omega) \exp[-i\omega t + i\mathbf{r} \cdot \mathbf{q}] \sin\left(\frac{n\pi z}{L}\right), \quad (47)$$

where $\alpha = x, y$. Here, \mathbf{r} is a vector in the $x - y$ plane and the corresponding wavevector is denoted by \mathbf{q} . We linearize the state of the system around a reference state with $v_\alpha = 0$, $v_z = 0$ and $\mathbf{p} = \hat{\mathbf{e}}_z$. The force balance equation together with the incompressibility condition and the constitutive Eq. (8) yield equations for the flow field

$$\begin{aligned} -\eta(q^2 + \frac{n^2\pi^2}{L^2})\tilde{v}_z^n(\mathbf{q}, t) &= \zeta\Delta\mu P_{z\beta} \frac{n\pi}{L} \tilde{p}_\beta^n + i\zeta\Delta\mu P_{zzq\beta} \tilde{p}_\beta^n - \frac{\nu_1}{2} P_{zz}(iq_\beta \tilde{h}_\beta - \frac{n\pi}{L} \tilde{h}_z^n) \\ &\quad - \frac{\nu_1}{2} \frac{n\pi}{L} (P_{z\beta} \tilde{h}_\beta^n - P_{zz} \tilde{h}_z^n) - \frac{1}{2} (iq_\beta \tilde{h}_\beta^n - \frac{n\pi}{L} \tilde{h}_z^n) + P_{z\beta} \tilde{\xi}_\beta^{\sigma,n} + P_{zz} \tilde{\xi}_z^{\sigma,n} \quad , \end{aligned}$$

$$\begin{aligned} -\eta(q^2 + \frac{n^2\pi^2}{L^2})\tilde{v}_\alpha^n(\mathbf{q}, t) &= \zeta\Delta\mu P_{\alpha\beta} \frac{n\pi}{L} \tilde{p}_\beta^n + i\zeta\Delta\mu P_{\alpha z q\beta} \tilde{p}_\beta^n - \frac{\nu_1}{2} (P_{\alpha\beta} \frac{n\pi}{L} \tilde{h}_\beta^n - P_{\alpha z} \frac{n\pi}{L} \tilde{h}_z^n) \\ &\quad - \frac{\nu_1}{2} P_{\alpha z} (iq_\beta \tilde{h}_\beta^n - \frac{n\pi}{L} \tilde{h}_z^n) + \frac{1}{2} \frac{n\pi}{L} \tilde{h}_\alpha^n + P_{\alpha\beta} \tilde{\xi}_\beta^{\sigma,n} + P_{\alpha z} \tilde{\xi}_z^{\sigma,n} \quad (48) \end{aligned}$$

where $\alpha, \beta = x$ or y . Here, we have introduced the transverse projection operators $P_{zz} = q^2 / (q^2 + n^2\pi^2/L^2)$, $P_{\alpha\beta} = \delta_{\alpha\beta} - q_\alpha q_\beta / (q^2 + n^2\pi^2/L^2) = P_{\beta\alpha}$, and $P_{\alpha z} = -iq_\alpha (n\pi/L) / (q^2 + n^2\pi^2/L^2) = P_{z\alpha}$ and the pressure P has already been eliminated. The noise terms $\tilde{\xi}_\alpha^{\sigma,n}$ have zero-mean with variance

$$\langle \tilde{\xi}_\alpha^{\sigma,n}(\mathbf{q}, \omega) \tilde{\xi}_\beta^{\sigma,m}(\mathbf{q}', \omega') \rangle = 2\eta k_B T (q^2 + \frac{n^2\pi^2}{L^2}) (2\pi)^3 \delta(\mathbf{q} + \mathbf{q}') \delta(\omega + \omega') \delta_{\alpha\beta} \delta_{nm} \quad (49)$$

where α and $\beta = x, y, z$.

The dynamic equation for the polarization field reads

$$-i\omega \tilde{p}_\alpha^n = -\tilde{\omega}_{\alpha z}^n - \frac{K}{\gamma_1} (q^2 + \frac{n^2\pi^2}{L^2}) \tilde{p}_\alpha^n - \frac{1}{\gamma_1} (h_\parallel \tilde{p}_\alpha^n - iq_\alpha \tilde{\phi}^n) - \nu_1 \tilde{u}_{\alpha z}^n + \tilde{\xi}_{\perp, \alpha}^n, \quad (50)$$

with $\tilde{u}_{\alpha z}^n = (-\frac{n\pi}{L}\tilde{v}_\alpha^n + iq_\alpha\tilde{v}_z^n)/2$, $\tilde{\omega}_{\alpha z}^n = -(\frac{n\pi}{L}\tilde{v}_\alpha^n + iq_\alpha\tilde{v}_z^n)$ and noise correlations

$$\langle \tilde{\xi}_{\perp,\alpha}^n(\mathbf{q}, \omega) \tilde{\xi}_{\perp,\beta}^m(\mathbf{q}', \omega') \rangle = \frac{2K_B T}{\gamma_1} (2\pi)^3 \delta(\mathbf{q} + \mathbf{q}') \delta(\omega + \omega') \delta_{\alpha\beta} \delta_{nm}. \quad (51)$$

Further, with $K_2 = K_3 = K$ we have $h_\alpha = -\frac{\delta F}{\delta p_\alpha} = K\nabla^2 p_\alpha + h_{\parallel} p_\alpha + \nabla_\alpha \phi$ in the real space. Elimination of the Lagrange multipliers h_{\parallel} and ϕ finally leads to [15]

$$-\eta(q^2 + \frac{n^2\pi^2}{L^2})\tilde{v}_z^n = P_{zz}\xi_z^{\sigma,n} + P_{z\beta}f_\beta^{\sigma,n}, \quad (52)$$

$$-\eta(q^2 + \frac{n^2\pi^2}{L^2})\tilde{v}_\alpha^n = \zeta\Delta\mu\frac{n\pi}{L}\tilde{p}_\alpha^n + \frac{\nu_1 - 1}{2}\frac{n\pi}{L}K(q^2 + \frac{n^2\pi^2}{L^2})\tilde{p}_\alpha^n + P_{\alpha\beta}\tilde{\xi}_\beta^{\sigma,n} + P_{\alpha z}\tilde{\xi}_z^{\sigma,n}, \quad (53)$$

$$\frac{\partial \tilde{p}_\alpha^n}{\partial t} = -\frac{K}{\gamma_1}(q^2 + \frac{n^2\pi^2}{L^2})\tilde{p}_\alpha^n + \frac{\nu_1 - 1}{2}\frac{n\pi}{L}\tilde{v}_\alpha^n + P_{\alpha\beta}\tilde{\xi}_{\perp,\beta}^n + P_{\alpha z}\tilde{\xi}_{\perp,z}^n. \quad (54)$$

Note that \tilde{v}_z^n decouples from \tilde{p}_α^n . Equations (53-54) may be used to obtain expressions for the fluctuations of \tilde{p}_α^n :

$$\left(\frac{\partial}{\partial t} + \frac{1}{\tilde{\tau}_q}\right)\tilde{p}_\alpha^n = -\frac{n\pi}{L}\frac{\nu_1 - 1}{2}\frac{P_{\alpha\beta}\tilde{\xi}_\beta^{\sigma,n} + P_{\alpha z}\tilde{\xi}_z^{\sigma,n}}{\eta(q^2 + \frac{n^2\pi^2}{L^2})} + P_{\alpha\beta}\tilde{\xi}_{\perp,\beta}^n + P_{\alpha z}\tilde{\xi}_{\perp,z}^n. \quad (55)$$

where we have identified an effective relaxation time $\tilde{\tau}_q$ of the polarization fluctuations \tilde{p}_α^n :

$$\tilde{\tau}_q = \left[\frac{K}{\gamma_1}(q^2 + \frac{n^2\pi^2}{L^2}) + \frac{\nu_1 - 1}{2} \left(\zeta\Delta\mu + \frac{\nu_1 - 1}{2}K(q^2 + \frac{n^2\pi^2}{L^2}) \right) \frac{n^2\pi^2}{L^2} \frac{1}{\eta(q^2 + \frac{n^2\pi^2}{L^2})} \right]^{-1}. \quad (56)$$

For the stability of the assumed oriented state of polarization one must have $\tilde{\tau}_q > 0$. Time-scale $\tilde{\tau}_q$ is the analog of the time-scale $t_p(q)$ that we extract from Eq. (31). This allows us to calculate the correlation function of p_α^n ($\alpha = x, y$): We find

$$\langle \sigma_{zz}^e \rangle_{z=L} = -\langle (\partial_z p_i)^2 \rangle_{z=L} = -\int \frac{d^2q}{(2\pi)^2} \frac{\pi}{L} \sum_n \frac{n^2\pi^2}{L^2} \frac{2k_B T}{\Delta_n} \left[\frac{1}{\gamma_1} + \frac{(\nu_1 - 1)^2}{4\eta(q^2 + \frac{n^2\pi^2}{L^2})} \frac{n^2\pi^2}{L^2} \right], \quad (57)$$

where

$$\Delta_n = K(q^2 + \frac{n^2\pi^2}{L^2}) \left[\frac{1}{\gamma_1} + \frac{(\nu_1 - 1)^2}{4\eta(q^2 + \frac{n^2\pi^2}{L^2})} \frac{n^2\pi^2}{L^2} + \frac{\xi\Delta\mu(\nu_1 - 1)}{2\eta\kappa(q^2 + \frac{n^2\pi^2}{L^2})^2} \frac{n^2\pi^2}{L^2} \right]. \quad (58)$$

Thus we obtain for the active Casimir stress in an orientationally ordered active fluid: Using (22)

$$\begin{aligned} C &= -\frac{K}{2} \langle (\partial_z p_i)^2 \rangle_{z=L} + \frac{K}{2} \langle (\partial_z p_i)^2 \rangle_{z=L, \Delta\mu=0} \\ &= \frac{K}{2} \int \frac{d^2q}{(2\pi)^2} \frac{\pi}{L} \sum_n \frac{n^2\pi^2}{L^2} \frac{2k_B T \xi \Delta\mu (\nu_1 - 1)}{2\eta K(q^2 + \frac{n^2\pi^2}{L^2}) \Delta_n}. \end{aligned} \quad (59)$$

This holds for both contractile and extensile active fluids and vanishes as $\Delta\mu$ is set to zero.

For a contractile active fluid with nematic order, C diverges when $\Delta_n = 0$, which can happen with a finite $\Delta\mu < 0$. The minimum thickness for which this can happen is given by the condition

$$\frac{K}{\gamma_1} \frac{\pi^2}{L_c^2} + \frac{(\nu_1 - 1)^2}{4\eta} K \frac{\pi^2}{L_c^2} = -\zeta \Delta\mu \frac{\nu_1 - 1}{2\eta}. \quad (60)$$

We evaluate the active contribution in (59) near the instability threshold (for a finite $\zeta \Delta\mu < 0$), i.e., as $L \rightarrow L_c$ from below. In this limit, only the $n = 1$ contribution diverges; the contributions with $n > 1$ are all finite. Therefore, we retain only the $n = 1$ contribution and evaluate it; we discard all higher- n contributions. Define $L = L_c(1 - \delta)$, $\delta > 0$ is a small dimensionless number. Keeping only the divergent term contribution as $\delta \rightarrow 0$, we obtain for the active contribution to the Casimir stress C as L approaches L_c from below.

$$C = k_B T \frac{1}{2L_c} \frac{\xi \Delta\mu (\nu_1 - 1)}{8\eta + \gamma_1 (\nu_1 - 1)^2} \ln \left| \frac{[2/\gamma_1 + (\nu_1 - 1)^2/4\eta] \gamma_1}{2\delta \Gamma(\nu_1 - 1)} \right|. \quad (61)$$

Substituting for $\xi \Delta\mu$ from (60), we find

$$C = k_B T \frac{-\pi^2}{2L_c^3} \frac{\Gamma \gamma_1}{8\eta + \gamma_1 (\nu_1 - 1)^2} \ln \left| \frac{[2/\gamma_1 + (\nu_1 - 1)^2/4\eta] \gamma_1}{2\delta \Gamma} \right|, \quad (62)$$

same as (23) as above. Thus, C approaches $-\infty$ as $\delta \rightarrow 0$. Thus, it is *attractive*, similar to the equilibrium contribution [3]. The equilibrium contribution may be evaluated in straightforward ways by following Ref. [3]: One finds, at $L \rightarrow L_c$,

$$C_E = -\frac{1}{8\pi} \frac{k_B T}{L_c^3} \zeta_R(3). \quad (63)$$

Thus, following the logic outlined in the main text, the total Casimir stress for an active fluid layer of thickness $L \rightarrow L_c$ from below is given by

$$C_{tot} = k_B T \frac{-\pi^2}{2L_c^3} \frac{\Gamma \gamma_1}{8\eta + \gamma_1 (\nu_1 - 1)^2} \ln \left| \frac{[2/\gamma_1 + (\nu_1 - 1)^2/4\eta] \gamma_1}{2\delta \Gamma} \right| - \frac{1}{8\pi} \frac{k_B T}{L_c^3} \zeta_R(3), \quad (64)$$

which is, of course, overall attractive.

The scaling of the active contribution to C with L changes drastically for $L \ll L_c$. We use (59) and focus on the second term on the right hand side of it which is the active contribution. We extract the $\mathcal{O}(\zeta \Delta\mu)$ contribution for small $\zeta \Delta\mu$ that yields the leading order active contribution to C for small $\zeta \Delta\mu$. We find

$$C = \frac{K}{2} \int \frac{d^2 q}{(2\pi)^2} \frac{\pi}{L} \sum_n \frac{n^2 \pi^2}{L^2} \frac{2k_B T \zeta \Delta\mu (\nu_1 - 1)}{2\eta K^2 (q^2 + \frac{n^2 \pi^2}{L^2})^2 \left[\frac{1}{\gamma_1} + \frac{(\nu_1 - 1)^2}{4\eta (q^2 + \frac{n^2 \pi^2}{L^2})} \frac{n^2 \pi^2}{L^2} \right]}. \quad (65)$$

This active contribution, being negative ($\zeta\Delta\mu < 0$), remains attractive and clearly scales as $1/L$, different from both the equilibrium contribution (that scales as $1/L^3$) and the contribution for $L \rightarrow L_c$ from below that shows a logarithmic divergence. This is consistent with the predictions from our simplified analysis above.

So far, we have considered only thermal noises above while averaging over the noise ensembles, keeping the active effects only in the deterministic parts of the dynamical model. In general, however, there are active noises present over and above the thermal noises. For simplicity, we supplement the thermal noise in (55) by an active noise that is assumed to be δ -correlated in space and time, with a variance that should scale with $\Delta\mu$. The precise amplitude of the variance should depend on the detailed nature of the stochasticity of the motor movements. We now refer to Eq. (57): then to the leading order in $\Delta\mu$, the active noises should generate an additional active contribution δC_A to C in (59) above near $L = L_c$. This is of the form

$$\delta C_A \sim -\frac{D_0\Delta\mu}{L_c^3}\zeta_R(3), \quad (66)$$

where D_0 is a dimensional constant. Thus, this additional contribution is attractive, has the same scaling with L as the equilibrium contribution C_E and has no divergence as $L \rightarrow L_c$ from below. We did not consider any active, multiplicative noises that may be important in cell biology contexts as illustrated in Ref. [6].

Our analyses above may be extended to obtain C just above the the threshold of the spontaneous flow instability [12]. Above the threshold, the steady reference state is given by $v_x = A \cos(z\pi/L)$, $p_z = 1$, $p_{x0} = \epsilon \sin(z\pi/L)$, $v_z = 0 = v_y$, $p_y = 0$, with $A = 4L\zeta\Delta\mu\epsilon/[\pi(4\eta + \gamma_1(\nu_1 + 1)^2)]$ and $\epsilon = \sqrt{1 - L_c/L}$, $L > L_c$ [12]. We discuss the case with $\epsilon \rightarrow 0$. We impose the same boundary conditions as above. The viscous contribution to C_{tot} continues to be zero by the same argument as above, since the spontaneous flow velocity v_x has no in-plane coordinate dependences. Defining δp_x as the fluctuation of p_x around p_{x0} , the new reference state, we note that the boundary condition on δp_x is same as that on p_x before, i.e., for no spontaneous flows; boundary conditions on p_y , having a zero value in the reference state, naturally remains unchanged from the previous case. We, thus, conclude that δp_x and p_y follow the same (linearized) equations (55) for p_x and p_y as in the previous case. Hence, the solutions for δp_x and p_y are identical to those of p_x and p_y in the previous case. It is now straightforward to see that the expression for the Casimir force C_{tot} as given in (64) now

has an additional contribution

$$\begin{aligned}
\delta C &= -\frac{K}{2} \langle \partial_z p_{x0} \partial_z p_{x0} \rangle |_{z=L} \\
&= -\frac{K}{2} \epsilon^2 \frac{\pi^2}{L^2} \cos^2(\pi z/L) |_{z=L} \\
&= -\frac{K}{2} \frac{L - L_c}{L} \frac{\pi^2}{L^2}.
\end{aligned} \tag{67}$$

We note that the additional contribution δC_{tot} depends on the Frank elastic constant K and has a negative sign, displaying its attractive nature. Further and not surprisingly, it vanishes as $(L - L_c)$ as $L \rightarrow L_c$, and hence is small just above the threshold. Thus, even above the threshold of the spontaneous flow instability, the dominant contribution to C still comes from (64), its value just below the threshold. Lastly, if we continue to use the above reference states for $L_c \lesssim L$ even for $L \gg L_c$, then δC scales as $1/L^2$ for $L \gg L_c$ and forms the dominant contribution in C .

In the above we have considered a contractile active fluid. For an extensile system with $\xi \Delta\mu > 0$, there are no divergences in (57) or (59) for any L . Expanding (59) in $\zeta \Delta\mu$, we extract an active contribution linear in $\zeta \Delta\mu$ that scales with L as $1/L$, different from the scaling of C in the contractile case, or from the equilibrium contribution C_E . We find for the leading order active contribution to the Casimir stress

$$C = \frac{K}{2} \int \frac{d^2 q}{(2\pi)^2} \frac{\pi}{L} \sum_n \frac{n^2 \pi^2}{L^2} \frac{2k_B T \zeta \Delta\mu (\nu_1 - 1)}{2\eta K^2 (q^2 + \frac{n^2 \pi^2}{L^2})^2 [\frac{1}{\gamma_1} + \frac{(\nu_1 - 1)^2}{4\eta (q^2 + \frac{n^2 \pi^2}{L^2})} \frac{n^2 \pi^2}{L^2}]} \sim \frac{K_B T \zeta \Delta\mu \gamma_1}{\eta K L}, \tag{68}$$

that scales with L as $1/L$; here only. Thus, the active contribution comes with a positive sign ($\zeta \Delta\mu > 0$), i.e., repulsive Casimir stress, a feature obtained in our simplified analysis above. Furthermore given that $C_{eq} < 0$, it is possible that $C_{tot} = C + C_{eq}$ changes sign as the thickness L or the activity parameter $\zeta \Delta\mu$ is varied, potentially creating an intriguing crossover between a repulsive and an attractive Casimir stress. Lastly, the differences in the active Casimir stress C for the contractile and extensile cases potentially open up experimental routes to distinguish contractile activity from extensile activity by measuring C .

[1] H. B. G. Casimir, *Proc. Kon. Ned. Akad. Wet.* **51**, 793 (1948); S. Lamoreaux, *Phys. Rev. Lett.* **78**, 5 (1996).

- [2] P.G. de Gennes and J. Prost *The Physics of Liquid Crystals* Oxford University Press, Oxford (1993).
- [3] A. Ajdari, B. Duplantier, D. Hone, L. Peliti and J. Prost, *J. Phys. II France* **2**, 487 (1992).
- [4] D. S. Dean and A. Gopinathan, *J. Stat. Mech. - Th. Exp.* (2009), online at stacks.iop.org/JSTAT/2009/L08001; D. S. Dean and A. Gopinathan, *Phys. Rev. E* **81**, 041126 (2010); P. Rodriguez-Lopez, R. Brito and R. Soto, *Phys. Rev. E* **83**, 031102 (2011); D. S. Dean, V. Démery, V. A. Parsegian and R. Podgornik, *Phys. Rev. E* **85**, 031108 (2012).
- [5] D. Bartolo, A. Ajdari, and J. B. Fournier, *Phys. Rev. E* **67**, 061112 (2003); R. Brito, R. Soto, and U. M. B. Marconi, *Gran. Matt.* **10**, 29 (2007); R. Brito, U. M. B. Marconi, and R. Soto, *Phys. Rev. E* **76**, 011113 (2007); A.-F. Bitbol and J.-B. Fournier, *Phys. Rev. E* **83**, 061107 (2011).
- [6] A. Singh, J.-F. Rupprecht, G.V. Shivashankar, J. Prost and M. Rao, arXiv: 1703.04401.
- [7] J.-F. Rupprecht, A. Singh, G.V. Shivashankar, M. Rao and J. Prost, arXiv: 1703:04395.
- [8] E. Makhija, D. S. Jokhun, and G. V. Shivashankar, *Proc. Nat. Acad. Sc. (USA)* **113**, 13189 (2015).
- [9] K. Kruse, J.F. Joanny, F. Jülicher, J. Prost, and K. Sekimoto, *Phys. Rev. Lett.* **92**, 078101 (2004).
- [10] R.A. Simha, S. Ramaswamy, *Phys. Rev. Lett.*, **89**, 058101 (2002); S. Ramaswamy, *Annu. Rev. Cond. Matt. Phys.*, **1**, 323 (2010).
- [11] M. C. Marchetti, J. F. Joanny, S. Ramaswamy, T. B. Liverpool, J. Prost, M. Rao, and R. A. Simha *Rev. Mod. Phys.* **85**, 1143 (2013).
- [12] R. Voituriez, J.F. Joanny, and J. Prost, *Eur. Phys. Lett.* **70**, 404 (2005).
- [13] S. Sankararaman and S. Ramaswamy, *Phys. Rev. Lett.*, **102**, 118107 (2009).
- [14] A. Basu, J.F. Joanny, F. Jülicher and J. Prost, *Eur. Phys. J E*, **27**, 149 (2008).
- [15] A. Basu, J.-F Joanny, F. Jülicher and J. Prost, *New J. Phys.* **14**, 115001 (2012).