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FUNDAMENTAL GROUPS, 3-BRAIDS, AND EFFECTIVE ESTIMATES OF INVARIANTS

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Abstract. We define invariants of braids rather than invariants of conjugacy classes of braids. For any pure 3-braid we give effective upper and lower bounds for these invariants. This is done in terms of a natural syllable decomposition of the word representing the image of the braid in the braid group modulo its center. The bounds differ by a multiplicative constant not depending on the word. Respective bounds are given for all 3-braids. We also obtain effective upper and lower bounds for the entropy of pure 3-braids in these terms. The proof leads to the study of the extremal length of classes of curves representing elements of the fundamental group of the twice punctured complex plane.

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1. Statement of the results

In this paper we define invariants of n-braids (rather than invariants of conjugacy classes of braids.) Note, that a popular invariant of braids, the entropy, is a conjugacy invariant, i.e. it does not distinguish different elements of a conjugacy class of braids. The present invariants may be compared from a conceptional point of view with the conformal module of conjugacy classes of braids which is inverse proportional to the entropy (see [5]).

For pure 3-braids we give effective upper and lower bounds for the invariants. This is done in terms of a natural syllable decomposition of the word representing the image of the braid in $\mathcal{B}_3 \diagup \mathcal{Z}_3$, the group of 3-braids modulo its center. The bounds differ by a universal multiplicative constant.

The estimates for the 3-braid invariants also give bounds for the entropy of arbitrary pure 3-braids in the mentioned terms, and also provide estimates of the invariants for arbitrary (maybe, not pure) 3-braids.

Notice that there is an algorithm which detects in principle whether a braid is a pseudo-Anosov braid and in this case it computes the entropy ([3]). This approach uses train tracks. For 3-braids which can be represented by short words the entropy is known and can be easily calculated explicitly without using train tracks. As far as we know despite these results explicit bounds for the entropy of arbitrary 3-braids in terms of representing words are new.

The braid invariants are defined as follows. For a subset A of the complex plane $\mathbb{C}$ we consider the configuration space $C_n(A) = \{(z_1, \ldots, z_n) \in A^n : z_i \neq z_j \text{ for } i \neq j\}$ of n particles moving along A without collision. Denote by $\mathcal{S}_n$ the symmetric group. Each permutation in $\mathcal{S}_n$ acts on $C_n(A)$ by permuting the coordinates. Consider the

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quotient $C_n(A)\diagup\mathcal{S}_n$. The quotient $C_n(A)\diagup\mathcal{S}_n$ is called the symmetrized configuration space related to A. The natural projection $C_n(\mathbb{C})\to C_n(\mathbb{C})\diagup\mathcal{S}_n$ is denoted by $\mathcal{P}_{\mathcal{S}_n}$.

Choose a base point $E_n\in C_n(\mathbb{C})\diagup\mathcal{S}_n$. Regard braids on n strands (n-braids for short) with base point E_n as homotopy classes of loops with base point E_n in the symmetrized configuration space, equivalently, as element of the fundamental group $\pi_1(C_n(\mathbb{C})\diagup\mathcal{S}_n, E_n)$ of the symmetrized configuration space with base point E_n.

The totally real subspace $\mathcal{E}_n^{tr}\overset{def}{=}C_n(\mathbb{R})\diagup\mathcal{S}_n$ of $C_n(\mathbb{C})\diagup\mathcal{S}_n$ is connected and simply connected. Indeed, the totally real subspace $C_n(\mathbb{R})$ of $C_n(\mathbb{C})$ is the union of the connected components $\{(x_1,\ldots,x_n)\in\mathbb{R}^n: x_{\sigma(1)}<x_{\sigma(2)}<\ldots<x_{\sigma(n)}\}$ over all permutations $\sigma\in\mathcal{S}_n$. Thus $C_n(\mathbb{R})$ is invariant under the action of $\mathcal{S}_n$ and the quotient is homeomorphic to $\{(x_1,\ldots,x_n)\in\mathbb{R}^n: x_1<x_2<\ldots<x_n\}$. Hence the claim.

The fundamental group $\pi_1(\,C_n(\mathbb{C})\diagup\mathcal{S}_n\,,\ E_n\,)$ is isomorphic to the relative fundamental group $\pi_1(\,C_n(\mathbb{C})\diagup\mathcal{S}_n\,,\ C_n(\mathbb{R})\diagup\mathcal{S}_n\,)$. The elements of the latter group are homotopy classes of arcs in $C_n(\mathbb{C})\diagup\mathcal{S}_n$ with endpoints in $C_n(\mathbb{R})\diagup\mathcal{S}_n$.

The isomorphism between the two groups is obtained as follows. Since the change of the base point leads to an isomorphism of the fundamental group with base point, we may assume that E_n is contained in the totally real subspace $C_n(\mathbb{R})\diagup\mathcal{S}_n$. Each element of $\pi_1(\,C_n(\mathbb{C})\diagup\mathcal{S}_n\,,\ E_n\,)$ is a subset of an element of $\pi_1(\,C_n(\mathbb{C})\diagup\mathcal{S}_n\,,\ C_n(\mathbb{R})\diagup\mathcal{S}_n\,)$. Vice versa, since $C_n(\mathbb{R})\diagup\mathcal{S}_n$ is connected and E_n is contained in $C_n(\mathbb{R})\diagup\mathcal{S}_n$ each class in $\pi_1(\,C_n(\mathbb{C})\diagup\mathcal{S}_n\,,\ C_n(\mathbb{R})\diagup\mathcal{S}_n\,)$ contains a class in $\pi_1(\,C_n(\mathbb{C})\diagup\mathcal{S}_n\,,\ E_n\,)$. Since $C_n(\mathbb{R})\diagup\mathcal{S}_n$ is simply connected, each class in $\pi_1(\,C_n(\mathbb{C})\diagup\mathcal{S}_n\,,\ C_n(\mathbb{R})\diagup\mathcal{S}_n\,)$ contains no more than one class of $\pi_1(\,C_n(\mathbb{C})\diagup\mathcal{S}_n\,,\ E_n\,)$. Indeed, if two loops in $C_n(\mathbb{C})\diagup\mathcal{S}_n$ with base point E_n are homotopic as loops in $C_n(\mathbb{C})\diagup\mathcal{S}_n$ with varying base point in $C_n(\mathbb{R})\diagup\mathcal{S}_n$ then they are homotopic as loops in $C_n(\mathbb{C})\diagup\mathcal{S}_n$ with fixed base point E_n.

Let R be an open rectangle in the complex plane $\mathbb{C}$. Unless said otherwise the considered rectangles will always have sides parallel to the coordinate axes. Denote the length of the horizontal sides of R by b and the length of the vertical sides by a. (For instance, we may consider $R=\{z=x+iy: 0<x<\mathsf{b},\, 0<\mathsf{y}<\mathsf{a}\}$.) The conformal module of the rectangle R introduced by Ahlfors [1] equals $m(R)=\frac{\mathsf{b}}{\mathsf{a}}$. The extremal length of R equals $\lambda(R)=\frac{\mathsf{a}}{\mathsf{b}}$, which is the inverse of the conformal module .

Let $b\in\mathcal{B}_n$ be a braid. Denote its image in the relative fundamental group $\pi_1(\,C_n(\mathbb{C})\diagup\mathcal{S}_n\,,\ C_n(\mathbb{R})\diagup\mathcal{S}_n\,)$ by b_{tr}. For a rectangle R as above let $f: R\to C_n(\mathbb{C})\diagup\mathcal{S}_n$ be a mapping which admits a continuous extension to the closure $\bar{R}$ (denoted again by f) which maps the (open) horizontal sides into $C_n(\mathbb{R})\diagup\mathcal{S}_n$. We say that the mapping represents b_{tr} if for each maximal vertical line segment contained in R (i.e. R intersected with a vertical line in $\mathbb{C}$) the restriction of f to the closure of the line segment represents b_{tr}.

We are now ready to define for any braid its extremal length with totally real boundary values (and the conformal module with totally real boundary values, respectively).

Definition 1. *Let $b\in\mathcal{B}_n$ be an n-braid. The extremal length $\Lambda_{tr}(b)$ with totally real horizontal boundary values is defined as*

$$\begin{aligned}\Lambda_{tr}(b)=&\inf\{\lambda(R): R \text{ a rectangle which admits a holomorphic map to}\\ &C_n(\mathbb{C})\diagup\mathcal{S}_n \text{ that represents } b_{tr}\}\,.\end{aligned}$$

The conformal module $\mathcal{M}_{tr}(b)$ of b with totally real horizontal boundary values, respectively, is defined as

$$\mathcal{M}_{tr}(b) = \sup\{m(R) : R \text{ a rectangle which admits a holomorphic map to } C_n(\mathbb{C})\diagup\mathcal{S}_n \text{ that represents } b_{tr}\}\,.$$

Note that the two invariants are inverse to each other. It is more convenient to work with the extremal length and we will mostly speak about the extremal length rather than about the conformal module.

The choice of the boundary values is motivated by real algebraic geometry. Following Arnold [2] a point in the symmetrized configuration space $C_n(\mathbb{C})\diagup\mathcal{S}_n$ can be considered as unordered n-tuple of pairwise distinct complex numbers and can be identified with the monic polynomial with these zeros. Parametrize the space $\mathfrak{P}_n$ of monic polynomials of degree n without multiple zeros by their coefficients. We obtain a biholomorphic map between $C_n(\mathbb{C})\diagup\mathcal{S}_n$ and $\mathbb{C}^n \setminus \{\mathsf{D}_n = 0\}$, where D_n is the discriminant.

The zero set of a polynomial with real coefficients consists of a set of real points and a set of pairs of complex conjugate points. A polynomial in $\mathfrak{P}_n$ all whose zeros are real can be identified with a point in $C_n(\mathbb{R})\diagup\mathcal{S}_n$. Thus, totally real boundary values correspond to one of the cases arising in real algebraic geometry. The set of polynomials in $\mathfrak{P}_n$ with real coefficients and $\ell \geq 2$ pairs of complex conjugate zeros is not simply connected. We will not develop the case of such boundary values in this paper.

However, for 3-braids it is convenient to consider the case of boundary values corresponding to one real root and a pair of complex conjugate roots. The set $\mathcal{E}^n_{pb}$ of polynomials in $\mathfrak{P}_3$ with one real zero and a pair of complex conjugate zeros is simply connected. ("*pb*" stands for perpendicular bisector. In fact, the real root lies on the perpendicular bisector of the line segment joining the two complex conjugate roots.) The braid group $\mathcal{B}_3$ is isomorphic to $\pi_1(C_3(\mathbb{C})\diagup\mathcal{S}_3, \mathcal{E}^3_{pb})$. For a braid $b \in \mathcal{B}_3$ we denote by b_{pb} its image in $\pi_1(C_3(\mathbb{C})\diagup\mathcal{S}_3, \mathcal{E}^3_{pb})$ under this isomorphism. The convention that a continuous mapping from a rectangle to symmetrized configuration space represents b_{pb} is made in the same way as the respective convention for b_{tr}.

We have the following definition.

Definition 2. *Let $b \in \mathcal{B}^3$ be a 3-braid. The extremal length $\Lambda_{pb}(b)$ of b with perpendicular bisector boundary values is defined as*

$$\Lambda_{pb}(b) = \inf\{\lambda(R) : R \text{ a rectangle which admits a holomorphic map to } C_3(\mathbb{C})\diagup\mathcal{S}_3 \text{ that represents } b_{pb}\}.$$

Recall the respective definition for closed braids. They can be identified with conjugacy classes of braids or with free isotopy classes of loops in $C_n(\mathbb{C})\diagup\mathcal{S}_n$. (See [5], [6].) The extremal length (conformal module, respectively) of a conjugacy class of braids is defined as follows. We say that a continuous mapping f of an annulus $A = \{z \in \mathbb{C} : r < |z| < R\}$, $0 \leq r < R \leq \infty$, into $C_n(\mathbb{C})\diagup\mathcal{S}_n$ represents a conjugacy class $\hat{b}$ of n-braids if for some (and hence for any) circle $\{|z| = \rho\} \subset A$ the loop $f : \{|z| = \rho\} \to C_n(\mathbb{C})\diagup\mathcal{S}_n$ represents the conjugacy class $\hat{b}$. According to Ahlfors' definition the conformal module of an annulus $A = \{z \in \mathbb{C} : r < |z| < R\}$ equals $m(A) = \frac{1}{2\pi}\log(\frac{R}{r})$, and the extremal length equals $\lambda(A)$ as $\frac{1}{m(A)}$. Associate to each conjugacy class of elements of the fundamental group of $C_n(\mathbb{C})\diagup\mathcal{S}_n$, or, equivalently, to each conjugacy class of n-braids, its extremal length, defined as follows.

Definition 3. *Let $\hat{b}$ be a conjugacy class of n-braids, $n \geq 2$. The extremal length $\Lambda(\hat{b})$ of $\hat{b}$ is defined as $\Lambda(\hat{b}) = inf_{A \in \mathcal{A}} \lambda(A)$, where $\mathcal{A}$ denotes the set of all annuli which admit a holomorphic mapping into $C_n(\mathbb{C})\diagup\mathcal{S}_n$ that represents $\hat{b}$.*

It is proved in [5] that the extremal length $\Lambda(\hat{b})$ is proportional to the entropy $h(\hat{b})$ of the conjugacy class, namely $h(\hat{b}) = \frac{\pi}{2}\Lambda(\hat{b})$.

Denote by Δ_n the Garside element in the braid group $\mathcal{B}_n$. (Δ_n^2 is a full twist). Note that the subgroup $\langle \Delta_n^2 \rangle$ of $\mathcal{B}_n$ generated by Δ_n^2 is the center $\mathcal{Z}_n$ of $\mathcal{B}_n$. We have the following lemma.

Lemma 1. *For each braid $b \in \mathcal{B}_n$ we have $\Lambda(\hat{b}) = \Lambda(\widehat{b\Delta_n^2})$ and $\Lambda_{tr}(b) = \Lambda_{tr}(b\Delta_n)$. For each braid $b \in \mathcal{B}_3$ we have $\Lambda_{pb}(b) = \Lambda_{pb}(b\Delta_3)$.*

Proof. Indeed, let $R = \{x + iy : x \in (0,1),\, y \in (0, \mathsf{a})\}$, and suppose a holomorphic map $f : R \to C_n(\mathbb{C}^n)\diagup\mathcal{S}_n$ represents b_{tr} (or b_{pb}, respectively, with $n = 3$). Let $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_n)) : R \to C_n(\mathbb{C}^n)$ be a lift of f to a mapping into $C_n(\mathbb{C}^n)$. The mapping $\zeta \to e^{\frac{\pi}{\mathsf{a}}\zeta}\tilde{f}(\zeta) = e^{\frac{\pi}{\mathsf{a}}\zeta}(\tilde{f}_1(\zeta), \ldots, \tilde{f}_n(\zeta))$ is holomorphic on R. For the canonical projection $\mathcal{P}_{\mathcal{S}_n} : C_n(\mathbb{C}^n) \to C_n(\mathbb{C}^n)\diagup\mathcal{S}_n$ the mapping $\zeta \to \mathcal{P}_{\mathcal{S}_n}(e^{\frac{\pi}{\mathsf{a}}\zeta}(\tilde{f}(\zeta)))$ represents $(b\,\Delta_n)_{tr}$ (or $(b\,\Delta_n)_{pb}$, respectively, with $n = 3$). The relation for conjugacy classes of braids is proved in the same way. □

The versions of the extremal length of braids are morally related. In this paper we will give details only for 3-braids.

We consider now pure 3-braids identified with elements of the fundamental group $\pi_1(C_3(\mathbb{C})\diagup\mathcal{S}_3, E_3)$ with base point $E_3 \in C_3(\mathbb{C})\diagup\mathcal{S}_3$. Denote the group of pure 3-braids by $\mathcal{PB}_3$.

The quotient $\mathcal{PB}_3\diagup\langle\Delta_3^2\rangle$ is a free group in two generators, the class of σ_1^2 and the class of σ_2^2. Denote the two generators by a_1 and a_2.

We will now state theorems on upper and lower bounds for the versions of the extremal length of any pure braid in terms of the word representing its image in $\mathcal{PB}_3\diagup\langle\Delta_3^2\rangle$. Take a non-trivial element of $\mathcal{PB}_3\diagup\langle\Delta_3^2\rangle$. Represent it as reduced word $w = w_1^{n_1} \cdot w_2^{n_2} \cdot \ldots$, where the n_j are non-zero integers and the w_j are alternately equal to either a_1 or a_2. We refer to the $w_j^{n_j}$ as the terms of the word. We will estimate the extremal length in terms of a decomposition of the word into syllables. We describe now the syllable decomposition of the word.

(1) Any term $w_j^{n_j}$ of the reduced word with $|n_j| \geq 2$ is a syllable.
(2) Any maximal sequence of at least two consecutive terms of the reduced word which have equal power equal to either $+1$ or -1 is a syllable.
(3) Each remaining term of the reduced word enters with power $+1$ or -1 and is characterized by the following property. The neighbouring term on the right (if there is one) and also the neighbouring term on the left (if there is one) has power different from that of the given one. Each term of this type is a syllable called singleton.

Define the degree of a syllable deg(syllable) to be the sum of the absolute values of the powers of terms entering the syllable.

For example, the syllables of the word $a_2^{-1}\,a_1^2\,a_2^{-3}\,a_1^{-1}\,a_2^{-1}\,a_1^{-1}\,a_2\,a_1^{-1}$ from left to right are the singleton a_2^{-1}, the syllable a_1^2 of degree 2, the syllable a_2^{-3} of degree 3, the syllable $a_1^{-1}\,a_2^{-1}\,a_1^{-1}$ of degree 3, the singleton a_2 and the singleton a_1^{-1}.

Label the syllables of a non-trivial word from left to right by consecutive integral numbers $j = 1, 2, \dots$. Let d_j be the degree of the j-th syllable $\mathfrak{s}_j$. Put

$$\mathcal{L}(w) \overset{def}{=} \sum_j \log(4d_j - 1). \tag{1}$$

If w is the identity we put $\mathcal{L}(w) = 0$. Notice that for the word consisting of the single syllable $\mathfrak{s}_j$ we have $\mathcal{L}(\mathfrak{s}_j) = \sum_j \log(4d_j - 1)$. Thus, $\mathcal{L}$ is an additive function of the syllables.

The following theorem holds.

Theorem 1. *Let $b \in \mathcal{PB}_3$ be a pure 3-braid and let w be the word representing its image in $\mathcal{PB}_3 \diagup \langle \Delta_3^2 \rangle$. Then*

$$\frac{1}{2\pi}\,\mathcal{L}(w) \le \Lambda_{tr}(b) = \frac{1}{\mathcal{M}_{tr}(b)} \le 300 \cdot \mathcal{L}(w), \tag{2}$$

except in the following cases: $w = a_1^n$ or $w = a_2^n$ for an integer n. In these cases $\Lambda_{tr}(b) = 0$, i.e $\mathcal{M}_{tr}(b) = \infty$.

Moreover,

$$\frac{1}{2\pi}\,\mathcal{L}(w) \le \Lambda_{pb}(b) = \frac{1}{\mathcal{M}_{pb}(b)} \le 300 \cdot \mathcal{L}(w), \tag{3}$$

except in the following case: each term in the reduced word w has the same power, which equals either $+1$ or -1. In these cases $\Lambda_{pb}(b) = 0$, i.e. $\mathcal{M}_{pb}(b) = \infty$.

The following propositions are immediate consequences of Theorem 1.

Proposition 1. *For a pure braid $b \in \mathcal{PB}_3$ which is not one of the exceptional cases of Theorem 1 the two versions of the extremal length are comparable:*

$$C_1\,\Lambda_{tr}(b) \;\le\; \Lambda_{pb}(b) \;\le\; C_2\,\Lambda_{tr}(b)$$

for positive constants C_1 and C_2 which do not depend on b.

Proposition 2. *For each element $b \in \mathcal{PB}_3$ whose image w in the pure braid group modulo its center is not a singleton the estimate*

$$\frac{1}{2\pi} \cdot \mathcal{L}(w) \le \Lambda_{tr}(b) + \Lambda_{pb}(b) \le 600 \cdot \mathcal{L}(w)$$

holds.

Let $b \in \mathcal{PB}_3$ be a pure 3-braid and let w be the word representing its image in $\mathcal{PB}_3 \diagup \langle \Delta_3^2 \rangle$. The conjugacy class $\hat{w}$ of elements in $\mathcal{PB}_3 \diagup \langle \Delta_3^2 \rangle$ corresponds to the conjugacy class $\hat{b}$. Any word $\tilde{w}$ obtained from w as follows will be called a cyclically syllable reduced conjugate of the word w. Write in reduced form the periodic word $\dots w\,w \dots$ which is infinite in both directions and obtained by repeating the entry w and no other entry. Cut off from the infinite word a word $\tilde{w}$ consisting of consecutive terms of the infinite word, so that $\tilde{w}$ is conjugate to w and the cuts are between two syllables, not inside a syllable. The following theorem holds.

Theorem 2. *Let $\hat{b}$ be a conjugacy class of pure 3-braids, let $\hat{w}$ be the conjugacy class of elements of $\mathcal{PB}_3 \diagup \langle \Delta_3^2 \rangle$ corresponding to $\hat{b}$ and let w be a cyclically syllable reduced word representing the conjugacy class $\hat{w}$. Then*

$$\frac{1}{2\pi} \cdot \mathcal{L}(w) \;\le\; \Lambda(\hat{b}) = \frac{2}{\pi} h(\hat{b}) = (\mathcal{M}(\hat{b}))^{-1} \;\le\; 300 \cdot \mathcal{L}(w),$$

with the following exceptions: $w = a_1^n$, $w = a_2^n$ *and* $w = (a_1 a_2)^n$. *In these cases* $\Lambda(\hat{b}) = h(\hat{b}) = 0$ *and* $\mathcal{M}(\hat{b}) = \infty$.

Notice, that the exceptional braids for Theorem 2 are exactly the reducible pure 3-braids. Notice also, that the word $w = (a_1 a_2)^n a_1$ is not cyclically syllable reduced. A cyclically syllable reduced conjugate is $a_1^2 (a_2 a_1)^{n-1} a_2$. Hence, for a braid b whose image in $\mathcal{PB}_3 \diagup \langle \Delta_3^2 \rangle$ equals w we obtain that $\Lambda(\hat{b})$ is positive and $\Lambda_{pb}(b) = 0$. However, $\Lambda_{tr}(b)$ is positive.

We want to point out here the following fact. Take any pure braid b whose image in $\mathcal{PB}_3 \diagup \langle \Delta_3^2 \rangle$ is a cyclically syllable reduced word which is not one of the exceptional cases of Theorem 1. Then the equality $\Lambda(\hat{b}) \geq \Lambda_{pb}(b)$ holds. However, the extremal length of $\hat{b}$ may be strictly larger than the extremal length of b_{pb}, see the remark after the proof of Theorem 3. There is a respective remark for Λ_{tr} replaced by Λ_{pb}.

We consider now arbitrary 3-braids (not necessarily pure braids). The following lemma holds.

Lemma 2. *Any braid* $b \in \mathcal{B}_3$ *which is not a power of* Δ_3 *can be written in a unique way in the form*

$$\sigma_j^k \, b_1 \, \Delta_3^\ell \tag{4}$$

where $j = 1$ *or* $j = 2$, $k \neq 0$ *is an integer,* ℓ *is a (not necessarily even) integer, and* b_1 *is a word in* σ_1^2 *and* σ_2^2 *in reduced form. If* b_1 *is not the identity, then the first term of* b_1 *is a non-zero even power of* σ_2 *if* $j = 1$, *and* b_1 *is a non-zero even power of* σ_1 *if* $j = 2$.

For an integer $j \neq 0$ we denote by $q(j)$ that even number closest to j, which is closest to zero. In other words, $q(j) = j$ for each even integer $j \neq 0$. For each odd integer j, $q(j) = j - \operatorname{sgn}(j)$, where $\operatorname{sgn}(j)$ for a non-zero integral number j equals 1 if j is positive, and -1 if j is negative. For a braid in form (4) we put $\vartheta(b) \overset{def}{=} \sigma_j^{q(k)} \, b_1$. The following theorem holds.

Theorem 3. *Let* $b \in \mathcal{B}_3$ *be not a (not necessarily) pure braid which is not a power of* Δ_3, *and let* w *be the word representing the image of* $\vartheta(b)$ *in* $\mathcal{B}_3 \diagup \langle \Delta_3^2 \rangle$. *Then*

$$\frac{1}{2\pi} \mathcal{L}(w) \leq \Lambda_{tr}(b) \leq 300 \cdot \mathcal{L}(w) \,,$$

except in the case when $b = \sigma_j^k \, \Delta_3^\ell$ *for* $j = 1$ *or* $j = 2$, $k \neq 0$ *is an integral number, and* ℓ *is an arbitrary integer. In this case* $\Lambda_{tr}(b) = 0$.

2. Coverings of $\mathbb{C} \setminus \{-1, 1\}$ and slalom curves

It will be convenient to work with homotopy classes of curves in a more general setting. Let X be a topological space, and let $\mathcal{E}_1$ and $\mathcal{E}_2$ be relatively closed subsets of X. Let $\mathsf{h} =_{\mathcal{E}_1} \mathsf{h}_{\mathcal{E}_2}$ be a homotopy class of curves in X with initial point in $\mathcal{E}_1$ and terminating point in $\mathcal{E}_2$. If $\mathcal{E}_1 = \mathcal{E}_2$ we write $\mathsf{h}_{\mathcal{E}_1}$ instead of $\mathsf{h} =_{\mathcal{E}_1} \mathsf{h}_{\mathcal{E}_2}$. A continuous mapping f from an open rectangle into X which admits a continuous extension to the closure of the rectangle (denoted again by f) is said to represent h if the lower open horizontal side is mapped to $\mathcal{E}_1$, the upper horizontal side is mapped to $\mathcal{E}_2$ and the restriction of f to the closure of each maximal vertical segment in the rectangle represents h. In case $X = C_n(\mathbb{C}) \diagup \mathcal{S}_n$ and $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_n^{tr}$ we write for short h_{tr} instead of $\mathsf{h}_{\mathcal{E}_n^{tr}}$, and in case $X = C_3(\mathbb{C}) \diagup \mathcal{S}_3$ and $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_n^{pb}$ we write for short h_{pb}.

We give the following general definition of extremal length for homotopy classes of curves.

Definition 4. *For an open subset X of a complex manifold $\mathcal{X}$, two relatively closed subsets $\mathcal{E}_1$ and $\mathcal{E}_2$ of X and a homotopy class $\mathsf{h} =_{\mathcal{E}_1} \mathsf{h}_{\mathcal{E}_1}$ of curves in X with initial point in $\mathcal{E}_1$ and terminating point in $\mathcal{E}_2$ the extremal length $\Lambda(\mathsf{h})$ is defined as*

$$\Lambda(\mathsf{h}) = \inf\{\lambda(R) : R \text{ a rectangle which admits a holomorphic map to } X \text{ that represents } \mathsf{h}\}\,.$$

Consider now again the full braid group $\mathcal{B}_3$ identified with the relative fundamental group $\pi_1(C_3(\mathbb{C})\diagup\mathcal{S}_3, C_3(\mathbb{R})\diagup\mathcal{S}_3)$. The elements of the relative fundamental group are represented by arcs with endpoints in the totally real space $C_3(\mathbb{R})\diagup\mathcal{S}_3$. The preimage $C_3(\mathbb{R})$ of the totally real set $C_3(\mathbb{R})\diagup\mathcal{S}_3$ under $\mathcal{P}_{\mathcal{S}_3}$ has several connected components.

Let $z = (z_1, z_2, z_3) \in C_3(\mathbb{C})$. Denote by M_z the Möbius transformation that maps z_1 to 0, z_3 to 1 and fixes ∞. Then $M_z(z_2)$ omits 0, 1 and ∞. Notice that $M_z(z_2)$ is equal to the cross ratio $(z_2, z_3; z_1, \infty) = \frac{z_2-z_1}{z_3-z_1} \cdot \frac{z_3-\infty}{z_2-\infty} = \frac{z_2-z_1}{z_3-z_1}$. For a point $z = (z_1, z_2, z_3) \in C_3(\mathbb{C})$ we put $\mathfrak{C}(z) = 2M_z(z_2) - 1$. The mapping $\mathfrak{C}$ maps $C_3(\mathbb{C})$ to $\mathbb{C} \setminus \{-1, 1\}$ and $C_3(\mathbb{R})^0 \overset{def}{=} \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 < x_2 < x_3\}$ to $(-1, 1)$.

Associate to a curve $\tilde\gamma(t) = (\tilde\gamma_1(t), \tilde\gamma_2(t), \tilde\gamma_3(t))$, $t \in [0, 1]$, in $C_3(\mathbb{C})$ the curve $\mathfrak{C}(\tilde\gamma)(t) \overset{def}{=} 2\,\frac{\tilde\gamma_2(t)-\tilde\gamma_1(t)}{\tilde\gamma_3(t)-\tilde\gamma_1(t)} - 1$, $t \in [0, 1]$, in $\mathbb{C}$ which omits the points -1 and 1.

Let γ be a curve in $C_3(\mathbb{C})\diagup\mathcal{S}_3$ with endpoints on the totally real set $C_3(\mathbb{R})\diagup\mathcal{S}_3$. Consider its lift $\tilde\gamma$ to $C_3(\mathbb{C})$ with initial point in a chosen connected component of $C_3(\mathbb{R})$. The curve $\mathfrak{C}(\tilde\gamma)$ is a curve in $\mathbb{C} \setminus \{-1, 1\}$ with initial point in a connected component of $\mathbb{R} \setminus \{-1, 1\}$. In this way we obtain mappings from homotopy classes b_{tr} of curves in $C_3(\mathbb{C})\diagup\mathcal{S}_3$ with endpoints in $C_3(\mathbb{R})\diagup\mathcal{S}_3$ to homotopy classes in $\mathbb{C} \setminus \{-1, 1\}$ with endpoints in $\mathbb{R} \setminus \{-1, 1\}$. Since each of the mappings is defined by a lift and a composition with a holomorphic mapping the extremal length of the obtained class of curves in the complex plane coincides with the extremal length of the class of curves in the symmetrized configuration space. This will allow us to study the extremal length of 3-braids with totally real boundary values by studying the extremal length of homotopy classes of curves with end points on $\mathbb{R} \setminus \{-1, 1\}$. For each homotopy class b_{tr} we may choose the most convenient among these mappings.

For a pure braid b_{tr} we usually choose the connected component $C_3^0(\mathbb{R})$ and denote by γ^0 the lift to $C_3(\mathbb{C})$ with initial point in $C_3^0(\mathbb{R})$ of a curve γ representing b_{tr}. The curve $\mathfrak{C}(\gamma^0)$ has endpoints on $(-1, 1)$.

In the case of pb boundary values we usually choose the connected component $\overset{\circ}{\mathcal{E}^3}_{pb} = \{(z_1, z_2, z_3) \in C_3(\mathbb{C}) : z_2 \in \mathbb{R}, z_1 \text{ and } z_3 \text{ complex conjugate}\}$ of the preimage $\mathcal{P}^{-1}(\mathcal{E}^3_{pb})$. The mapping $\mathfrak{C}$ maps $\overset{\circ}{\mathcal{E}^3}_{pb}$ onto $i\mathbb{R}$.

The following lemma states that the mapping $\gamma \to \mathfrak{C}(\gamma^0)$ induces a homomorphism $\mathfrak{C}_*$ from $\mathcal{PB}_3$ to the fundamental group of $\mathbb{C} \setminus \{-1, 1\}$ whose kernel is the subgroup $\langle\Delta_3^2\rangle$ of $\mathcal{PB}_3$ generated by Δ_3^2.

Lemma 3. *The group $\mathcal{PB}_3\diagup\langle\Delta_3^2\rangle$ is isomorphic to the fundamental group of $\mathbb{C} \setminus \{-1, 1\}$ with base point.*

Proof. Identify the pure braid group $\mathcal{PB}_3$ with a subgroup of the fundamental group of $C_3(\mathbb{C})\diagup\mathcal{S}_3$ with base point in the totally real set. For curves γ representing elements of this subgroup we consider the lift γ^0. In this way we may identify the pure braid group with base point in the totally real set with the fundamental group of $C_3(\mathbb{C})$ with base point in $C_3(\mathbb{R})^0$. Choose the base point $(-1,0,1) \in C_3(\mathbb{C})$ and identify $\mathcal{PB}_3$ with $\pi_1(C_3(\mathbb{C}),(-1,0,1))$.

Let $\overset{\circ}{\gamma}(t) = (\overset{\circ}{\gamma}_1(t),\overset{\circ}{\gamma}_2(t),\overset{\circ}{\gamma}_3(t))$, $t \in [0,1]$, be a curve in $C_3(\mathbb{C})$ and let $\mathfrak{C}(\overset{\circ}{\gamma})$ be the associated curve in $\mathbb{C}\setminus\{-1,1\}$. If $\overset{\circ}{\gamma}$ is a loop with base point $\overset{\circ}{\gamma}(0) = (-1,0,1) \in C_3(\mathbb{C})^0$, then $\mathfrak{C}(\overset{\circ}{\gamma})$ is a loop with base point $\mathfrak{C}(\overset{\circ}{\gamma})(0) = 0$. The homotopy class of $\mathfrak{C}(\overset{\circ}{\gamma})$ in $\mathbb{C}\setminus\{-1,1\}$ with base point 0 depends only on the homotopy class of $\overset{\circ}{\gamma}$ with base point $(-1,0,1)$ in the configuration space $C_3(\mathbb{C})$. We obtain a homomorphism $\mathfrak{C}_*$ from the fundamental group $\pi_1(C_3(\mathbb{C}),(-1,0,1))$ of $C_3(\mathbb{C})$ with base point $(-1,0,1)$ to the fundamental group $\pi_1 \overset{def}{=} \pi_1(\mathbb{C}\setminus\{-1,1),0)$ of the twice punctured complex plane with base point 0.

The braids represented by the two loops $\overset{\circ}{\gamma}$ and $\tilde{\gamma}$, $\tilde{\gamma}(t) \overset{def}{=} (-1,\mathfrak{C}(\overset{\circ}{\gamma})(t),1)$, $t \in [0,1]$, differ by a power Δ_3^{2N} of the full twist Δ_3^2. The number N can be interpreted as the linking number of the first and the third strands of the geometric braid $\overset{\circ}{\gamma}(t)$, $t \in [0,1]$. This linking number is obtained as follows. Discard the second strand. The resulting braid equals σ^{2N} where N is the mentioned linking number. For the geometric braid $\tilde{\gamma}$ the linking number of the first and third strand is zero. It follows that $\mathfrak{C}_*$ is surjective and its kernel equals $\langle\Delta_3^2\rangle$, the subgroup of $\mathcal{B}_3$ generated by the full twist. We obtain an isomorphism from $\mathcal{PB}_3\diagup\langle\Delta_3^2\rangle$ to the fundamental group $\pi_1 = \pi_1(\mathbb{C}\setminus\{-1,1),0)$ of the twice punctured complex plane with base point 0. $\square$

Notice that the mapping $\mathfrak{C}$ maps free homotopy classes of loops in $C_3(\mathbb{C})\diagup\mathcal{S}_3$ corresponding to conjugacy classes of pure braids to free homotopy classes of loops in $\mathbb{C}\setminus\{-1,1\}$ corresponding to conjugacy classes of elements of the fundamental group of $\mathbb{C}\setminus\{-1,1\}$.

Further, for an arc $\overset{\circ}{\gamma}$ in $C_3(\mathbb{C})$ with both endpoints in $C_3(\mathbb{R})^0$ the arc $\mathfrak{C}(\overset{\circ}{\gamma})$ in $\mathbb{C}\setminus\{-1,1\}$ has endpoints in $(-1,1)$. This gives an isomorphism from $\pi_1(C_3(\mathbb{C}),C_3(\mathbb{R})^0)$ to $\pi_1^{tr} \overset{def}{=} \pi_1(\mathbb{C}\setminus\{-1,1\},(-1,1))$ which we denote again by $\mathfrak{C}_*$ if no confusion arises. In some cases it may be more convenient to consider the homomorphism obtained by using instead of $C_3(\mathbb{R})^0$ another connected component of $C_3(\mathbb{R})$.

If $\overset{\circ}{\gamma}$ has endpoints on the perpendicular bisector space $\overset{\circ}{\mathcal{E}}{}^3{}_{pb}$ then $\mathfrak{C}(\overset{\circ}{\gamma})$ has endpoints on the imaginary axis $i\,\mathbb{R}$. We obtain a homomorphism, denoted again by $\mathfrak{C}_*$ if no confusion arises, from the relative fundamental group $\pi_1(\,C_3(\mathbb{C})\,,\ \overset{\circ}{\mathcal{E}}{}^3{}_{pb}\,)$ to the relative fundamental group $\pi_1^{pb} \overset{def}{=} \pi_1(\,\mathbb{C}\setminus\{-1,1\}\,,\ i\,\mathbb{R}\,)$. In each case the kernel of the homomorphism is the subgroup $\langle\Delta_3^2\rangle$ of $\mathcal{PB}_3$ generated by Δ_3^2. Notice that π_1, π_1^{tr} and π_1^{pb} are isomorphic.

We will also consider the relative fundamental groups ${}^{pb}\pi_1^{tr}$ and ${}^{tr}\pi_1^{pb}$ with mixed horizontal boundary values (and the respective relative fundamental groups of the configuration space). Here the elements of ${}^{pb}\pi_1^{tr}$ are homotopy classes ${}_{i\mathbb{R}}\mathsf{h}_{(-1,1)}$ and the elements of ${}^{tr}\pi_1^{pb}$ are homotopy classes ${}_{(-1,1)}\mathsf{h}_{i\mathbb{R}}$ in the space $X = \mathbb{C}\setminus\{-1,1\}$.

The following theorem holds for mixed boundary values. Note that for mixed boundary the estimate of the extremal length holds always while for pb or tr boundary values there are exceptional cases.

Theorem 1′. *For all $w \in \pi_1$ the following inequalities hold*

$$\frac{1}{2\pi}\,\mathcal{L}(w) \le \Lambda({}_{tr}(w)_{pb}) \le 300 \cdot \mathcal{L}(w), \tag{5}$$

$$\frac{1}{2\pi}\,\mathcal{L}(w) \le \Lambda({}_{pb}(w)_{tr}) \le 300 \cdot \mathcal{L}(w). \tag{6}$$

Lemma 1 and Lemma 3 together with the arguments above imply the following lemma.

Lemma 4. *The invariants Λ_{tr} Λ_{pb} descend to invariants of the quotient $\mathcal{PB}_3 \diagup \langle \Delta_3^2 \rangle$. For any pure braid $b \in \mathcal{PB}_3$ the following equalities hold*

$$\Lambda_{tr}(b) = \Lambda(\mathfrak{C}_*(b_{tr}))\,,$$
$$\Lambda_{pb}(b) = \Lambda(\mathfrak{C}_*(b_{pb}))\,,$$
$$\Lambda(\hat{b}) = \Lambda(\mathfrak{C}_*(\hat{b}))\,.$$

For the proof of the theorems we will work with the fundamental group of the twice punctured complex plane rather than with the fundamental group of the symmetrized configuration space. We will lift mappings with image in $\mathbb{C} \setminus \{-1, 1\}$ to mappings with image in a suitable covering of the twice punctured plane. It will be convenient to use instead of the universal covering of $\mathbb{C} \setminus \{-1, 1\}$ two different coverings of $\mathbb{C} \setminus \{-1, 1\}$ by $\mathbb{C} \setminus i\mathbb{Z}$. Though the universal covering is well studied, factorizing the universal covering through two different coverings by $\mathbb{C} \setminus i\mathbb{Z}$ has the advantage to make the contribution of the syllables to the extremal length transparent. This is especially crucial for the proof of the upper bound.

To obtain the first covering $\mathbb{C} \setminus i\mathbb{Z} \to \mathbb{C} \setminus \{-1, 1\}$ we take the universal covering of the twice punctured Riemann sphere $\mathbb{P}^1 \setminus \{-1, 1\}$ (the logarithmic covering) and remove all preimages of ∞ under the covering map. Geometrically the logarithmic covering of $\mathbb{P}^1 \setminus \{-1, 1\}$ can be described as follows. Take copies of $\mathbb{P}^1 \setminus [-1, 1]$ labeled by the set $\mathbb{Z}$ of integer numbers. Attach to each copy two copies of $(-1, 1)$, the $+$-edge (the accumulation set of points of the upper half-plane) and the $-$-edge (the accumulation set of points of the lower half-plane). For each $k \in \mathbb{Z}$ we glue the $+$-edge of the k-th copy to the $-$-edge of the $k+1$-st copy (using the identity mapping on $(-1, 1)$ to identify points on different edges). Denote by $U_{\log}$ the set obtained from the described covering by removing all preimages of ∞.

Denote by $\alpha_j(t), t \in [0, 1]$, $j = 1, 2$, curves in $\mathbb{C} \setminus \{-1, 1\}$ which represent the generators $a_j, j = 1, 2$, of the fundamental group $\pi_1(\mathbb{C} \setminus \{-1, 1\}, 0\,\}$ and have the following properties. The initial point and the terminating point of each of the curves are the only points of the curve on the interval $(-1, 1)$ and are also the only points of the curve on the imaginary axis. The following proposition holds.

Proposition 3. *The set $U_{\log}$ is conformally equivalent to $\mathbb{C} \setminus i\,\mathbb{Z}$. The mapping $f_1 \circ f_2$, $f_2(z) = \frac{e^{\pi z}-1}{e^{\pi z}+1}$, $z \in \mathbb{C} \setminus i\,\mathbb{Z}$, $f_1(w) = \frac{1}{2}(w + \frac{1}{w})$, $w \in \mathbb{C} \setminus \{-1, 0, 1\}$, is a covering map from $\mathbb{C} \setminus i\,\mathbb{Z}$ to $\mathbb{C} \setminus \{-1, 1\}$.*

For each $k \in \mathbb{Z}$ the lift of α_1 with initial point $\frac{-i}{2} + ik$ is a curve which joins $\frac{-i}{2} + ik$ with $\frac{-i}{2} + i(k+1)$ and is contained in the closed left half-plane. The only points on the imaginary axis are the endpoints.

The lift of α_2 with initial point $\frac{-i}{2} + ik$ is a curve which joins $\frac{-i}{2} + ik$ with $\frac{-i}{2} + i(k-1)$ and is contained in the closed right half-plane. The only points on the imaginary axis are the endpoints.

Figure 1 shows the curves α_1 and α_2 which represent the generators of the fundamental group $\pi_1(\mathbb{C} \setminus \{-1,1\}, 0)$ and their lifts under the covering maps f_1 and $f_2 \circ f_1$. For $j = 1, 2$ the curves α_j' and α_j'' are the two lifts of α_j under the double covering $f_1 : \mathbb{C} \setminus \{-1,0,1\} \to \mathbb{C} \setminus \{-1,1\}$. The curve $\hat{\alpha}_1'$ is the lift of α_1' under the mapping f_2 with initial point $\frac{-i}{2}$, the curve $\hat{\alpha}_1''$ is the lift of α_1'' under the mapping f_2 with initial point $\frac{i}{2}$, the curve $\hat{\alpha}_2'$ lifts α_2' and has initial point $\frac{i}{2}$, and the curve $\hat{\alpha}_2''$ lifts α_2'' and has initial point $-\frac{i}{2}$.

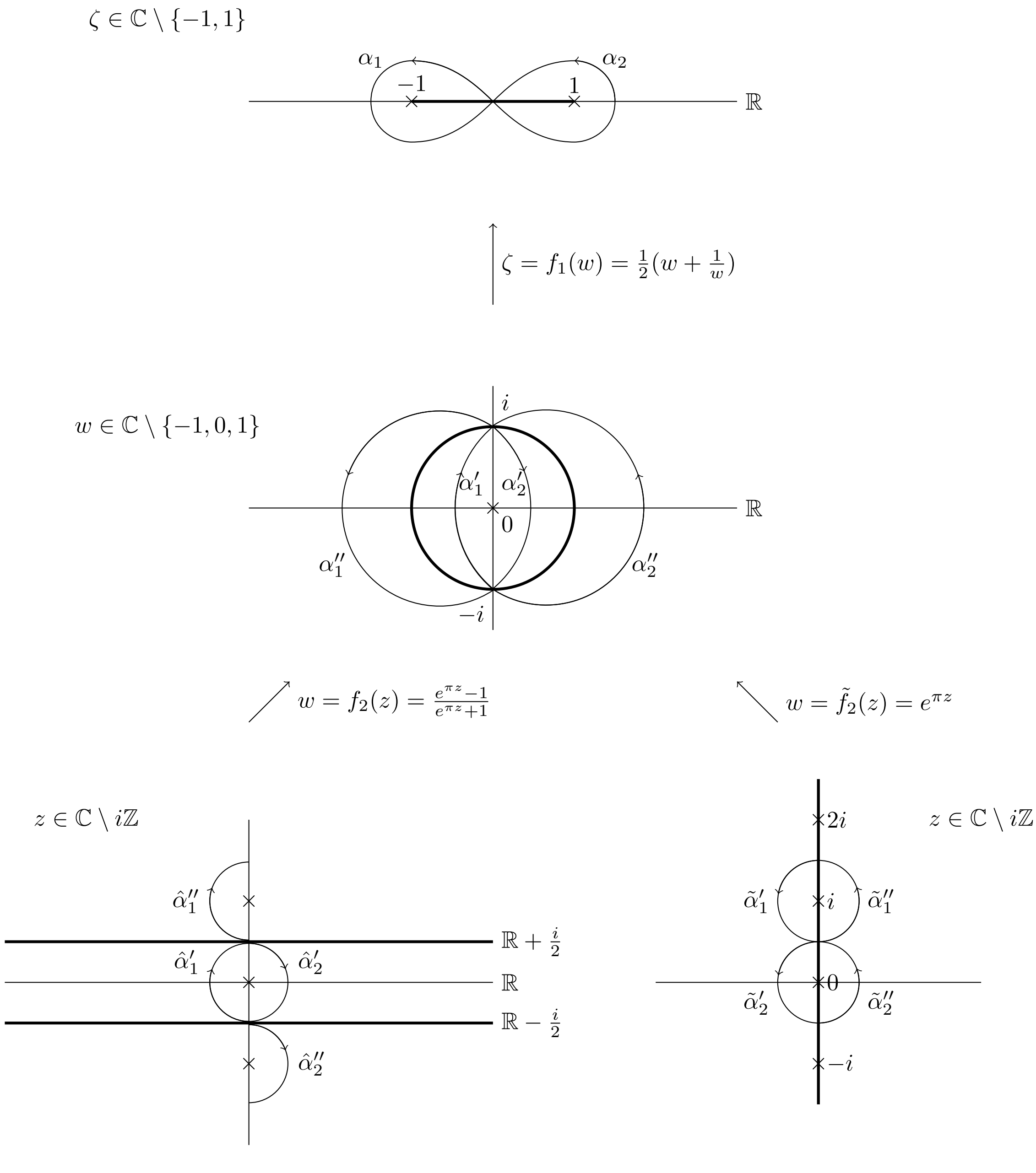

Figure 1

Proof. The mapping f_1 is the restriction of the Zhukovsky function to $\mathbb{C} \setminus \{-1,0,1\}$. The Zhukovski function defines a double branched covering of the Riemann sphere $\mathbb{P}^1$ with branch locus $\{-1,1\}$. In particular, the Zhukovsky function provides a conformal mapping from the unit disc $\mathbb{D}$ onto $\mathbb{P}^1 \setminus [-1,1]$. It maps -1 to -1, 1 to 1 and 0 to ∞. The upper half-circle is mapped onto the $-$-edge, the upper half-disc is mapped onto the lower half-plane, the lower half-circle is mapped onto the $+$-edge and the lower half-disc is mapped onto the upper half-plane. Similarly, it provides a conformal mapping of the exterior of the closed unit disc onto $\mathbb{P}^1 \setminus [-1,1]$ which preserves the upper half-plane and also preserves the lower half-plane.

The mapping f_2 extends to an infinite covering of $\mathbb{P}^1 \setminus \{-1,1\}$ by $\mathbb{C}$. By an abuse of notation we denote this extension also by f_2. (The extension of) f_2 provides a conformal mapping f_2^0 from the strip $\{z \in \mathbb{C} : -\frac{1}{2} < \mathrm{Im} z < \frac{1}{2}\}$ onto the unit disc, which maps the real axis onto the segment $(-1,1)$, such that $\lim_{x\in\mathbb{R},x\to-\infty} = -1$, $\lim_{x\in\mathbb{R},x\to+\infty} = 1$. Further, f_2 maps $\frac{i}{2}$ to i, $-\frac{i}{2}$ to $-i$, and 0 to 0. The line $\{z \in \mathbb{C} : \mathrm{Im} z = \frac{1}{2}\}$ is mapped onto the upper half-circle, the upper half-strip $\{z \in \mathbb{C} : 0 < \mathrm{Im} z < \frac{1}{2}\}$ is mapped onto the upper half-disc, the line $\{z \in \mathbb{C} : \mathrm{Im} z = -\frac{1}{2}\}$ is mapped onto the lower half-circle and the lower half-strip is mapped onto the lower half-disc.

We obtain the following properties for the composition $f = f_1 \circ f_2 : \mathbb{C} \setminus i\mathbb{Z} \to \mathbb{C} \setminus \{-1,1\}$. The mapping f maps the punctured strip $\{z \in \mathbb{C} \setminus \{0\} : -\frac{1}{2} < \mathrm{Im} z < \frac{1}{2}\}$ onto $\mathbb{C} \setminus [-1,1]$, so that the upper half-strip $\{z \in \mathbb{C} : 0 < \mathrm{Im} z < \frac{1}{2}\}$ is mapped onto the lower half-plane and the lower half-strip is mapped onto the upper half-plane. Since both mappings f_1 and f_2 map points in the left half-plane to points in the left half-plane, the mapping f maps the left half-strip $\{z \in \mathbb{C} : -\frac{1}{2} < \mathrm{Im} z < \frac{1}{2}, \mathrm{Re} z < 0\}$ onto the subset $\{z \in \mathbb{C} : \mathrm{Re} z < 0\} \setminus [-1,0]$ of the left half-plane, respectively, it maps the right half-strip $\{z \in \mathbb{C} : -\frac{1}{2} \le \mathrm{Im} z \le \frac{1}{2}, \mathrm{Re} z > 0\}$ onto the subset $\{z \in \mathbb{C} : \mathrm{Re} z > 0\} \setminus (0,1]$ of right half-plane. This implies that the lift of the curve α_1 under $f_1 \circ f_2$ with initial point $\frac{-i}{2}$ is a curve which joins $\frac{-i}{2}$ with $\frac{+i}{2}$ and is contained in the intersection of the closed left half-plane with the strip $\{z \in \mathbb{C} : -\frac{1}{2} \le \mathrm{Im} z \le \frac{1}{2}, \}$. The only points on the imaginary axis are the endpoints. Respectively, the lift of the curve α_2 with initial point $\frac{+i}{2}$ is a curve which joins $\frac{i}{2}$ with $\frac{-i}{2}$ and is contained in the right half-strip $\{z \in \mathbb{C} : -\frac{1}{2} \le \mathrm{Im} z \le \frac{1}{2}, \mathrm{Re} z \ge 0\}$. Again, the only points on the imaginary axis are the endpoints.

The mapping $f = f_1 \circ f_2$ has period i. Indeed, f_2 has period $2i$, $f_2(z+i) = \frac{1}{f_2(z)}$, and $f_1(\frac{1}{w}) = f_1(w)$.

We proved the statement concerning the lift of curves under $f_1 \circ f_2$.

To see that $U_{\log}$ is conformally equivalent to $\mathbb{C} \setminus i\,\mathbb{Z}$ we identify the set $\mathbb{C} \setminus [-1,1]$ with the sheet of $U_{\log}$ labeled by 0. The conformal map $f \mid \{z \in \mathbb{C} \setminus \{0\} : -\frac{1}{2} < \mathrm{Im} z < \frac{1}{2}\}$ (whose image is $\mathbb{C} \setminus [-1,1]$) extends by Schwarz's reflection principle through the line $\{z \in \mathbb{C} : \mathrm{Im} z = \frac{1}{2}\}$ which is mapped onto the $-$-edge of $\mathbb{C} \setminus [-1,1]$. The reflected mapping maps $\{z \in \mathbb{C} \setminus \{i\} : \frac{1}{2} < \mathrm{Im} z < \frac{3}{2}\}$ conformally onto $\mathbb{C} \setminus [-1,1]$). We identify the image of the punctured strip $\{z \in \mathbb{C} \setminus \{i\} : \frac{1}{2} < \mathrm{Im} z < \frac{3}{2}\}$ with the sheet of $U_{\log}$ labeled by -1.

Induction on reflection through the lines $\{z \in \mathbb{C} : \mathrm{Im} z = \frac{1}{2} + j\}$, $j \in \mathbb{Z}$, gives the conformal mapping from $C \setminus i\mathbb{Z}$ onto $U_{\log}$. □

The second covering is given by the map $f_1 \circ \tilde{f}_2 : \mathbb{C} \setminus i\mathbb{Z} \to \mathbb{C} \setminus \{-1,1\}$, where f_1 is as before and $\tilde{f}_2$ is the exponential map, $\tilde{f}_2(z) = e^{\pi z}$. Recall that each curve α_j, $j = 1,2$, has two lifts α_j' and α_j'' under f_1. For an illustration of the following proposition see the right part of Figure 1.

Proposition 4. *The mapping $\tilde{f}_2$ maps $\mathbb{C} \setminus i\mathbb{Z}$ to $\mathbb{C} \setminus \{-1, 0, 1\}$, and f_1 maps the latter set to $\mathbb{C} \setminus \{-1, 1\}$. The lifts $\tilde{\alpha}'_j$ and $\tilde{\alpha}''_j$, $j = 1, 2$, of α'_j and α''_j under $\tilde{f}_2$ have the following properties. The lifts $\tilde{\alpha}'_j$, $j = 1, 2$, are contained in the closed left half-plane and are directed downwards (i.e in the direction of decreasing y), the lifts $\tilde{\alpha}''_j$, $j = 1, 2$, are contained in the closed right half-plane and directed upwards. The initial point of $\tilde{\alpha}'_1$ is $i + \frac{1}{2}i$, the initial point of $\tilde{\alpha}''_1$ and $\tilde{\alpha}'_2$ is $\frac{1}{2}i$, the initial point of $\tilde{\alpha}''_2$ is $-\frac{1}{2}i$. All other lifts are obtained by translation by an integral multiple of $2i$.*

The straightforward proof is left to the reader.

Consider the curve α_1^n, $n \in \mathbb{Z} \setminus \{0\}$. It runs n times along the curve α_1 if $n > 0$, and $|n|$ times along the curve α_1 with inverted orientation if $n < 0$. It is homotopic in $\mathbb{C} \setminus \{-1, 1\}$ with base point 0 to a curve whose interior (i.e. the complement of its endpoints) is contained in the open left half-plane. We call such a curve a standard representative of a_1^n with respect to the covering map $f_1 \circ f_2$. In the same way we define standard representatives of a_2^n. For each $k \in \mathbb{Z}$ the curve α_1^n lifts under $f_1 \circ f_2$ to a curve with initial point $\frac{-i}{2} + ik$ and terminating point $\frac{-i}{2} + ik + in$ which is contained in the closed left half-plane and omits the points in $i\mathbb{Z}$. Respectively, α_2^n, $n \in \mathbb{Z} \setminus \{0\}$, lifts under $f_1 \circ f_2$ to a curve with initial point $\frac{+i}{2} + ik$ and terminating point $\frac{+i}{2} + ik - in$ which is contained in the closed right half-plane and omits the points in $i\mathbb{Z}$. The mentioned lifts are homotopic through curves in $\mathbb{C} \setminus i\mathbb{Z}$ with endpoints on $i\,\mathbb{R} \setminus i\mathbb{Z}$ to curves with interior contained in the open (right, respectively, left) half-plane. Standard representatives of a_1^n and a_2^n lift to such curves.

Definition 5. *A simple curve in $\mathbb{C} \setminus i\mathbb{Z}$ with endpoints on different connected components of $i\mathbb{R} \setminus i\mathbb{Z}$ is called an elementary slalom curve if its interior is contained in one of the open half-planes $\{z \in \mathbb{C} : Rez > 0\}$ or $\{z \in \mathbb{C} : Rez < 0\}$.*

A curve in $\mathbb{C} \setminus i\mathbb{Z}$ is called an elementary half slalom curve if one of the endpoints is contained in a horizontal line $\{z \in \mathbb{C} : Imz = k + \frac{1}{2}\}$ for an integer k and the union of the curve with its (suitably oriented) reflection in the line $\{z \in \mathbb{C} : Imz = k + \frac{1}{2}\}$ is an elementary slalom curve.

A slalom curve in $\mathbb{C} \setminus i\mathbb{Z}$ is a curve which can be divided into a finite number of elementary slalom curves so that consecutive elementary slalom curves are contained in different half-planes.

A curve which is homotopic to a slalom curve (elementary slalom curve, respectively) in $\mathbb{C} \setminus i\mathbb{Z}$ through curves with endpoints in $i\mathbb{R} \setminus i\mathbb{Z}$ is called a homotopy slalom curve (elementary homotopy slalom curve, respectively).

A curve which is homotopic to an elementary half-slalom curve in $\mathbb{C} \setminus i\mathbb{Z}$ through curves with one endpoint in $i\mathbb{R} \setminus i\mathbb{Z}$ and the other endpoint on the line $\{z \in \mathbb{C} : Imz = k + \frac{1}{2}\}$ for an integer k is called a homotopy half-slalom curve.

We call an elementary slalom curve non-trivial if its endpoints are contained in intervals $(ik, i(k+1))$ and $(i\ell, i(\ell+1))$ with $|k - \ell| \geq 2$. Note that the union of an elementary half-slalom curve with its reflection in the horizontal line that contains one endpoint is always a non-trivial elementary slalom curve (i.e. an elementary half-slalom curve is "half" of a non-trivial elementary slalom curve). We saw that the lifts under $f_1 \circ f_2$ of representatives of terms $_{pb}(a_j^n)_{pb} \in \pi_1^{pb}$ with $|n| \geq 2$ are non-trivial elementary homotopy slalom curves.

The lifts of representatives of elements of π_1^{pb} under $f_1 \circ \tilde{f}_2$ look different. The representatives of a_j^n, $|n| \geq 1$, lift to curves which make $|n|$ half-turns around a point in

$i\mathbb{Z}$ (positive half-turns if $n > 0$, and negative half-turns if $n < 0$). Hence, each such representative lifts under $f_1 \circ \tilde{f}_2$ to the composition of $|n|$ trivial elementary slalom curves.

Take any syllable of from (2), i.e. any maximal sequence of at least two consecutive terms of the word which enter with equal power being either 1 or -1. Recall that d denotes the sum of the absolute values of the powers of the terms of the syllable. There is a representing curve that makes d half-turns around the interval $[-1, 1]$ (positive half-turns, if the exponents of terms in the syllable are 1, and negative half-turns otherwise). The lift under $f_1 \circ \tilde{f}_2$ of this representative is a non-trivial elementary slalom curve.

We will call homotopy classes of homotopy slalom curves for short slalom classes.

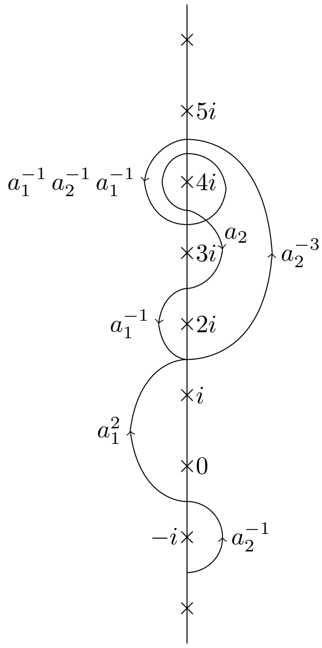

Figure 2a

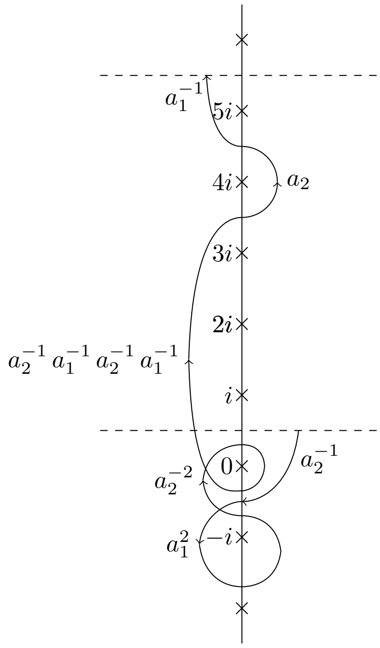

Figure 2b

Figure 2

Figure 2 shows two slalom curves. The curve in Figure 2a is a lift under $f_1 \circ f_2$ of a curve in $\mathbb{C} \setminus \{-1, 1\}$ with initial point and terminating point equal to $0 \in i\mathbb{R}$ representing the word $a_2^{-1}\, a_1^2\, a_2^{-3}\, a_1^{-1}\, a_2^{-1}\, a_1^{-1}\, a_2\, a_1^{-1}$ in the relative fundamental group $\pi_1(\mathbb{C} \setminus \{-1, 1\}, i\mathbb{R})$. The curve in figure 2b is a lift under $f_1 \circ \tilde{f}_2$ of a curve with initial and terminating point in $i\mathbb{R} \setminus \{0\}$ representing the same word. For each elementary piece of the slalom curves in Figure 2 we indicate the element of π_1 which lifts under the considered mapping to the respective elementary slalom class.

Non-trivial elementary slalom classes and elementary half-slalom classes have positive extremal length (in the sense of Definition 4) which can be effectively estimated from above and from below. In this sense homotopy classes of curves in $\mathbb{C} \setminus \{-1, 1\}$ whose lifts under $f_1 \circ f_2$ or $f_1 \circ \tilde{f}_2$ are non-trivial slalom classes or half-slalom classes serve as building blocks. We obtained the following fact. For an element of a relative fundamental group of the twice punctured complex plane the pieces representing syllables of form (1) with *pb* boundary values lift to non-trivial elementary slalom classes under $f_1 \circ f_2$, while the pieces representing syllables of form (2) with tr boundary values lift to non-trivial elementary slalom classes under $f_1 \circ \tilde{f}_2$. For singletons we may select pieces of representing curves which lift to non-trivial elementary homotopy half-slalom curves. These facts will be used to obtain a lower bound for the extremal length of elements of the relative fundamental groups. Recall that the extremal length of a homotopy class of curves is equal to the extremal length of the class of their lifts under a holomorphic covering.

The method to obtain the upper bound is roughly to patch together in a quasiconformal way the holomorphic mappings of rectangles representing syllables and to perturb the obtained quasiconformal mapping to a holomorphic mapping.

We conclude this section with relating the two explicitly given coverings of $\mathbb{C} \setminus \{-1, 1\}$ by $\mathbb{C} \setminus i\mathbb{Z}$ to the universal covering of the twice punctured plane. Denote by U the universal covering of $\mathbb{C} \setminus i\mathbb{Z}$. Geometrically it is obtained in the following way. Consider the left half-plane $\mathbb{C}_\ell$ and call it the Riemann surface of generation 0. The first step is the following. For each integer k we take a copy of the right half-plane $\mathbb{C}_r$ and glue it to $\mathbb{C}_\ell$ along the interval $(ki, (k+1)i)$ (using the identity mapping for gluing). We obtain a Riemann surface with a natural projection to $\mathbb{C} \setminus i\mathbb{Z}$ called the Riemann surface of first generation. At the second step we consider the Riemann surface of first generation and proceed similarly by gluing the left half-plane along each copy of intervals $(ki, (k+1)i)$ which is an end of the Riemann surface of first generation. By induction we obtain the universal covering of $\mathbb{C} \setminus i\mathbb{Z}$.

Denote by $\widetilde{\mathbb{C}}_\ell$ the lift of the left half-plane to the first sheet of U over $\mathbb{C}_\ell$. Let $\mathbb{C}_\ell^{Cl}$ be the closure of $\mathbb{C}_\ell$ in $\mathbb{C} \setminus i\mathbb{Z}$, let $\widetilde{\mathbb{C}}_\ell^{Cl}$ be the closure of $\widetilde{\mathbb{C}}_\ell$ in the universal covering of $\mathbb{C} \setminus i\mathbb{Z}$ and let $\widetilde{(ki, (k+1)i)} \subset \widetilde{\mathbb{C}}_\ell^{Cl}$ be the lift of the intervals $(ki, (k+1)i)$.

Denote by $\mathfrak{D}_k^\ell$ the half-disc $\{z \in \mathbb{C}_\ell : |z - i(k + \frac{1}{2})| < \frac{1}{2}\}$ with diameter $(ik, i(k+1))$ which is contained in the left half-plane and by ρ_k the respective open half-circle $\{z \in \mathbb{C}_\ell : |z - i(k + \frac{1}{2})| = \frac{1}{2}\}$. We call the ρ_k half-circles of generation 0.

Lemma 5. . *There is a conformal mapping $\varphi : U \to \mathbb{C}_\ell$ which maps $\widetilde{\mathbb{C}}_\ell$ onto $\mathbb{C}_\ell \setminus \bigcup_{k=-\infty}^{\infty} \overline{\mathfrak{D}_k^\ell}$ so that $\widetilde{(ki, (k+1)i)}$ is mapped onto ρ_k for each k.*

Proof. Consider the half-strip $\mathfrak{H}_0 \overset{def}{=} \{z \in \mathbb{C}_\ell : 0 < \mathrm{Im} z < 1\}$. By a theorem of Caratheodory ([4],Chapter II.3, Theorem 4 and Theorem 4′, and also [8] , Theorem 2.24 and Theorem 2.25) each conformal mapping which maps its lift $\widetilde{\mathfrak{H}}_0$ onto the set $\mathfrak{H}_0 \setminus \overline{\mathfrak{D}_0^\ell}$, extends continuously to a homeomorphism between closures. Let φ_0 be the conformal mapping whose extension to the boundary maps the point $\tilde{0}$ over 0 to 0, the point $\tilde{1}$ to 1 and the point $\tilde{\infty}$ (considered as prime end of $\widetilde{\mathfrak{H}}_0$) to ∞. The extension of the conformal map to the boundary maps the lift $\widetilde{\{z \in \mathbb{C}_\ell : \mathrm{Im} z = 1\}}$ onto $\{z \in \mathbb{C}_\ell : \mathrm{Im} z = 1\}$, it maps $\widetilde{\{z \in \mathbb{C}_\ell : \mathrm{Im} z = 0\}}$ onto $\{z \in \mathbb{C}_\ell : \mathrm{Im} z = 0\}$, and maps $\widetilde{(0, i)}$ onto ρ_0. By induction we extend φ_0 by Schwarz's reflection principle across the half-lines $\widetilde{\{z \in \mathbb{C}_\ell : \mathrm{Im} z = k\}}$. We obtain a conformal mapping of $\widetilde{\mathbb{C}}_\ell$ onto $\mathbb{C}_\ell \setminus \bigcup_{k=-\infty}^{\infty} \overline{\mathfrak{D}_k^\ell}$, denoted again by φ_0, whose extension to $\widetilde{\mathbb{C}}_\ell^{Cl}$ maps the segment $\widetilde{(ik, i(k+1))}$ onto the half-circle ρ_k for each integer number k .

Schwarz's reflection principle across each segment $\widetilde{(ik, i(k+1))}$ provides extension of the conformal mapping φ_0 to the Riemann surface of first generation. Note that for each

k_0 the image of the half-circles ρ_k, $k \neq k_0$, under reflection of $\mathbb{C}_\ell \setminus \bigcup_{k=-\infty}^{\infty} \overline{\mathfrak{D}_k^\ell}$ across ρ_{k_0} are half-circles with diameter on the imaginary axis. We call them half-circles of the first generation. We obtained a conformal mapping of the Riemann surface of first generation to the unbounded connected component of the left half-plane with the half-circles of first generation removed.

Apply the reflection principle by induction to the Riemann surface of generation n and all copies of intervals $(ik, i(k+1))$ in its "boundary". We obtain the Riemann surface of generation $n+1$, half-circles of generation $n+1$ and a conformal mapping of this Riemann surface onto the unbounded connected component of the left half-plane with the half-circles of generation $n+1$ removed.

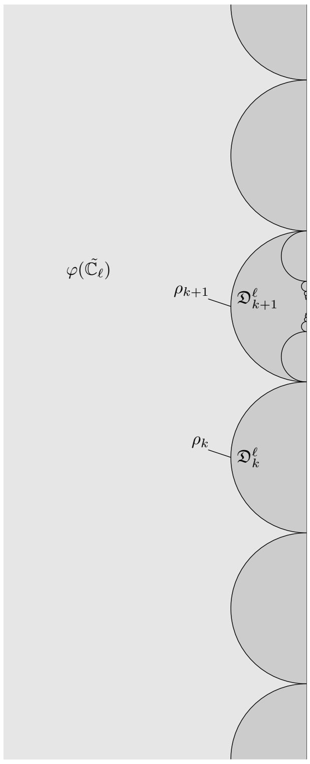

Figure 3

The supremum of the diameters of the half-circles of generation n does not exceed 2^{-n}. Indeed, the half-circles of generation n are obtained as follows. Take a half-circle ρ of generation $n-1$ and reflect all other half-circles of generation $n-1$ across it. Each half-circle of generation $n-1$ different from ρ has diameter on one side of the diameter of ρ. Hence, the image of each half-circle of generation $n-1$ different from ρ under reflection across ρ is contained in a quarter disc, hence has diameter not exceeding half of the diameter of ρ. The statement is obtained by induction. By the statement we obtain in the limit a conformal mapping φ from U onto $\mathbb{C}_\ell$. □

3. Building blocks and lower bound of their extremal length

Consider an elementary slalom curve in the left half-plane with initial point in the interval $(ik, i(k+1))$ and with terminating point in the interval $(ij, i(j+1))$. Suppose that $|k-j| \geq 2$, i.e. the slalom curve is non-trivial. After a translation by a (half-integer or integer) multiple of i we obtain a curve that has endpoints in the intervals

$(-i(M+1),-iM)$ and $(iM,i(M+1))$ with $M=\frac{|k-j|-1}{2}\geq\frac{1}{2}$. Assume that $j-k\geq 2$, i.e. the curve is oriented "upwards". For this case we denote the curve by $\gamma_{\ell,M}$. Notice that with the chosen value of M the curve $\gamma_{\ell,M}$ is not an elementary slalom curve for even $|k-j|$, but the curve $\gamma_{\ell,M}+iM$ is always an elementary slalom curve. It will be convenient to work with the normalized curve $\gamma_{\ell,M}$ for estimating the extremal length.

The general case of elementary slalom curves in either the left or the right half-plane with initial point in the interval $(ik,i(k+1))$ and with terminating point in the interval $(ij,i(j+1))$ and either $k>j$ or $k<j$ is treated in the same way and leads to the same estimates related to the lower bound for the extremal length.

Let M be as chosen above. Let $\gamma_{\ell,M}$ be a curve with interior contained in the left half-plane $\mathbb{C}_\ell$ with initial point in the interval $(-i(M+1),-iM)$ and terminating point in $(iM,i(M+1)$. Consider the class $\gamma^*_{\ell,M}$ of curves which are homotopic to $\gamma_{\ell,M}$ through curves in $\mathbb{C}\setminus i(\mathbb{Z}-M)$ with initial point in $(-i(M+1),-iM)$ and terminating point in $(iM,i(M+1))$. The class $\gamma^*_{\ell,M}$ is represented by the conformal mapping of an open rectangle onto the open left half-plane with the following property. The conformal mapping admits a continuous extension to the closed rectangle which maps the open lower side to $(-i(M+1),-iM)$, and the open upper side to $(iM,i(M+1))$, respectively. Hence, the extremal length of the class $\gamma^*_{\ell,M}$ is bounded from above by the extremal length of this rectangle.

Respectively, let $\gamma_{r,M}$ be a curve with interior contained in the right half-plane $\mathbb{C}_r$ with initial point in the interval $(-i(M+1),-iM)$ and terminating point in $(iM,i(M+1)$. Consider the class $\gamma^*_{r,M}$ of curves which are homotopic to $\gamma_{r,M}$ through curves in $\mathbb{C}\setminus i(\mathbb{Z}-M)$ with initial point in $(-i(M+1),-iM)$ and terminating point in $(iM,i(M+1))$.

The conformal mappings representing $\gamma^*_{\ell,M}$ and $\gamma^*_{r,M}$ are related to elliptic integrals. With a suitable normalization the inverse of the conformal mapping representing $\gamma^*_{\ell,M}$ is equal to the elliptic integral

$$\begin{aligned}\mathcal{F}_M(z)&=\int_0^z\frac{d\zeta}{\sqrt{(\zeta^2-(iM)^2)(\zeta^2-(i(M+1))^2)}}\\&=\frac{i}{M+1}\int_0^{\frac{z}{iM}}\frac{dw}{\sqrt{(1-w^2)((1-(\frac{M}{M+1})^2w^2)}},\ z\in\mathbb{C}_\ell.\end{aligned}\tag{7}$$

We use the branch of the square root which is positive on the positive real axis. The function $\mathcal{F}_M$ extends holomorphically along any path in $\mathbb{C}\setminus\{\pm i(M+1),\pm iM\}$, and extends continuously to the imaginary axis (the integral converges). The extended map maps the closed left half-plane homeomorphically to the closure of a rectangle contained in the left half-plane. The points $-i(M+1)$, $-iM$, iM and $i(M+1)$ are mapped by the extension of $\mathcal{F}_M$ to the vertices of the rectangle.

Note that

$$\frac{M+1}{i}\mathcal{F}_M(iM)=\int_0^1\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}=K(k)\tag{8}$$

with $k=\frac{M}{M+1}$ is a complete elliptic integral of first kind. Its values can be found in tables.

It is convenient to use elliptic integrals for obtaining an upper bound for the extremal length of arbitrary slalom curves by suitably normalizing and patching together pieces in a quasiconformal way. (See section 5.)

As for the lower bound, the representation of $\gamma^*_{\ell,M}$ by the inverse of (7) does not give a lower bound for the extremal length of the elementary slalom curve by two reasons. First, the representing mappings of rectangles are not supposed to be conformal, they are merely holomorphic, and secondly, their images are not required to be contained in the half-plane, but merely, they are contained in $\mathbb{C}\setminus i(\mathbb{Z}-M)$ and therefore the mappings lift to the universal covering of $\mathbb{C}\setminus i(\mathbb{Z}-M)$.

The following simple lemma deals with the difficulties.

Lemma 6. *Let R and R' be rectangles with sides parallel to the axes. Suppose S' is the vertical strip bounded by the two vertical lines which are prolongations of the vertical sides of the rectangle R'. Let $f: R \to S'$ be a holomorphic map with continuous extension to the closure that maps the two horizontal sides of R into different horizontal sides of R'. Then*

$$\lambda(R) \geq \lambda(R')\,.$$

Equality holds if and only if the mapping is a surjective conformal map from R to R'.

The following lemma will be applied to classes of curves corresponding to conjugacy classes of elements of the fundamental group.

Lemma 7. *Let A and A' be two annuli and let f be a holomorphic mapping from A into A' which induces an isomorphism on fundamental groups. Then*

$$\lambda(A) \geq \lambda(A')\,.$$

Equality holds if and only if the mapping is a conformal mapping from A onto A'.

Proof of Lemma 6. Normalize the rectangles and the mapping so that $R = \{x+iy : x \in (0,1), y \in (0,\mathsf{a})\}$ and $R' = \{x+iy : x \in (0,1), y \in (0,\mathsf{a}')\}$. Denote the continuous extension of f to the closure of R again by f. We may assume that f maps the upper side of R to the upper side of R' and the lower side of R to the lower side of R'. Put $u = \mathrm{Re} f$ and $v = \mathrm{Im} f$. Then for each $x \in (0,1)$

$$\begin{aligned} \mathsf{a}' = \int_0^1 \mathsf{a}'\,dx = \int_0^1 (v(x,a) - v(x,0))dx = \int_0^1 dx \int_0^a dy \frac{\partial}{\partial y} v(x,y) \qquad (9)\\ = \int_0^a dy \int_0^1 dx \frac{\partial}{\partial x} u(x,y) = \int_0^a dy\,(u(1,y) - u(0,y)) \leq \int_0^a 1\,dy = \mathsf{a}. \end{aligned}$$

Hence, $\mathsf{a}' \leq \mathsf{a}$. We used the Cauchy-Riemann equations.

If $\mathsf{a}' = \mathsf{a}$ then $u(1,y) - u(0,y) = 1$ for each $y \in (0,a)$. Hence, the left side of R is mapped to the left side of R' and the right side of R is mapped to the right side of R'. Since also the lower side of R is mapped to the lower side of R' and the upper side of R is mapped to the upper side of R' the image of the positively oriented boundary curve of R has index 1 with respect to any point of R' and index 0 with respect to each point in $\mathbb{C}\setminus \bar{R}$. By the argument principle $f(R) = R'$ and f takes each value in R' exactly once. Hence, f is a conformal map of R onto R'. $\square$

Proof of Lemma 7. Assume the annuli have center 0, smaller radius 1 and larger radius r and r', respectively. The set $A\setminus(0,\infty)$ is conformally equivalent to the rectangle $R = \{\xi + i\eta : \xi \in (0,\log r), \eta \in (0,2\pi)\}$. The exponential function covers the annulus A' by the strip $S' = \{\xi + i\eta : \xi \in (0,\log r'), \eta \in \mathbb{R}\}$. We obtain a holomorphic mapping $g =$

$U+iV$ from R to S' for which either $V(\xi,2\pi)=V(\xi,0)+2\pi$ or $V(\xi,2\pi)=V(\xi,0)-2\pi$ for $\xi\in(0,\log r)$. Assume without loss of generality that the first option holds. Then

$$\begin{aligned}2\pi\log r &= \int_0^{\log r}(V(\xi,2\pi)-V(\xi,0))d\xi = \int_0^{\log r}d\xi\int_0^{2\pi}d\eta\frac{\partial}{\partial\eta}V(\xi,\eta) \\ &= \int_0^{\log r}d\xi\int_0^{2\pi}d\eta\frac{\partial}{\partial\xi}U(\xi,\eta) = \int_0^{2\pi}(U(\log r,\eta)-U(0,\eta))d\eta \\ &\le \int_0^{2\pi}\log r' d\eta = 2\pi\log r'.\end{aligned} \tag{10}$$

Equality $r=r'$ holds iff f maps the bigger circle of A to the bigger circle of A' and maps the smaller circle of A to the smaller circle of A'. Since the map f induces an isomorphism of fundamental groups an application of the argument principle shows that f is a conformal mapping of A onto A'. □

Lemma 6 has the following two consequences.

Corollary 1. *The extremal length of* $\gamma^*_{\ell,M}$ *is equal to the extremal length of a rectangle which admits a conformal mapping* f *onto* $\mathbb{C}_\ell^M \overset{def}{=} \mathbb{C}_\ell\setminus\big(\{|z-i(M+\frac12)|\le\frac12\}\cup\{|z+i(M+\frac12)|\le\frac12\}\big)$, *which maps open horizontal sides to the two half-circles in the boundary of* $\mathbb{C}_\ell^M$.

Corollary 2. *The extremal length of an element of each of the relative fundamental groups* π_1^{pb}, π_1^{tr}, ${}^{tr}\pi_1^{pb}$, *and* ${}^{pb}\pi_1^{tr}$ *is realized on a locally conformal mapping of a rectangle representing the element. The extremal mapping extends locally conformally across the open horizontal sides of the rectangle. The extremal length of a conjugacy class of elements of the fundamental group of the twice punctured plane is realized on a locally conformal mapping of an annulus into the twice punctured plane.*

Before proving the two corollaries we recall Ahlors' definition of extremal length of a family of curves in the complex plane.

Let Γ be a family each member of which consists of the union of no more than countably many (connected) locally rectifiable curves in the complex plane. (We do not require that this union reparametrizes to a single (connected) curve.) In this context we will call also the elements of Γ "curves". Ahlfors defined the extremal length of the family Γ as follows. For a non-negative measurable function ϱ in the complex plane he defines $A(\varrho)=\iint\varrho^2$. For an element in $\gamma\in\Gamma$ and such a function ϱ he puts $L_\gamma(\varrho)=\int_\gamma\varrho|dz|$, if ϱ is measurable on γ with respect to arc length and $L(\varrho)=\infty$ otherwise. Put $L(\varrho)=\inf_{\gamma\in\Gamma}L_\gamma(\varrho)$. The extremal length of the family Γ is the following value

$$\lambda(\Gamma)=\sup_\varrho\frac{L(\varrho)^2}{A(\varrho)},$$

where the supremum is taken over all non-negative measurable functions ϱ for which $A(\varrho)$ is finite and does not vanish.

It is not hard to see from this definition that the extremal length is invariant under conformal mappings ((Theorem 3 in [1]). Further, a small computation shows that the previously mentioned definition of the extremal length of a rectangle is equal to the extremal length of the family of curves contained in the rectangle which join the opposite horizontal sides (see Example 1 in [1]). Similarly, the extremal length of an annulus is equal to the extremal length of the family of curves which are contained in the annulus and represent the conjugacy class of the positively oriented generator of the fundamental group of the annulus.

Further, by Corollary 1 the extremal length of $\gamma^*_{\ell,M}$ in the sense of Definition 4 is equal to the extremal length of the family of curves contained in the half-plane with two half-discs removed and joining the two half-circles that are contained in the boundary of the half-discs. We have chosen the general form of Definition 4 because it applies to classes of curves in complex manifolds of arbitrary dimension. Recall that the family $\gamma^*_{\ell,M}$ was obtained from a family in a 3-dimensional manifold by holomorphic mappings and lifting. Notice that the curves in the family $\gamma^*_{\ell,M}$ are not required to be contained in $\mathbb{C}^M_\ell$, the half-plane with two half-discs removed.

For later use we formulate two theorems of Ahlfors.

For two families Γ_1 and Γ_2 as above the following relation is introduced by Ahlfors: $\Gamma_1 < \Gamma_2$ if each "curve" $\gamma_2 \in \Gamma_2$ contains a "curve" $\gamma_1 \in \Gamma_1$.

Ahlfors defines the sum $\Gamma_1+\Gamma_2$ of two such families as follows. Each element of $\Gamma_1+\Gamma_2$ is the union of a "curve" $\gamma_1 \in \Gamma_1$ and a "curve" $\gamma_2 \in \Gamma_2$. The set $\Gamma_1 + \Gamma_2$ contains all possible such unions.

The following theorems were proved by Ahlfors .

Theorem A.([1], Ch.1 Theorem 2) *If* $\Gamma' < \Gamma$ *then* $\lambda(\Gamma') < \lambda(\Gamma)$.

Theorem B.([1], Ch.1 Theorem 4) *If the families* Γ_j *are contained in disjoint measurable sets then* $\sum \lambda(\Gamma_j) \leq \lambda(\Gamma)$.

Proof of Corollary 1. Lift the class $\gamma^*_{\ell,M} + iM$ (obtained from $\gamma^*_{\ell,M}$ by translation by iM) to the class $\widetilde{\gamma^*_{\ell,M} + iM}$ on the universal covering U of $\mathbb{C} \setminus i\mathbb{Z}$. Consider the composition $\varphi \circ (\widetilde{\gamma^*_{\ell,M} + iM})$ where φ is the conformal mapping from U onto $\mathbb{C}_\ell$. The obtained class is the class of curves in $\mathbb{C}_\ell$ which joins the half-circles $\{z \in \mathbb{C}_\ell : |z+\frac{i}{2}| = \frac{1}{2}\}$ and $\{z \in \mathbb{C}_\ell : |z - i(2M + \frac{1}{2})| = \frac{1}{2}\}$. Hence, any holomorphic mapping of a rectangle into $\mathbb{C} \setminus i\mathbb{Z}$ which represents the class of curves $\gamma^*_{\ell,M} + iM$ results after lifting to U and composing with a conformal mapping in a holomorphic mapping of the rectangle to $\mathbb{C}_\ell$ which represents the class of curves in $\mathbb{C}_\ell$ which join the half-circles $\rho^- \overset{def}{=} \{z \in \mathbb{C}_\ell : |z + i(M + \frac{1}{2})| = \frac{1}{2}\}$ and $\rho^+ \overset{def}{=} \{z \in \mathbb{C}_\ell : |z - i(M + \frac{1}{2})| = \frac{1}{2}\}$.

The proof follows now from Lemma 6. To see this, identify $\mathbb{C}^M_\ell$ with a rectangle R_M by the conformal mapping $\mathfrak{c}$ from R_M onto $\mathbb{C}^M_\ell$ whose extension to the boundary maps the horizontal sides to the two half-circles. Denote by λ_M the extremal length of R_M. Notice that λ_M is positive. Extend the mapping $\mathfrak{c}$ by Schwarz's reflection principle across the horizontal sides of R_M to a rectangle $3R_M$ of vertical side length equal to 3 times the vertical side length of R_M. The rectangle added on top of R_M is mapped to the domain obtained by reflection of $\mathbb{C}^M_\ell$ in the half-circle $\rho_1^+ = \{z \in \mathbb{C}_\ell : |z - i(M + \frac{1}{2})| = \frac{1}{2}\}$. The half-circle $\rho_1^- = \{z \in \mathbb{C}_\ell : |z + i(M + \frac{1}{2})| = \frac{1}{2}\}$ is mapped under this reflection to a half-circle ρ_2^-. The respective fact holds for the rectangle added on the bottom of R_M. It follows that the rectangle $3R_M$ is mapped conformally onto the component of $\mathbb{C}_\ell \setminus (\rho_2^+ \cup \rho_2^-)$ which contains $\mathbb{C}^M_\ell$. This component is the left half-plane with two half-discs removed. The half-circles ρ_2^+ and ρ_2^- are symmetric with respect to the real axis.

After repeated application of the reflection principle we obtain a conformal mapping from a rectangle $3^n\, R_M$ onto a domain Ω_n which is equal to the left half-plane with two half-discs removed. The half-discs are symmetric with respect to the real axis. The domains Ω_n are increasing. The diameter of the removed half-discs at step n tends to zero for $n \to \infty$ since the extremal length of $3^n\, R_M$ tends to ∞. We obtain a conformal mapping of an infinite strip S' onto the left half-plane which we denote again by $\mathfrak{c}$.

Let f be any holomorphic mapping from a rectangle to the left half-plane whose extension to the closure maps the upper side to ρ_1^+ and the lower side to ρ_1^-. The corollary follows by applying Lemma 6 to the mapping $\mathfrak{c}^{-1} \circ f$. The form of the extremal mapping in Lemma 6 shows that the extremal mapping of the corollary extends locally holomorphically across the horizontal sides of the rectangle. $\square$

Proof of Corollary 2. For this proof it is more convenient to consider the fundamental group of the complex plane punctured at 0 and 1 rather than at -1 and 1, and to consider the upper half plane $\mathbb{C}_+$ as universal covering of $\mathbb{C} \setminus \{0, 1\}$.

Each holomorphic mapping of a rectangle into $\mathbb{C} \setminus \{0, 1\}$ that represents an element of the fundamental group lifts to the universal covering. The lift maps the open horizontal sides of the rectangle to certain half-circles with diameter on the imaginary axis (maybe, after a conformal self-map of the half-plane) and represents the class of curves that are contained in the half-plane and join the two half-circles. As in the proof of Corollary 1 the extremal length is realized on a conformal mapping of a rectangle onto the half-plane with two deleted half-discs such that the horizontal sides are mapped onto the half-circles. Composing with the covering map we obtain a locally conformal mapping that extends locally conformally across the open horizontal sides of the rectangle.

Suppose a holomorphic mapping of an annulus into $\mathbb{C} \setminus \{0, 1\}$ represents a conjugacy class of elements of the fundamental group of $\mathbb{C} \setminus \{0, 1\}$ with base point. Each element of the fundamental group corresponds to a covering transformation. Each covering transformation is a holomorphic self-homeomorphism of the universal covering $\mathbb{C}_+$ that extends to a broken linear transformation $T(z) = \frac{az+b}{cz+d}$ of the Riemann sphere with integral coefficients a, b, c, d, such that $ad - bc = 1$. Moreover, T is either parabolic (i.e. T has one fixed point and is conjugate to the mapping $z \to z + b'$ for a constant b'), or T is hyperbolic (i.e., T has two fixed points and is conjugate to $z \to \kappa z$ for a positive real number κ), or T is elliptic (i.e., T has two complex fixed points symmetric with respect to the imaginary axis and is conjugate to $z \to e^{i\theta} z$ for a real number θ). See [7], Chapter II, 9D and 9E.

Let $\hat{a}$ be a conjugacy class of elements of the fundamental group of $\mathbb{C} \setminus \{0, 1\}$ and let a be an element of the fundamental group that represents $\hat{a}$. Denote by T_a the covering transformation corresponding to a, and by $\langle T_a \rangle$ the subgroup of the group of covering transformations generated by T_a. Then the quotient $\mathbb{C}_+ \diagup \langle T_a \rangle$ is an annulus. It has extremal length 0 if T_a is parabolic or elliptic and has positive extremal length if T_a is hyperbolic. If $f : A \to \mathbb{C} \setminus \{0, 1\}$ is a holomorphic map of an annulus A to $\mathbb{C} \setminus \{0, 1\}$ that represents $\hat{a}$, then f lifts to a holomorphic map of A into $\mathbb{C}_+ \diagup \langle T_a \rangle$. The lift represents the class of a generator of the fundamental group of $\mathbb{C}_+ \diagup \langle T_a \rangle$. The corollary follows from Lemma 6. $\square$

The following proposition gives effective upper and lower bounds for the extremal length of $\gamma^*_{\ell,M}$ in dependence on the number $M \geq 2$.

Proposition 5. *Let M be a positive number. The extremal length $\Lambda(\gamma^*_{\ell,M})$ of the class of curves $\gamma^*_{\ell,M}$ satisfies the following inequalities:*

$$\frac{2}{\pi} \log(4M + 1) \leq \Lambda(\gamma^*_{\ell,M}) \leq \frac{2}{\pi} \log(4M + 3)\,. \tag{11}$$

Proof. By Corollary 1 the extremal length $\Lambda(\gamma^*_{\ell,M})$ equals the extremal length of a rectangle which admits a conformal mapping f onto $\mathbb{C}^M_\ell \overset{def}{=} \mathbb{C}_\ell \setminus \big(\{|z - i(M+\frac{1}{2})| \le \frac{1}{2}\} \cup \{|z + i(M+\frac{1}{2})| \le \frac{1}{2}\}\big)$, which maps horizontal sides to the two half-circles in the boundary of $\mathbb{C}^M_\ell$.

Put $y^-_M \overset{def}{=} \frac{1}{(4M+1)(4M+2)-1}$ and $y^+_M \overset{def}{=} \frac{1}{(4M+3)(4M+2)-1}$. Notice that $y^+_M < y^-_M$. Denote by D^ℓ_M the open half-disc contained in the left half-plane with diameter $(-y^-_M\, i, y^+_M\, i)$ and by D^ℓ_1 the open unit half-disc in the left half-plane. We will prove that there is a conformal self-map of the left half-plane which maps $\mathbb{C}^M_\ell$ onto $\Omega_M \overset{def}{=} D^\ell_1 \setminus \overline{D^\ell_M}$.

Indeed, consider the conformal self-mapping $\phi_{1,M}(z) = \frac{1}{2z+i(2M+1)}$ of the left half-plane. Its extension to the boundary maps iM to $-\frac{i}{4M+1}$, maps $i(M+1)$ to $-\frac{i}{4M+3}$, it maps $-iM$ to $-i$ and $-i(M+1)$ to i. Hence, $\phi_{1,M}$ maps the half-circle $\{|z+i(M+\frac{1}{2})| = \frac{1}{2}, \mathrm{Re} z < 0\}$ to the unit half-circle $\{|z| = 1, \mathrm{Re} z < 0\}$. Since ∞ is mapped to 0, $\phi_{1,M}$ maps $\mathbb{C}_\ell \setminus \{|z + i(M+\frac{1}{2})| \le \frac{1}{2}\}$ onto D^ℓ_1. Further $\phi_{1,M}$ maps the half-circle $\{|z - i(M+\frac{1}{2})| = \frac{1}{2}, \mathrm{Re} z < 0\}$ onto the half-circle with diameter $(-\frac{i}{4M+1}, -\frac{i}{4M+3})$.

The mapping $\phi_{2,M}(z) = \frac{z+\frac{i}{4M+2}}{1-z\frac{i}{4M+2}}$ maps the left half-plane conformally onto itself and preserves the unit half-disc D^ℓ_1. It maps the point $-\frac{i}{4M+1}$ to the point $-iy^-_M$, and it maps the point $-\frac{i}{4M+3}$ to the point iy^+_M. Hence, $\phi_{2,M}$ maps the half-disc with diameter $(-\frac{i}{4M+1}, -\frac{i}{4M+3})$ onto the half-disc with diameter $(-iy^-_M, iy^+_M)$, i.e. onto D^ℓ_M. Hence, the composition $\phi_{2,M} \circ \phi_{1,M}$ maps $\mathbb{C}^M_\ell$ conformally onto $\Omega_M = D^\ell_1 \setminus \overline{D^\ell_M}$.

Consider the family Γ_M of curves in $\overline{\Omega_M}$ which join the two half-circles that are contained in the boundary of Ω_M. By Corollary 1 the extremal length of Γ_M in the sense of Ahlfors equals $\Lambda(\gamma^*_{\ell,M})$.

The following inclusions hold for the domain Ω_M:

$$D^\ell_1 \setminus \{|z| \le y^-_M\} \subset \Omega_M \subset D^\ell_1 \setminus \{|z| \le y^+_M\}\,.$$

The first set is conformally equivalent to a rectangle with horizontal side length π and vertical side length $\log((4M+1)(4M+2)-1)$, the second is conformally equivalent to a rectangle with horizontal side length π and vertical side length $\log((4M+2)(4M+3)-1)$. Ahlfors's Theorem A implies the proposition. $\square$

Let $\tilde{\gamma}^*_{\ell,M}$ be the set of curves in $\mathbb{C} \setminus i\mathbb{Z} - iM$ that are homotopic through curves joining the interval $(-i(M+1), -iM)$ with the real axis to a curve $\tilde{\gamma}$ whose union with its reflection in the real axis belongs to $\gamma^*_{\ell,M}$. The following lemma holds.

Lemma 8.

$$\Lambda(\tilde{\gamma}^*_{\ell,M}) = \frac{1}{2}\Lambda(\gamma^*_{\ell,M})\,. \tag{12}$$

Proof. It follows from Lemma 6 in the same way as Corollary 1 that $\Lambda(\tilde{\gamma}^*_{\ell,M})$ equals the extremal length of a rectangle which admits a conformal mapping onto $\{z \in \mathbb{C} : \mathrm{Re} z < 0, \mathrm{Im} z < 0\} \setminus \{|z+i(M+\frac{1}{2})| \le \frac{1}{2}\}$ such that the open lower side is mapped onto the half-circle in the boundary of the domain and the open upper side is mapped onto the negative real half-axis. Apply Schwarz's reflection principle. By reflection in the upper side of the rectangle the mapping extends to a conformal mapping from the doubled rectangle (i.e. from the rectangle with the same horizontal side length and double vertical side length) onto the set $\mathbb{C}_M$, which equals the half-plane with two removed half-discs. By

Lemma 6 the conformal mapping of the doubled rectangle realises the extremal length of $\gamma^*_{\ell,M}$. Hence,

$$\Lambda(\gamma^*_{\ell,M}) = 2\Lambda(\tilde{\gamma}^*_{\ell,M}) .$$

The Lemma is proved. $\square$

The following proposition considers elements of the fundamental group π_1 which have the form of syllables and gives upper and lower bounds for the extremal length of the corresponding classes of curves in the relative fundamental groups with *pb*, *tr* or mixed boundary values.

Proposition 6. *The following statements hold for curves representing elements of the relative fundamental groups.*

1. **Syllables of form (1) with *pb*-boundary values.** *The class ${}_{pb}(a_1^n)_{pb} \in \pi_1^{pb}$, $d = |n| \geq 2$, lifts under the covering $f_1 \circ f_2$ to a class of elementary homotopy slalom curves which is mapped by translation by an integral or half-integral multiple of i to the class $\gamma^*_{\ell,M}$ with $M = \frac{d-1}{2}$ if $n > 0$ and to this class with inverted orientation if $n < 0$. The respective statement holds for ${}_{pb}(a_2^n)_{pb} \in \pi_1^{pb}$, $d = |n| \geq 2$ with $\gamma^*_{\ell,M}$ replaced by $\gamma^*_{r,M}$ with inverted orientation. (See Figure 4a for a_2^{-3}.) For the extremal length the following inequalities hold:*
$$\frac{2}{\pi}\log(2d-1) \leq \Lambda({}_{pb}(a_j^n)_{pb}) \leq \frac{2}{\pi}\log(2d+1) .$$
2. **Syllables of form (1) or (3) with mixed boundary values.** *For any $d = |n| \geq 1$ the class ${}_{pb}(a_1^n)_{tr} \in {}^{pb}\pi_1^{tr}$ lifts under the covering $f_1 \circ f_2$ to a class of elementary homotopy half-slalom curves. For $n > 0$ it is mapped by translation by an integral or half-integral multiple of i to the class $\tilde{\gamma}^*_{\ell,M}$ with $M = d - \frac{1}{2}$. For $n < 0$ we obtain the "upper half" of the class $\gamma^*_{\ell,M}$ with the orientation reversed.*

   The respective statements hold for ${}_{pb}(a_2^n)_{tr} \in {}^{pb}\pi_1^{tr}$, $d = |n| \geq 1$, with the respective halfs of $\tilde{\gamma}^*_{r,M}$ with suitable orientation. For the classes ${}_{tr}(a_j^n)_{pb}$ respective statements hold with the classes of translated homotopy slalom curves reflected in the real axis. (See Figure 4b for ${}_{pb}(a_2^{-1})_{tr}$.)

   For the extremal length the following inequalities hold:
$$\frac{1}{\pi}\log(4d-1) \leq \Lambda({}_{pb}(a_j^n)_{tr}) \leq \frac{1}{\pi}\log(4d+1) .$$
   The same inequalities hold for the remaining cases of mixed horizontal boundary values.
3. **Syllables of form (2) with mixed boundary values.** *Let ${}_{pb}\mathfrak{s}_{tr}$ be a syllable of type (2) of degree $d \geq 2$ with mixed boundary values. It lifts under $f_1 \circ \tilde{f}_2$ to a class of elementary homotopy half-slalom curves a suitable translation of which is either $\tilde{\gamma}^*_{\ell,M}$ or a reflection of this class in the real axis or in the imaginary axis or in both, or one of these classes with inverse orientation. (See Figure 4c for ${}_{tr}(a_2a_1a_2a_1a_2)_{pb}$.) The number M equals $M = d - \frac{1}{2}$. The extremal length satisfies the inequalities*
$$\frac{1}{\pi}\log(4d-1) \leq \lambda({}_{pb}(\mathfrak{s})_{tr}) \leq \frac{1}{\pi}\log(4d+1) .$$
4. **Syllables of form (2) with *tr* boundary values.** *Let ${}_{tr}(\mathfrak{s})_{tr}$ be a syllable of type (2) with totally real boundary values. Then it lifts under $f_1 \circ \tilde{f}_2$ to a*

homotopy class of elementary slalom curves with $M = \frac{d-1}{2}$. The extremal length of the class satisfies the inequalities

$$\frac{2}{\pi}\log(2d-1) \le \lambda({}_{tr}(\mathfrak{s})_{tr}) \le \frac{2}{\pi}\log(2d+1)\,.$$

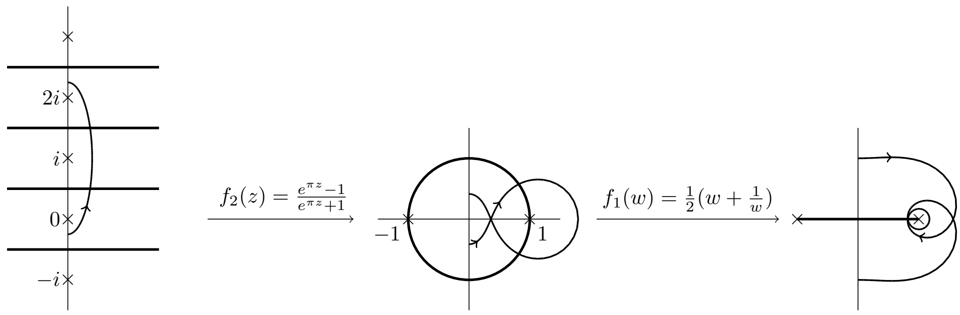

Figure 4a

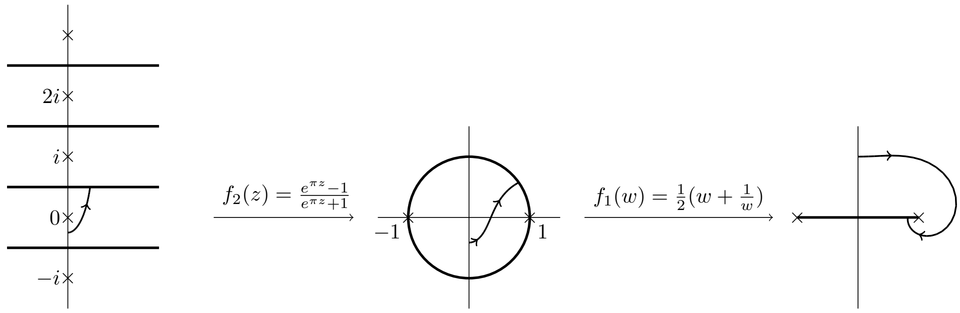

Figure 4b

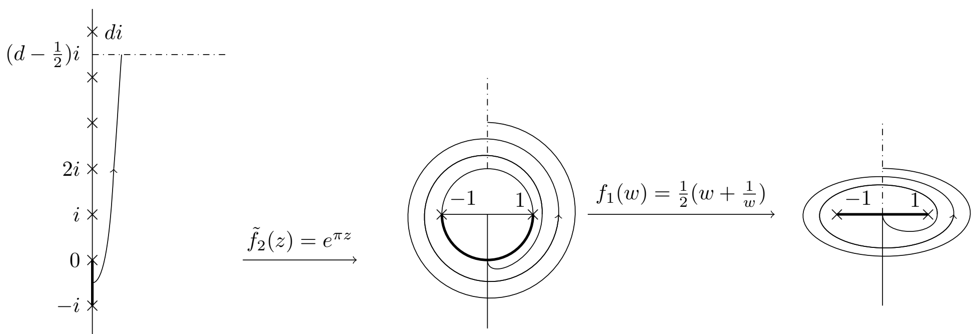

Figure 4c

Figure 4

Proof. Consider a curve α_1 with interior in the left half-plane which represents a_1 in the fundamental group with base point 0. The curve α_1^n represents a_1^n with pb boundary values and also with tr and mixed boundary values. Make the respective choice for α_2.
Proof of statement 1. Assume first that $n = d > 0$. The curve α_1^n lifts under $f_1 \circ f_2$ to a curve in the closed left half-plane with initial point $i(k - \frac{1}{2})$ and terminating point

$i(k+n-\frac{1}{2})$ for an integer number k. A translation of the lifted curve represents $\gamma^*_{\ell,M}$ with $M=\frac{d-1}{2}$. If $n<0$ we obtain the class $\gamma^*_{\ell,M}$ with inverted orientation. For a_1 replaced by a_2 we obtain the respective classes in the right half-plane. See Figure 4a for an elementary slalom curve with $d=3$ that is the lift under $f_1\circ f_2$ of a representative of ${}_{pb}(a_2^{-3})_{pb}$ with initial point not equal to 0.

The estimate for the extremal length follows from formula (11).

Proof of statement 2. For $n=d>0$ the curve α_1^n lifts under $f_1\circ f_2$ to an elementary homotopy half-slalom curve in $\mathbb{C}_\ell$ with initial point $i(k+\frac{1}{2})\in(ik,i(k+1))$ and terminating point $i(k+d+\frac{1}{2})\in i(k+d+\frac{1}{2})+\mathbb{R}$, i.e the lift represents $\tilde{\gamma}^*_{\ell,M}+i(d+k+\frac{1}{2})$ with $M=d-\frac{1}{2}$. See Figure 4b for ${}_{pb}(a_2^{-1})_{tr}$ with $d=1$. The remaining cases ($n<0$, or a_1 replaced by a_2, or both) can be obtained from $\tilde{\gamma}^*_{\ell,M}$ by suitable choices of inverting orientation, reflection in the real line or reflection in the imaginary line. The estimate for the extremal length follows by lemma 8 and (11).

Proof of statement 3. Let α_1 and α_2 be as before. The curve $\alpha_1\,\alpha_2\ldots$ represents ${}_{tr}(\mathfrak{s})_{pb}={}_{tr}(a_1\,a_2\ldots)_{pb}$. Its lift under $f_1\circ\tilde{f}_2$ with initial point $i(2k-\frac{1}{2})\in((2k-1)i,2ki)$ for an integer number k can be seen as an elementary homotopy half-slalom curve in the closed left half-plane with terminating point $i(2k-d-\frac{1}{2})\in i(2k-d-\frac{1}{2})+\mathbb{R}$ where d is the number of letters in $\mathfrak{s}$. Here $M=d-\frac{1}{2}$. Notice that the lifts of $(-1,1)$ under $f_1\circ\tilde{f}_2$ are the intervals $(ik,i(k+1))$ for integers k. The lift of $\alpha_1\,\alpha_2\ldots$ under $f_1\circ\tilde{f}_2$ with initial point $i(2k+\frac{1}{2})\in(2ki,(2k+1)i)$ for an integer number k is a curve in the closed right half-plane with terminating point $i(2k+d+\frac{1}{2})\in i(2k+d+\frac{1}{2})+\mathbb{R}$.

Let the first letter in the syllable be a_2 instead of a_1. The lift of the curve $\alpha_2\,\alpha_1\ldots$ under $f_1\circ\tilde{f}_2$ with initial point $i(2k-\frac{1}{2})\in((2k-1)i,2ki)$ for an integer number k is a curve in the right half-plane with terminating point $i(2k+d-\frac{1}{2})\in i(2k+d-\frac{1}{2})+\mathbb{R}$. See Figure 4c with number of half-turns equal to $d=5$, and also proposition 4 and its proof, as well as Figure 1. The lift with initial point $i(2k+\frac{1}{2})$ is contained in the left half-plane and "moves down".

If all generators enter with power -1 the orientation is reversed. In all cases the estimate of the extremal length follows from (11).

The proof of statement 4 is related to the proof of statement 3 in the same way as the proof of statement 1 is related to the proof of statement 2. We leave it to the reader. □

4. The extremal length of arbitrary words in π_1. Lower bound

Take an element of a relative fundamental group of the twice punctured complex plane. We will break representing curves into pieces. The pieces will be chosen so that we have a good lower bound of the extremal length of the homotopy class of each piece. Ahlfors' theorem B will give a lower bound for the extremal length of the element of the relative fundamental group by the sum of the extremal lengths of the classes of the pieces.

Lemma 9 below treats curves with *pb* boundary values and is given in terms of their lifts under $f_1\circ f_2$ to slalom curves. Lemma 10 is given in terms of relative fundamental groups of $\mathbb{C}\setminus\{-1,1\}$ and is more general.

We will use the symbol # for the boundary values if we are free to choose either *pb* or *tr* boundary values.

Lemma 9. *Each homotopy slalom curve is the composition of elementary homotopy slalom curves.*

Let v_1 and v_2 be words in π_1. For the word ${}_{\#}(v)_{\#} = {}_{\#}(v_1\, v_2)_{\#} \in {}^{\#}\pi_1{}^{\#}$ we say that there is a sign change of exponents for the pair (v_1, v_2) if the sign of the exponent of the last term of v_1 is different from the sign of the exponent of the first term of v_2.

Lemma 10. *We have the following statements about breaking curves into elementary pieces.*

1. ***(pb-boundary values between terms, any powers.)*** *Write an element $w \in \pi_1$ as reduced word $w = w_1^{n_1}\, w_2^{n_2} \ldots w_k^{n_k}$, where each w_j is one of the generators and the generators in consecutive terms are different, and the n_j are integral numbers different from zero. Let ${}_{\#}(w)_{\#}$ be the element of the relative fundamental group with tr, pb or mixed boundary values corresponding to w. Then any smooth curve that represents w in the chosen relative fundamental group is the composition of curves β_j with the following property. For all j, $1 < j < k$, β_j represents ${}_{pb}(w_j^{n_j})_{pb}$. The left boundary values of the first curve β_1 and the right boundary values of the last curve β_k are prescribed by the choice of the relative fundamental group, the remaining boundary values of β_1 and β_k are also pb.*
2. ***(tr-boundary values between terms, sign change.)*** *Let w_1 be one of the standard generators a_1 or a_2 of π_1 and let w_2 be the other standard generator. If for non-zero integers n_1 and n_2 there is a sign change of exponents for the pair $(w_1^{n_1}, w_2^{n_2})$, then any representative of ${}_{\#}(w_1^{n_1}\, w_2^{n_2})_{\#}$ is the composition of curves representing ${}_{\#}(w_1^{n_1})_{tr}$ and ${}_{tr}(w_2^{n_2})_{\#}$, respectively.*
3. ***(tr-boundary values between terms, one power of absolute value bigger than 1.)*** *Let w_1 be one of the standard generators a_1 or a_2 of π_1 and let w_2 be the other standard generator. If $|n_1| > 1$ then any representative of ${}_{\#}(w_1^{n_1}\, w_2^{n_2})_{\#}$ is the composition of curves representing ${}_{\#}(w_1^{(|n_1|-1)sgn(n_1)})_{tr}$ and ${}_{\#}(w_1^{sgn(n_1)}\, w_2^{n_2})_{\#}$, respectively. If $|n_2| > 1$ then any representative of ${}_{\#}(w_1^{n_1}\, w_2^{n_2})_{\#}$ is the composition of curves representing ${}_{\#}(w_1^{n_1}\, w_2^{sgn(n_2)})_{tr}$ and ${}_{tr}(w_2^{(|n_2|-1)\, sgn(n_2)})_{\#}$, respectively.*

Proof of statement 1 of Lemma 10. Let β^1 be any curve representing ${}_{\#}w_{\#}$. We may assume that the curve is smooth (that is parametrized by a smooth mapping with nowhere vanishing derivative), avoids 0 and is transversal to $(-1, 1) \cup i\mathbb{R}$.

The argument is the following. First, we may assume that there are finitely many intersection points with $(-1, 1) \cup i\mathbb{R}$. Indeed, if a piece of the curve between two intersection points with $(-1, 1) \cup i\mathbb{R}$ stays in a small neighbourhood of this set, then the piece is homotopic to a constant in the class of curves with endpoints on this set and we may change the piece between the two intersection points by a piece that avoids 0 and has finitely many intersection points with $(-1, 1) \cup i\mathbb{R}$. Further, we may approximate the paramatrizing mapping for the curve by a smooth mapping which is transversal to $(-1, 1) \cup i\mathbb{R}$, so that the initial point of the original curve and that of the approximating curve coincide, also the terminating points of the two curves coincide, and the intersection points with $(-1, 1) \cup i\mathbb{R}$ and the parameters of these points are the same for the two curves. Suppose the approximation is good enough. Then for each pair of intersection points with $(-1, 1) \cup i\mathbb{R}$ the piece of the original curve and the piece of the approximating curve between these two points are homotopic to each other (with fixed endpoints). We will assume now that the curve β^1 is smooth.

Consider the case when β^1 has pb boundary values. The other cases are similar and are left to the reader (though some small hints are given in the present proof). Consider a smooth curve β^0 which represents ${}_{pb}(w)_{pb}$ and lifts under $f_1 \circ f_2$ to a smooth slalom curve (not just to a homotopy slalom curve) which intersects the imaginary axis transversely. This means that the lift of β^0 is the composition of elementary slalom curves. The collection of the elementary slalom curves projects to a collection β_j^0 of curves representing the terms $w_j^{n_j}$ of w in π_1^{pb}. (Note that in case some boundary values of w are tr the respective piece of β^0 lifts to an elementary half-slalom curve.)

Consider a smooth homotopy h which joins β^0 with β^1. In other words, there is a closed rectangle and a smooth mapping h from the rectangle to $\mathbb{C} \setminus \{-1, 1\}$ whose restriction to the left vertical side equals β^0, whose restriction to the right vertical side equals β^1, and whose values on the horizontal sides are contained in the imaginary axis $i\mathbb{R}$. (If some boundary values of w are tr then the values on the respective horizontal sides are contained in $(-1, 1)$ instead.) Since β^0 and β^1 are smooth curves that intersect the imaginary axis transversely we may assume that zero is a regular value of $\operatorname{Re} h$. Indeed, there is a neighbourhood U of the two closed vertical sides of the rectangle such that 0 is a regular value of $\operatorname{Re} h$ on U, and we may assume that in a neighbourhood U' of the closed horizontal sides this is so. Let Ψ be a non-negative smooth function on the closed rectangle which equals 1 outside $U \cup U'$ and equals zero near the boundary of the rectangle. If $\varepsilon > 0$ is a sufficiently small regular value of $\operatorname{Re} h$ on the closed rectangle, then $h - \varepsilon\Psi$ is another smooth homotopy joining β^0 with β^1 and zero is a regular value of $\operatorname{Re}(h - \varepsilon\Psi)$ on the closed rectangle.

The level set $\operatorname{Re} h = 0$ on the open rectangle is a relatively closed curve. Each of its connected components either closes up to a closed arc that joins two boundary points, or closes up to a circle or it is a circle. Consider the closed arcs. The intersection points of all such arcs with the left side of the rectangle divide the side into connected components. The restrictions of h to these components are the β_j^0. Each of the closed arcs has at most one endpoint on the open left side of the rectangle. Indeed, assume the contrary. Then the interval I on the left side of the rectangle between the two endpoints of the arc contains either an interval with $\operatorname{Re} h > 0$ or an interval with $\operatorname{Re} h < 0$. That means that the restriction of h to the interval I lifts under $f_1 \circ f_2$ to a slalom curve and, hence, this restriction is not homotopic to a constant through curves in $\mathbb{C} \setminus \{-1, 1\}$ with end points in the imaginary axis. On the other hand, the existence of an arc in the level set $\operatorname{Re} h = 0$ joining the two endpoints of I would provide a relative homotopy in $\mathbb{C} \setminus \{-1, 1\}$ (with endpoints in the imaginary axis) joining the restriction of h to the interval I with a curve contained in the imaginary axis, which would be a contradiction.

The same reasoning shows that an arc in the level set $\{\operatorname{Re} h = 0\} \cap R$ with one endpoint on the open left side of the rectangle cannot have its other endpoint on a closed horizontal side of R. Indeed, otherwise the restriction of h to the interval on the vertical side between the endpoint of the arc and the vertex of the rectangle belonging to the closed horizontal side would be homotopic to a constant through curves with endpoints on the imaginary axis. (In case the respective boundary values of w are tr, the homotopy to a constant is through curves with endpoints in $(-1, 1) \cup i\mathbb{R}$.) We may ignore the arcs with both endpoints on the same closed horizontal side and will also ignore the circles in the level set of $\operatorname{Re} h = 0$ on the closure $\bar{R}$.

We consider the arcs in R contained in the level set $\operatorname{Re} h = 0$ that have one endpoint on the open left side of the rectangle and the other point on the open right side. These arcs divide the rectangle into curvilinear rectangles which are in bijective correspondence to the intervals of division on the left side. Take the curvilinear rectangle whose left side corresponds to β_j^0. The restriction of h to this curvilinear rectangle provides a homotopy with boundary values in the imaginary axes joining β_j^0 to the restriction β_j^1 of h to the right side of the curvilinear rectangle. (In case some boundary values of w are tr the homotopy may be with mixed boundary values, respectively.) Since the division of the rectangle into curvilinear rectangles induces a division of the right side of the rectangle into intervals, the curve β^1 is the composition of the curves $\beta_j \overset{def}{=} \beta_j^1$. Assertion 1 is proved.

Proof of statement 2. Consider the case when n_1 is positive and $w_1 = a_1$. The remaining cases are similar. Let α_j be standard representatives of the a_j, $j = 1, 2$. A lift of $\alpha_1^{n_1}$ under $f_1 \circ f_2$ joins the point $i(k+\frac{1}{2})$with the point $i(k+n_1+\frac{1}{2})$, the respective lift of $\alpha_2^{n_2}$ joins $i(k+n_1+\frac{1}{2})$ with $i(k+n_1-n_2+\frac{1}{2})$. In the condition of the statement $n_2 < 0$. Hence the lift of each representing curve intersects the line $\{\mathrm{Im} z = k + n_1 + \frac{1}{2}\}$, so that one obtains a decomposition into two pieces representing ${}_{\#}{a_1^{n_1}}_{tr}$, and ${}_{tr}{a_2^{n_2}}_{\#}$ respectively.

Proof of statement 3. Assume without loss of generality that $w_1 = a_1$ and $w_2 = a_2$. By assertion 1 any curve representing ${}_{\#}(w_1^{n_1}\, w_2^{n_2})_{\#}$ is the composition of a curve representing ${}_{\#}(w_1^{n_1})_{pb}$ and a curve representing ${}_{pb}(w_2^{n_2})_{\#}$.

If $|n_1| > 1$ and $k \in \mathbb{Z}$ the line $\{\mathrm{Im} z = k + n_1 - \frac{1}{2}\mathrm{sgn}\,(n_1)\}$ does not meet the interval $(ik, i(k+1))$. The lift with initial point in $(ik, i(k+1))$ of any curve representing ${}_{\#}(w_1^{n_1})_{pb}$ intersects the line $\{\mathrm{Im} z = k + n_1 - \frac{1}{2}\mathrm{sgn} n_1\}$. Hence, any curve representing ${}_{\#}(w_1^{n_1})_{pb}$ is the composition of a curve representing ${}_{\#}(w_1^{(|n_1|-1)\mathrm{sgn}(n_1)})_{tr}$ and a curve representing ${}_{tr}(w_1^{\mathrm{sgn}(n_1)})_{pb}$. Hence, any curve representing ${}_{\#}(w_1^{n_1}\, w_2^{n_2})_{\#}$ is the composition of a curve representing ${}_{\#}(w_1^{(|n_1|-1)\mathrm{sgn}(n_1)})_{tr}$ and a curve representing ${}_{tr}(w_1^{\mathrm{sgn}(n_1)}\, w_2^{n_2})_{\#}$. The argument for $|n_2| > 1$ is similar. □

The following lemma is a key part for the proof of the lower bound of the extremal length of slalom classes, respectively, of elements of the relative fundamental group of $\mathbb{C} \setminus \{-1, 1\}$.

Let R be a rectangle. By a curvilinear rectangle contained in R we mean a simply connected domain in R whose boundary looks as follows. It consist of two vertical segments, one in each vertical side of R, and either two simple arcs with interior in R and endpoints on opposite open vertical sides of R, or one such arc and a horizontal side of the rectangle. The rectangle R itself may also be considered as a curvilinear rectangle contained in R. The arcs in R with endpoints on opposite open vertical sides are called curvilinear horizontal sides of the curvilinear rectangle. Each curvilinear rectangle admits a conformal map onto a true rectangle that maps curvilinear horizontal sides to horizontal sides and vertical sides to vertical sides.

Lemma 11. *Consider a word $w \in \pi_1$ with at least two syllables. Take a locally conformal mapping $g : R \to \mathbb{C} \setminus \{-1, 1\}$ of a rectangle R into the twice punctured plane which represents the word w with pb, tr or mixed boundary values. Assume that g extends locally conformally to a neighbourhood of the open horizontal sides.*

The following statements hold.

1. *R can be divided into a collection of pairwise disjoint curvilinear rectangles R_j which are in bijection to the terms of w in such a way that the restriction of g to R_j represents the j-th term with pb boundary values if the term is not at the*

(right or left) end of the word and, possibly, with mixed boundary values if the term is at the end of the word.

2. *If R contains at least one syllable of form (1) then R contains a collection of mutually disjoint curvilinear rectangles R'_j, $j \in J$, which are in bijective correspondence to the collection of syllables $\mathfrak{s}_j$ of w that are not of the form (1) such that either the restriction of g to R'_j represents the respective syllable $\mathfrak{s}_j$ with mixed boundary values, or the restriction represents a syllable of form (2) with one more letter than $\mathfrak{s}_j$ and mixed boundary values.*
3. *If all syllables $\mathfrak{s}_j$, $j = 1, \ldots, N$, of the word w are of form (2) or (3) then there is a division of R into curvilinear rectangles R'_j, $j = 1, \ldots, N$, such that for $j = 1, \ldots, N-1$, the restriction $g \mid R'_j$ represents the j-th syllable with mixed boundary values. In the same way there is a decomposition of R into curvilinear rectangles R''_j, $j = 1, \ldots, N$, such that for $j = 2, \ldots, N$, the restriction $g \mid R''_j$ represents the j-th syllable with mixed boundary values.*

Proof. We start with the **proof of statement 1.** Shrinking the rectangle slightly in the horizontal direction we may assume that the mapping g extends locally conformally to a neighbourhood of the closed rectangle. The 0-level $L_0 \overset{def}{=} \{z \in R : \mathrm{Re} g(z) = 0\}$ of the real part of g is a relatively closed smooth real one-dimensional submanifold of R. The connected components are arcs with two different endpoints on the boundary of R. Indeed, the closure of L_0 does not contain circles by the maximum principle applied to the functions $z \to exp(\pm g(z))$.

We consider the connected components of L_0 that have at least one endpoint on the open left side of the rectangle. The word w has at least two syllables. Hence, there is a point in the closure of L_0 that is contained in the open left side of R and divides the left side into two connected components with the following property. The restriction of g to each of the connected components is not homotopic to a constant in $(-1,1) \cup i\mathbb{R}$. Let L_0^0 be the connected component of L_0 the closure of which contains this point. By the same reasoning as in the proof of Lemma 10 the other endpoint of the arc L_0^0 is contained in the open right side of R.

Notice that in case w has *pb* boundary values no connected component of L_0 can have an endpoint on a closed horizontal side since g extends locally holomorphically to a neighbourhood of the closure $\bar{R}$. If w has mixed or *tr* boundary values then a connected component of L_0 with one endpoint on the open left side can have its other endpoint on the open left side, or on a closed horizontal side, or on the open right side.

We call the arcs in L_0 which have endpoints on different open vertical sides dividing arcs. We saw that in case the word has at least two syllables there is at least one dividing arc.

To each non-dividing arc in L_0 with one endpoint on the open left side of R we associate a topological disc in R. This disc is that component of the complement of the arc which does not contain dividing arcs. We call it the inessential disc associated to the arc. Note, that the restriction of g to the part of the boundary of R which is contained in the boundary of the inessential disc is homotopic to a constant through curves in $\mathbb{C} \setminus \{-1, 1\}$ with endpoints in $i\mathbb{R}$ (or in $(-1,1) \cup i\mathbb{R}$ in case some boundary values of w are *tr*) . Inclusion defines a partial order among inessential discs. If there is an inessential disc associated to a non-dividing arc the arc is called maximal if the associated disc is maximal with respect to inclusion among inessential discs.

For each maximal non-dividing arc we take a relatively closed arc in R which approximates the non-dividing arc well enough and has the following property. If no endpoint of the original arc is on the closed right side the approximating arc shall not have an endpoint on the right side either and shall not intersect the closure of the inessential disc associated to the original non-dividing arc.

If the original arc has one endpoint on the closed right side of R this point shall also be an endpoint of the approximating arc and this shall be the only intersection point of the approximating arc with the closure of the inessential disc associated to the original non-dividing arc. We may choose the approximating arcs so that they are pairwise disjoint and contained in the set $\{\mathrm{Re} z \neq 0\}$ except, maybe, for an endpoint on the right side of R.

If the new arc has both endpoints on the left side of R we call the segment of the left side between the endpoints of the approximating arc a replaceable segment and we call the approximating arc a replacing arc associated to the segment.

Suppose the approximating arc has one endpoint p on the open left side of R and the other endpoint on a closed horizontal side of R. Let q be the endpoint of the left side of the rectangle which is contained in the boundary of the inessential disc associated to the original arc. We call the segment $[p, q]$ of the left side of R a replaceable segment and we call the approximating arc a replacing arc associated to the segment.

Take the left side of the rectangle R and replace each replaceable segment by the associated replacing arc. We obtain a curve denoted by γ.

If the replacing arcs approximate the non-dividing arcs well enough then the restriction of g to the left side of the rectangle and its restriction to the obtained curve γ represent the same element in the respective relative fundamental group.

Apply Lemma 10, statement 1, to the restriction of g to the curve γ. We obtain a collection of points on the interior of the curve γ which belong to L_0 and divide γ into parts s_j such that the restriction $g \mid s_j$ represents the j-th term of the word with the required boundary values. Each point in this collection lies on the left side of R and is an endpoint of a dividing arc.

The collection of the thus obtained dividing arcs divides the rectangle R into curvilinear rectangles such that the restriction $g \mid R_j$ represents the j-th term with the required boundary values. Statement 1 is proved.

Proof of statement 2. Choose a syllable $\mathfrak{s}_k$ of form (1). Suppose there are syllables on the left of $\mathfrak{s}_k$ which are not of form (1). Consider them in the order from left to right. If the left boundary values of w are pb then we consider the most left syllable $\mathfrak{s}_{j_1}$ which is not of form (1). Suppose $j_1 + 1 < k$. If the next syllable $\mathfrak{s}_{j_1+1}$ to the right of $\mathfrak{s}_{j_1}$ is also not of form (1) there is a sign change of exponents for the pair $(\mathfrak{s}_{j_1}, \mathfrak{s}_{j_1+1})$ of consecutive syllables. For the curvilinear rectangles R_j of statement 1 the restriction of g to $R_{j_1,j_1+1} \overset{def}{=} \mathrm{Int}(\bar{R}_{j_1} \cup \bar{R}_{j_1+1})$ represents ${}_{pb}(\mathfrak{s}_{j_1}\mathfrak{s}_{j_1+1})_{pb}$. (Here $\mathrm{Int}X$ denotes the interior of a subset X of a topological space.) By Lemma 10 statement 2 and a similar argument as in the proof of statement 1 of Lemma 11 the curvilinear rectangle R_{j_1,j_1+1} can be divided into two curvilinear rectangles such that the restriction of g to them represents the syllables ${}_{pb}(\mathfrak{s}_{j_1})_{tr}$ and ${}_{tr}(\mathfrak{s}_{j_1+1})_{pb}$. We represented the two syllables $\mathfrak{s}_{j_1}$ and $\mathfrak{s}_{j_1+1}$ which are both not of form (1) as required.

Suppose the next syllable $\mathfrak{s}_{j_1+1}$ to the right of $\mathfrak{s}_{j_1}$, is of form (1), i.e it equals $w_{j_1+1}^n$ for an integer $n \geq 2$ with w_{j_1+1} being a standard generator of π_1. If there is a sign change of exponents for the pair $(\mathfrak{s}_j, w_{j+1}^n)$ the preceding argument applies.

Suppose there is no sign change. By Lemma 10 statement 3 and a similar argument as in the proof of statement 1 of Lemma 11 the curvilinear rectangle R_{j_1,j_1+1} can be divided into two curvilinear rectangles such that the restriction of g to them represents the syllables ${}_{pb}(\mathfrak{s}_{j_1}\, w_{j_1+1})_{tr}$ and ${}_{tr}(w_{j_1+1}^{n-1})_{pb}$ with mixed boundary values.

If the left boundary values of w are tr then $g \mid R_1$ represents $\mathfrak{s}_1$ with mixed boundary values. Consider the most left syllable $\mathfrak{s}_{j_1}$ with $j_1 > 1$ which is not of form (1) and proceed as in the previous case.

The statement 2 holds now for all syllables not of form (1) with label $j \le j_1 + 1$. The disjoint curvilinear rectangles contained in R used for representing these syllables (or the syllables with one letter more) are contained in the union of the closure of the rectangles R_j, $j \le j_1 + 1$, of statement 1 of the lemma.

We proceed by induction as follows. Suppose for some $j' < k$ we achieved the following. We found disjoint curvilinear rectangles contained in $\bigcup_{j \le j'} \overline{R}_j$ which are in bijective correspondence to all syllables $\mathfrak{s}_j$ with $j \le j'$ that are not of form (1) such that the restrictions of g to these rectangles represent the syllable, or the syllable with one more letter, with mixed boundary values.

Consider the first syllable $j_2 < k$ on the right of $\mathfrak{s}_{j'}$ with $\mathfrak{s}_{j_2}$ not of form (1) (if there is any). If $j_2 + 1 < k$ we proceed with $\mathfrak{s}_{j_2}$ in the same way as it was done for $\mathfrak{s}_{j_1}$ in the case $j_1 + 1 < k$ and continue the process.

Suppose there is an $\mathfrak{s}_{j_2}$ but the right neighbour of $\mathfrak{s}_{j_2}$ is the syllable $\mathfrak{s}_k$ of form (1), say it equals $w_k^{n_k}$. Then we represent ${}_{pb}(\mathfrak{s}_{j_2}\, w_k)_{tr}$ and stop the process.

Make the same procedure from right to left until each syllable not of form (1) on the right of $\mathfrak{s}_k$ is represented in the desired way. The curvilinear rectangles obtained by the construction which starts from the left do not intersect the curvilinear rectangles obtained by the construction which starts from the right because $\mathfrak{s}_k = w_k^{n_k}$ with $|n_k| \ge 2$. Statement 2 is proved.

Proof of statement 3. Under the conditions of statement 3 there is a sign change of exponents for any pair of consecutive syllables.

If the left boundary values of w are pb we may consider curvilinear rectangles $R_{2j-1,2j}$, $j = 1, 2, \ldots$, such that the restriction of g to $R_{2j-1,2j}$, $j = 1, 2, \ldots$, represents ${}_{pb}(\mathfrak{s}_{2j-1}\, \mathfrak{s}_{2j})_{pb}$. Divide $R_{2j-1,2j}$ by an arc in $\operatorname{Im} g = 0$ into two curvilinear rectangles such that the restriction of g to the first curvilinear rectangle R'_{2j-1} represents the syllable $\mathfrak{s}_{2j-1}$ with right tr boundary values and the restriction of g to the second curvilinear rectangle R'_{2j} represents the syllable with left tr boundary values. In this way each syllable except, maybe, the last one is represented with mixed boundary values by restricting g to a member of the collection of the obtained pairwise disjoint curvilinear rectangles. If the left boundary values of w are tr then $g \mid R_1$ represents $\mathfrak{s}_1$ with mixed boundary values and we consider instead the rectangles $R_{2j,2j+1}$, $j = 1, \ldots$. Repeating the procedure from right to left gives the rectangles R''_j. This finishes the proof of statement 3. $\square$

We will now give the proof of the lower bound in Theorem 1 and in Theorem 1′.

Figure 5 shows a mapping from a rectangle R to $\mathbb{C} \setminus i\mathbb{Z}$ which represents the lift of an element $w \in \pi_1$ with pb boundary values. The rectangle R is covered by curvilinear rectangles. The right part of the figure shows the image of R in $\mathbb{C} \setminus i\mathbb{Z}$ and a slalom curve that lifts a representative of w. For the elementary pieces of the slalom curve we indicate the element of π_1 a lift of which the piece represents. For the curvilinear rectangles contained in R we indicate the element of the relative fundamental group with respective boundary values, a lift of which is represented by the restriction of the mapping. Notice that for some choices of the boundary values the curvilinear rectangles may intersect.

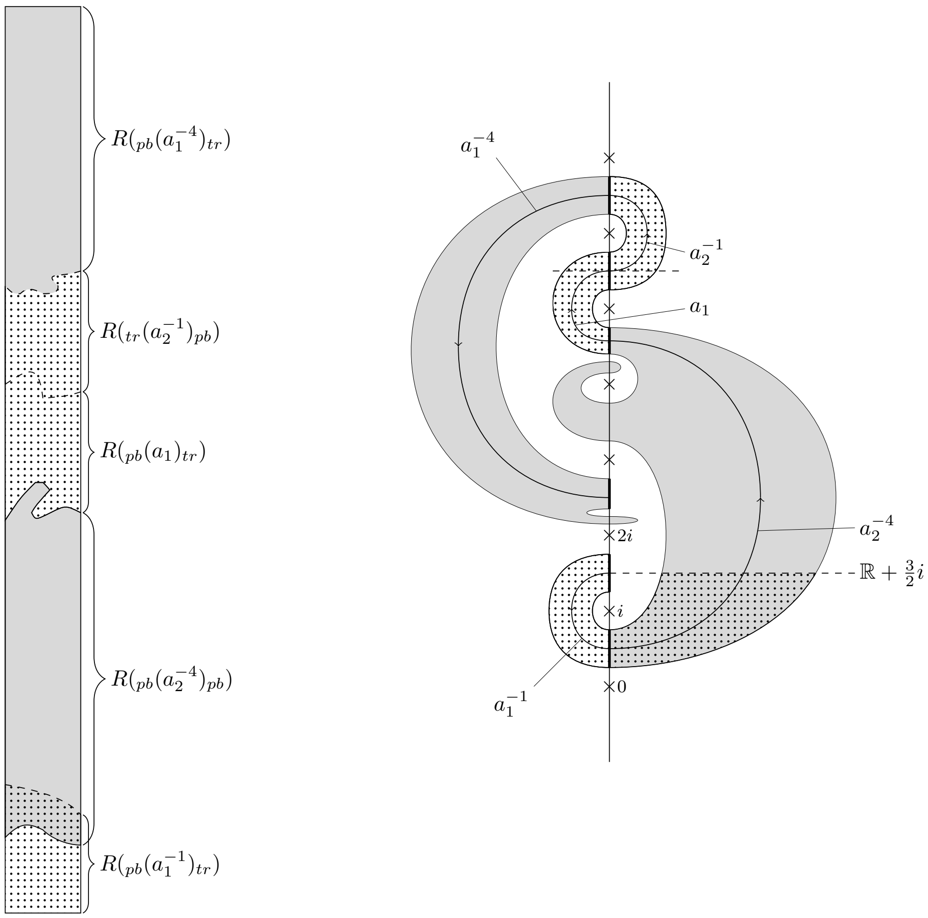

Figure 5

Proof of the lower bound in Theorem 1 and in Theorem $1'$**.** Identify w with $\mathfrak{C}_*(b) \in \pi_1$. Let g be a locally conformal mapping from a rectangle R with sides parallel to the axes into $\mathbb{C} \setminus \{-1, 1\}$ which represents the word w in the relative fundamental group of the twice punctured complex plane with tr, pb, or mixed horizontal boundary values. We assume first that the word has at least two syllables. Let Γ be the family of rectifiable curves in R which join the two horizontal sides.

Suppose first that the word has at least one syllable of form (1). Let I be the set of natural numbers for which the j-th term of w is a syllable of form (1) and let R_j be the curvilinear rectangles of statement (1) of Lemma 11. For $j \in I$ we denote by Γ_j the family of rectifiable arcs in R_j which join the two horizontal curvilinear sides of R_j. In Ahlfors' notation we have $\sum_{j \in I} \Gamma_j < \Gamma$. By Ahlfors' Therorems A and B we obtain

$$\sum_{j \in I} \lambda(\Gamma_j) \leq \lambda(\Gamma)\,.$$

The extremal length $\lambda(\Gamma_j)$ is estimated from below as follows. Let $\mathcal{R}_j$ be a rectangle with sides parallel to the axis which is conformally equivalent to R_j (i.e. there is a conformal mapping $\psi_j : R_j \to \mathcal{R}_j$ which maps vertical sides to vertical sides and curvilinear horizontal sides to horizontal sides). Let Γ_j^* be the push forward of Γ_j under the conformal mapping. The extremal length is a conformal invariant (see [1]) hence $\lambda(\Gamma_j^*) = \lambda(\Gamma_j)$, and $\lambda(\Gamma_j^*)$ is the extremal length of the rectangle $\mathcal{R}_j$. The rectangle $\mathcal{R}_j$ admits a holomorphic map $(g \mid R_j) \circ (\psi_j)^{-1}$ to $\mathbb{C} \setminus \{-1, 1\}$ which represents the syllable corresponding to R_j with *pb* boundary values if the syllable is not at the end of the word, and, possibly, with mixed boundary values if the syllable is at the end of the word.

Hence, by Proposition 6 the extremal length of $\mathcal{R}_j$ is not smaller than $\frac{1}{\pi}\log(4d_j - 1)$ or $\frac{2}{\pi}\log(2d_j - 1) > \frac{1}{\pi}\log(4d_j - 1)$ for $d_j \geq 2$. Here d_j is the degree of the respective syllable.

Hence,

$$\frac{1}{2}\lambda(\Gamma) \geq \frac{1}{2}\sum_{j\in I}\lambda(\Gamma_j) \geq \sum_{j\in I}\frac{1}{2\pi}\log(4d_j - 1)\,. \tag{13}$$

Suppose again that w contains at least one syllable of form (1). Denote by J the set of all natural numbers j for which $\mathfrak{s}_j$ is not of form (1). For $j \in J$ we denote by R_j' the curvilinear rectangle of statement 2 of Lemma 11 corresponding to $\mathfrak{s}_j$ (or to a syllable with one more letter than $\mathfrak{s}_j$). Let Γ_j' be the family of rectifiable curves in R_j' which join the two curvilinear horizontal sides.

By Ahlfors' Theorems

$$\sum_{j\in J}\lambda(\Gamma_j') \leq \lambda(\Gamma)\,.$$

By the same arguments as before and by Proposition 6 the extremal length of R_j' is not smaller than $\frac{1}{\pi}\log(4d_j - 1)$ since $g \mid R_j'$ has mixed boundary values. (If R_j' represents a syllable with one letter more than $\mathfrak{s}_j$ then the lower bound is $\frac{1}{\pi}\log(4(d_j+1) - 1)$.) As before $d_j \geq 1$ is the degree of the syllable $\mathfrak{s}_j$. Hence,

$$\frac{1}{2}\lambda(\Gamma) \geq \frac{1}{2}\sum_{j\in J}\lambda(\Gamma_j') \geq \sum_{j\in J}\frac{1}{2\pi}\log(4d_j - 1)\,. \tag{14}$$

For the case when w contains syllables of form (1) and syllables not of form (1) we add the two inequalities (13) and (14). We obtain

$$\lambda(\Gamma) \geq \sum_{j\in J}\frac{1}{2\pi}\log(4d_j - 1) + \sum_{j\in I}\frac{1}{2\pi}\log(4d_j - 1) = \sum_{\mathfrak{s}_j}\frac{1}{2\pi}\log(4d_j - 1)\,. \tag{15}$$

The last sum is extended over all syllables $\mathfrak{s}_j$ of the word w, and d_j is the degree of $\mathfrak{s}_j$.

Suppose that the word does not contain syllables of form (1) and the syllables are labeled from left to right by $j = 1, \ldots, N$. Let R_j' be the rectangles from statement 3 of Lemma 11 and let Γ_j' be the family of curves in R_j' joining the pair of horizontal sides of R_j'.

Then by the same arguments as before

$$\lambda(\Gamma) \geq \sum_{j=1}^{N-1}\lambda(\Gamma_j') \geq \sum_{j=1}^{N-1}\frac{1}{\pi}\log(4d_j - 1)\,. \tag{16}$$

On the other hand, with the curvilinear rectangles R_j'' from statement 3 of Lemma 11 and the family Γ_j'' of curves in R_j'' joining the pair of horizontal sides of R_j'' we obtain the inequality

$$\lambda(\Gamma) \geq \sum_{j=2}^{N} \lambda(\Gamma''_j) \geq \sum_{j=2}^{N} \frac{1}{\pi} \log(4d_j - 1)\,. \tag{17}$$

It follows that for any word $w \in \pi_1$ with at least two syllables for the respective family Γ the inequality

$$\lambda(\Gamma) \geq \sum_{\mathfrak{s}_j} \frac{1}{2\pi} \log(4d_j - 1) \tag{18}$$

holds.

If the word consists of a single syllable Proposition, 6 implies (18) in the case of mixed boundary values as well as in the non-exceptional cases with both boundary values being tr or pb.

Consider the exceptional cases. If $w = a_1^n$ with $n > 0$ the mapping $\zeta \to -1+e^{\zeta}$, $\zeta \in R$, with $R = \{\xi + i\eta,\ \xi \in (-\infty, 0),\ \eta \in (0, 2\pi n)\}$, represents w_{tr}. Hence, $\Lambda(w_{tr}) = 0$.

If $w = a_1 a_2 \ldots$ and has degree $d \geq 2$ then the mapping $\zeta \to e^{\zeta}$, $\zeta \in R$, with $R = \{\xi + i\eta,\ \xi \in (1, \infty),\ \eta \in (\frac{\pi}{2}, \frac{\pi}{2} + \pi d)\}$, represents w_{pb}. Hence, $\Lambda(w_{pb}) = 0$.

The other exceptional cases are similar. The lower bound in Theorem 1 and in Theorem $1'$ is proved. □

Proof of the lower bound in Theorem 2. Identify the class $\hat{w}$ with a conjugacy class of elements of the fundamental group π_1 of $\mathbb{C} \setminus \{-1, 1\}$ with base point 0. Let $\hat{w}$ be not among the exceptional cases of Theorem 2. Take any representative $w_1 \in \pi_1$ of $\hat{w}$ and write the infinite word $\ldots, w_1 w_1 w_1, \ldots$ in reduced form. The infinite word consists of different syllables. Take a locally conformal mapping $\hat{g}$ from an annulus A into the twice punctured complex plane $\mathbb{C} \setminus \{-1, 1\}$ that represents $\hat{w}$. The lift of $\hat{g}$ to the infinite strip S that covers A represents the infinite word. Apply Lemma 11 statement 1 to the infinite word and the infinite strip and project to the annulus A. We see that there is a connected component L_0^0 of the level set $L_0 = \{z \in A : \mathrm{Re} g(z) = 0\}$ that joins the two boundary circles of A such that $A \setminus L_0^0$ is a curvilinear rectangle and the restriction of $\hat{g}$ to it represents the syllable reduced word w with pb boundary values. Lemma 11 applies to the restriction $\hat{g} \mid (A \setminus L_0^0)$. As in the proof of Theorem 1 we obtain building blocks for estimating the lower bound of the extremal length. The estimates of the extremal length of the building blocks are put together in the same way as in the proof of Theorem 1.

Similarly as in the proof of the lower bound in Theorem 1 we see that in the exceptional cases the extremal length of $\hat{b}$ equals zero. □

Proof of Lemma 2. Consider the permutation $\tau_3(b)$ where τ_3 is the natural homomorphism from the braid group $\mathcal{B}_3$ to the symmetric group $\mathcal{S}_3$. When $\tau_3(b)$ is the identity then the braid is pure and the statement is clear.

Consider first the case when $\tau_3(b)$ is a transposition. If $\tau_3(b) = (13)$ then $b\Delta_3^{-1}$ is a pure braid, hence if b is not a power of Δ_3 it can written in the form (4) with k even and ℓ odd. If $\tau_3(b) = (12)$ then $\sigma_1^{-1} b$ is a pure braid. Hence, b can be written in the form (4) with $j = 1$, k odd and ℓ even. If $\tau_3(b) = (23)$ then $\sigma_2^{-1} b$ is a pure braid. Hence, b can be written in the form (4) with $j = 2$, k odd and ℓ even.

Consider the case when $\tau_3(b)$ is a cycle. Suppose, $\tau_3(b) = (123)$. Since $(123)(13) = (12)$ the relation $\tau_3(b)(13)^{-1} = (12)$ holds. Hence, $\sigma_1^{-1} b \Delta_3^{-1}$ is a pure braid and b has the form (4) with $j = 1$, and k and ℓ odd. If $\tau_3(b) = (132)$ then, since $(132)(13) = (23)$ the braid $\sigma_2^{-1} b \Delta_3^{-1}$ is a pure braid and b can be written in the form (4) with $j = 2$, and k and ℓ odd. $\square$

Proof of the lower bound in Theorem 3. Since $\Lambda(b_{tr}) = \Lambda((b\Delta_3)_{tr})$ we may suppose that $b = \sigma_j^k b_1$ for a pure braid b_1 which is a reduced word in σ_1^2 and σ_2^2. We may suppose that k is an odd integer. The case when k is even is contained in Theorem 1. Let $j = 1$. Consider a holomorphic mapping g of a rectangle to $C_3(\mathbb{C}) \diagup \mathcal{S}_3$ that represents the braid and lift it to a mapping $\tilde{g}$ into $C_3(\mathbb{C})$ so that the extension of the map to the closed rectangle maps the open lower side to $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 < x_1 < x_3\}$. Compose the mapping with the mapping $\mathfrak{C}$. We obtain a holomorphic mapping $\mathfrak{C}(\tilde{g})$ of the rectangle to $\mathbb{C} \setminus \{-1, 1\}$ whose continuous extension to the closure maps the open lower side of the rectangle to $(-\infty, -1)$ and the open upper side to $(-1, 1)$.

Consider first the case when $|k| \geq 3$. By Lemma 10 the restriction of the mapping $\mathfrak{C}(\tilde{g})$ to the closure of an arbitrary maximal vertical segment in the rectangle is the composition of the following two curves: an arc that is homotopic in $\mathbb{C} \setminus \{-1, 1\}$ to a half-circle in the upper half-plane if $k > 0$, or in the lower half-plane if $k < 0$, which joins a point in $(-\infty, -1)$ with a point in $(-1, 1)$, and a curve representing the element $\mathfrak{C}_*(\vartheta(b)_{tr}) \in \pi_1^{tr}$. Hence, by Lemma 11 the rectangle contains a curvilinear rectangle such that the restriction of the mapping $\mathfrak{C}(\tilde{g})$ to it represents $\mathfrak{C}_*(\vartheta(b)_{tr})$. Therefore $\Lambda(b_{tr}) \geq \Lambda(\vartheta(b)_{tr})$. If $\vartheta(b) \neq \sigma_1^{2k'}$ for an integer k', in other words, if $\vartheta(b)$ is not among the exceptional cases of Theorem 1 the lower bound holds.

Suppose $|k| = 1$. If $b \neq \sigma_1^{\pm 1}$ then as in Lemma 10 the restriction of the mapping $\mathfrak{C}(\tilde{g})$ to the closure of each maximal vertical segment in the rectangle is the composition of an arc that is homotopic in $\mathbb{C} \setminus \{-1, 1\}$ to a quarter-circle in the upper or lower half-plane which joins a point in $(-\infty, -1)$ with a point in $i\mathbb{R}$, and a curve that represents ${}_{pb}\mathfrak{C}_*(\vartheta(b)_{tr}$. As in Lemma 11 the rectangle contains a curvilinear rectangle such that the restriction of the mapping $\mathfrak{C}(\tilde{g})$ to it represents ${}_{pb}\mathfrak{C}_*(\vartheta(b))_{tr}$. The lower bound for this case follows from Theorem $1'$.

The proof for the case $b = \sigma_2^k b_1$ (with b_1 a reduced word in σ_1^2 and σ_2^2) is similar and is left to the reader. In this case the lift to $C_3(\mathbb{C})$ of a representing mapping for b_{tr} is chosen with initial point in $\{x_1 < x_3 < x_2\}$.

The lower bound of Theorem 3 is proved. $\square$

Remark. *For each conjugacy class $\hat{w}$ of elements of π_1 with at least two syllables and any syllable reduced representative $w \in \pi_1$ of $\hat{w}$ the inequality*

$$\Lambda(w_{pb}) \leq \Lambda(\hat{w}) \tag{19}$$

holds, but the inequality may be strict.

Indeed, let $\hat{g} : A \to \mathbb{C} \setminus \{-1, 1\}$ be a locally conformal mapping representing $\hat{w}$. As in the proof of Theorem 2 there is a connected component L_0^0 of the level set $L_0 = \{z \in A : \text{Re} g(z) = 0\}$ that joins the two boundary circles of A such that $R \stackrel{def}{=} A \setminus L_0^0$ is a curvilinear rectangle and the restriction of $\hat{g}$ to R represents the syllable reduced word w with pb boundary values.

Denote by Γ_A the set of loops in A representing the positively oriented generator of the fundamental group of A and by Γ_R the set of arcs in R whose extensions to the boundary of R join the lower side coming from L_0^0 with the respective upper side. Then the relation $\Gamma_R < \Gamma_A$ holds, but not vice versa. (The elements of Γ_R may not close up to circles!) We obtain (19), but there are examples when the inequality is strict.

There is a statement analogous to the remark that concerns the extremal length of w_{tr} (instead of w_{pb}). Take any conjugacy class $\hat{w}$ of elements of π_1 except conjugates of $(a_1\, a_2)^n$ for an integer $n \neq 0$. Then each loop representing $\hat{w}$ intersects the interval $(-1, 1)$. Let $\hat{g} : A \to \mathbb{C} \setminus \{-1, 1\}$ be a locally conformal mapping on an annulus A representing $\hat{w}$. As in the proof of Theorem 2 there is a connected component L_0^0 of the level set $L_0 = \{z \in A : g(z) \in (-1, 1)\}$ that joins the two boundary circles of A. Let w be the element of π_1^{tr} represented by the restriction $\hat{g} \mid (A \setminus L_0^0)$. Then

$$\Lambda(\hat{w}) \geq \Lambda(w_{tr})$$

and the inequality may be strict.

5. The upper bound for the extremal length. Plan of proof.

Consider first the case of the extremal length with pb boundary values. Take a word w in the relative fundamental group ${}^{pb}\pi_1^{pb}$ and consider its lift under $f_1 \circ f_2$ to a homotopy class of slalom curves. We will represent this slalom class by a holomorphic mapping of a rectangle into $\mathbb{C} \setminus i\mathbb{Z}$. The extremal length of the rectangle provides an upper bound of the extremal length of w with pb boundary values.

For each syllable $\mathfrak{s}_j$ of w (labeled from left to right) we represent the lift under $f_1 \circ f_2$ of ${}_{pb}(\mathfrak{s}_j)_{pb}$ by a holomorphic mapping $\overset{\circ}{g}_j$ of a rectangle $\overset{\circ}{R}_j$ into $\mathbb{C} \setminus i\mathbb{Z}$. The rectangle $\overset{\circ}{R}_j$ is chosen as always with sides parallel to the axes. We choose two points $\overset{\circ}{p}_j{}^-$ and $\overset{\circ}{p}_j{}^+$ in the boundary of $\overset{\circ}{R}_j$. The rectangle $\overset{\circ}{R}_j$ will be normalized as follows. If a standard representative of the syllable, or a standard representative of the first term of the syllable, respectively, is contained in the right half-plane, then the derivative $\overset{\circ}{g}_j{}'$ at $\overset{\circ}{p}_j{}^-$ maps i to the positive unit vector. It maps i to the negative unit vector in the remaining case. Similarly, the derivative $\overset{\circ}{g}_j{}'$ at $\overset{\circ}{p}_j{}^+$ maps i to the negative unit vector if a standard representative of the syllable, or a standard representative of the last term of the syllable, respectively, is contained in the right half-plane, and maps i to the positive unit vector in the remaining case.

Each rectangle $\overset{\circ}{R}_j$ will be shifted by a complex number c_j so that $\overset{\circ}{p}_j{}^+ + c_j = \overset{\circ}{p}_{j+1}{}^- + c_{j+1}$. Put $p_j{}^\pm = \overset{\circ}{p}_j{}^\pm + c_j$ and $R_j = \overset{\circ}{R}_j + c_j$. Consider the "shifted" functions g_j, $g_j(z + c_j) = \overset{\circ}{g}_j(z) + m_j$. The integers m_j are chosen so that $g_j(p_j^+) = g_{j+1}(p_{j+1}^-)$. Since for each j a standard representative of the (last term of the) syllable $\mathfrak{s}_j$ and a standard representative of the (first term of the) syllable $\mathfrak{s}_{j+1}$ are in different half-planes the derivatives of g_j and g_{j+1} at $p_j^+ = p_{j+1}^-$ coincide. The shifted functions g_j will be patched together in a quasiconformal way using an estimate for the second derivatives of the g_j. The obtained function is denoted by g. The union of the closures of the rectangles $\bar{R}_j$ will contain a curvilinear rectangle of the form $R_{J,\Phi,\frac{1}{18}}$. Here for a real C^1-function Φ on a given open interval J and a positive number b we consider the curvilinear rectangle which is defined as $R_{J,\Phi,\mathsf{b}} \overset{def}{=} \{x + iy \in \mathbb{C} : y \in J,\, x \in (\Phi(y), \Phi(y) + \mathsf{b})\}$. The construction produces a quasiconformal mapping of $R_{J,\Phi,\frac{1}{18}}$ into $\mathbb{C} \setminus i\mathbb{Z}$ that represents the lift of ${}_{pb}(w)_{pb}$ under $f_1 \circ f_2$. An upper bound of the vertical length of each R_j, an estimate of the quasiconformal dilatation of the function g and the following lemma will provide an upper bound of the extremal length of $R_{J,\Phi,\frac{1}{18}}$, and, hence of $\Lambda_p(w)$.

Lemma 12. *Let Φ be a real C^1-function on an open interval J and let b be a positive number. Denote by Γ_Φ the set of curves in the rectangle $R_{J,\Phi,\mathsf{b}} = \{x + iy \in \mathbb{C} : y \in J,\, x \in (\Phi(y), \Phi(y) + \mathsf{b})\}$ which join the two horizontal sides. Suppose the absolute value $|\Phi'|$ of the derivative of Φ is bounded by the constant C. Then*

$$\lambda(\Gamma_\Phi) \le (1 + C^2)\lambda(\Gamma_0),$$

where Γ_0 is the family corresponding to the function which is identically equal to zero.

Proof. The proof is similar to Example 1 in chapter 1 of [1]. For any measurable function ϱ on $\mathbb{C}$ and any $x \in (0, \mathsf{b})$ we have

$$\int_J \varrho(x + \Phi(y) + iy)\sqrt{1 + \Phi'(y)^2}dy \ge L_{\Gamma_\Phi}(\varrho).$$

Integrate over the interval $(0, \mathsf{b})$ and apply Fubini's Theorem and Hölder's inequality. Using the bound for $|\Phi'|$ we obtain

$$\Big(\int\int_{R_{\Phi,\mathsf{b}}} dm_2 \int\int_{R_{\Phi,\mathsf{b}}} \varrho^2 \cdot (1 + C^2)dm_2\Big)^{\frac{1}{2}} \ge \mathsf{b}L_{\Gamma_\Phi}(\varrho).$$

Denote by $|J|$ the length of the interval J. We obtain

$$\mathsf{b}|J|(1 + C^2)A(\varrho) \ge \mathsf{b}^2 L_{\Gamma_\Phi}(\varrho)^2.$$

Hence,

$$\frac{L_\Phi(\varrho)^2}{A(\varrho)} \le \frac{|J|}{\mathsf{b}} \cdot (1 + C^2) = \lambda(\Gamma_0)(1 + C^2).$$

Taking the supremum over all measurable functions ϱ with finite non-vanishing integral we obtain

$$\lambda(\Gamma_\Phi) \le (1 + C^2)\lambda(\Gamma_0).$$

The lemma is proved. □

The basis for the construction of the rectangles R_j and the mappings g_j is the elliptic integral $\mathcal{F}_M$ (see equality (7) in section 3) for integers and half-integers M. Recall that we take the branch of the square root which is positive on the positive real half-axis. The mapping $\mathcal{F}_M$ maps the left half-plane to a rectangle R_M contained in the left half-plane with sides parallel to the axis . The mapping extends to a homeomorphism of the union of the closed half plane with ∞ which maps the interval $[-i(M + 1), -iM]$ to the lower side and $[iM, i(M + 1)]$ to the upper side of the rectangle, and it maps the interval $[-iM, iM]$ into the imaginary axis. Further, 0 is mapped to 0 which is the midpoint of one vertical side of the rectangle, and ∞ is mapped to the midpoint of the other vertical side. Moreover,

$$\mathcal{F}_M(-z) = -\mathcal{F}_M(z) \text{ for } z \in (-iM, iM).$$

By the reflection principle, $\mathcal{F}_M$ extends holomorphically across each of the intervals $(-i(M + 1), -iM)$, $(-iM, iM)$, $(iM, i(M + 1))$, and across the complement of $[-i(M + 1), i(M + 1)]$ in $\mathbb{R} \cup \infty$, to a mapping from the right half-plane to the respective reflected rectangle. We will denote the extension of $\mathcal{F}_M$ also by the same letter $\mathcal{F}_M$.

Similarly, the integral, given by (7) defines a conformal mapping from the right half-plane onto a rectangle contained in the right half-plane. Again, we use the branch of the square root which is positive on the positive real half-axis. The rectangle is the reflection in the imaginary axis of the rectangle obtained in the previous case. We denote the mapping given by (7) on the right half-plane by $\mathcal{F}^r_M$.

The following lemma provides an estimate of the horizontal side length of the rectangle R_M.

Lemma 13. *For $M \geq \frac{1}{2}$ the following inequalities hold*

$$\frac{\pi}{2(M+1)} \leq |\mathcal{F}_M(i(M+1)) - \mathcal{F}_M(iM)| \leq \frac{\pi}{2M}. \tag{20}$$

Proof. For $X \in (0,1)$ we have

$$\begin{aligned}\mathcal{F}_M(i(M+X)) - \mathcal{F}_M(iM) &= \int_{iM}^{i(M+X)} \frac{d\zeta}{\sqrt{(iM-\zeta)(iM+\zeta)(i(M+1)+\zeta)(i(M+1)-\zeta)}} \\ &= i\int_M^{M+X} \frac{dw}{\sqrt{(M-w)(M+1-w)(M+w)(M+1+w)}} \\ &= i\int_0^X \frac{dx}{\sqrt{x(x-1)c(M,x)}}. \end{aligned} \tag{21}$$

Here $c(M,x) \stackrel{\text{def}}{=} (2M+x)(2M+1+x)$. Integration is along the interval $(0,X) \subset (0,1)$ and we have chosen the branch of the square root which is positive on the positive part of the real axis. The term $c(M,x)$ is positive and its square root is contained in the interval $(2M, 2M+2)$. With the choice of the branch of the square root we have $\sqrt{x(x-1)} = -i\sqrt{x(1-x)}$, $x \in (0,1)$.

For $X = 1$ we obtain

$$\mathcal{F}_M(i(M+1)) - \mathcal{F}_M(iM) = -\int_0^1 \frac{dx}{\sqrt{x(1-x)c(M,x)}}$$

Since

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = 2\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{x(1-x)}} = 2\int_0^{\sqrt{\frac{1}{2}}} \frac{2udu}{\sqrt{u^2(1-u^2)}} =$$
$$4\int_0^{\sqrt{\frac{1}{2}}} \frac{du}{\sqrt{1-u^2}} = 4\arcsin\sqrt{\frac{1}{2}} = \pi,$$

we obtain (20). □

The following lemma gives an estimate of the vertical side length $\frac{2}{i}\mathcal{F}_M(iM)$ of the rectangle R_M for the case $M \geq 2$.

Lemma 14. *For $M \geq 2$ the inequalities*

$$\frac{1}{2}\log(2M+1) < \frac{M+1}{i}\mathcal{F}_M(iM) = K(\frac{M}{M+1}) < \frac{1}{2}\big(\frac{12}{5} + \log(2M+1)\big) \tag{22}$$

hold.

Proof. Use the following estimate for the denominator in (8)

$$(1-x)^2 < \sqrt{(1-x^2)(1-(\frac{M}{M+1})^2x^2)} < 1 - (\frac{M}{M+1})^2x^2$$

with $x \in (-1,1)$. The lower bound in (22) is obtained as follows.

$$\frac{M+1}{i}\mathcal{F}_M(iM) > \int_0^1 \frac{dx}{1-(\frac{M}{M+1})^2x^2} = \frac{1}{2}\int_0^1 (\frac{dx}{1-\frac{M}{M+1}x} + \frac{dx}{1+\frac{M}{M+1}x})$$

$$= \frac{1}{2\frac{M}{M+1}}(\log(1+\frac{M}{M+1}) - \log(1-\frac{M}{M+1})) = \frac{M+1}{2M}\log(2M+1)$$
$$> \frac{1}{2}\log(2M+1). \tag{23}$$

For the estimate from above we write

$$\frac{M+1}{i}\mathcal{F}_M(iM) = \int_0^{1-\frac{1}{M+1}} + \int_{1-\frac{1}{M+1}}^1 \overset{\text{def}}{=} \mathcal{I}_1 + \mathcal{I}_2.$$

We obtain for the term $\mathcal{I}_1$

$$\mathcal{I}_1 < \int_0^{1-\frac{1}{M+1}} \frac{dx}{1-x^2} = \frac{1}{2}\int_0^{1-\frac{1}{M+1}} (\frac{dx}{1-x} + \frac{dx}{1+x}) =$$
$$\frac{1}{2}(-\log(1-(1-\frac{1}{M+1})) + \log(1+(1-\frac{1}{M+1}))) =$$
$$\frac{1}{2}\log(2M+1)$$

For the estimate of $\mathcal{I}_2$ we use the fact that for $1-\frac{1}{M+1} \le x \le 1$ the denominator $\sqrt{(1-x^2)(1-(\frac{M}{M+1}x)^2)}$ is not smaller than

$$\sqrt{1-x}\sqrt{2-\frac{1}{M+1}}\sqrt{1-(\frac{M}{M+1})^2}.$$

Hence,

$$\mathcal{I}_2 \le \frac{\sqrt{(M+1)^3}}{\sqrt{2M+1}\sqrt{(M+1)^2-M^2}} \int_{1-\frac{1}{M+1}}^1 \frac{dx}{\sqrt{1-x}}$$
$$= \frac{\sqrt{(M+1)^3}}{2M+1} \cdot 2 \cdot \sqrt{\frac{1}{M+1}} = 2 \cdot \frac{M+1}{2M+1} \le 2 \cdot \frac{3}{5} \tag{24}$$

for $M \ge 2$. For the last inequality we used that for $M \ge 2$ the inequality $\frac{M+1}{2M+1} \le \frac{3}{5}$ holds.

The lemma is proved. □

The inverse mappings of the versions of the elliptic integral represent classes of curves as follows. The integral (7) which defines $\mathcal{F}_M(z)$ with $z \in \mathbb{C}_\ell$ is the inverse of a conformal mapping of a rectangle R_M representing the class $\gamma^*_{\ell,M}$. The class of curves $\gamma^*_{\ell,M,-}$ obtained from $\gamma^*_{\ell,M}$ by inverting the orientation is represented by the inverse of the mapping $z \to \mathcal{F}^-_M(z) \overset{def}{=} -\mathcal{F}_M(z) - |\mathcal{F}_M(i(M+1)) - \mathcal{F}_M(iM)|$, $z \in \mathbb{C}_\ell$, which maps the left half-plane to the same rectangle R_M. (Here $|\mathcal{F}_M(i(M+1) - \mathcal{F}_M(iM)|$ is the horizontal side length of the rectangle $R_M = \mathcal{F}_M(\mathbb{C}_\ell)$.)

Consider the function $\mathcal{F}^r_M$ represented by the integral (7) with $z \in \mathbb{C}_r$. The inverse $(\mathcal{F}^r_M)^{-1}$ of $\mathcal{F}^r_M$ is a conformal mapping onto the right half-plane from a rectangle R^r_M in the right half-plane with one vertical side on the imaginary axis. The mapping $(\mathcal{F}^r_M)^{-1}$ represents the class $\gamma^*_{r,M,-}$ which is obtained from $\gamma^*_{r,M}$ by inverting orientation. The class $\gamma^*_{r,M}$ is represented by the inverse of the mapping $z \to \mathcal{F}^{r,-}_M(z) \overset{def}{=} -\mathcal{F}^r_M(z) + |\mathcal{F}_M(i(M+1) - \mathcal{F}_M(iM)|$, $z \in \mathbb{C}_r$, which is defined on the same rectangle R^r_M.

6. Holomorphic maps representing lifts of syllables with pb boundary values

1. Syllables of form (1) with pb boundary values of degree at least 5. We consider first syllables of the form a^n with pb boundary values where a is one of the standard generators or its inverse and $n \geq 5$. Recall that $d = n$ is the degree of the syllable in this case. Assume $a = a_1$. All other cases are treated in the same way. According to Proposition 6 the class of representatives of the syllable with pb boundary values lifts to the class $\gamma^*_{M,\ell} + iM$ with $M = \frac{d-1}{2}$. Notice that with the choice of n the parameter M is at least 2. Let $\mathcal{F}_M, z \in \mathbb{C}_\ell$, be the mapping (7) defined in section 3 and let $\mathcal{F}_M^{-1}$ be its inverse. The derivative $\mathcal{F}'_M(\pm i(M+\frac{1}{2}))$ is equal to

$$\begin{aligned}\mathcal{F}'_M(\pm i(M+\frac{1}{2})) &= \frac{1}{\sqrt{((i(M+\frac{1}{2}))^2-(iM)^2)((i(M+\frac{1}{2}))^2-(i(M+1))^2)}}\\ &= \frac{1}{\sqrt{((M+\frac{1}{2})^2-M^2)((M+\frac{1}{2})^2-(M+1)^2)}}\\ &= \frac{\pm i}{\sqrt{(M+\frac{1}{4})(M+\frac{3}{4})}}\,. \end{aligned}\tag{25}$$

Put

$$\begin{aligned} r_{(1),M} &\overset{def}{=} \sqrt{(M+\frac{1}{4})(M+\frac{3}{4})}\,,\\ R_{(1),M} &\overset{def}{=} r_{(1),M}\,R_M\,,\\ f_{(1),M} &\overset{def}{=} r_{(1),M}\,\mathcal{F}_M \qquad \text{, and}\\ g_{(1),M} &\overset{def}{=} (f_{(1),M})^{-1}. \end{aligned}\tag{26}$$

Let $\xi^{\pm}_M$ be the point $f_{(1),M}(\pm i(M+\frac{1}{2}))$ on the upper (lower, respectively) horizontal side of the rectangle. Then $(g_{(1),M})'(\xi^{\pm}_M) = \frac{1}{(f_{(1),M})'(\pm i(M+\frac{1}{2}))} = \mp i$. Thus $g'_{(1),M}(\xi^+_M)$ maps i to the unit vector in positive direction and $g'_{(1),M}(\xi^-)$ maps i to the unit vector in negative direction. Moreover, $g_{(1),M}$ maps $R_{(1),M}$ onto $\mathbb{C}_\ell$ so that the upper horizontal side is mapped to $(iM, i(M+1))$, the lower horizontal side is mapped to $(-i(M+1), iM)$, and the points $\xi^{\pm}_M$ are mapped to $\pm i(M+\frac{1}{2})$.

The following lemma holds.

Lemma 15. *For $\zeta \in \mathbb{C}$, $|\zeta \pm i(M+\frac{1}{2})| < \frac{1.03\sqrt{2}}{18}$, the inequality*

$$|\frac{1}{(f_{(1),M})'(\zeta)}| < 1.03$$

holds.

Proof. Put $\zeta = i(M+\frac{1}{2}) + z$ with $|z| < \frac{1.03\sqrt{2}}{18}$. For the function $\mathcal{F}_M$ (see (7)) we have

$$(\frac{1}{\mathcal{F}'(i(M+\frac{1}{2})+z)})^2$$

$$
\begin{aligned}
&= ((i(M+\frac{1}{2})+z)^2-(iM)^2)((i(M+\frac{1}{2})+z)^2-(i(M+1))^2)\\
&= ((M+\frac{1}{2}-iz)^2-M^2)((M+\frac{1}{2}-iz)^2-(M+1)^2)\\
&= (M+\frac{1}{2}-iz-M)(M+\frac{1}{2}-iz+M)(M+\frac{1}{2}-iz-(M+1))(M+\frac{1}{2}-iz+(M+1))\\
&= (\frac{1}{2}-iz)(-\frac{1}{2}-iz)\big((2M+\frac{1}{2})-iz\big)\big((2M+\frac{3}{2})-iz\big). \qquad (27)
\end{aligned}
$$

We obtained

$$
\frac{1}{\mathcal{F_M}'(i(M+\frac{1}{2})+z)} = -i\sqrt{(\frac{1}{4}+z^2)\big(2M+\frac{1}{2}-iz\big)\big(2M+\frac{3}{2}-iz\big)}, \qquad (28)
$$

hence, for $|z| < \frac{1.03\sqrt{2}}{18}$

$$
\begin{aligned}
&|\frac{1}{(f_{(1),M})'(i(M+\frac{1}{2})+z)}|\\
&\leq \frac{\sqrt{\frac{1}{4}+2\frac{1.03^2}{18^2}}\sqrt{2M+\frac{1}{2}+\frac{1.03\cdot\sqrt{2}}{18}}\sqrt{2M+\frac{3}{2}+\frac{1.03\cdot\sqrt{2}}{18}}}{\sqrt{(M+\frac{1}{4})(M+\frac{3}{4})}}. \qquad (29)
\end{aligned}
$$

The expression on the right hand side of (29) is decreasing in M, hence for $M \geq 2$ it does not exceed its value at 2, i.e.

$$
\begin{aligned}
|\frac{1}{(f_{(1),M})'(i(M+\frac{1}{2})+z)}| &\leq 4\frac{\sqrt{\frac{1}{4}+2\frac{1.03^2}{18^2}}\sqrt{\frac{9}{2}+\frac{1.03\sqrt{2}}{18}}\sqrt{\frac{11}{2}+\frac{1.03\sqrt{2}}{18}}}{\sqrt{99}}\\
&< 1.0296 < 1.03\,. \qquad (30)
\end{aligned}
$$

The estimate for $i(M+\frac{1}{2})$ replaced by $-i(M+\frac{1}{2})$ is the same.

Corollary 3. *The inverse $g_{(1),M} = (f_{(1),M})^{-1}$ of $f_{(1),M}$ maps the sets $\{|\xi \mp \xi_M^{\pm}| < \frac{\sqrt{2}}{18}\}$ into the sets $\{|\zeta \mp i(M+\frac{1}{2})| < \frac{1.03\cdot\sqrt{2}}{18}\}$. The following estimate holds for the derivative of the inverse*

$$
|(g_{(1),M})'(\xi)| < 1.03 \qquad (31)
$$

for $|\xi - \xi_M^{\pm}| < \frac{\sqrt{2}}{18}$.

Proof. If for $\zeta = g_{(1),M}(\xi)$ the inequality $|\zeta \mp i(M+\frac{1}{2})| < \frac{1.03\sqrt{2}}{18}$ holds we have $|(g_{(1),M})'(\xi)| = |\frac{1}{(f_{(1),M})'((f_{(1),M})^{-1}(\xi))}| < 1.03$. Suppose (31) is not true for some ξ, $|\xi - \xi_M^{\pm}| < \frac{\sqrt{2}}{18}$. Since the inequality (31) holds at $\xi_M^{\pm}$ there is a point $\xi'_{\pm}$, $|\xi'_{\pm} - \xi_M^{\pm}| < \frac{\sqrt{2}}{18}$, of closest distance to $\xi_M^{\pm}$ among points for which (31) is violated. Then $|(g_{(1),M})'(\xi'_{\pm})| = 1.03$ and (31) is satisfied for all ξ with $|\xi - \xi_M^{\pm}| < |\xi'_{\pm} - \xi_M^{\pm}|$. Write $g_{(1),M}(\xi'_{\pm}) - g_{(1),M}(\xi_M^{\pm}) = \int_{\xi_M^{\pm}}^{\xi'_{\pm}} (g_{(1),M})'(\xi)d\xi$ where integration in the last integral is along the straight line segment $[\xi_M^{\pm}, \xi'_{\pm}]$ joining the two points. The estimate of the length of the segment and the fact that (31) is satisfied on the open segment of integration yields $|g_{(1),M}(\xi'_{\pm}) - g_{(1),M}(\xi_M^{\pm})| < 1.03 \cdot \frac{\sqrt{2}}{18}$ which contradicts the assumption. The corollary is proved. □

The corollary implies that $g_{(1),M}$ maps the sets $E^{\pm}_{(1),M} \overset{def}{=} \{\xi \in \mathbb{C} : |\xi - \xi^{\pm}_M| < \frac{\sqrt{2}}{18}\}$ into $\mathbb{C} \setminus i\mathbb{Z}$. Indeed, the image $g_{(1),M}$ of a point $\xi \in E^{\pm}_{(1),M}$ has distance less than $\frac{1.03\sqrt{2}}{18} < \frac{1}{2}$ from $\pm i(M + \frac{1}{2})$.

To estimate the second derivative of $g_{(1),M}$ we need the following simple lemma.

Lemma 16. *For any conformal mapping f on a small disc Δ and its inverse f^{-1} the following formula holds*

$$(f^{-1})''(f(z)) = \frac{1}{2}\Big(\frac{1}{f'(z)^2}\Big)'.$$

Proof. Since $(f^{-1})'(f(z)) = \frac{1}{f'(z)}$, we obtain for the second derivative

$$(f^{-1})''(f(z)) \cdot f'(z) = (\frac{1}{f'(z)})'.$$

Hence

$$(f^{-1})''(f(z)) = \frac{1}{f'(z)} \cdot (\frac{1}{f'(z)})' = \frac{1}{2}\Big(\frac{1}{f'(z)^2}\Big)'.$$

□

Lemma 17. *For $\xi \in E^{\pm}_{(1),M}$ the following inequality holds for the second derivative of $g_{(1),M}$*

$$|(g_{(1),M})''(\xi)| < 2.75. \tag{32}$$

Proof. Under the condition of the lemma $\xi = f_{(1),M}(\pm i(M + \frac{1}{2}) + z)$ with $|z| < \frac{1.03\,\sqrt{2}}{18}$. By lemma 16

$$\begin{aligned}
&(g_{(1),M})''(\xi) = ((f_{(1),M})^{-1})''(f_{(1),M}(\pm i(M + \frac{1}{2}) + z))) \\
&= \frac{1}{2}\frac{1}{(r_{(1),M})^2} \cdot \Big(((M + \frac{1}{2} - iz)^2 - M^2)((M + \frac{1}{2} - iz)^2 - (M+1)^2)\Big)' \\
&= \frac{1}{2}\frac{1}{(r_{(1),M})^2} \cdot \frac{1}{((\mathcal{F}_M)'(\pm i(M + \frac{1}{2}) + z))^2} \cdot \Big(\frac{-i}{\frac{1}{2} - iz} + \frac{i}{\frac{1}{2} + iz} - \frac{i}{2M + \frac{1}{2} - iz} - \frac{i}{2M + \frac{3}{2} - iz}\Big).
\end{aligned}$$

Hence, (31) implies for $M \geq 2$

$$|(g_{(1),M})''(\xi)| \leq \frac{1}{2}1.03^2\Big(\frac{2}{\frac{1}{2} - \frac{1.03\sqrt{2}}{18}} + \frac{1}{\frac{9}{2} - \frac{1.03\sqrt{2}}{18}} + \frac{1}{\frac{11}{2} - \frac{1.03\sqrt{2}}{18}}\Big) < 2.75. \tag{33}$$

The lemma is proved. □

The relations (22) and (26) imply the following upper bound for the vertical side length $\mathrm{vsl}(R_{(1),M})$ of the normalized rectangle $R_{(1),M} = r_{(1),M}R_M$ with $M = \frac{d-1}{2} \geq 2$

$$\begin{aligned}
\mathrm{vsl}(R_{(1),M}) &\leq (\frac{12}{5} + \log(2M+1))\frac{\sqrt{(M + \frac{1}{4})(M + \frac{3}{4})}}{M+1} < \frac{12}{5} - \log(4 - \frac{1}{5}) + \log((4 - \frac{1}{5})d) \\
&< 0.362 \cdot \log 19 + \log(4d - 1) < 1.362 \cdot \log(4d - 1). \qquad (34)
\end{aligned}$$

Here $d = 2M + 1 \geq 5$, $(4 - \frac{1}{5})d < 4d - 1$ for $d \geq 5$. We used the inequality $\frac{12}{5} - \log(4 - \frac{1}{5}) < 0.362 \cdot \log 19$.

2. Syllables of form (2) with pb boundary values of degree at least 5. Consider first syllables of degree at least 5. Without loss of generality we consider the syllables $(a_2^{-1}a_1^{-1})^k a_2^{-1}$ or $(a_2^{-1}a_1^{-1})^{k+1}$ for $k \geq 2$ with pb boundary values. The degree d is equal to $2|k|+1$ or $2|k|+2$ respectively, and is at least 5. The other syllables of form (2) of degree at least 5 are treated in the same way. The lift under $f_1 \circ f_2$ of a suitable representative is contained in the punctured unit disc $\mathbb{D} \setminus \{0\}$, has endpoints on the imaginary axis and makes d half-turns around 0. This lift is a slalom curve which is the composition of $d \geq 5$ trivial elementary slalom curves.

We construct now a holomorphic mapping on a curvilinear rectangle which represents the homotopy class of this slalom curve. Take $M = \frac{d+1}{2}$. Restrict $\mathcal{F}_M$ to the domain $\{z \in \mathbb{C}_\ell : |\mathrm{Im} z| < M - \frac{1}{2}\}$. Recall that $\mathcal{F}_M$ maps $\mathbb{C}_\ell$ onto a rectangle R_M whose right vertical side is contained in the imaginary axis and has midpoint 0. The extension of $\mathcal{F}_M$ maps the point ∞ to the midpoint of the left vertical side of R_M. Each of the two lines $\{z \in \mathbb{C}_\ell : \mathrm{Im} z = \pm(M - \frac{1}{2})\}$ is mapped to a curve in R_M with one endpoint being a point on the right vertical side. The other endpoint is the image of ∞. Put $X_M \overset{def}{=} \mathcal{F}_M(\{z \in \mathbb{C}_\ell : |\mathrm{Im} z| < M - \frac{1}{2}\})$. (See also Figure 7 below.)

Let $r_{(2),M}$ be a positive constant that will be specified later. Put

$$
\begin{aligned}
&R_{(2),M} \overset{def}{=} r_{(2),M} R_M,\ \tilde{R}_{(2),M} \overset{def}{=} r_{(2),M} X_M \cap \{-\frac{1}{18} < \mathrm{Re}\xi < 0\},\\
&f_{(2),M} \overset{def}{=} r_{(2),M} \cdot \mathcal{F}_M \qquad \text{and}\\
&g_{(2),M}(\xi) \overset{def}{=} \exp(\pi((f_{(2),M})^{-1}(\xi) + i(M-1))). \qquad (35)
\end{aligned}
$$

Note that with these definitions $r_{(2),M} X_M = f_{(2),M}(\{z \in \mathbb{C}_\ell : |\mathrm{Im} z| < M - \frac{1}{2}\})$.

Lemma 18. *The set $\tilde{R}_{(2),M}$ is a curvilinear rectangle with horizontal sides $f_{(2),M}(\{z \in \mathbb{C}_\ell : Im z = \pm(M - \frac{1}{2})\}) \cap \{-\frac{1}{18} < Re\xi < 0\}$ being graphs over the a segment of the real axis of functions with absolute value of the derivative not exceeding* 0.05.

We postpone the proof of the lemma.

Restrict the mapping $(f_M)^{-1}$ to the curvilinear rectangle $\tilde{R}_{(2),M}$.

The mapping $\zeta \to \exp(\pi(\zeta + i(M-1)))$ maps the point $-i(M-\frac{1}{2})$ to $-i$. For each $x \in (-\infty, 0)$ the curve $y \to \exp(\pi(x + iy + i(M-1)))\,, y \in (-(M-\frac{1}{2}), (M-\frac{1}{2}))\,,$ runs along the circle of radius $e^x < 1$ and center 0. It has initial point in $(-i, 0)$ and makes $d = 2M-1$ half-turns around zero. Hence, the composition $\xi \to g_{(2),M}(\xi) = \exp(\pi(f_{(2),M})^{-1}(\xi) + i(M-1)),\ \xi \in \tilde{R}_{(2),M},$ represents a lift of the syllable. The image $g_{(2),M}(\tilde{R}_{(2),M})$ is contained in the punctured unit disc.

Put $\eta^\pm \overset{def}{=} f_{(2),M}(\pm i(M-\frac{1}{2}))$. Note that

$$
\begin{aligned}
(g_{(2),M})'(f_{(2),M}(\zeta)) &= \pi \exp(\pi((f_{(2),M})^{-1}(f_{(2),M}(\zeta)) + i(M-1))) \cdot ((f_{(2),M})^{-1})'(f_{(2),M}(\zeta))\\
&= \pi \exp(\pi(\zeta + i(M-1))) \cdot \frac{1}{(f_{(2),M})'(\zeta)} \qquad (36)
\end{aligned}
$$

The derivative of $g_{(2),M}$ at $\eta^\pm$ equals

$$
(g_{(2),M})'(\eta^\pm) = \pi \exp(\pm\pi i((M-\frac{1}{2}) + (M-1))) \cdot \frac{1}{r_{(2),M}(\mathcal{F}_M)'(\pm i(M-\frac{1}{2}))}.
$$

Here

$$\frac{1}{(\mathcal{F}_M)'(\pm i(M-\frac{1}{2}))} = \sqrt{((M-\frac{1}{2})^2 - M^2)((M-\frac{1}{2})^2-(M+1)^2)}$$
$$= \sqrt{3}\sqrt{(M-\frac{1}{4})(M+\frac{1}{4})}.$$

Put

$$r_{(2),M} = \pi\sqrt{3}\sqrt{(M-\frac{1}{4})(M+\frac{1}{4})}. \tag{37}$$

Then, $(g_{(2),M})'(\eta^-) = -i$ and $(g_{(2),M})'(\eta^+)$ equals i if d is odd and equals $-i$ if d is even.

Lemma 19. *For $\zeta \in \mathbb{C}$, $|\zeta \mp i(M-\frac{1}{2})| < \frac{0.4\sqrt{2}}{18}$, the inequality*

$$\frac{1}{|(f_{(2),M})'(\zeta)|} < 0.3343$$

holds.

Proof. Put $\zeta = \pm i(M-\frac{1}{2}) + z$. Then

$$\frac{1}{\mathcal{F}'_M(\zeta)^2} =$$
$$((M-\frac{1}{2}-iz)^2 - M^2)((M-\frac{1}{2}-iz)^2-(M+1)^2) =$$
$$(-\frac{1}{2}-iz)(-\frac{3}{2}-iz)(2M-\frac{1}{2}-iz)(2M+\frac{1}{2}-iz). \tag{38}$$

Hence for $|z| < \frac{0.4\sqrt{2}}{18}$

$$\frac{1}{|(f_{(2),M})'(\zeta)|} < \frac{2\sqrt{\frac{1}{2}+\frac{0.4\sqrt{2}}{18}}\sqrt{\frac{3}{2}+\frac{0.4\sqrt{2}}{18}}\sqrt{M+\frac{1}{4}+\frac{0.2\sqrt{2}}{18}}\sqrt{M-\frac{1}{4}+\frac{0.2\sqrt{2}}{18}}}{\pi\sqrt{3}\sqrt{(M+\frac{1}{4})(M-\frac{1}{4})}}. \tag{39}$$

Since the right hand side is decreasing in M, the expression on the right for $M \geq 2$ does not exceed the expression for $M = 2$, i.e. for $M \geq 2$

$$\frac{1}{|(f_{(2),M})'(\zeta)|} < \frac{8\sqrt{\frac{1}{2}+\frac{0.4\sqrt{2}}{18}}\sqrt{\frac{3}{2}+\frac{0.4\sqrt{2}}{18}}\sqrt{\frac{9}{4}+\frac{0.2\sqrt{2}}{18}}\sqrt{\frac{7}{4}+\frac{0.2\sqrt{2}}{18}}}{\pi\sqrt{3}\sqrt{63}} < 0.3343.$$

$\square$

The proof of the following corollary is the same as the proof of corollary 3.

Corollary 4. *The inverse $(f_{(2),M})^{-1}$ of $f_{(2),M}$ maps the sets $\{|\xi - \eta_M^\pm| < \frac{\sqrt{2}}{18}\}$ into the sets $\{|\zeta \mp i(M-\frac{1}{2})| < \frac{0.3343\cdot\sqrt{2}}{18}\}$. The derivative of the inverse satisfies the following inequality*

$$|((f_{(2),M})^{-1})'(\xi)| < 0.3343 \tag{40}$$

for $|\xi - \eta_M^\pm| < \frac{\sqrt{2}}{18}$.

The corollary says that the images $f_{(2),M}(\{|\zeta \mp i(M-\frac{1}{2})| < \frac{0.3343\cdot\sqrt{2}}{18}\})$ cover the sets $\{|\xi - \eta_M^\pm| < \frac{\sqrt{2}}{18}\}$. Since $0.3343 \cdot \frac{\sqrt{2}}{18} < 0.0263 < 0.03$, Lemma 18 is an immediate consequence of the following lemma.

Lemma 20. *For ζ in a $\frac{3}{100}$-neighbourhood of $\pm i(M-\frac{1}{2})$ the (principal branch) of the argument of $f'_{(2),M}(\zeta)$ satisfies the inequality*

$$|\arg(f'_{M,(2)}(\zeta))| < \arctan(0.05).$$

Proof of Lemma 20. We have

$$\arg(f'_{M,(2)}(\zeta)) = \frac{1}{2}\arg(f'_{M,(2)}(\zeta))^2 = -\frac{1}{2}\arg(f'_{M,(2)}(\zeta))^{-2}.$$

Also, $(f'_{M,(2)}(\zeta))^{-2}$ is the product of four complex numbers, hence its argument is the sum of the arguments of the factors provided all these arguments are small. For $|z| < \frac{3}{100}$ and $M \geq 2$ we have

$$|\arg(-\frac{1}{2}-iz)| \leq \arctan(\frac{3}{100}\frac{1}{\frac{1}{2}-\frac{3}{100}}) \leq 0.06375,$$

$$|\arg(-\frac{3}{2}-iz)| \leq \arctan(\frac{3}{100}\frac{1}{\frac{3}{2}-\frac{3}{100}}) \leq 0.02041,$$

$$|\arg(2M-\frac{1}{2}-iz)| \leq \arctan(\frac{3}{100}\frac{1}{2M-\frac{1}{2}-\frac{3}{100}}) \leq \arctan(\frac{3}{100}\frac{1}{4-\frac{1}{2}-\frac{3}{100}}) \leq 0.00865,$$

and

$$|\arg(2M+\frac{1}{2}-iz)| \leq \arctan(\frac{3}{100}\frac{1}{4+\frac{1}{2}-\frac{3}{100}}) \leq 0.006712.$$

The sum of the four numbers does not exceed 0.0996. Hence,

$$|\arg(f'_M(\zeta)| < \frac{0.0996}{2} = 0.0498 < \arctan(0.05).$$

□

Lemma 21. *For $|\xi-\eta_M^{\pm}| < \frac{\sqrt{2}}{18}$ the following inequality holds for the second derivative of $g_{(2),M}$*

$$|(g_{(2),M})''(\xi)| < 1.863. \tag{41}$$

Proof. Put $\xi = f_{(2),M}(\zeta)$. By formulas (36) and (38) and by lemma 16

$$\begin{aligned}
&(g_{(2),M})''(f_{(2),M}(\zeta)) = \\
&\pi^2\exp(\pi((f_{(2),M})^{-1}(f_{(2),M}(\zeta))+i(M-1)))\cdot\frac{1}{((f_{(2),M})'(\zeta))^2}+ \\
&\pi\exp(\pi((f_{(2),M})^{-1}(f_{(2),M}(\zeta))+i(M-1)))\cdot((f_{(2),M})^{-1})''(f_{(2),M}(\zeta)) = \\
&\pi^2\exp(\pi(\zeta+i(M-1)))(\frac{1}{(f_{(2),M})'(\zeta)})^2+\pi\exp(\pi(\zeta+i(M-1)))\frac{1}{2}((\frac{1}{(f_{(2),M})'(\zeta)})^2)' = \\
&\pi^2\exp(\pi(\zeta+i(M-1)))(\frac{1}{(f_{(2),M})'(\zeta)})^2\cdot \\
&\big(1+\frac{1}{2\pi}(\frac{i}{\frac{1}{2}+iz}+\frac{i}{\frac{3}{2}+iz}-\frac{i}{2M-\frac{1}{2}-iz}-\frac{i}{2M+\frac{1}{2}-iz})\big)
\end{aligned} \tag{42}$$

If $|\xi-\eta_M^{\pm}| < \frac{\sqrt{2}}{18}$ then by Corollary 4

$$|(f_{(2),M})^{-1}(\xi)-(f_{(2),M})^{-1}(\eta_M^{\pm})| = |\zeta \mp i(M-\frac{1}{2})| < 0.3343\frac{\sqrt{2}}{18} < 0.4\frac{\sqrt{2}}{18}.$$

Hence, by (42) and Corollary 4 for $M \geq 2$

$$\begin{aligned} |(g_{(2),M})''(\xi)| <& \pi^2 \exp(\frac{0.4\sqrt{2}}{18}\pi) \cdot (0.3343)^2 \cdot \\ & (1+\frac{1}{2\pi}(\frac{1}{\frac{1}{2}-\frac{0.4\sqrt{2}}{18}}+\frac{1}{\frac{3}{2}-0.4\frac{\sqrt{2}}{18}}+\frac{1}{\frac{7}{2}-\frac{0.4\sqrt{2}}{18}}+\frac{1}{\frac{9}{2}-\frac{0.4\sqrt{2}}{18}})) < 1.863. \end{aligned} \tag{43}$$

□

The lemma implies that $g_{(2),M}$ maps the sets $E^{\pm}_{(2),M} \overset{def}{=} \{\xi \in \mathbb{C} : |\xi - \eta^{\pm}_M| < \frac{\sqrt{2}}{18}\}$ into $\mathbb{C} \setminus i\mathbb{Z}$. Indeed, by the lemma for $\xi \in E^{\pm}_{(2),M}$

$$|(g_{(2),M})'(\xi)| \leq 1 + 1.863 \cdot \frac{\sqrt{2}}{18} < 1.15.$$

Hence, the image $g_{(2),M}(\xi)$ of a point $\xi \in E^{\pm}_{(2),M}$ has distance less than $\frac{1.15\sqrt{2}}{18} < \frac{1}{2}$ from $\pm i(M - \frac{1}{2}) = g_{(2),M}(\eta^{\pm}_M)$.

The inequality (22) and the equation (37) imply the following upper bound for the vertical side length $\mathrm{vsl}(R_{(2),M})$ of the normalized rectangle $R_{(2),M} = r_{(2),M} R_M$ with $M = \frac{d+1}{2}$ and $d \geq 5$

$$\mathrm{vsl}(R_{(2),M}) \leq (\frac{12}{5} + \log(2M+1)) \frac{\pi\sqrt{3}\sqrt{(M-\frac{1}{4})(M+\frac{1}{4})}}{M+1} < 1.504 \cdot \pi\sqrt{3}\log(4d-1). \tag{44}$$

The last inequality follows from the inequalities

$$\frac{12}{5} - \log 2.5 < 0.504 \log 19 < 0.504 \log(4d-1)$$

for $d \geq 5$ and

$$\log(\frac{5}{2}(d+2)) \leq \log(4d-1)$$

for $d \geq 5$.

Short syllables.

a)Syllables of form (1) of degree at most 4 **and of form (3)**. Without loss of generality we assume that the syllable has the form a_1^n where n is a natural number at most equal to 4. The other cases are treated symmetrically. Put $M = \frac{d-1}{2} \leq \frac{3}{2}$ (with $d = n$). Consider the rectangle $R_{(1),M}$ in the plane with vertices $\pm i\frac{\pi}{2}(M + \frac{1}{2})$ on the imaginary line and vertices $(M + \frac{1}{2}) \log \frac{M}{M+1} \pm i\frac{\pi}{2}(M + \frac{1}{2})$ in the open left half-plane. Take $\xi^{\pm}_M = -(M + \frac{1}{2}) \log \frac{M+\frac{1}{2}}{M} \pm i\frac{\pi}{2}(M + \frac{1}{2})$. Put

$$g_{(1),M}(\xi) = -M \exp(\frac{-\xi}{M + \frac{1}{2}}). \tag{45}$$

The mapping $g_{(1),M}$ maps the rectangle $R_{(1),M}$ conformally onto a half-annulus in the left half-plane. It maps the vertices $\pm\frac{\pi i}{2}(M + \frac{1}{2})$ of $R_{(1),M}$ contained the imaginary axis to $\pm iM$ and the vertices of $R_{(1),M}$ in the open left half-plane to $\pm i(M + 1)$. Hence, the mapping $g_{(1),M} : R_{(1),M} \to \mathbb{C} \setminus i\mathbb{Z}$ represents a lift of a_1^n.

The points $\xi_M^\pm$ are mapped to $\pm i(M+\frac{1}{2})$. The derivative of $g_{(1),M}$ at the points $\xi_M^\pm$ equals $\mp i$. The second derivative of $g_{(1),M}$ can be estimated by

$$|(g_{(1),M})''(\xi)| \le \frac{M+1}{(M+\frac{1}{2})^2} \le \frac{3}{2} \tag{46}$$

for $\xi \in R_{(1),M}$ and $M \ge \frac{1}{2}$.

By the same reasoning as above $g_{(1),M}$ maps the sets $E^\pm_{(1),M} \overset{def}{=} \{\xi \in \mathbb{C} : |\xi - \xi_M^\pm| < \frac{\sqrt{2}}{18}\}$ into $\mathbb{C} \setminus i\mathbb{Z}$.

For the considered values M, $\frac{1}{2} < M < \frac{3}{2}$, the horizontal side length of the rectangle $R_{(1),M}$ is at least $(M+\frac{1}{2})\log(1+\frac{1}{M}) \ge \log\frac{5}{3} \ge 0.5 > \frac{1}{18}$. The vertical side length of the rectangle $R_{(1),M}$ equals

$$\mathrm{vsl}(R_{(1),M}) = \pi(M+\frac{1}{2}) = \frac{\pi}{2}d. \tag{47}$$

Since

$$\frac{\pi}{2} < 1.43 \cdot \log 3, \quad \frac{2\pi}{2} < 1.615 \cdot \log 7, \quad \frac{3\pi}{2} < 1.97 \cdot \log 11, \quad \frac{4\pi}{2} < 2.33 \cdot \log 15, \tag{48}$$

we obtain the estimate

$$\mathrm{vsl}(R_{(1),M}) < \frac{5}{2} \cdot \log(4d-1) \tag{49}$$

for $d = 1,2,3,4$ and $M = \frac{d-1}{2}$.

b) Short syllables of form (2) of degree 2, 3, and 4. Consider the syllables $a_2^{-1}a_1^{-1}$, $a_2^{-1}a_1^{-1}a_2^{-1}$, and $a_2^{-1}a_1^{-1}a_2^{-1}a_1^{-1}$. Other syllables of this form are treated similarly. Put $M = \frac{d}{2}$. Consider the rectangle $\tilde{R}_{(2),M}$ with vertices 0 and $id\frac{\pi}{2}$ on the imaginary axis and vertices $\log\frac{1}{2}$ and $\log\frac{1}{2} + id\frac{\pi}{2}$ in the open left half-plane. $\tilde{R}_{(2),M}$ is contained in the rectangle $R_{(2),M}$ with vertices $-i\frac{1}{18}$, $i(d\frac{\pi}{2}+\frac{1}{18})$, $\log\frac{1}{2} - i\frac{1}{18}$, and $\log\frac{1}{2} + i(d\frac{\pi}{2}+\frac{1}{18})$. The horizontal side length of $R_{(2),M}$ equals $\log 2 > \frac{1}{18}$.

Let $\eta_M^- = 0$ and $\eta_M^+ = id\frac{\pi}{2}$. Put

$$g_{(2),M}(\xi) = -\frac{i}{2}\exp(2\xi), \ \xi \in R_{(2),M}. \tag{50}$$

$g_{(2),M}$ maps the rectangle $R_{(2),M}$ into the annulus $\{\frac{1}{4} < |z| < \frac{1}{2}\}$. Denote the extension of $g_{(2),M}$ to the closed rectangle by the same letter. We obtain $g_{(2),M}(\eta_M^-) = -\frac{i}{2}$ and $g'_{(2),M}(\eta_M^-) = -i$. Further, $g_{(2),M}(\eta_M^+) = -\frac{i}{2}$ if $d=2$ and $d=4$, and $+\frac{i}{2}$ if $d=3$, and $(g_{(2),M})'(\eta_M^+) = -i$ if $d=2$ and $d=4$, and $+i$ if $d=3$. The restriction of $g_{(2),M}$ to each maximal vertical segment in $\tilde{R}_{(2),M}$ makes $d = 2M$ half-turns around 0. Hence the mapping $g_{(2),M} : \tilde{R}_{(2),M} \to \mathbb{C} \setminus i\mathbb{Z}$ represents a lift of the syllable.

For the second derivative we have the estimate

$$|(g_{(2),M})''(\xi)| \le 2 \tag{51}$$

on $R_{(2),M}$.

By the same reasoning as above $g_{(2),M}$ maps the sets $E^\pm_{(2),M} = \{\xi \in \mathbb{C} : |\xi - \xi^\pm| < \frac{\sqrt{2}}{18}\}$ into $\mathbb{C} \setminus i\mathbb{Z}$.

The vertical side length of the rectangle $R_{(2),M}$ equals

$$\frac{\pi}{2}d + \frac{1}{9}. \tag{52}$$

Since $\pi + \frac{1}{9} < 1.672 \cdot \log 7$, $\frac{3}{2}\pi + \frac{1}{9} < 2.0116 \cdot \log 11$, and $2\pi + \frac{1}{9} < 2.362 \cdot \log 15$ we obtain the estimate

$$\mathrm{vsl}(R_{(2),M}) < \frac{5}{2}\log(4d-1) \tag{53}$$

for $d = 2, 3, 4$ and $M = \frac{d}{2}$.

7. Quasiconformal gluing

We give the proof of the upper bound in Theorem 1 and Theorem 1′. Identify w with $\mathfrak{C}(b) \in \pi_1$. Assume first that w has at least two syllables and we are interested in pb boundary values. We associated to each syllable $\mathfrak{s}_j$, $j = 1, \ldots, N$, of the word w a rectangle $\overset{\circ}{R}_j$ of the form $R_{(1),M}$ or $R_{(2),M}$ for a number $M = M_j$. If the rectangle is of the form $R_{(1),M}$ we consider the holomorphic mapping $g_{(1),M} + iM$ defined on it (see (26) and (45))and put $\overset{\circ}{p}{}_j^{\pm} = \xi_M^{\pm}$, otherwise we consider the mapping $g_{(2),M} + iM$ on $\overset{\circ}{R}_j$ (see (35)and (50)) and put $\overset{\circ}{p}{}_j^{\pm} = \eta_M^{\pm}$. Denote the chosen mapping on $\overset{\circ}{R}_j$ by $\overset{\circ}{g}_j$. The mapping $\overset{\circ}{g}_j$ maps the rectangle $\overset{\circ}{R}_j$ to $\mathbb{C} \setminus i\mathbb{Z}$. In the first case the mapping represents a lift of ${}_{pb}(\mathfrak{s}_j)_{pb}$ under $f_1 \circ f_2$, in the second case a restriction of the mapping to a curvilinear rectangle contained in $\overset{\circ}{R}_j$ has this property.

Choose complex numbers c_j so that $\overset{\circ}{p}{}_j^{+} + c_j = \overset{\circ}{p}{}_{j+1}^{-} + c_{j+1}$ for all except the last number j. Put $p_j^{\pm} = \overset{\circ}{p}{}_j^{\pm} + c_j$. Since for all j the real part of $\overset{\circ}{p}{}_j^{+}$ coincides with the real part of $\overset{\circ}{p}{}_j^{-}$ we may choose the numbers c_j so that the $p_j^{\pm}$ lie on the imaginary axis for all j. Denote by R_j the translated rectangles $\overset{\circ}{R}_j + c_j$. The mappings $g_j : R_j \to \mathbb{C} \setminus i\mathbb{Z}$ are called normalized representatives of the lifts of the syllables $\mathfrak{s}_j$. See figure 6.

We describe now the choice of a closed curvilinear rectangle $\bar{R}$ contained in the union of the closed rectangles $\overline{R_j}$, see Figure 6. For each rectangle R_j of the form $R_{(1),M}$ we consider the rectangle R_j^- contained in R_j of the same vertical side length as R_j and of horizontal side length $\frac{1}{18}$, for which $p_j^{\pm}$ are the midpoints of the horizontal sides. For $j = 2, \ldots, N-1$, we denote by R_j^0 the subset of R_j^- that is contained in the strip $\{\mathrm{Im} p_j^- + \frac{1}{18} < \mathrm{Im}\xi < \mathrm{Im} p_{j+1}^+ - \frac{1}{18}\}$.

For each rectangle R_j of the form $R_{(2),M}$ we consider the rectangle R_j^- contained in R_j of the same vertical side length as R_j and of horizontal side length $\frac{1}{18}$, one vertical side of which is contained in the imaginary axis. Notice that the points $p_j^{\pm}$ lie on this side. For $1 < j < N$ we denote by R_j^0 the subset of this rectangle that is contained in the horizontal strip $\{\mathrm{Im} p_j^- + \frac{1}{18} < \mathrm{Im}\xi < \mathrm{Im} p_j^+ - \frac{1}{18}\}$. Put $R_1^0 = R_1^- \cap \{\mathrm{Im}\xi < \mathrm{Im} p_j^+ - \frac{1}{18}\}$ and $R_N^0 = R_N^- \cap \{\mathrm{Im} p_j^- + \frac{1}{18} < \mathrm{Im}\xi\}$.

The rectangles R_j^0 are contained in the $\frac{1}{18}$-neighbourhood of the imaginary axis. Let q_j^- be the left endpoint of the lower side of R_j^0, and let q_j^+ be the left endpoint of the upper side of R_j^0, respectively. For $j < N$ the inequality $|\mathrm{Re}(q_j^+ - q_{j+1}^-)| \le \frac{1}{18}$ holds. Hence, for each $\varepsilon > 0$ in each horizontal strip $\{\mathrm{Im} p_j^+ - \frac{1}{18} < \mathrm{Im}\xi < \mathrm{Im} p_j^+ + \frac{1}{18}\}$ around the $p_j^+ = p_{j+1}^-$, $j = 1, \ldots, N-1$, there is a closed curvilinear rectangle whose horizontal sides are the upper side of R_j^0 and the lower side of R_{j+1}^0 which has the following two properties. Each horizontal line contained in $\{|\mathrm{Im}(\xi - p_j^+)| \le \frac{1}{18}\}$ intersects the curvilinear rectangle along a segment of length $\frac{1}{18}$. The vertical curvilinear sides are graphs of smooth functions over the segment $[i(\mathrm{Im} p_j^+ - \frac{1}{18}), i(\mathrm{Im} p_j^+ + \frac{1}{18})]$ of the imaginary axis whose derivatives vanish near the endpoints and have absolute value not exceeding $\frac{1}{2} + \varepsilon$. The number $\varepsilon > 0$ will be chosen later as close to 0 as needed. The closed curvilinear rectangle $\bar{R}$ is the union of the $\overline{R_j^0}$ and all obtained closed curvilinear rectangles. See Figure 6.

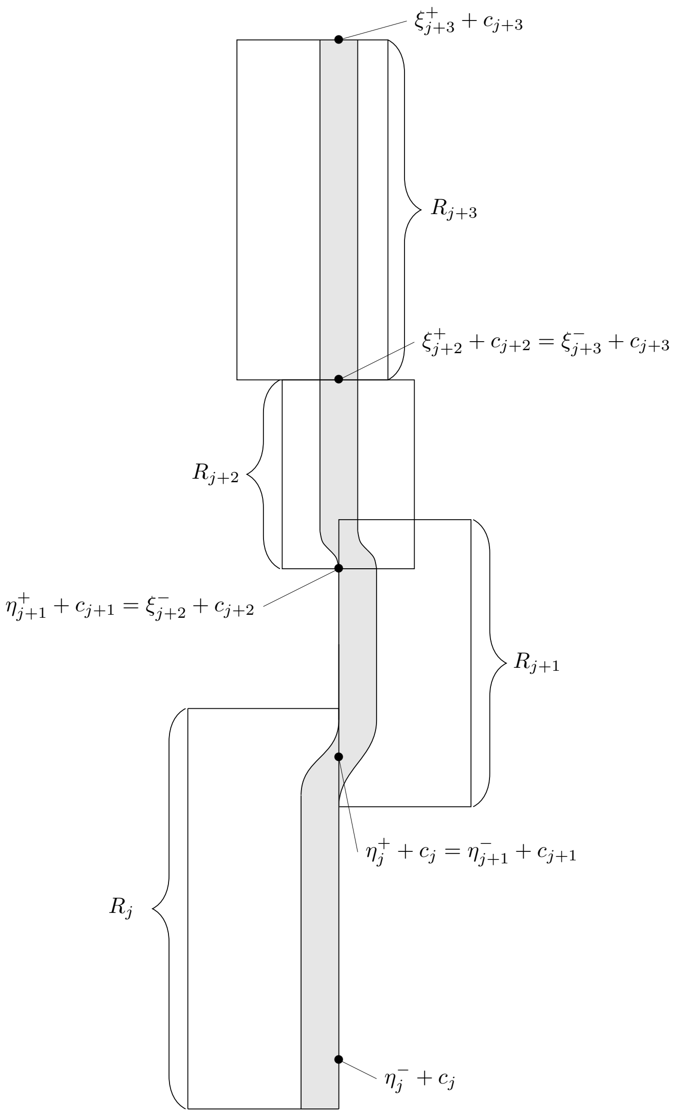

Figure 6

Consider the functions g_j, $g_j(\xi + c_j) = \overset{\circ}{g}_j(\xi) + im_j$ on R_j and their holomorphic extensions to a $\frac{\sqrt{2}}{18}$-neighbourhood of $p_j^\pm$. The integers m_j are chosen in such a way that for all j (except the last one) the values of g_j and of g_{j+1} at $p_j^+ = p_{j+1}^-$ coincide and are equal to an imaginary half-integer which is not an imaginary integer. Moreover, by the normalization of the $\overset{\circ}{R}_j$ the values of their derivatives at the point p_j^+ coincide. Both functions, g_j and g_{j+1} map the $\frac{\sqrt{2}}{18}$-neighbourhood of the point into a $\frac{1}{2}$-neighbourhood of an imaginary half-integer which is not an imaginary integer.

Consider the C^1-function χ_0 on the interval $[0,1]$, $\chi_0(t) = 6\int_0^t \tau(1-\tau)d\tau$. Then $\chi_0(0) = 0$, $\chi_0(1) = 1$, and $0 \le \chi_0'(t) \le 6t(1-t) \le \frac{3}{2}$. Put $\chi(t) = \chi_0(9t)$.

Define a function g on R as follows. Each point ξ in R for which $|\mathrm{Im}(\xi - p_j^+)| > \frac{1}{18}$ for all $j < N$ belongs to a single rectangle $\overline{R_j}$ (depending on ξ) and we put $g(\xi) \overset{def}{=} g_j(\xi)$ for such a point.

Fix a number $j < N$ and consider the set $Q_j \overset{def}{=} \{\xi : |\mathrm{Im}(\xi - p_j^+)| \le \frac{1}{18}\}$. Put $\chi_j(\xi) = \chi(\mathrm{Im}(\xi - p_j^+) + \frac{1}{18})$ for $\xi \in Q_j$. Let $g \overset{def}{=} (1-\chi_j)g_j + \chi_j g_{j+1}$ on $\bar{R} \cap Q_j$. For $\{\xi \in Q_j : \mathrm{Im}\xi = p_j^+ - \frac{1}{18}\}$ the equalities $\chi_j(\xi) = \chi_0(0) = 0$ and $\chi_j'(\xi) = \chi_0'(0) = 0$ hold. Hence, the function g is C^1 smooth near such points ξ. Further, for $\{\xi \in Q_j : \mathrm{Im}\xi = p_j^+ + \frac{1}{18}\}$ the equalities $\chi(\mathrm{Im}(\xi - p_j^+) + \frac{1}{18}) = \chi_0(1) = 1$ and $\chi_j'(\xi) = \chi_0'(1) = 0$ hold, hence, the function g is C^1 smooth near such ξ. Since both functions g_j and g_{j+1} map the $\frac{\sqrt{2}}{18}$-neighbourhood of p_j^+ into a $\frac{1}{2}$-neighbourhood of an imaginary half-integer which is not an imaginary integer, the convex combination g of these two functions has the same property. Make the same definition for all but the last number j. We obtain a smooth mapping g from the curvilinear rectangle $\bar{R}$ to $\mathbb{C} \setminus i\mathbb{Z}$ which represents a lift of ${}_{pb}(w)_{pb}$ under $f_1 \circ f_2$.

Lemma 22. *The mapping g is a quasiconformal mapping from R onto its image with Beltramy differential μ_g of absolute value $|\mu_g| < 0.1712$.*

Proof. Put $\xi = u + iv$. If $\xi \in R$, $|\mathrm{Im}(\xi - p_j^+)| \le \frac{1}{18}$ for some j, $1 \le j < N$, then the Beltrami differential at ξ equals

$$\mu_g(\xi) = \frac{\frac{\partial}{\partial \bar{\xi}} g(\xi)}{\frac{\partial}{\partial \xi} g(\xi)} = \frac{\frac{i}{2}(\frac{\partial}{\partial v}\chi_j \cdot (g_{j+1} - g_j))(\xi)}{(\frac{-i}{2}\frac{\partial}{\partial v}\chi_j \cdot (g_{j+1} - g_j) + (1-\chi_j) \cdot g_j' + \chi_j \cdot g_{j+1}')(\xi)}.$$

On the rest of the rectangle R the function is analytic.

By the Lemmas 17 and 21 and inequalities (46) and (51) the estimate

$$\max\{|g_j''(\xi)|, |g_{j+1}''(\xi)|\} < 2.75$$

holds for the considered points ξ. Since $g_{j+1} - g_j$ vanishes together with its derivative at p_j^+, the estimate

$$|(g_{j+1} - g_j)(\xi)| \le 2 \cdot \frac{1}{2} \cdot (\frac{\sqrt{2}}{18})^2 \cdot 2.75$$

holds for ξ in the $\frac{\sqrt{2}}{18}$-neighbourhood of p_j^+. Further

$$|\frac{\partial}{\partial v}\chi_j| \le \frac{3}{2} \cdot 9,$$

on Q_j, and

$$\max\{|g_j' - g_j'(p_j^+)|, |g_{j+1}' - g_j'(p_j^+)|\} < 2.75 \cdot \frac{\sqrt{2}}{18}$$

on the $\frac{\sqrt{2}}{18}$-neighbourhood of p_j by the estimate for the second derivative of the g_k, since $g_j'(p_j^+) = g_{j+1}'(p_j^+)$. Hence,

$$k = \sup_R |\mu_g(\xi)| < \frac{\frac{1}{24} \cdot 2.75}{1 - \frac{1}{24} \cdot 2.75 - \frac{\sqrt{2}}{18} \cdot 2.75} < 0.1712.$$

We used that $|g_j'(p_j^+)| = 1$. □

The quasiconformal dilatation $K = \frac{1+k}{1-k}$ does not exceed 1.414.

Let ω be the normalized solution of the Beltrami equation

$$\frac{\partial}{\partial \overline{z}}\omega = \tilde{\mu}_g \frac{\partial}{\partial z}\omega$$

on the complex plane. Here $\tilde{\mu}_g$ equals μ_g on $\bar{R}$ and equals 0 outside $\bar{R}$. ω is a Hölder continuous self-homeomorphism of the complex plane. The mapping $g \circ \omega^{-1}$ is holomorphic on $\omega(R)$ (see [1], Chapter I C). The image $\omega(R)$ can be considered as a curvilinear rectangle. The curvilinear sides are the images of the sides of R. By [1] (chapter I, Theorem 3) the extremal length of $\omega(R)$ does not exceed $K \cdot \lambda(R)$. In other words, there is a conformal mapping ψ of a true rectangle $\mathcal{R}$ of extremal length not exceeding $K \cdot \lambda(R)$ onto $\omega(R)$, which maps the sides of $\mathcal{R}$ to the respective curvilinear sides of $\omega(R)$. The mapping $g \circ \omega^{-1} \circ \psi : \mathcal{R} \to \mathbb{C} \setminus i\mathbb{Z}$ is a holomorphic mapping from the rectangle $\mathcal{R}$ of extremal length not exceeding $K \cdot \lambda(R)$ to $\mathbb{C} \setminus i\mathbb{Z}$ that represents a lift of ${}_{pb}(w)_{pb}$.

Estimate the extremal length $\lambda(R)$ of R. By Lemma 12

$$\lambda(R) \le (1 + C^2) \sum \frac{\mathrm{vsl}(R_j)}{\mathsf{b}}.$$

Here $\mathsf{b} = \frac{1}{18}$ and the rectangle $\bar{R}$ was chosen so that C does not exceed $\frac{1}{2} + \varepsilon$.

We obtain

$$\lambda(\mathcal{R}) \le 1.414 \cdot (1 + (\frac{1}{2} + \varepsilon)^2) \cdot 18 \cdot \sum \mathrm{vsl}(R_j). \tag{54}$$

Further, by (34), (44), (49), and (53)

$$\mathrm{vsl}(R_j) \le 1.504 \cdot \pi\sqrt{3}\log(4d_j - 1), \tag{55}$$

where d_j is the degree of the j-th syllable. Hence,

$$\lambda(\mathcal{R}) \le 1.414 \cdot (1 + (\frac{1}{2} + \varepsilon)^2) \cdot 18 \cdot 1.504 \cdot \sqrt{3}\pi \cdot \sum \log(4d_j - 1) \tag{56}$$

Since $1.414 \cdot (1 + \frac{1}{4}) \cdot 18 \cdot 1.504 \cdot \sqrt{3}\pi < 260.4$, the number ε may be chosen so that the right hand side of (56) does not exceed $300 \cdot \sum \log(4d_j - 1)$, i.e.

$$\lambda(\mathcal{R}) \le 300 \cdot \sum \log(4d_j - 1). \tag{57}$$

The upper bound is proved for *pb* boundary values.

Suppose now, for instance, that the left boundary values are tr. We assume again that w has at least two syllables. Suppose the first syllable $\mathfrak{s}_1$ is of form (1) or (3), say, without loss of generality, $\mathfrak{s}_1 = a_1^n$ for $n \ge 1$. A lift of ${}_{tr}(\mathfrak{s}_1)_{pb}$ is a half-slalom class with $M = d - \frac{1}{2}$ (see Proposition 6). Hence, the lift can be represented by the restriction of a mapping of the form $g_{(1),M}$ to a rectangle $R^{\#}_{(1),M}$ of half of the vertical side length of $R_{(1),M}$. Shift the rectangle $R^{\#}_{(1),M}$ and the restriction of $g_{(1),M}$ in the needed way and denote the obtained rectangle by R_j and the function by g_j.

If $n \ge 3$ then $M \ge 2$, and by the first inequality in (34) the vertical side length of $R_{(1),M}$ does not exceed $\frac{12}{5} + \log(2M+1) = \frac{12}{5} + \log(2d)$. Hence,

$$\mathrm{vsl}(R_j) \le \frac{1}{2}(\frac{12}{5} + \log(2d)) \,.$$

Since $\frac{12}{5} < 2.7 < \log 16$ we obtain for $d \geq 3$

$$\mathrm{vsl}(R_j) \leq \frac{1}{2}(\frac{12}{5} + \log(2d)) < \frac{1}{2}\log(32d) < \log(4d-1). \tag{58}$$

If $n = d \leq 2$ then by (47) the vertical side length of $R_{(1),M}$ does not exceed $\pi(M+\frac{1}{2}) = \pi d$. Hence by (48)

$$\mathrm{vsl}(R_j) \leq \frac{1}{2}\pi d < 2\log(4d-1)$$

for $d = 1$ and 2.

Suppose the first syllable ${}_{tr}(\mathfrak{s}_1)_{pb}$ is of form (2). We may assume that it has the form $(a_2^{-1}a_1^{-1})^k$ or $(a_2^{-1}a_1^{-1})^k a_2^{-1}$ for $k \geq 1$.

Consider first the case when $d \geq 5$. With $d = 2M-1$ the curve $y \to \frac{1}{2}\exp(\pi(iy+i(M-1)))$, $y \in [-(M-\frac{1}{2}), (M-\frac{1}{2})]$, represents a lift of the syllable ${}_{tr}(\mathfrak{s}_1)_{pb}$ to $\mathbb{C}\setminus i\mathbb{Z}$. Further, for any curve α with interior in $\{z \in \mathbb{C}_\ell : -(M-\frac{1}{2}) < \mathrm{Im} z < (M-\frac{1}{2})\}$, with initial point in $[\log\frac{1}{2}, 0) - i(M-\frac{1}{2})$ and with endpoint in $(-\infty, 0) + i(M-\frac{1}{2})$, the curve $\exp(\pi\alpha + i(M-1))$ intersects $-\frac{i}{2} + \mathbb{R}$. (See Figure 7.) For $x \in [\log\frac{1}{2}, 0]$ we let $y(x)$ be the first parameter of intersection of the curve $y \to \exp(\pi(x+iy+i(M-1)))$, $y \in [-(M-\frac{1}{2}), (M-\frac{1}{2})]$, with $-\frac{i}{2} + \mathbb{R}$. We obtain an arc β, $x \to \beta(x) = (x, y(x))$, $x \in [\log\frac{1}{2}, 0]$, with left endpoint equal to $\log\frac{1}{2} - i(M-\frac{1}{2})$ and right endpoint on $(-i(M-\frac{1}{2}), i(M-\frac{1}{2}))$. The interior of the arc is contained in $\mathbb{C}_\ell \cap \{|\mathrm{Im}\zeta| < M - \frac{1}{2}\}$. The image of β under $f_{(2),M}$ is an arc with interior in $f_{(2),M}(\{z \in \mathbb{C}_\ell : -(M-\frac{1}{2}) < \mathrm{Im} z < (M-\frac{1}{2})\})$, with initial point equal to $f_{(2),M}(\log\frac{1}{2} - i(M-\frac{1}{2}))$ and right endpoint on $(-\eta_M^-, \eta_M^+)$. (See Figure 7.)

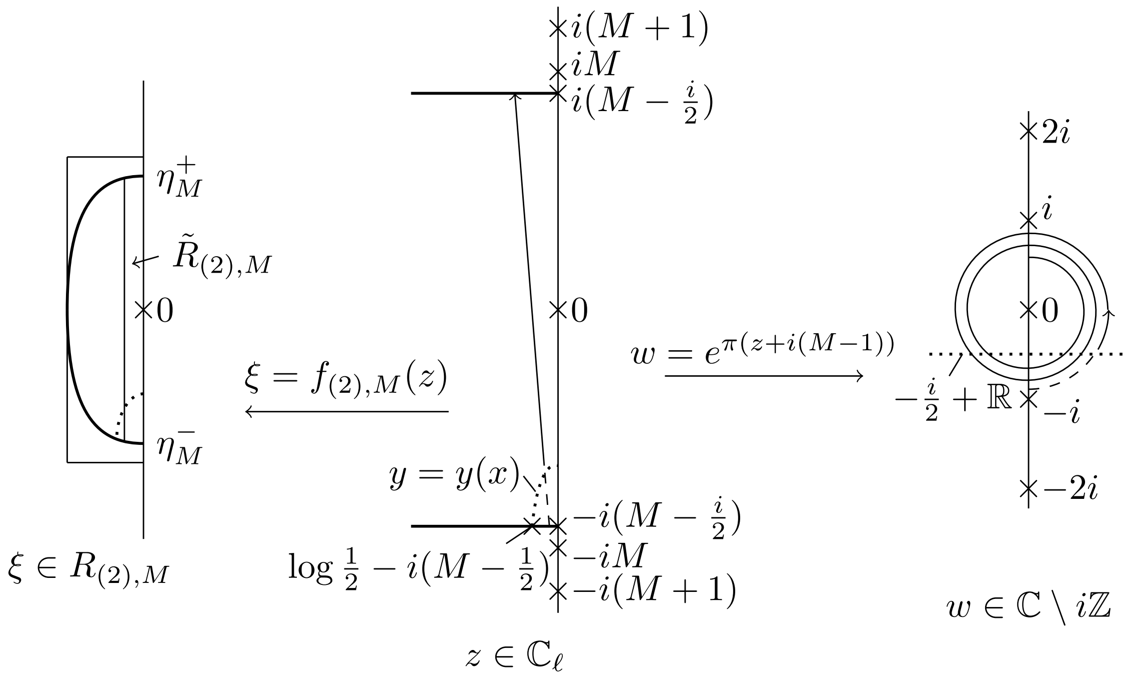

Figure 7

The initial point of $f_{(2),M} \circ \beta$ is not contained in the closure $\overline{\tilde{R}_{(2),M}}$ of $\tilde{R}_{(2),M}$. Indeed, the point lies on the curve $f_{(2),M}((-\infty, 0) - i(M-\frac{1}{2}))$. If it was contained in the closure

$\overline{\tilde R_{(2),M}}$ of $\tilde R_{(2),M}$ then by Lemma 20 it would have distance not exceeding $\frac{1}{18\cos(0.05)}$ from η_M^- and by Corollary 4 the distance of $\log\frac12 - i(M-\frac12)$ from $i(M-\frac12)$ would not exceed $\frac{1}{18\cos(0.05)}\cdot 0.3343$. But the latter distance equals $\log 2 > \frac{1}{18\cos(0.05)}\cdot 0.3343$.

As a consequence, the arc $f_{(2),M}\circ\beta$ intersects $\tilde R_{(2),M}$ along a union of relatively closed arcs. There is exactly one connected component $\tilde\beta$ of the intersection that has an endpoint on $(-\eta_M^-, \eta_M^+)$. The other endpoint of the component $\tilde\beta$ lies on the left side of $\tilde R_{(2),M}$. The arc $\tilde\beta$ divides $\tilde R_{(2),M}$ into two connected components. Denote by $\tilde R^0_{(2),M}$ the component of $\tilde R_{(2),M}\setminus\tilde\beta$ whose closure contains η_M^+. The restriction of $g_{(2),M}$ to $\tilde R^0_{(2),M}$ represents the syllable ${}_{tr}(\mathfrak{s}_j)_{pb}$. The vertical side length of $\tilde R^0_{(2),M}$ does not exceed the vertical side length of $R_{(2),M}$.

The case when the degree of the syllable is either 2, 3, or 4 can be treated similarly but is simpler because the mapping $g_{(2),M}$ is simpler. Also in this case we obtain a rectangle $\tilde R^0_{(2),M}\subset\tilde R_{(2),M}$ such that the restriction $g_{(2),M}|\tilde R^0_{(2),M}$ represents the syllable ${}_{tr}(\mathfrak{s}_j)_{pb}$. The vertical side length of $\tilde R^0_{(2),M}$ does not exceed the vertical side length of $R_{(2),M}$.

The case when w has totally real right boundary values or both boundary values are totally real, is treated in the same way. The quasiconformal gluing is done as in the case of pb boundary values.

If w consists of a single syllable the upper bound in the case of mixed boundary values or in the non-exceptional cases of tr or pb boundary values follows directly from Proposition 6. The exceptional cases were treated in the proof of the lower bound in Theorem 1.

Theorem 1 and Theorem 1′ are proved. □

Proof of the upper bound in Theorem 2. Take any syllable reduced word w representing the conjugacy class $\mathfrak{C}_*(\hat b)$. Consider all syllables $\mathfrak{s}_j$, $j=1,\ldots,N$, of w, labeled from left to right, with pb boundary values. Consider for each j the rectangle $\overset{\circ}{R}_j$ and the holomorphic mapping $\overset{\circ}{g}_j$ on it. Extend the finite sequence $\mathfrak{s}_j$ of syllables to an infinite sequence of syllables $\mathfrak{s}_j$, $j\in\mathbb{Z}$, with $\mathfrak{s}_j=\mathfrak{s}_{j+N}$ for all j. Put also $\overset{\circ}{R}_j=\overset{\circ}{R}_{j+N}$ and $\overset{\circ}{g}_j=\overset{\circ}{g}_{j+N}$ for integers j and N. In the same way as in the case of finite sequences of syllables we consider for each j the normalized representative $g_j: R_j\to\mathbb{C}\setminus i\mathbb{Z}$ of the lift under $f_1\circ f_2$ of the syllable ${}_{pb}(\mathfrak{s}_j)_{pb}$. The normalization is done in such a way that the value and the first derivative of g_j and g_{j+1} coincide at $p_j^+=p_{j+1}^-$. Recall that all $p_j^\pm$ are on the imaginary axis. Hence, $R_{N+j}=R_j+i\mathsf{a}$ for all j and a positive number a, and $g_{j+N}(\xi+i\mathsf{a})=g_j(\xi)+im$ for all $j\in\mathbb{Z}$ and an integer m.

Do quasiconformal gluing for all $j\in\mathbb{Z}$ by the same procedure as described before. We may perform quasiconformal gluing in such a way that for the obtained quasiconformal mapping g on the infinite curvilinear strip R_∞ of width $\frac{1}{18}$ the following relation

$$g(\xi+i\mathsf{a})=g(\xi)+mi$$

holds. The quasiconformal dilatation K of the mapping g on the infinite strip has the same estimate $K\le 1.414$ as the quasiconformal dilatation of the mapping constructed in the proof of Theorem 1. Since $f_1\circ f_2$ has period i the composition $G=f_1\circ f_2\circ g$ has period $i\mathsf{a}$.

Let the annulus A be the quotient of R_∞ by the equivalence relation $\xi\sim\xi+i\mathsf{a}$. The function G descends to a function $\hat G: A\to\mathbb{C}\setminus\{-1,1\}$ which has quasiconformal dilatation K and represents $\hat{b}$. The annulus A is conformally equivalent to the annulus $A' \subset \mathbb{C}$, which is the image of R_∞ under the mapping $\xi \to \exp(\frac{2\pi\xi}{\mathsf{a}})$. Consider the infinite strip of width $\frac{1}{18}$, that is bounded by two vertical lines. Let A_0 be the quotient of this strip by the equivalence relation $\xi \sim \xi + i\mathsf{a}$. The inequality for the extremal length $\lambda(A') \le (1 + C^2)\lambda(A_0)$ is obtained by the same arguments that are used for the proof of Lemma 12.

Putting together the estimates for the quasiconformal dilatation of $\hat{G}$, for the constant C, and for the estimate of the extremal length of A_0 through the vertical side length's of the R_j, we obtain the same upper bound $300 \cdot \sum \log(4d_j - 1)$ as in Theorem 1. Theorem 2 is proved. $\square$

Proof of the upper bound of Theorem 3. We prove the upper bound for 3-braids that are not pure and not among the exceptional cases. Write such a braid in the form (4). We may assume that the braid equals $\sigma_j^k b_1$ with $j = 1$ or $j = 2$, an odd integral number k and a pure braid b_1 as in Lemma 2 which is not the identity. Indeed, multiplying by a power of Δ_3 does not change the extremal length. Suppose $j = 1$. Then each curve γ in $C_3(\mathbb{C})\diagup\mathcal{S}_3$ representing b_{tr} can be decomposed into two curves, γ_0 representing ${}_{tr}(\sigma_1^k)_{pb}$, and γ_1 representing ${}_{pb}(b_1)_{tr}$. This can be seen by lifting γ to a curve $\tilde{\gamma}$ in $C_3(\mathbb{C})$ with initial point on $\{x_2 < x_1 < x_3\}$, and applying the analog of Lemma 10 to $\mathfrak{C}(\tilde{\gamma})$. Note that the terminating point of $\tilde{\gamma}_0$ (the lift of γ_0) is on $C_3(\mathbb{C})^0 = \{x_1 < x_2 < x_3\}$, and $\tilde{\gamma}_1$ (the lift if γ_1) has initial and terminating point on $C_3(\mathbb{C})^0$.

The arc $\mathfrak{C}(\tilde{\gamma}_0)$ is an arc in $\mathbb{C} \setminus \{-1, 1\}$ with initial point in $(-\infty, -1)$ and terminating point in the imaginary axis. Recall that for each integer k' the mapping $f_1 \circ f_2$ maps the interval $(-\infty, 0) + ik'$ onto $(-\infty, -1)$, and for each $j' \in \mathbb{Z}$ the mapping $f_1 \circ f_2$ maps the interval $(ij', i(j' + 1))$ onto the imaginary axis. (See also Figure 1.) More precisely, put $k = 2\ell + 1$ for an integer ℓ. Assume ℓ is non-negative. (The case of negative integers ℓ is treated similarly.) Take for Γ_0 the lift of $\mathfrak{C}(\tilde{\gamma}_0)$ to $\mathbb{C} \setminus i\mathbb{Z}$ with initial point in $(-\infty, 0)$. Then the terminating point of the lift is contained in $(i\ell, i(\ell + 1))$.(See also Figure 1.)

Associate to $\mathfrak{C}({}_{tr}(\sigma_1^k)_{pb})$ a rectangle $\overset{\circ}{R}_0$ and a holomorphic function $\overset{\circ}{g}_0 : \overset{\circ}{R}_0 \to \mathbb{C} \setminus i\mathbb{Z}$ that represents the class of curves that are homotopic in $\mathbb{C} \setminus i\mathbb{Z}$ to Γ_0 with initial point in $(-\infty, 0)$ and terminating point in $(i\ell, i(\ell + 1))$. For $\ell \ge 1$ the class of Γ_0 can be represented by the restriction of the mapping $g_{(1),M}$ to the upper half of the normalized rectangle $R_{(1),M}$ with $M = \ell = \frac{k-1}{2}$. If $\ell \ge 3$ the vertical side length of this "half-rectangle" does not exceed

$$\frac{1}{2}\left(\frac{12}{5} + \log(2\ell + 1)\right) < \frac{1}{2}(\log(32\ell + 16)) < \log(4\ell - 1). \tag{59}$$

(We used the first inequality in (34) and the relation $\frac{12}{5} < 2.7 < \log 16$.)

If $\ell = 1$ or 2 the vertical side length of this "half-rectangle" does not exceed

$$\frac{\pi}{2}\left(M + \frac{1}{2}\right) = \frac{\pi}{2}\left(\ell + \frac{1}{2}\right) < 3\log(4\ell - 1). \tag{60}$$

(We used (47) and the fact that $\frac{\pi}{2}\frac{3}{2} < 3\log 3$ and $\frac{\pi}{2}\frac{5}{2} < 3\log 7$, see also (48).)

Suppose $k = 1$ (i.e. $\ell = 0$). In this case the class of Γ_0 can be represented by a conformal mapping of a rectangle to a quarter of an annulus. Indeed, consider the normalized rectangle with vertices $\frac{\pi i}{4}$ and 0 on the real axis and vertices $\frac{\pi i}{4} - \log 2$ and $-\log 2$ in the open left half-plane. The mapping $\xi \to -\frac{1}{4}e^{-2\xi}$ maps $\frac{\pi i}{4}$ to $\frac{i}{4}$, 0 to $-\frac{1}{4}$,

$\frac{\pi i}{4} - \log 2$ to i and $-\log 2$ to -1. The image of the rectangle under this mapping is the upper left quarter of the annulus $\{\frac{1}{4} < |z| < 1\}$.

The mapping represents the class of Γ_0. Put $\xi_0^+ = -\frac{1}{2}\log 2 + \frac{\pi i}{4}$. Then the derivative at this point equals $-i$. We obtained a normalized rectangle $\overset{\circ}{R}_0$ and a mapping $\overset{\circ}{g}_0$. The normalized rectangle has vertical side length $\frac{\pi}{4}$.

Prove the theorem first in the case when $k = 1$ or $k = -1$. Then the class ${}_{tr}b_{tr}$ is the product of ${}_{tr}(\sigma_1^{\pm 1})_{pb}$ and ${}_{pb}(b_1)_{tr}$. The class ${}_{tr}(\sigma_1^{\pm 1})_{pb}$ can be represented by a holomorphic mapping on a normalized rectangle of vertical side length not exceeding $\frac{\pi}{4}$. The class ${}_{pb}(b_1)_{tr}$ is a pure braid which can be decomposed into parts that can be treated as in the proof of the upper bound of Theorem $1'$. Quasiconformal gluing as in Theorem $1'$ gives the estimate

$$\Lambda(b_{tr}) \leq 1.414 \cdot (1 + (\frac{1}{2} + \varepsilon)^2) \cdot 18 \cdot (1.504 \cdot \sqrt{3}\pi \cdot \sum_{\mathfrak{s}_j} \log(4d(\mathfrak{s}_j) - 1) + \frac{1}{4}\pi), \tag{61}$$

where the $\mathfrak{s}_j$ run over the syllables of the image of $\vartheta(b)$ in the braid group modulo its center. Since $\mathfrak{C}_*(\vartheta(b))$ is not the identity the expression $\mathcal{L}(\mathfrak{C}_*(\vartheta(b))) = \sum_{\mathfrak{s}_j} \log(4d(\mathfrak{s}_j)-1)$ is not smaller than $\log 3$. Further, the inequality

$$\frac{\pi}{4} \leq 0.715 \cdot \log 3 \leq 0.715 \cdot \mathcal{L}(\mathfrak{C}_*(\vartheta(b))), \tag{62}$$

holds. Since $1.414 \cdot \frac{5}{4} \cdot 18 \cdot (1.504 \cdot \sqrt{3}\pi + 0.715) < 260.4 + 0.715 < 300$, the number ϵ can be chosen small so that

$$\begin{aligned}\Lambda(b_{tr}) &\leq 1.414 \cdot (1 + (\frac{1}{2} + \varepsilon)^2)(\cdot 18 \cdot 1.504 \cdot \sqrt{3}\pi + 0.715) \cdot \mathcal{L}(\mathfrak{C}_*(\vartheta(b))) \\ &< 300 \cdot \mathcal{L}(\mathfrak{C}_*(\vartheta(b))).\end{aligned} \tag{63}$$

Consider the case when $|k| \geq 3$ is an odd integer and $\mathfrak{C}_*(\sigma_1^{q(k)})$ is a syllable of $\mathfrak{C}_*(\vartheta(b))$. By our estimates (59) and (60) for the vertical side length of the normalized rectangle $\overset{\circ}{R}_0$ associated to $\mathfrak{C}_*({}_{tr}(\sigma_1^{q(k)})_{pb})$, quasiconformal gluing gives us the same estimate for $\Lambda(b_{tr})$ as Theorem 1 gives for $\Lambda({}_{tr}(\vartheta(b))_{tr})$. Hence we obtain again

$$\Lambda(b_{tr}) < 300 \cdot \mathcal{L}(\mathfrak{C}_*(\vartheta(b))). \tag{64}$$

It remains to consider the case when the term $\mathfrak{C}_*(\sigma_1^{q(k)})$ is not the identity and not a syllable of $\mathfrak{C}_*(\vartheta(b))$. In this case the exponent k equals ± 3. The vertical length of the normalized rectangle corresponding to $\mathfrak{C}_*({}_{tr}(\sigma_1^{\pm 3})_{pb})$ does not exceed $3\log 3$ (see (60)). Since the term $\mathfrak{C}_*(\sigma_1^{q(k)}) = \mathfrak{C}_*(\sigma_1^{\pm 2})$ is not a syllable of $\mathfrak{C}_*(\vartheta(b))$, the first syllable $\mathfrak{s}_1$ of $\mathfrak{C}_*(\vartheta(b))$ has the form $\mathfrak{C}_*(\sigma_1^2\sigma_2^2 \ldots)$ or $\mathfrak{C}_*(\sigma_1^{-2}\sigma_2^{-2} \ldots)$ and has degree at least two. Hence $\mathcal{L}(\mathfrak{C}_*(\vartheta(b))) \geq \log 7$.

Suppose first the syllable $\mathfrak{s}_1$ equals $\mathfrak{s}_1 = \mathfrak{C}_*(\sigma_1^2\sigma_2^2)$ or $\mathfrak{C}_*(\sigma_1^{-2}\sigma_2^{-2})$. The normalized rectangle corresponding to ${}_{pb}(\sigma_2^{\pm 2})_{pb}$ or ${}_{pb}(\sigma_2^{\pm 2})_{tr}$ has vertical side length not exceeding $3\log 3$ (see (49)). Hence, as in Theorem 1 and Theorem $1'$ quasiconformal gluing of the mappings representing $\mathfrak{C}_*({}_{tr}(\sigma_1^{\pm 3})_{pb})$, $\mathfrak{C}_*({}_{pb}(\sigma_2^2)_{\#})$, and syllables $\mathfrak{s}_j$, $j \geq 2$, of the word $\mathfrak{C}_*(\vartheta(b))$ (if there are such) gives the estimate

$$\Lambda(b_{tr}) \leq 1.414 \cdot (1 + (\frac{1}{2} + \varepsilon)^2) \cdot 18 \cdot (6 \cdot \log 3 + 1.504 \cdot \sqrt{3} \cdot \sum_{j \geq 2} \log(4d(\mathfrak{s}_j) - 1)), \tag{65}$$

where $\mathfrak{s}_j$ runs over the syllables of $\mathfrak{C}_*(b_1)$. Since

$$6 \log 3 \leq 0.414 \cdot 1.504 \cdot \sqrt{3} \cdot \pi \cdot \log 7 < 1.504 \cdot \sqrt{3}\pi \cdot \log(4d(\mathfrak{s}_1) - 1), \tag{66}$$

we obtain the upper bound (64) of Theorem 3 for this case.

For the case when the degree of $\mathfrak{s}_1$ equals 3 the same arguments give the estimate

$$\Lambda(b_{tr}) \leq 1.414 \cdot (1 + (\frac{1}{2} + \varepsilon)^2) \cdot 18 \cdot (3 \log 3 + \pi + \frac{1}{9} + 1.504 \cdot \sqrt{3} \cdot \sum_{j \geq 2} \log(4d(\mathfrak{s}_j) - 1)). \tag{67}$$

In case the degree of $\mathfrak{s}_1$ equals 4 the estimate is

$$\Lambda(b_{tr}) \leq 1.414 \cdot (1 + (\frac{1}{2} + \varepsilon)^2) \cdot 18 \cdot (3 \log 3 + \frac{3}{2}\pi + \frac{1}{9} + 1.504 \cdot \sqrt{3} \cdot \sum_{j \geq 2} \log(4d(\mathfrak{s}_j) - 1)). \tag{68}$$

(See (52) for $d = 2$ and $d = 3$ and the argument in the proof of Theorem 1 for other than pb boundary values.) Since

$$3 \log 3 + \pi + \frac{1}{9} \leq 0.334 \cdot 1.504 \cdot \sqrt{3} \cdot \pi \cdot \log 11 < 1.504 \cdot \sqrt{3}\pi \cdot \log(4d(\mathfrak{s}_1) - 1), \tag{69}$$

in the first case, and

$$3 \log 3 + \frac{3}{2}\pi + \frac{1}{9} \leq 0.3664 \cdot 1.504 \cdot \sqrt{3} \cdot \pi \cdot \log 15 < 1.504 \cdot \sqrt{3}\pi \cdot \log(4d(\mathfrak{s}_1) - 1), \tag{70}$$

in the second case, we obtain the upper bound in Theorem 3 also in these cases.

In the case when the degree of $\mathfrak{s}_1$ is at least $d_1 = 5$, we use that the vertical length of the normalized rectangle corresponding to the expression $\mathfrak{C}_*({}_{pb}(\sigma_2^{\pm 2}\sigma_1^{\pm 2}\ldots)_{\#})$ (of degree $d_1 - 1$) does not exceed the vertical length of the rectangle corresponding to $\mathfrak{s}_1$. We obtain

$$\Lambda(b_{tr}) \leq 1.414 \cdot (1 + (\frac{1}{2} + \varepsilon)^2) \cdot 18 \cdot (3 \log 3 + 1.504 \cdot \sqrt{3} \cdot \sum_{j \geq 1} \log(4d(\mathfrak{s}_j - 1)). \tag{71}$$

Since $\log(4d(\mathfrak{s}_1) - 1) \geq \log 19$ and

$$1.414 \cdot \frac{5}{4} \cdot 18 \cdot (1.504\pi\sqrt{3} + \frac{3 \log 3}{\log 19}) \leq 296, \tag{72}$$

we obtain the upper bound in Theorem 3 also in this case.

Theorem 3 is proved . □

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