Hydrodynamic attractors in phase space

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Hydrodynamic attractors have recently gained prominence in the context of early stages of ultra-relativistic heavy-ion collisions at the RHIC and LHC. We critically examine the existing ideas on this subject from a phase space point of view. In this picture the hydrodynamic attractor can be seen as a special case of the more general phenomenon of dynamical dimensionality reduction of phase space regions. We quantify this using Principal Component Analysis. Furthermore, we generalize the well known slow-roll approximation to this setting. These techniques generalize easily to higher dimensional phase spaces.

Introduction – The physics of strong interactions studied in heavy-ion collisions at the RHIC and LHC (see e.g. Ref. [1] for a concise contemporary review) has been a remarkable source of inspiration for the study of complex systems far from equilibrium. The phenomenological success of relativistic hydrodynamics, together with studies of microscopic models based on holography and kinetic theory, have inspired several novel research directions. One such direction is centered on the notion of hydrodynamic attractors. These were introduced in Ref. [2] with the aim of capturing universal features of non-equilibrium physics beyond the limitations of the gradient expansion and were subsequently explored in many works, including Refs. [3–28].

In the context of reproducing the spectra of soft particles in ultra-relativistic heavy-ion collisions, the underlying observable of interest is the expectation value of the energy-momentum tensor $\langle T^{\mu\nu} \rangle$. Ideally, one would like to have a way of predicting its behaviour as a function of time directly in QCD. Such calculations remain beyond reach, but have been pursued in quantum field theories possessing a gravity dual [29–31] in a number of works, see Ref. [32] for a review. Another line of development replaces QCD by its effective kinetic theory description [33], as reviewed in Ref. [34].

In general, $\langle T^{\mu\nu} \rangle$ has 10 components. In high-energy QCD it is reasonable to impose the simplifying assumption of conformal symmetry, which reduces this to 9. They are connected by 4 conservation equations $\nabla_\mu \langle T^{\mu\nu} \rangle = 0$ and their evolution depends on the underlying quantum state. This is captured by the metric in holography, or by the initial distribution function in the effective kinetic theory description. After some time, the vast majority of this information about initial conditions is effectively lost, and $\langle T^{\mu\nu} \rangle$ evolves hydrodynamically, i.e., it satisfies hydrodynamic constitutive relations specified by the truncated expansion in gradients of 4 functions: the local energy density $\mathcal{E}$ and the local flow velocity u^μ. This series is divergent in settings of interest to heavy-ion collisions, as shown in Ref. [35] and further explored in Refs. [2–4].

These considerations have provided motivation to systematize the notion of information loss in ultra-relativistic heavy-ion collisions as seen by $\langle T^{\mu\nu} \rangle$ and have led to the idea of hydrodynamic attractors. For highly-symmetric flows in strongly-coupled (holographic) gauge theories, well before the onset of local equilibrium, the pressure anisotropy of longitudinally expanding matter can exhibit universal behaviour when time is measured in units of local energy density [36]. Building on this finding, the hydrodynamic attractor was identified in Ref. [2] in a class of hydrodynamic theories [37–39] as a specific solution of their equations of motion – one which generic solutions approach when initialized at arbitrarily small times.

It was also observed there that a "slow-roll" condition akin to what is used in inflationary cosmology (see, e.g., Refs. [40, 41]) leads to an accurate approximation of this attractor. Subsequent works (starting with Ref. [5]) have found a way to apply these ideas to holographic and kinetic theory plasmas and have lead interesting phenomenological applications to ultra-relativistic heavy-ion collisions ranging from improved hydrodynamic descriptions, see, e.g. Refs. [2, 5, 16], to entropy production far-from-equilibrium [42].

Despite these developments, there are three important yet largely unexplored issues. The first addresses the different concepts of attractor, the second concerns their relevance for the dynamics of initial states of interest and the third is the question of their existence beyond highly-symmetric settings.

We address these points with the goal of clearing the ground for new developments, in particular for generalisations to more realistic flows with less symmetry. Our approach is based on the phase space picture, i.e., the space of variables needed to parametrize the dynamics underlying $\langle T^{\mu\nu} \rangle$, which was introduced in this context in Refs. [11, 43].

Dissipation of initial state information – There are two key features of the dynamics following an ultra-relativistic heavy-ion collision: the expansion which

drives the system away from equilibrium and interactions which favour equilibration [13]. The simplest model of this is Bjorken flow which assumes one-dimensional expansion and boost-invariance along the expansion axis. This is conveniently described in terms of proper time τ and spacetime rapidity y, in which the Minkowski metric takes the form

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + h(\tau)\,\mathrm{d}y^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \quad (1)$$

with $h(\tau) = \tau^2$ and the post-collision region corresponds to $\tau \geq 0$. In interacting conformal theories, at asymptotically late times the system is described by a scaling solution for local (effective) temperature [44]

$$T(\tau) = \frac{\Lambda}{(\Lambda\tau)^{1/3}} + \dots \,. \quad (2)$$

In this equation, the dimensionful constant Λ is the only trace of initial conditions. Both in hydrodynamic theories and in some microscopic models, corrections to the above equation come in two forms: higher-order power-law terms in the gradient expansion, which are sensitive only to Λ, and exponential corrections which bring in more information contained in the initial conditions [2, 4, 12, 35, 45, 46]. This asymptotic form is usually referred to as a transseries [47].

Even in the simplest of dynamical settings we should expect there to be more freedom of choosing an initial state than just a single scale. A generic set of initial conditions will occupy a (possibly high dimensional) volume on the initial time slice of phase space. Given the above late time behaviour, this implies a rather dramatic loss of dimensionality of this set. The simplest context where this can be observed explicitly is furnished by models formulated in the language of hydrodynamics.

Models of hydrodynamics– We focus on hydrodynamic theories (see Ref. [48] for a review), which despite their name include transient non-hydrodynamic excitations needed to avoid acausality. In such models, $\langle T^{\mu\nu}\rangle$ is decomposed into a perfect fluid term and a "dissipative" part denoted by $\pi^{\mu\nu}$:

$$\langle T^{\mu\nu}\rangle = (\mathcal{E}+\mathcal{P})\,u^\mu u^\nu + \mathcal{P}\,g^{\mu\nu} + \pi^{\mu\nu}, \quad (3)$$

where $u_\mu u^\mu = -1$, $u_\mu \pi^{\mu\nu} = 0$, and the energy density $\mathcal{E}$ and pressure $\mathcal{P}$ are related by the thermodynamic equation of state. In the conformal case considered here, $\mathcal{P} = \frac{\mathcal{E}}{3}$. Conservation equations of $\langle T^{\mu\nu}\rangle$ provide the four equations of motion for $\mathcal{E}$ and u^μ. Hydrodynamic models, building on the original ideas of Refs. [37, 38], provide the remaining equations for $\pi^{\mu\nu}$ in terms of relaxation-type dynamics that ensure matching to the hydrodynamic gradient expansion of any microscopic model.

In this Letter, we consider two classes of models. The first one is the Müller-Israel-Stewart (MIS) model [37, 38] with the additional equations of motion

$$\tau_\pi\,\mathcal{D}\,\pi^{\mu\nu} = -\pi^{\mu\nu} + \eta\,\sigma^{\mu\nu}, \quad (4)$$

where $\mathcal{D}\,\pi^{\mu\nu} = u^\alpha\nabla_\alpha\pi^{\mu\nu} + \dots$ is the Weyl-covariant derivative in the co-moving direction [49, 50] and $\sigma^{\mu\nu} = 2\mathcal{D}^{(\mu}u^{\nu)}$ is the shear tensor. One can supplement Eq. (4) with additional terms defining the so-called Baier-Romatschke-Son-Starinets-Stephanov (BRSSS) model [39] and in the following we will refer to it as MIS/BRSSS. In the conformal case the shear viscosity η and the relaxation time τ_π depend on thermodynamic quantities as

$$\eta = C_\eta\frac{\mathcal{E}+\mathcal{P}}{T} \quad\text{and}\quad \tau_\pi = \frac{C_{\tau_\pi}}{T} \quad\text{with}\quad \mathcal{E}\sim T^4, \quad (5)$$

where C_η, C_{τ_π} are dimensionless constants and T is defined as the temperature of an equilibrium state at the same energy density $\mathcal{E}$. Eq. (4) implies relaxation phenomena on a time scale defined by the relaxation time τ_π.

For the Bjorken flow, the combined equations for the MIS/BRSSS model reduce to a second order ordinary differential equation (ODE) for the effective temperature $T(\tau)$, see eq. (7.17) in Ref. [48]. The hydrodynamic attractor was originally observed in this model in Ref. [2] using a special scale-invariant parametrization involving pressure anisotropy (note $\pi_2^2 = \pi_3^3$)

$$\mathcal{A} = \frac{\pi_2^2 - \pi_y^y}{\mathcal{P}} = 6 + 18\,\tau\frac{\dot{T}(\tau)}{T(\tau)} \quad (6)$$

understood as a function of time measured by

$$w = \tau\,T(\tau)\,. \quad (7)$$

This variable is just proper time measured in units of the relaxation time, which in conformal theories at vanishing charges is inversely proportional to the effective temperature (see eq. (5)). We denote derivatives with respect to τ with a dot and derivatives with respect to w with a prime. In contrast with $T(\tau)$, $\mathcal{A}(w)$ satisfies a first order ODE, see eq. (7.18) in Ref. [48].

The second model of interest is the Heller-Janik-Spalinski-Witaszczyk (HJSW) model [50], in which case one supplements the conservation equations with

$$\{(\frac{1}{T}\mathcal{D})^2 + \frac{2}{T^2\,\tau_\pi}\mathcal{D}\}\,\pi^{\mu\nu} =$$
$$-T^{-2}\,(\tau_\pi^{-2}+\omega^2)\,\{\pi^{\mu\nu} + \eta\,\sigma^{\mu\nu}\}. \quad (8)$$

This structure again ensures relaxation phenomena on a time scale τ_π, but here, inspired by holographic gauge theories [51], they occur in an oscillatory manner with frequency $\omega = C_\omega T$. This model leads to a third order ODE for the effective temperature $T(\tau)$ or a second order ODE for $\mathcal{A}(w)$, see eq. (7.26) in Ref. [48]. This model provides a workable setting with a richer set of initial conditions than offered by MIS/BRSSS.

Hydrodynamic attractors– The hydrodynamic attractor reported in Ref. [2] arose through studying a range of solutions for $\mathcal{A}(w)$ corresponding to different initial conditions in MIS/BRSSS. It was noted there that all

solutions converge and then evolve to the final destination – local thermal equilibrium.

This behaviour is known in the mathematical literature [52] as a *forward attractor*. Intuitively, forward attractors are solutions that attract nearby sets as $w \to \infty$. In MIS/BRSSS every solution forward attractor. Conversly, by considering solutions initialized at earlier and earlier times we observe that generic solutions, which diverge at $w = 0$, decay to a specific solution which is regular there. Such behaviour is known as a *pullback attractor*. These two notions of attractors, introduced in this context in [43] are concerned with different regimes: the forward attractor describes asymptotically late times while the pullback attractor is a statement about asymptotically early times (in the present setting $w = 0$ is the earliest time one can consider). In MIS/BRSSS there is a unique solution [2] which is both a pullback attractor and a forward attractor; we denote this solution by $\mathcal{A}_\star(w)$.

The slow-roll approximation employed also by Ref. [2] and defined by seeking solutions that evolve slowly has a priori no relation to the above definitions (as it does not refer to any asymptotic limit), but the $\mathcal{A}(w)$ obtained this way is very close to $\mathcal{A}_\star(w)$.

These different characterizations of universal behaviour are logically independent, but they all capture different aspects of loss of information about the initial state. The reason why these notions are important and useful is because they give insight into the physics even though we cannot find explicit solutions. Not also that they all involve the idea of convergence of different solutions, which implicitly assumes a notion of distance in solutions space. If this space is parameterized in a different way or a different notion of proximity were to be employed, attractor behaviour in $\mathcal{A}(w)$ would of course not disappear, but it might no longer be manifest – e.g. if one instead looked at $w^4 \mathcal{A}(w)$. It is not known how to identify universal quantities such as $\mathcal{A}(w)$ for general flows, so it is important to develop techniques which capture universal features without relying on the special properties of the boost-invariant MIS/BRSSS system. This is the main goal of this Letter.

Attractors and phase space– The above discussion was devoted to a specific physical quantity, the pressure anisotropy, as a function of a particular clock variable. This brings in two concerns. The first is that were one to visualize a range of solutions as a two-dimensional plot $T(\tau)$, the hydrodynamic attractor would not be apparent and yet it is encoded there since $\mathcal{A}(w)$ can be clearly derived from $T(\tau)$. If a theory were to show attractor behaviour in a different quantity, how would one find it? This issue must be clarified if one aims to generalize the attractor to more complicated models. The second issue stems from the fact that forward and pullback attractors strongly depend on either asymptotically late or asymptotically early time dynamics. Such asymptotic regimes may be inaccessible or unphysical. For example, hydrodynamic theories [37–39, 50] do not share early-time behaviour with strongly-coupled gauge theories [53, 54] or with color glass condensate [55].

The first issue is addressed by considering the full *phase space*. The phase space variables are $(\tau, T, \dot{T})$ for MIS/BRSSS and $(\tau, T, \dot{T}, \ddot{T})$ for HJSW. We include proper time as one of the phase space variables because we are dealing with a non-autonomous system, i.e., equations of motion depend explicitly on τ. A solution of the equations of motion is a curve in phase space. Note that in the above considerations one can replace $T, \dot{T}, \ddot{T}$ by respectively $w, \mathcal{A}, \mathcal{A}'$. This corresponds to a different way of parametrizing solutions in which case the equations of motion decouple: one can first solve for $\mathcal{A}(w)$ and then use Eq. (6) to solve for $T(\tau)$.

The second problem mentioned above and related to dependence on asymptotic behaviour at early or late time is addressed by focusing on the *local dynamics* on families of constant τ slices of phase space (see Figs. 1 and 3). On these slices, solutions of the equations of motion appear as points. However, any particular solution $\mathcal{A}(w)$ corresponds to a different curve on each slice, as dictated by eq. (6). On the basis of earlier studies it is expected that as τ increases, different solutions (points on constant-τ slices) will collapse to the curves representing the attractor $\mathcal{A}_*(w)$. Indeed, this is what one *eventually* sees in Figs. 1 (bottom) and 3 (bottom-right).

However, numerical studies of phase space histories reveal a much finer picture. The process of information loss can be split into three phases: local dimensionality reduction, approach to the hydrodynamic attractor loci (red curves in Fig. 1), and finally the evolution toward equilibrium along the attractor. Consider a finite "cloud" of initial states, such as any one of the three colored sets of points shown in Fig. 1. Each cloud contracts and becomes one-dimensional. When this happens depends on the initial conditions. For example, the brown cloud in Fig. 1 (the one initialized at smallest value of $\tau_0 T$) loses a dimension quite early and rather far from the attractor, while the other two clouds do this much later. Note also that the time when the attractor is reached is a remnant of the information about the initial data.

Quantifying dimensionality reduction– We have argued that dimensionality reduction in phase space is an important feature of hydrodynamic attractors, but so far we have not provided a working recipe to quantify this process. A promising direction, which we only begin to explore here, follows from recognizing that dimensionality reduction is one of the basic tasks of machine learning. For the simple cases considered here, Principal Component Analysis (PCA) is quite effective at identifying dimensionality reduction in phase space, see Refs. [56–60] for other applications of PCA to problems in ultra-relativistic heavy-ion collisions. Intuitively, PCA
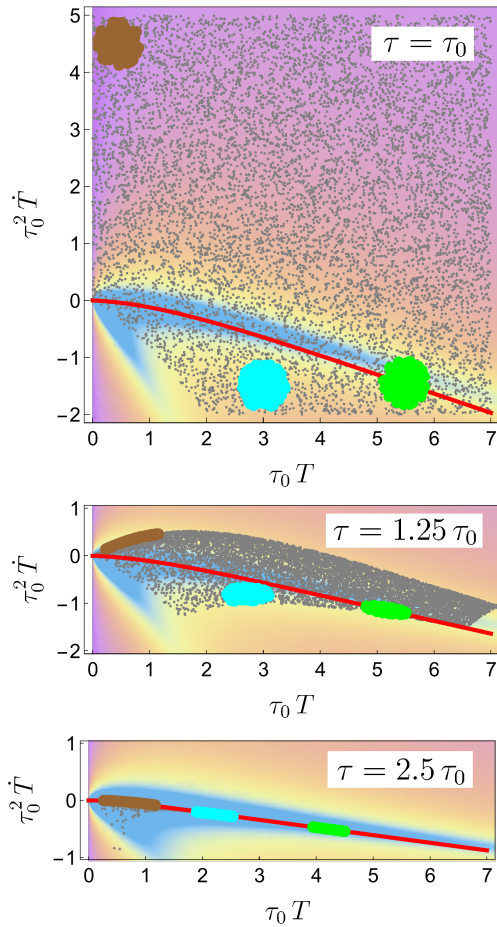
FIG. 1. Three snapshots of evolution in MIS/BRSSS phase space of a cloud of about 9000 random states in the region determined by the ranges of the top plot (the empty left-bottom corner on the top plot corresponds to an instability of equations of motion, see Ref. [2]). The red curve denotes the family of solutions corresponding to the pullback attractor of $\mathcal{A}_\star(\tau T)$. The background color represents the speed at which the points move in phase space, with magenta being faster than blue. For the purposes of visualizing local dimensionality reduction, see also Fig. 2, we track three initially spherical regions. The plots were made for $C_\eta = 0.75$ and $C_{\tau_\pi} = 1$, see Eqns. (4) and (5), and τ_0 denotes initialization time.
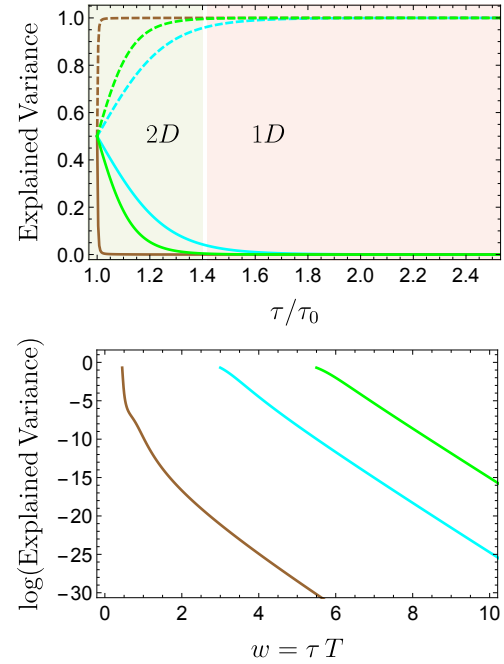

FIG. 2. Top: Evolution of explained variance of each principal component in MIS/BRSSS for circles (radius: 10^{-4}) of initial conditions with centres lying in the middle of initial dots of corresponding color in Fig. 1. Bottom: Logarithm of decaying principal components plotted as a function of w. For large enough values of w one clearly sees persistent exponential decay. The associated decay rate is consistent with twice the one of the transient mode contribution to $\mathcal{A}(w)$.

quantifies the variations of a data set in different directions. It identifies a set of directions and associates an *explained variance* to each. Directions in which the data set extends significantly will be associated with a large explained variance. The approximate vanishing of a component signals a reduction in dimensionality.

We start by applying PCA to the two-dimensional phase space of MIS/BRSSS. On the initial time slice at some value of τ_0 we pick a state $(T, \dot{T})$ and consider a random set of points within a disc around it. For this set of points, the two principal components are approximately equal in magnitude. At each timestep in τ, we recompute the principal components for the set of states we started with (see Fig. 1) and their evolution in time is shown

in Fig. 2. We declare that a given "cloud" has reduced dimensionality when one of the principal components is much smaller than the other one.

Note that PCA is less effective when the dimensionally reduced data set has curvature, though there are more sophisticated non-linear techniques for manifold learning (e.g. kernel-PCA) which can handle this case. Here, we focus on small enough sets so that the curvature does not play a role. This analysis extends easily to phase spaces of arbitrary dimension. As an interesting example, we turn to the three-dimensional phase space of HJSW. The evolution of principal components is shown in the top part of Fig. 3. There are evidently three different regimes, with different dimensionality. The first reduction, from three to two dimensions is analogous to the earliest phase of the collapse in MIS/BRSSS, most likely coming from the expansion. In contrast to that case, there are oscillations which eventually dissipate away resulting in one-dimensional evolution. This pattern is a consequence of the oscillations characteristic of the non-hydrodynamic sector of HJSW.

Slow-roll in phase space– The basic intuition is that the attractor locus should correspond to a region where the flow in phase space is slowest – earlier on we referred
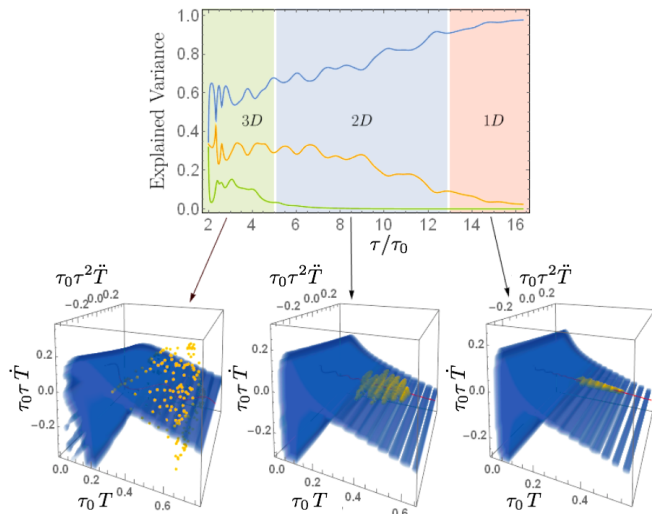
FIG. 3. In HJSW, the evolution of a cloud in phase space can be split into three stages, corresponding to the dimensionality of the cloud. In the top figure, the explained variance of the three principal components is plotted as a function of τ. The reduction from three to two dimensions corresponds to a collapse onto the slow region (blue region in plots). After the collapse from two to one dimensions, the pullback attractor from $\mathcal{A}(w)$ is a good description. The plot is made with $C_\eta = 0.75$, $C_{\tau_\pi} = 1.16$ and $C_\omega = 9.8$.

to it as the slow region. One way to motivate why the slow region should behave as an attractor is to use an argument inspired by thermodynamics: a system is likely to be found in a large entropy macrostate because such states cover the majority of phase space. In our setting, the system is likely to be in a slow region because it takes a long time to escape it, while the fast regions can be quickly traversed.

We have found that in the case of MIS/BRSSS this idea correctly identifies the attractor on any given time slice. Let $\vec{X}(\tau) = \left(\tau_0 T(\tau), \tau_0^2 \dot{T}(\tau) \right)$ be a point in a slice of phase space at time τ and τ_0 denotes initialization time. This point moves with the velocity vector $\vec{V} = \tau_0 \frac{\partial \vec{X}}{\partial \tau}$ found directly from the equations of motion, where we introduced an additional factor of τ_0 to render it dimensionless. The slow region can be defined by looking at the Euclidean norm V of this vector, where one uses the equations of motion to express $\ddot{T}$ in terms of $\dot{T}$, T and τ. This function has a minimum at asymptotically late times when the system approaches local thermal equilibrium. However, we should not only care about the point where V is minimal, but rather about a whole region where it is small.

In Fig. 1, the background color is determined by V, where bluer color implies lower speed. There is a slow region stretching out from local thermal equilibrium, and the attractor $\mathcal{A}_*(w)$ lies along it. This slow region can also be approximated by a slow-roll approximation where

one neglects $\ddot{T}$ in the equations of motion. This generalizes directly to phase spaces of any dimension; in particular, for the case of HJSW the slow regions are shown also in blue in the bottom row of Fig. 3.

Note that in these non-autonomous systems, the slow region changes with time. If it were to evolve faster than the solutions do, it would not be useful to characterize the attractor. This is however not the case here: in both Fig. 1 and Fig. 3 we observe that once solutions reach the slow-region, they stay inside and evolve with it. This is analogous to the adiabatic evolution observed in the case of the Boltzmann equation in the relaxation time approximation considered in Ref. [24].

Summary and outlook– The pressure anisotropy $\mathcal{A}(w)$ has been observed to exhibit universal behaviour in various models. This has been characterized by different authors in terms of different concepts such as a pullback attractor, a forward attractor or slow-roll, all of which capture come aspects of loss of information in dissipative systems while making use of special features of the models under consideration. We would like to emphasise that these notions are really conceptually different and it is not clear at present which will be most useful for applications to heavy-ion physics.

In this work we attempted to describe hydrodynamic attractors using concepts which allow for generalizations to more realistic theories and flows. We have shown that this can be achieved by considering the full (or in practice probably significant-enough projection of) the phase space of the theory. In this way there is no need to know in advance what particular quantity makes attractor behaviour manifest.

From this perspective, the hydrodynamic attractor is associated with a reduction of dimensionality, although the latter is more general. We have used PCA as a simple method to quantify this and observed that it works well in the cases of MIS/BRSSS [37–39] and HJSW [50] models. Furthermore, the slow-roll characterization of the attractor is successful in simple models, and has been explored in other cases [5]. This idea naturally generalizes to higher-dimensional phase spaces.

These notions depend explicitly on how one chooses to parametrize phase space as well as on the choice of a metric on constant time slices of phase space. In models with higher dimensional phase spaces (or even infinite-dimensional, as in kinetic theory or holography), the proper choice of metric is not obvious, but the physical interpretation of attractors depends on it.

In realistic settings one may not be able to consider the full phase space. However, the situation is actually not so grim, because in these cases one makes constant time slices of phase space finite dimensional by discretizing the system. For example, in hydrodynamic models or in holography one uses finite number of grid points to solve associated partial differential equations and in

kinetic theory one is naturally led to truncating high moments of the distribution function. Also, physically-relevant initial conditions are not specified by arbitrary real numbers on a grid but those that lead to smooth functions or sufficiently fast decaying distribution functions. This means that effective dimensionality of phase space in interesting cases might be after all manageable.

Finally, let us remark that the our discussion applies also to other situations where attractor behaviour was observed, such as non-thermal fixed points [61–63].

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