FORMAL ASPECTS OF NON-PERTURBATIVE QUANTUM FIELD THEORY VIA AN OPERATOR THEORETIC SETTING

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ABSTRACT. The article builds the original foundations of a new operator theoretic setting for the study of quantum dynamics of non-perturbative aspects originated from Green's functions in Quantum Field Theory with strong couplings.

CONTENTS

1. INTRODUCTION

In this research article we obtain some new applications of renormalization Hopf algebra, Noncommutative Geometry and Functional Analysis to deal with complicated physical processes in Quantum Field Theory beyond the perturbation theory. At the first step, we formulate a new multi-scale non-perturbative renormalization group on the set of Dyson–Schwinger equations which encodes the dynamics of these non-perturbative type of equations under changing the scales of momenta (or running coupling constants derived from dimensional regularization and renormalization schemes) and bare strong coupling constants (which are independent from regularization processes and renormalization maps). At the second step, we construct a new class of spectral triples in Noncommutative Geometry which are originated from fixed point equations of Green's functions such that our study offers a new spectral geometric approach to deal with the geometry of quantum motions. At the third step, we discuss the concept of evolution of finite formal expansions of Feynman diagrams (which converge to the unique solution of a given Dyson–Schwinger equation) under the Gelfand transform in functional analysis and generalized Dyson series (formulated by Johnson and Lapidus). The achievements of this research work move forward our knowledge about dynamics and geometry of quantum motions in modern Quantum Field Theories with strong couplings such as models of non-perturbative asymptotic freedom.

The reconstruction of geometries under algebraic settings is useful to obtain some new computational tools for the study of quantum systems. The story begins with passing from

Key words and phrases. Dyson–Schwinger equations; Non-perturbative renormalization group methods; Noncommutative Geometry and Quantum Field Theory.

Preprint for MPIM Library, Published version: https://www.worldscientific.com/doi/abs/10.1142/S0219887819501925.

mathematical structures which describe Classical Mechanics such as manifolds, groups and points to mathematical structures which describe semi-Classical Mechanics such as Poisson manifolds, Poisson Lie groups and 0-leaf and then developing concepts such as noncommutative algebra, noncommutative Hopf algebras and homomorphisms to discover the foundations of Quantum Mechanics. Noncommutative Geometry is capable to encode (semi-)Classical and Quantum Mechanics in terms of modern algebraic and geometric tools where the notion of ”distance” is interpreted in the context of Dirac operator which leads us to encode the geometry of a physical theory in the language of spectral triples. [3, 10, 11, 22]

We can study physical models in the language of Lagrangian density such as Quantum Field Theory in terms of Green's functions originated from the interaction part of the Lagrangian. Each term in Green's functions can have an ill-defined iterated integral which requires renormalization to generate finite values. Dealing with Feynman integrals, which have nested or overlapping sub-divergencies, is one of the most important challenges and in this direction, renormalization of this class of integrals has been formulated in the language of Feynman diagrams and the Bogoliubov–Zimmermann's forest formula. In addition, thanks to the Kreimer's renormalization coproduct, the Connes–Kreimer renormalization Hopf algebra of Feynman diagrams enables us to encode the perturbative renormalization machinery under some general mathematical concepts which is useful to find some new computational tools. Furthermore, the geometric interpretation of the modified Standard Model has shown the essential role of Noncommutative Geometry and its potential for the description of modern physical theories. [2, 3, 17, 18, 21, 22, 31, 30, 35]

In Quantum Field Theories with strong couplings, we need to deal with even more complicated problems originated from infinite formal expansions of Feynman integrals (or Feynman diagrams) under running and bare coupling constants. Fixed point equations of Green's functions are the original tools to classify these expansions where we study Dyson–Schwinger equations. Generally speaking, Green's functions are infinite formal expansions of Feynman diagrams such that most of the series in this formalism are divergent or at most, asymptotic rather than convergent. The situations beyond perturbation theory are determined by general expressions such as

$$P(g) = X_0 + gX_1 + g^2X_2 + ... + g^nX_n + ... \tag{1.1}$$

such that g is the (bare) coupling constant and each term X_n is the symbol for the class of Feynman diagrams which contribute to the order n of the perturbative expansion. In physical theories with very small g, it is enough to concern only the first terms of the above expansion but in physical theories with strong coupling g, we can observe the appearance of nonperturbative phenomena which are encoded in the context of Dyson–Schwinger equations. [6, 8, 14, 24, 25]

In the second section our main effort focuses on the understanding of non-perturbative Quantum Field Theory from the viewpoint of the Connes–Kreimer renormalization Hopf algebra and Hochschild cohomology theory where we formulate a new multi-scale non-perturbative renormalization group on the set of all Dyson–Schwinger equations of a given physical theory. This new construction enables us to study the dynamics of non-perturbative situations on the basis of changing the scales of momenta and bare strong coupling constants. This method enables us to study a given Dyson–Schwinger equation under strong bare or running coupling constant in terms of the cut-distance convergent limit of a sequence of Dyson–Schwinger equations under weaker couplings. In the third section we plan to build a new Noncommutative Geometry model to encode Dyson–Schwinger equations where we explain the structure of a particular class of spectral triples which are capable to describe quantum motions. This methodology leads us to obtain some new investigations about the geometry of Quantum Field Theory beyond the perturbation theory. In the fourth section we focus on the concept of ”evolution” at the level of combinatorial

Dyson–Schwinger equations where thanks to the Gelfand transform, we formulate a modified version of the Fourier transformation for the measure space of Feynman diagrams of a given physical theory. In addition, we apply the generalized Dyson series (formulated by Johnson and Lapidus [15]) to find another alternative machinery to describe the evolution of large graphs in the language of Dyson series.

2. A MULTI-SCALE RENORMALIZATION GROUP ON THE SET OF ALL DYSON–SCHWINGER EQUATIONS OF A PHYSICAL THEORY

In this part we are going to build two classes of renormalization groups which act on the set of all Dyson–Schwinger equations of a given physical theory with respect to changing the scales of momentum and bare strong coupling constants, separately and then we will formulate a new multi-scale non-perturbative renormalization group which is capable to control the behavior of dynamics of Dyson–Schwinger equations under changing the scales of those two parameters.

The renormalization coproduct on Feynman diagrams is given by the relation

$$\Delta(\Gamma) = \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma \tag{2.1}$$

where the sum is over all disjoint unions of 1PI superficially divergent proper subgraphs [2, 18]. This coproduct can be reformulated in terms of the grafting operator B^+ as a linear operator which replaces a vertex in a given Feynman diagram with a whole graph in terms of the type of the vertex in the first graph and types of external edges in the second graph. Consider the chain complex $C = \{C_n\}_{n\geq 0}$ such that for each n, C_n is the set of all linear maps T from $H = H_{\mathrm{FG}}(\Phi)$ to $H^{\otimes n}$ and the coboundary operator is given by

$$\mathbf{b}T := (\mathrm{id} \otimes T)\Delta + \sum_{i=1}^{n} (-1)^i \Delta_i T + (-1)^{n+1} T \otimes \mathbb{I}. \tag{2.2}$$

The operator B^+ generates an important class of one cocycles with respect to the Hochschild cohomology theory on $(C, \mathbf{b})$. Thanks to the relation (2.2), for a given family $\{\gamma_n\}_{n\in\mathbb{N}}$ of (1PI) primitive Feynman diagrams, $\{B^+_{\gamma_n}\}_{n\in\mathbb{N}}$ is the corresponding family of Hochschild one cocycles. In addition, the Feynman rules characters are generally of the form

$$\varphi(B^+_{\gamma_n}(\Gamma))(\{q\}) = \int d_{\gamma_n}(\{k\},\{q\})\varphi(\Gamma)(\{k\}) \tag{2.3}$$

such that $d_{\gamma_n}(\{k\},\{q\})$ is a measure determined by primitive (1PI) graph γ_n which depends on internal loop momenta $\{k\}$ and external momenta $\{q\}$. Now apply the Mellin transformation of the form

$$F_{\gamma_n}(s) = \int \varphi(B^+_{\gamma_n}(\mathbb{I}))[k^2]^{-s} \tag{2.4}$$

enables us to reformulate Dyson–Schwinger equations as a class of combinatorial equations in $H[[g]]$ of the form

$$X = \mathbb{I} + \sum_{n\geq 1} g^n \omega_n B^+_{\gamma_n}(X^{n+1}). \tag{2.5}$$

It has the unique solution $X = \sum_{n\geq 0} g^n X_n$ which belongs to the completed topological Hopf algebra of Feynman diagrams respect to the cut-distance topology such that for each n,

$$X_n = \sum_{j=1}^{n} \omega_j B^+_{\gamma_j}\Big(\sum_{k_1+...+k_{j+1}=n-j,\ k_i\geq 0} X_{k_1}...X_{k_{j+1}}\Big) \in H. \tag{2.6}$$

This unique solution determines a graded commutative (finite type) Hopf subalgebra $H_{\rm DSE}$ of the renormalization Hopf algebra of Feynman diagrams which is free algebraically generated by objects X_n's. [17, 29, 30, 36]

Example 1. Consider a decorated version of the Connes–Kreimer renormalization Hopf algebra $H_{\rm CK}(I)$ of non-planar rooted trees decorated by $I = \{a, b\}$. The unique solution of the Dyson–Schwinger equation

$$X = \mathbb{I} + gB_a^+(X^2) + g^2 B_b^+(X^3) \tag{2.7}$$

with the coupling constant $g = 1$ in $H_{\rm CK}$ is given by the infinite formal expansion $X = \sum_{n\geq 0} X_n$. The operator B_a^+ acts on forests of rooted trees where it adds a new root labeled by a and then connects it with each roots of trees in each forest. We have

$$\begin{gathered} X_0 = \mathbb{I}, \ \text{empty tree}, \\ X_1 = B_a^+(\mathbb{I}) = a, \ \text{single vertex labeled by "a"}, \\ X_2 = 2B_a^+(a) + b, \\ X_3 = 2B_a^+(X_2) + B_a^+(aa) + 3B_b^+(a), \\ X_4 = 2B_a^+(X_1X_2) + 2B_a^+(X_3) + 3B_b^+(X_1X_1) + 3B_b^+(X_2), \\ X_5 = 2B_a^+(X_1X_3)+B_a^+(X_2X_2)+2B_a^+(X_4)+6B_b^+(X_1X_2)+B_b^+(X_1X_1X_1)+3B_b^+(X_3), \\ ... \end{gathered} \tag{2.8}$$

Each component X_n is a linear combination of rooted trees with the maximum overall vertex number n which contribute to the recursive equation (2.7). The complicated graph X does not belong to $H_{\rm CK}(I)$ and we need an enrichment of this Hopf algebra to restore X. The pixel picture presentations of rooted trees can be applied to translate the topology of graphons to rooted trees and equip $H_{\rm CK}(I)$ with the cut-distance topology to obtain a compact topological Hopf algebra denoted by $H_{\rm CK}^{\rm cut}(I)$. It is then shown that X is the cut-distance convergent limit of the sequence $\{X^{(n)}\}_{n\geq 0}$ of its partial sums [30].

Thanks to the Wilson's approach to renormalization group, which is on the basis of the re-scaling of the momentum parameter at the cut-off propagator level, we plan to obtain an abstract formulation of non-perturbative renormalization group to study the changes of the dynamics of Dyson–Schwinger equations when the scale of the momentum parameter is changed. Consider a given Lagrangian with the general form

$$L = \int (1/2a(\nabla\phi)^2 + 1/2m^2\phi^2 + I(\phi))d^4x \tag{2.9}$$

such that $a \in \mathbb{R}_+$, $I(\phi) = \sum_{k\geq 2} I_k(\phi)$ and for any $N \geq 0$, $I_k = O(g^N)$ for almost all k. The re-scaling procedure allows us to study an effective Lagrangian at the scale $\mu \leq \Lambda$ of a given Lagrangian L at the initial scale Λ where at the end of the day, we can determine a semigroup $\mathbb{R}_+^{\leq 1}$ of scales which acts on the space of physical theories. For a given (bare) coupling constant g, let $\mathcal{X}$ be the space of all Lagrangians of ϕ with coefficients in the ring $\mathbb{R}[[g]]$ and invariant under the change $\phi \longrightarrow -\phi$. For each k, define the space F_k as the set of smooth functions on the hyperplane $\sum_{i=1}^k v_i = 0$ in $(V^*)^{\oplus k}$ such that V is the Euclidean 4-dimensional space-time and then set $F = \prod_{i=1}^\infty F_{2i}$. A general form of Green's functions is given as maps such as

$$\mathcal{G} : \mathcal{X} \times M_{\rm m} \longrightarrow F, \ \ \mathcal{G} = (\mathcal{G}_2, \mathcal{G}_4, ...) \tag{2.10}$$

such that M_m is the set of scales of momenta, the value $\mathcal{G}_k$ at (L, Λ) is called the $k-$point correlation function of the Lagrangian L at the scale Λ and for each k, $\mathcal{G}_k$ is the sum of amplitudes of all Feynman diagrams with k external edges. Define

$$\widetilde{G}_\Lambda(p) = \frac{P(p^2/\Lambda^2)}{p^2 + m^2} \tag{2.11}$$

as the cut-off propagator in momentum space such that P is a smooth function on $\mathbb{R}_{>0}$ which is 1 on $(0, 1/2]$, 0 on $(2, +\infty)$ and varies on $(1/2, 2)$. Apply cut-off propagator for internal edges and free propagator for external edges. [7]

Fixed point equations for general Green's functions in a Quantum Field Theory Φ have the form

$$\mathcal{G} = 1 + \int I_\gamma \mathcal{G} \tag{2.12}$$

, which produce an abstract presentation for Dyson–Schwinger equations with the combinatorial form (2.5) where I_γ is an integral kernel with respect to the 1PI Feynman diagram γ. Set $\mathcal{S}^\Phi$ as the family of all Dyson–Schwinger equations originated from general Green's functions (2.12) in Φ.

Definition 2.1. An equation in $\mathcal{S}^\Phi$ is called effective at the scale Λ of the original equation DSE at the scale Λ_0, if the fixed point equation which generates DSE in the Green's function $\mathcal{G}(L^\Phi, \Lambda_0)$ coincides with the fixed point equation which generates DSE in the Green's function $\mathcal{G}(L^\Phi, \Lambda)$.

Lemma 2.2. *Let Φ be Quantum Field Theory with the Lagrangian $L^\Phi \in \mathcal{X}$. For any equation DSE in $\mathcal{S}^\Phi$ and any scales Λ_0, Λ for the momentum parameter, there exists a unique effective equation in $\mathcal{S}^\Phi$ at the scale Λ for the original equation DSE at the scale Λ_0.*

Proof. A homogenous interaction of degree k is a complex valued continuous homogenous Poincaré invariant polynomial functional such as J on the space of Schwartz functions on V. Its general formulation is given by

$$J(\phi) = \int K_J(x_1, ..., x_k)\phi(x_1)...\phi(x_k)dx_1...dx_k \tag{2.13}$$

such that it is called quasilocal if the Poincaré invariant distribution K_J has rapid decay at infinity which means that the Fourier transformation of K_J will be

$$\widetilde{K}_J = \delta(p_1 + ... + p_k)\widetilde{F}_J(p_1, ..., p_{k-1}). \tag{2.14}$$

It is shown that any quasilocal interaction J of degree $k \geq 1$ can be represented in terms of a Taylor expansion type formula such as

$$J(\phi) = \sum_{j=0}^{s-1} \int_V D_j[\phi(x_1)...\phi(x_{k-1})]|_{x_1=...=x_{k-1}=x}\phi(x)dx + J^s(\phi) \tag{2.15}$$

where

- J^s is of order larger than s,
- For all $j \geq 0$, D_j is the homogenous differential operator of order J which can be identified uniquely. [7]

Now consider the original Lagrangian L^Φ at the scale Λ_0 and then apply a re-scaling process to generate a new Lagrangian L^Φ_Λ at the scale Λ which has the property $\mathcal{G}(L^\Phi_\Lambda, \Lambda) = \mathcal{G}(L^\Phi, \Lambda_0)$ and is called effective. The smooth function $\widetilde{F}_{J_k}$ for the effective Lagrangian can be defined in terms of the sum of amplitudes of all connected Feynman diagrams with k external edges and in addition for the propagator we have

$$\widetilde{G}_{\Lambda_0\Lambda}(p) := \widetilde{G}_{\Lambda_0}(p) - \widetilde{G}_\Lambda(p). \tag{2.16}$$

On the other hand, each Lagrangian is determined uniquely on the basis of its correlation functions and in addition, Dyson–Schwinger equations come from fixed point equations of Green's functions. Thanks to these investigations and Definition 2.1, for any given equation DSE in $\mathcal{S}^\Phi$ at the scale Λ_0, we can generate uniquely its corresponding effective equation at the scale Λ in terms of the effective Lagrangian. □

Lemma 2.2 explains the machinery which re-scales the momenta of internal edges of each term of an infinite formal expansion of Feynman diagrams which are generated by the unique solution of a given Dyson–Schwinger equation. This changing of the scale has been performed by the cut-off propagator (2.11).

Proposition 2.3. *For a given Quantum Field Theory Φ with the Lagrangian $L^{\Phi} \in \mathcal{X}$, there exists a renormalization group on $\mathcal{S}^{\Phi}$ which encodes the dynamic of Dyson–Schwinger equations under the re-scaling of the momentum parameter.*

Proof. Let $M_{\rm m}$ be the set of scales of the momentum parameter which is non-canonically isomorphic to $\mathbb{R}^{+}$ and consider $\mathcal{S}^{\Phi}$ as the family of all Dyson–Schwinger equations of the form (2.5) originated from Green's functions in the (renormalizable) Quantum Field Theory Φ. For any scales $\Lambda_1, \Lambda_2, \Lambda_3 \in M_{\rm m}$ such that $\Lambda_1 < \Lambda_2 < \Lambda_3$, define the scale map $R^{\rm m}_{\Lambda_1 \Lambda_2}$ on $\mathcal{S}^{\Phi}$ which obeys the property

$$R^{\rm m}_{\Lambda_1 \Lambda_2} R^{\rm m}_{\Lambda_2 \Lambda_3} = R^{\rm m}_{\Lambda_1 \Lambda_3}. \tag{2.17}$$

Now thanks to Proposition 2.2 for each equation DSE in $\mathcal{S}^{\Phi}$ define the new equation $R^{\rm m}_{\Lambda_1 \Lambda_2}{\rm DSE}$ whose belongs to $\mathcal{S}^{\Phi}$ as the effective Dyson–Schwinger equation at the scale Λ_2 of the equation DSE at the initial scale Λ_1. As the result, we enable to define an action of the semi-group $\mathbb{R}^{+}_{\leq 1}$ on the space $\mathcal{S}^{\Phi} \times M_{\rm m}$ which is given by

$$\lambda \circ ({\rm DSE}, \Lambda) := (R^{\rm m}_{\Lambda, \lambda\Lambda}{\rm DSE}, \lambda\Lambda). \tag{2.18}$$

This construction provides an abstract model for the renormalization group on Dyson–Schwinger equations of the theory Φ which depends on the changing the scales of the momentum parameter. □

We name the renormalization group defined by Proposition 2.3 as the momentum type non-perturbative renormalization group. This renormalization group proposes a new way to study the dynamics of Dyson–Schwinger equations in terms of their corresponding equations under the running couplings derived from the chosen regularization schemes and scaled momenta. It is actually a modification of the standard running coupling type renormalization group which acts on Dyson–Schwinger equations.

Proposition 2.4. *For a given Quantum Field Theory Φ with the Lagrangian $L^{\Phi} \in \mathcal{X}$, there exists a renormalization group on $\mathcal{S}^{\Phi}$ which encodes the dynamics of Dyson–Schwinger equations under the re-scaling of the bare coupling constants.*

Proof. As we know the parameter g (as the bare coupling constant) has been included in the interaction part $I(\Phi)$ of the Lagrangian where infinite formal expansions of Feynman diagrams will appear. This interaction part is indeed the part which contribute to formulating Dyson–Schwinger equations via fixed point equations of Green's functions. Work on changing the scale of the coupling constant (before regularization schemes and renormalization maps) enables us to study the asymptotic behavior of Green's functions and related fixed point equations at the original level. Therefore we require to focus on changing scales such as g to Λg in the interaction part of the original Lagrangian L^{Φ}.

Let $M_{\rm b}$ be the set of scales of bare coupling constants which is non-canonically isomorphic to $\mathbb{R}^{+}$ and consider $\mathcal{S}^{\Phi}$ as the family of all Dyson–Schwinger equations of the form (2.5) originated from Green's functions in Φ. For any scales $\tau_1, \tau_2, \tau_3 \in M_{\rm b}$ such that $\tau_1 < \tau_2 < \tau_3$, define the scale map $R^{\rm b}_{\tau_1 \tau_2}$ on $\mathcal{S}^{\Phi}$ which obeys the property

$$R^{\rm b}_{\tau_1 \tau_2} R^{\rm b}_{\tau_2 \tau_3} = R^{\rm b}_{\tau_1 \tau_3}. \tag{2.19}$$

Now for each equation DSE in $\mathcal{S}^{\Phi}$ define the new equation $R^{\rm b}_{\tau_1 \tau_2}{\rm DSE} \in \mathcal{S}^{\Phi}$ as the re-scaled Dyson–Schwinger equation at the scale τ_2 of the equation DSE at the initial scale τ_1. This allows us to define an action of the semi-group $\mathbb{R}^{+}_{\leq 1}$ on the space $\mathcal{S}^{\Phi} \times M_{\rm b}$ given by

$$\lambda \circ ({\rm DSE}, \tau) := (R^{\rm b}_{\tau, \lambda\tau}{\rm DSE}, \lambda\tau). \tag{2.20}$$

This construction provides an abstract model for the renormalization group on Dyson–Schwinger equations of the theory Φ which depends on the changing the scales of the bare coupling constant parameter. □

We name the renormalization group defined by Proposition 2.4 as the bare type non-perturbative renormalization group. This renormalization group proposes a new way to study the dynamics of Dyson–Schwinger equations with strong bare couplings in terms of their corresponding equations at the smaller couplings independent of the regularization and renormalization procedures. It enables us to re-scale a bare strong coupling constant to produce a decreasing sequence of re-scaled couplings where Dyson–Schwinger equations at the level of weaker couplings can be handled under perturbative treatments. As the result, we will have the chance to study a given Dyson–Schwinger equation at the strong bare coupling in terms of a sequence of equations in weaker couplings.

Corollary 2.5. *For a given Quantum Field Theory Φ with the Lagrangian $L^{\Phi} \in \mathcal{X}$, there exists a multi-scale renormalization group on $\mathcal{S}^{\Phi}$ which encodes the dynamics of Dyson–Schwinger equations under the re-scalings of the bare coupling constant and the momentum parameter.*

Proof. Running couplings do not have any clear physical meanings and we can change them in terms of the cut-off propagator and the re-scaling of the momentum parameter. Therefore the scale of the bare coupling constants (as a physical parameter) has more priority than the scale of the running couplings. Consider the expression $(\mathrm{DSE}, \tau, \Lambda_\tau)$ in the space $\mathcal{S}^{\Phi} \times M_{\mathrm{b}} \times M_{\mathrm{m}}$ which presents a Dyson–Schwinger equation in $\mathcal{S}^{\Phi}$ such that it is defined in terms of the bare coupling constant with the initial scale τ and the momentum parameter with the initial scale Λ_τ. Thanks to Proposition 2.3 and Proposition 2.4, define a new action of the semi-group $\mathbb{R}^{+}_{\leq 1}$ on the space $\mathcal{S}^{\Phi} \times M_{\mathrm{b}} \times M_{\mathrm{m}}$ given by

$$\lambda \circ (\mathrm{DSE}, \tau, \Lambda_\tau) := (R^{\mathrm{multi}}_{(\tau,\Lambda_\tau),(\lambda\tau,\lambda\Lambda_\tau)} \mathrm{DSE}, (\lambda\tau, \lambda\Lambda_\tau)) \tag{2.21}$$

such that $R^{\mathrm{multi}}_{(\tau,\Lambda_\tau),(\lambda\tau,\lambda\Lambda_\tau)} \mathrm{DSE}$ is a new multi-scaled Dyson–Schwinger equation which is defined on the basis of the scaled bare coupling constant parameter of the size $\lambda\tau$ and the scaled momentum parameter $\lambda\Lambda_\tau$. □

Corollary 2.5 provides a mathematical machinery to produce running couplings in terms of choosing the scale of the bare couplings before perturbative renormalization procedure.

For example in QCD, there exist eight classes of 1PI Green's functions with respect to the types of fields e_1, e_2, e_3 namely, quark, ghost, gluon and five possible types of interactions $v_1, ..., v_5$ among those fields. By applying the bare type non-perturbative renormalization group R^{b} (i.e. Proposition 2.4) and the momentum type non-perturbative renormalization group R^{m} (i.e. Proposition 2.3), we can generate Green's functions in all possible scales of the bare coupling constant and the momentum parameter. We use the phrases

$$G^{e_i}_{(\tau,\Lambda_\tau)} = 1 - \sum_{\mathrm{res}(\Gamma)=e_i} (\tau g)^{L(\Gamma)} \frac{\Gamma}{\mathrm{Sym}(\Gamma)}, \ \ i = 1,2,3, \tag{2.22}$$

$$G^{v_j}_{(\tau,\Lambda_\tau)} = 1 + \sum_{\mathrm{res}(\Gamma)=v_j} (\tau g)^{L(\Gamma)} \frac{\Gamma}{\mathrm{Sym}(\Gamma)}, \ \ j = 1,...,5, \tag{2.23}$$

such that τg are the results of changing the scale of the bare coupling constant g, Λ_τ is the scale of the momentum parameter. The restriction of the sum to graphs at loop order $L(\Gamma) = l$ can be presented by $G^{r,l}_{(\tau,\Lambda_\tau)}$ where $r \in \{e_i, v_j\}_{i,j}$.

Consider $\mathcal{S}^{\mathrm{QCD}}$ as the collection of all Dyson–Schwinger equations as the results of fixed point equations of Green's functions $G^{e_i}_{(\tau,\Lambda_\tau)}, G^{v_j}_{(\tau,\Lambda_\tau)}$ which could cover Dyson–Schwinger equations under different running couplings in QCD. At relatively lower energy levels, these Green's functions behave non-perturbatively because of strong running couplings and we need to deal infinite formal expansions of Feynman diagrams as coefficients of powers of those running couplings. The multi-scale renormalization group given by Corollary 2.5 enables us to study the behavior of Green's functions of the type (2.22), (2.23) in $\mathcal{S}^{\mathrm{QCD}}$ with respect to each other while running couplings have been changed in terms of energy levels. In addition, thanks to the topology of graphons, it is also possible to describe the inverse of these Green's functions at infinite loop orders which enables us to provide some new non-perturbative computational methods. For example in [29], we have formulated a class of β-functions which control the behavior of solutions of Dyson–Schwinger equations under changing the scales of running coupling constants.

3. Non-perturbative spectral triples

The previous section has provided a new mathematical description for the dynamics of non-perturbative situations in Quantum Field Theory on the basis of changing the scales of the bare coupling constant and the momentum parameter which contribute to Dyson–Schwinger equations. In this section we study the geometry of quantum motions via a new application of Noncommutative Geometry to Quantum Field Theory. We will build a new class of spectral triples which are capable to encode the geometry of Dyson–Schwinger equations.

Theory of spectral triples has been developed to interesting complicated structures such as AF C^*-algebras and fractal sets ([4, 5, 9, 27]) and in this part we plan to provide another progress in this direction and explain the structure of a new class of spectral triples which are originated from Dyson–Schwinger equations.

Suppose we have a countable family $\{(A_m, \mathbb{H}_m, D_m)\}_{m\geq 1}$ of spectral triples (which are not necessarily unital) with the corresponding family of representations $\{\pi_m\}_{m\geq 1}$. Moreover, let $\{\alpha_m\}_{m\geq 1}$ be a sequence of non-zero real numbers such that $||(1+\alpha_m^2 D_m^2)^{\frac{-1}{2}}||_m$ converges to zero whenever m tends to infinity where $||.||_m$ is the norm on $\mathbb{H}_m$. Now set

$\mathbb{H}^{\oplus} := \bigoplus_{m\geq 1} \mathbb{H}_m$,

$D^{\oplus} := \bigoplus_{m\geq 1} \alpha_m D_m$ with the corresponding self-adjoint extension $\overline{D^{\oplus}}$,

$$A^{\oplus} := \{(a_m)_{m\geq 1} \in \prod_m A_m :$$

$$\sup_{m\geq 1} ||\pi_m(a_m)||_m < +\infty, \quad \sup_{m\geq 1} ||[\alpha_m D_m, \pi_m(a_m)]||_m < +\infty\}.$$

For each $a^{\oplus} \in A^{\oplus}$, $\pi^{\oplus}(a^{\oplus}) := \bigoplus_{m\geq 1} \pi_m(a_m)$. It is shown in [9] that the information $(A^{\oplus}, \mathbb{H}^{\oplus}, \overline{D^{\oplus}})$ is a spectral triple which is not necessarily unital. Generally speaking, this spectral triple is originated from five-tuples with the general form $(A_m, \mathbb{H}_m, D_m, \pi_m, \alpha_m)$ for each $m \geq 1$.

We are going to apply this model to the level of Green's functions to discover a new knowledge about the geometry of non-perturbative Quantum Field Theory. For this purpose, we need to determine the required mathematical structures for a spectral triple with respect to each Dyson–Schwinger equation.

For a given commutative Hopf algebra H, let $\mathrm{Spec}(H)$ be the set of all prime ideals of H equipped with the Zariski topology and the structure sheaf. The Hopf algebraic structure of H generates a product operation on $\mathrm{Spec}(H)$ which satisfies the properties of a group structure. The resulting space is called affine group scheme. Under a categorical setting, Spec is a contravariant functor from the category of commutative algebras to the category of topological spaces such that the object $\mathbb{G} = \mathrm{Spec}(H)$ is a covariant representable functor from the category of commutative algebras to the category of groups. [23]

For each commutative algebra A, the group $\mathbb{G}(A) = \mathrm{Spec}(H)(A)$ is the set of morphisms of the form

$$\varphi \in \mathbb{G}(A): \ \ \varphi: H \longrightarrow A, \ \ \varphi(h_1 h_2) = \varphi(h_1)\varphi(h_2), \ \ \varphi(1_h) = 1_A, \tag{3.1}$$

equipped with the convolution product $\varphi_1 * \varphi_2(h) := m \circ (\varphi_1 \otimes \varphi_2) \circ \Delta(h)$.

The space GL_n is the fundamental example of an affine group scheme for us which corresponds to the Hopf algebra $H_{\mathrm{GL}_n} = k[x_{i,j}, t]_{i,j=1,\ldots,n}/\det(x_{i,j})t-1$ with the coproduct $\Delta(x_{i,j}) = \sum_s x_{i,s} \otimes x_{s,j}$. It is shown that if the Hopf algebra H is finitely generated as an algebra, then its corresponding affine group scheme is a linear algebraic group embedded as a Zariski closed subset of some GL_n. [23]

Lemma 3.1. *For a given collection $\{H_n\}_{n\geq 0}$ of finitely generated commutative Hopf sub-algebras of a given Hopf algebra H, suppose $H = \bigcup_{n\geq 0} H_n$ such that for all n and m, there exists a k such that $H_n \cup H_m \subset H_k$. Then for each n, there exists the corresponding linear algebraic groups of the form $\mathbb{G}_n(\mathbb{C}) = \mathrm{Spec}(H_n)(\mathbb{C}) < \mathrm{GL}_{m_n}(\mathbb{C})$ for some m_n. These algebraic groups generate the affine group scheme $\mathbb{G}$ associated to the Hopf algebra H via the projective limit $\mathbb{G} = \lim_{\longleftarrow n} \mathbb{G}_n$.* [23]

Proposition 3.2. *There exists a class of spectral triples which provides an algebraic reconstruction of the geometry of quantum motions in a given Quantum Field Theory Φ.*

Proof. We consider Dyson–Schwinger equations as the original sources of quantum motions. For each given Dyson–Schwinger equation DSE, its corresponding Hopf sub-algebra is graded of finite type which means that $H_{\mathrm{DSE}}(\Phi) = \bigcup_{n\geq 0} H^n_{\mathrm{DSE}}(\Phi)$. For each n, the finite dimensional component $H^n_{\mathrm{DSE}}(\Phi)$, as a finite dimensional vector space, could determine the finite dimensional complex Lie group $\mathbb{G}^{(n)}_{\mathrm{DSE}}(\mathbb{C})$ which can be embedded as a closed subset of the linear algebraic group $\mathrm{GL}_{m_n}(\mathbb{C})$ for some m_n with respect to the Zariski topology. Thanks to Lemma 3.1, we can describe the complex Lie group $\mathbb{G}_{\mathrm{DSE}}(\mathbb{C})$ as the projective limit of $\mathbb{G}^{(n)}_{\mathrm{DSE}}(\mathbb{C})$'s as closed subsets of $\mathrm{GL}_{m_n}(\mathbb{C})$'s while for each m_n, the finite dimensional Riemannian manifold $\mathrm{GL}_{m_n}(\mathbb{C})$ can be described by the spectral triple

$$\mathcal{S}^{(m_n)} := (C^\infty(\mathrm{GL}_{m_n}(\mathbb{C})), L^2(\mathrm{GL}_{m_n}(\mathbb{C}), S), D_{\mathrm{GL}_{m_n}(\mathbb{C})}). \tag{3.2}$$

Restrictions of the spectral triple (3.2) could determine spectral triples corresponding to Lie sub-groups of $\mathrm{GL}_{m_n}(\mathbb{C})$. For each n, we present the spectral triple of the Lie sub-group $\mathbb{G}^{(n)}_{\mathrm{DSE}}(\mathbb{C})$ by

$$\mathcal{S}^{(n)}_{\mathrm{DSE}} = (A^{(n)}_{\mathrm{DSE}}, \mathbb{H}^{(n)}_{\mathrm{DSE}}, D^{(n)}_{\mathrm{DSE}}). \tag{3.3}$$

Now consider the family $\{\mathcal{S}^{(n)}_{\mathrm{DSE}}\}_n$ of countable number of spectral triples originated from components of the graduation structure of the Hopf sub-algebra $H_{\mathrm{DSE}}(\Phi)$ generated by the equation DSE. Moreover, let $\{\alpha_n\}_n$ be a sequence of non-zero real numbers such that $||(1+\alpha_n^2(D^{(n)}_{\mathrm{DSE}})^2)^{\frac{-1}{2}}||_n$ converges to zero whenever n tends to infinity where $||.||_n$ is the norm on $\mathbb{H}^{(n)}_{\mathrm{DSE}}$. Thanks to the explained construction in the previous part, the information

$$\mathcal{S}^{\oplus}_{\mathrm{DSE}} := (A^{\oplus}_{\mathrm{DSE}}, \mathbb{H}^{\oplus}_{\mathrm{DSE}}, \overline{D^{\oplus}_{\mathrm{DSE}}}) \tag{3.4}$$

is a spectral triple which is originated from five-tuples

$$(A^{(n)}_{\mathrm{DSE}}, \mathbb{H}^{(n)}_{\mathrm{DSE}}, D^{(n)}_{\mathrm{DSE}}, \pi^{(n)}_{\mathrm{DSE}}, \alpha_n) \tag{3.5}$$

for each n. □

We name the spectral triple $\mathcal{S}^{\oplus}_{\mathrm{DSE}}$ as non-perturbative spectral triple with respect to the equation DSE which has the following properties:

- The norm of the Hilbert space $\mathbb{H}^{\oplus}_{\mathrm{DSE}}$ is given by

$$||.||^{\oplus} := \sup_n ||.||_n. \tag{3.6}$$

- Thanks to the structure of $A^{\oplus}_{\mathrm{DSE}}$, we can see that the representation $\pi^{\oplus}_{\mathrm{DSE}}$ and the commutator $[D^{\oplus}_{\mathrm{DSE}}, \pi^{\oplus}_{\mathrm{DSE}}(A^{\oplus}_{\mathrm{DSE}})]$ are bounded.

- The sequence $\{\alpha_n\}_n$ has been applied to control the behavior of the sequence $\{D^{(n)}_{\mathrm{DSE}}\}_n$ which means that

$$\sum_n \dim(\mathrm{Ker} D^{(n)}_{\mathrm{DSE}}) < \infty. \tag{3.7}$$

It is possible to embed the renormalization Hopf algebra of Feynman diagrams of a gauge field theory into a decorated version of the Connes–Kreimer Hopf algebra of non-planar rooted trees which enables us to deal with Dyson–Schwinger equations in this combinatorial Hopf algebra . Vertices in trees are symbols for loops or (1PI) primitive Feynman diagrams and edges between vertices in trees encode the positions of loops with respect to each other in complicated Feynman diagrams. Each Feynman diagram with nested or independent loops can be presented by a decorated non-planar rooted tree while each Feynman diagram with overlapping divergences should be presented by a linear combination of decorated rooted trees. [1, 2, 18, 16]

Example 2. Let us build the noncommutative spectral triple for the non-linear Dyson–Schwinger equation

$$X = \mathbb{I} + gB^+_\gamma(X^2) \tag{3.8}$$

in the Connes–Kreimer renormalization Hopf algebra $H_{\mathrm{FG}}(\Phi)$ of Feynman diagrams of the physical theory Φ. The analytic version of this equation can be formulated by applying Feynman rules of the physical theory encoded by some characters ϕ of the renormalization Hopf algebra where the linear operator B^+_γ is the sum of all possible ways that we can insert the primitive 1PI Feynman diagram γ into a Feynman diagram. We have

$$\phi B^+_\gamma(\mathbb{I})[z] = \int_0^\infty k(x,z)dx \tag{3.9}$$

as the Fourier transform of a homogeneous kernel k with respect to the multiplicative group $\mathbb{R}_+$. The corresponding regularized Feynman rules characters can be given by

$$\phi B^+_\gamma(\Gamma)[z] = \int_0^\infty (k(x,z) - k(x,1))\phi(\Gamma)[x]dx. \tag{3.10}$$

If we apply the Feynman rules character to the equation (3.8), then we have its corresponding integral equation

$$\phi(X)[z,g] = 1 + g\int_0^\infty (\phi(X)[x,g])^2(k(x,z) - k(x,1))dx$$

$$= 1 + \int_0^\infty g_\tau \phi(X)[x,g](k(x,z) - k(x,1))dx \tag{3.11}$$

such that g is the bare coupling constant and $g_\tau = g\phi(X)[x,g]$ is a running coupling constant. $\phi(X)$ is well-defined in terms of the convergence of the sequence $\{\phi(X^{(n)})\}_{n\geq 0}$ with respect to partial sums of X. In addition, the solutions of these equations have a general form $G = \exp(-\sum_{j\geq 1} u_j(g)\ln^j(z))$ [1, 30]. Therefore we can produce a collection of equations by changing scales of the bare coupling constant. If the running coupling is constant, then the non-linear equation (3.11) can turn into a linear equation which is easier to solve. But dealing with non-linear equations is the main challenge and our built multi-scale renormalization group (i.e. Corollary 2.5) enables us to study the behavior of the original equation (3.11) in terms of changing the scales of the bare coupling g and running couplings.

Rooted tree representations of Feynman diagrams allow us to simplify the formulations of Dyson–Schwinger equations as some recursive equations in the combinatorial Hopf algebra $H_{\mathrm{CK}}(\Phi)$ of non-planar rooted trees decorated by primitive (1PI) Feynman diagrams of the physical theory. In this setting the operator B^+_γ is replaced by the Hochschild one cocycle map $B^+_{a_\gamma}$ which adds a new root labeled by a_γ together with an edge to the rooted tree representation of each Feynman diagram. The unique solution of the equation (3.8) is given by $X = \sum_{n=0}^{\infty} g^n X_n$ such that

$$X_0 = \mathbb{I}, \quad X_{n+1} = \sum_{k=0}^{n} B^+_{a_\gamma}(X_k X_{n-k}). \tag{3.12}$$

Formal expansions X_ns, which are actually weighted sums of rooted trees with vertex fertility bounded by two, play the role of algebraic generators of a new Hopf subalgebra H_X of $H_{\mathrm{CK}}(\Phi)$ which is graded in terms of number of vertices. For each $n \geq 0$, $H_X^{(n)}$ is a finite dimensional vector space of rooted trees with n vertices or products of rooted trees with overall vertex number n and with vertex fertility bounded by two. For each n, $H_X^{(n)}$ is a subspace of $H_{CK}^{(n)}$ and therefore the linear algebraic group $G_X^{(n)}(\mathbb{C})$ as dual to $H_X^{(n)}$ can be embedded in $\mathrm{GL}_n(\mathbb{C})$. For each n, define $A_X^{(n)} = C^\infty(G_X^{(n)}(\mathbb{C}))$, $\mathbb{H}_X^{(n)} = L^2(G_X^{(n)}(\mathbb{C}), S)$ and determine the Dirac operator $D_{G_X^{(n)}(\mathbb{C})}$ by the restriction of the domain of the Dirac operator $D_{\mathrm{GL}_n(\mathbb{C})}$. This information provides the spectral triple $\mathcal{S}_X^n$ with respect to the subalgebra $H_X^{(n)}$. Thanks to Proposition 3.2, the sequence $\{\mathcal{S}_X^n\}_{n\geq 0}$ could determine the infinite spectral triple $\mathcal{S}_X^{\oplus}$ corresponding to the Hopf subalgebra H_X.

The next result provides an alternative representation of those spectral triples which encode the geometry of non-perturbative Dyson–Schwinger equations in QCD.

Corollary 3.3. *The geometry of Dyson–Schwinger equations in QCD can be described by formal diffeomorphisms in five dimensions.*

Proof. Set $\bar{\mathrm{Diff}}(\mathbb{C}^5, 0)$ as the complex Lie group of formal diffeomorphisms tangent to the identity in five variables that leave the five axis-hyperplanes invariant and equipped with the composition as the multiplication. Each $f \in \bar{\mathrm{Diff}}(\mathbb{C}^5, 0)$ has a general form

$$f(x) = (f_1(x), ..., f_5(x))$$

$$f_i(x) = x_i(\sum a^{(i)}_{n_1...n_5}(f) x_1^{n_1}...x_5^{n_5}), \;\; a^{(i)}_{0,...,0} = 1, \;\; x = (x_1, ..., x_5). \tag{3.13}$$

Thanks to [34], the Hopf algebra generated by the coefficients $a^{(i)}_{n_1...n_5}$ could be mapped to the Connes–Kreimer renormalization Hopf algebra H_{QCD} of Feynman diagrams generated by 1PI Green's functions $G^r_{(\tau,\Lambda_\tau)}$, $r \in \{e_i, v_j\}_{i,j}$ in QCD. The fixed point equation corresponding to each Green's function $G^r_{(\tau,\Lambda_\tau)}$ generates a Hopf subalgebra $H_{G^r_{(\tau,\Lambda_\tau)}}$ of H_{QCD} which is dual to a subgroup of $\bar{\mathrm{Diff}}(\mathbb{C}^5, 0)$. Therefore for each Dyson–Schwinger equation DSE_r, $r \in \{e_i, v_j\}_{i,j}$ in QCD, there exists a surjective map $\bar{\rho}_{\mathrm{DSE}_r} : \bar{\mathrm{Diff}}(\mathbb{C}^5, 0) \to \mathbb{G}_{\mathrm{DSE}_r}(\mathbb{C})$ of Lie groups which enables us to represent the spectral triple of $\mathbb{G}_{\mathrm{DSE}_r}(\mathbb{C})$ in terms of a sub-spectral triple of the infinite dimensional spectral triple of $\bar{\mathrm{Diff}}(\mathbb{C}^5, 0)$. □

Noncommutative Geometry enables us to deal with the theory of spectral geometry on the basis of an operator theoretic setting where the fundamental integral can be explained by the Dixmier trace which extends the Wodzicki residue from pseudodifferential operators on a manifold to a general framework which concern spectral triples [12]. We have $⨍ T := \mathrm{Res}_{s=0} \mathrm{Tr}(T|D|^{-s})$.

Now thanks to the discussed topics in this section, for each n, $\mathcal{S}^{(n)}_{\mathrm{DSE}}$ is a finite dimensional spectral triple which is the result of the restriction of the spectral triple associated to the complex Lie group $\mathrm{GL}_{m_n}(\mathbb{C})$ for some m_n. This means that for each n, the functional $a \longmapsto \mathrm{Tr}^+(a|D^{(n)}_{\mathrm{DSE}}|^{-m_n})$ (as the usual Riemannian integral) provides a differential calculus theory and spectral geometry with respect to the Riemannian volume form for $\mathcal{S}^{(n)}_{\mathrm{DSE}}$. This construction explains the geometric nature of a quantum motion which is approximated by partial sums of the unique solution X_{DSE} of the corresponding equation DSE.

In addition, Proposition 3.2 opens a new way to work on the formulation of a spectral geometry for $\mathcal{S}^{\oplus}_{\mathrm{DSE}}$ on the basis of the Connes–Dixmier traces such that the noncommutative integral at this level, which have the general form

$$a^{\oplus} \longmapsto \mathrm{Tr}_{\omega}(a^{\oplus}|D^{\oplus}_{\mathrm{DSE}}|^{-p}) \tag{3.14}$$

for some $p \geq 1$ and state ω, is capable to describe the geometry of Dyson–Schwinger equations.

4. The evolution of Green's functions under a mathematical setting

In this section we plan to modify the concept of "evolution" for the study of infinite formal expansions of Feynman diagrams which contribute to non-perturbative solutions of fixed point equations in Green's functions in a given Quantum Field Theory with strong coupling constants. We will apply the Gelfand transform [12] to obtain a new modification of the Fourier transformation for the space of Feynman diagrams. In addition, we will study the generalized Dyson series, which was formulated by Johnson and Lapidus [15], to improve it for the level of Dyson–Schwinger equations.

For a given gauge field theory Φ, set $\mathcal{H}^{\Phi}$ as the collection of all Feynman diagrams and their formal expansions which contribute to Green's functions and all other graphs produced by insertion, quotient and shrinking. Let $V(\Phi)$ be the countable set of decorated vertices which represent all possible interactions among elementary particles in Φ. All graphs in $\mathcal{H}^{\Phi}$ are decorated by some objects of $V(\Phi)$. Fix a bijection α between edges of the infinite complete graph $K_{V(\Phi)}$ and the set $\mathbb{N}$ of natural numbers. For a given real number $c > 1$, we can define a metric structure on $\mathcal{H}^{\Phi}$ given by

$$d^{\alpha}_c : \mathcal{H}^{\Phi} \times \mathcal{H}^{\Phi} \longrightarrow [0, \infty)$$

$$d^{\alpha}_c(\Gamma_1, \Gamma_2) := \sum_{e \in \Gamma_1 \bigodot \Gamma_2} c^{-\alpha(e)} \tag{4.1}$$

such that $\bigodot$ is the symmetric difference as a binary operation on graphs in $\mathcal{H}^{\Phi}$ with respect to external or internal edges. Thanks to [19, 31], we can show that d^{α}_c is a translation-invariant metric such that for real numbers $c_1, c_2 > 1$, the resulting metrics $d^{\alpha}_{c_1}$ and $d^{\alpha}_{c_2}$ generate the equivalent topology. On the other hand, there is a natural one to one and onto correspondence between the set $\mathcal{H}^{\Phi}$ and the set $\{0, 1\}^{K_{V(\Phi)}}$ of all functions from $K_{V(\Phi)}$ to $\mathbb{Z}_2$ which supports the existence of a commutative topological $\mathbb{Z}_2$-algebra structure on $\mathcal{H}^{\Phi}$ via pointwise addition and multiplication. The binary operation $\bigodot$ gives an abelian compact Hausdorff topological group structure such that the empty graph is the zero element for this group. The product type σ-algebra $\sum_{\mathrm{prod}}$ can be applied to build a new Haar measure μ_{Haar} on $\mathcal{H}^{\Phi}$. We use the notation

$$\mathcal{G}^{\Phi} := (\mathcal{H}^{\Phi}, d^{\alpha}_c, \bigodot, \sum_{\mathrm{prod}}, \mu_{\mathrm{Haar}}) \tag{4.2}$$

for the resulting space of graphs which contribute to the physical theory Φ.

4.1. Evolution via Fourier transformation.

The space $\mathcal{G}^{\Phi}$ might contain some graphs which are not necessarily Feynman diagrams but they are the results of some mathematical operations on Feynman diagrams. In addition, infinite formal expansions of Feynman diagrams which contribute to solutions of Dyson–Schwinger equations belong to this measure space. In this part we plan to formulate a generalization of the Fourier transformation for the study of Green's functions in terms of μ_{Haar}-integrable functions on $\mathcal{G}^{\Phi}$.

Consider the space $L^1(\mathcal{G}^{\Phi}, \mu_{\mathrm{Haar}})$ of all complex valued μ_{Haar}-integrable functions on $\mathcal{G}^{\Phi}$ which is a complex commutative Banach algebra with respect to the convolution product

$$f_1 *_B f_2(\Gamma_1) = \int_{\mathcal{G}^\Phi} f_1(\Gamma_2) f_2(\Gamma_2^{-1} \bigodot \Gamma_1) d\mu_{\mathrm{Haar}}(\Gamma_2), \quad f_1, f_2 \in L^1(\mathcal{G}^\Phi, \mu_{\mathrm{Haar}}) \tag{4.3}$$

such that Γ_2^{-1} is the inverse of the graph with respect to the group structure. We apply the infinitesimal delta function δ as the unit for this Banach algebra where it obeys the condition

$$\int_{\mathcal{G}^\Phi} f(\Gamma)\delta(\Gamma) d\mu_{\mathrm{Haar}}(\Gamma) = f(\mathbb{I}) \tag{4.4}$$

for each $f \in L^1(\mathcal{G}^\Phi, \mu_{\mathrm{Haar}})$ and each graph $\Gamma \in \mathcal{G}^\Phi$ such that $\mathbb{I}$ is the empty graph.

Lemma 4.1. *For any $f \in L^1(\mathcal{G}^\Phi, \mu_{\mathrm{Haar}})$, the spectrum*

$$\mathrm{sp}(f) := \{\lambda \in \mathbb{C} : f - \lambda\delta \ \text{ not invertible}\}$$

is non-empty.

Proof. It is trivial if $f = 0$. For a given non-zero μ-integrable function f, suppose $\mathrm{sp}(f) = \emptyset$ which means that the function $R : \mathbb{C} \longrightarrow L^1(\mathcal{G}^\Phi, \mu_{\mathrm{Haar}}), \ \ \lambda \longmapsto (f - \lambda\delta)^{-1}$ is well-defined and it is holomorphic, non-constant and bounded.

Now we adapt the proof of Theorem 1 in [28] for our framework. For any bounded linear functional F on $L^1(\mathcal{G}^\Phi, \mu_{\mathrm{Haar}})$ define a new function G on $\mathbb{R}^2$ given by $G(x, y) := F(R(xe^{iy}))$ which is continuously differentiable with respect to variables x and y. Now by differentiation under the integral sign from the holomorphic bounded function $K(x) := \int_0^{2\pi} G(x, y)dy$, we have $K'(x) = 0$. So K is a constant function.

This fact shows the contradiction produced by our initial assumption which means that $\mathrm{sp}(f)$ should be non-empty. □

Thanks to the Hilbert's Nullstellensatz, it is possible to formulate a natural one to one correspondence between the set of maximal ideals of the Banach algebra $L^1(\mathcal{G}^\Phi, \mu_{\mathrm{Haar}})$ and the set of characters on the space $L^1(\mathcal{G}^\Phi, \mu_{\mathrm{Haar}})$. This correspondence is determined by the ideal generated from kernel of any character.

Lemma 4.2. *The space $\Omega(L^1(\mathcal{G}^\Phi, \mu_{\mathrm{Haar}}))$ of all characters of the complex Banach algebra $L^1(\mathcal{G}^\Phi, \mu_{\mathrm{Haar}})$ is a compact Hausdorff topological space.*

Proof. Each $\psi \in \Omega(L^1(\mathcal{G}^\Phi, \mu_{\mathrm{Haar}}))$ is an algebra homomorphism from $L^1(\mathcal{G}^\Phi, \mu_{\mathrm{Haar}})$ to $\mathbb{C}$ such that $\psi(\delta) = 1$. First we show that ψ is continuous of norm 1. Suppose it is not; i.e. there exists a function $f \in L^1(\mathcal{G}^\Phi, \mu_{\mathrm{Haar}})$ such that $||f|| < 1$ and $\psi(f) = 1$. Set $g := \sum_{n\geq 1} f^n$ with respect to the convolution product. From the equation $g = f + fg$ we will have

$$\psi(g) = \psi(f) + \psi(f)\psi(g) = 1 + \psi(g) \tag{4.5}$$

which shows a contradiction. So the norm of ψ is less than or equal to 1 and $\psi(\delta) = 1$ which implies that $||\psi|| = 1$. Thanks to this fact, it is possible to show that $\Omega(L^1(\mathcal{G}^\Phi, \mu_{\mathrm{Haar}}))$ is a closed subset of the unit ball of the dual space $L^1(\mathcal{G}^\Phi, \mu_{\mathrm{Haar}})^*$ which is a compact Hausdorff space with respect to the *-weak topology. Therefore $\Omega(L^1(\mathcal{G}^\Phi, \mu_{\mathrm{Haar}}))$ is a compact Hausdorff topological space. □

Our goal is to build a mathematical formalism for the description of the evolution of infinite number of interactions among elementary particles which are encoded under the solution of a given Dyson–Schwinger equation. Actually, the objects of the complex Banach algebra $L^1(\mathcal{G}^\phi, \mu_{\mathrm{Haar}})$ together with a modified version of the Fourier transformation will be the original tools to achieve our goal.

The Gelfand transform

$$L^1(\mathcal{G}^\Phi, \mu_{\mathrm{Haar}}) \longrightarrow C_0(\Omega(L^1(\mathcal{G}^\Phi, \mu_{\mathrm{Haar}}))) \tag{4.6}$$

given by $f \longmapsto \widetilde{f}$ such that $\widetilde{f}(\psi) := \psi(f)$ is a norm decreasing algebraic homomorphism such that its image separates μ-integrable functions on $\mathcal{G}^{\Phi}$. In addition, we have

$$||\widetilde{f}||_{\infty} = \max\{|\lambda| : \ \lambda \in \mathrm{sp}(f)\}. \tag{4.7}$$

Thanks to the Pontryagin duality Theorem [26], the Fourier transformation for locally compact abelian groups has a close relation with the Gelfand transform. It is shown that there is a correspondence between elements of the topological space $\Omega(L^1(\mathcal{G}^{\Phi}, \mu_{\mathrm{Haar}}))$ and elements of the Pontryagin dual. Therefore the canonical isomorphism of the form $\mathrm{ev}_{L^1(\mathcal{G}^{\Phi},\mu_{\mathrm{Haar}})}(\Gamma)(\rho) = \rho(\Gamma) \in S^1 \subset \mathbb{C}$ enables us to define the Fourier transformation on $L^1(\mathcal{G}^{\Phi}, \mu_{\mathrm{Haar}})$ by the following way

$$\widehat{f}(\rho) = \int_{\mathcal{G}^{\Phi}} f(\Gamma)\overline{\rho(\Gamma)} d\mu_{\mathrm{Haar}}(\Gamma). \tag{4.8}$$

The convolution product (4.3) on $L^1(\mathcal{G}^{\Phi}, \mu_{\mathrm{Haar}})$ and the transformation (4.8) show that $\mathcal{F}\{f *_B g\} = \mathcal{F}\{f\}\mathcal{F}\{g\}$.

The generalized Fourier transformation (4.8) allows us to analyze a function $f \in L^1(\mathcal{G}^{\Phi}, \mu_{\mathrm{Haar}})$ defined on infinite formal expansion of Feynman diagrams in terms of simple μ_{Haar}-measurable functions with respect to partial sums. While the Gelfand transform separates functions of the Banach algebra $L^1(\mathcal{G}^{\Phi}, \mu_{\mathrm{Haar}})$, the application of the generalized Fourier transformation (4.8) enables us to study the evolutions of finite expansions of Feynman diagrams (as partial sums) which converges to the unique solution X_{DSE} of a non-perturbative equation DSE in the direction of a given μ-integrable function f.

Perturbative QCD is only capable to concern Green's functions as convergent geometric series in sufficiently small couplings to generate some information. In low energy level QCD we need to deal with the divergent version of Green's functions $G^r_{(\tau,\Lambda_\tau)}$, $r \in \{e_i, v_j\}_{i,j}$ which contains Feynman diagrams with infinite loop numbers and increasing powers of strong coupling constants $g \geq 1$. Thanks to the built mathematical machinery, now it is possible to provide a new analytic description of these divergent Green's functions.

Corollary 4.3. *The Fourier transformation of the characteristic function* $\chi_{G^r_{(\tau,\Lambda_\tau)}}$ *of each Green's function* $G^r_{(\tau,\Lambda_\tau)}$, $r \in \{e_i, v_j\}_{i,j}$, $i = 1, 2, 3,\ j = 1, ..., 5$ *in QCD is well-defined.*

Proof. $\widehat{\chi}_{G^r_{(\tau,\Lambda_\tau)}}$ is the convergent limit of the sequence $\{\widehat{\chi}_{G^{r,l}_{(\tau,\Lambda_\tau)}}\}_{l\geq 0}$ of the Fourier transformations of the characteristic functions of Green's functions $G^{r,l}_{(\tau,\Lambda_\tau)}$ as functions in the Banach algebra $L^1(\mathcal{G}^{\Phi}, \mu_{\mathrm{Haar}})$. □

4.2. **Evolution via Dyson series.** Determining potential quantities among particles (such as heavy quark and antiquark in QCD) is known as a difficult challenge for the phenomenological description of interplay of perturbative and non-perturbative regions in gauge field theories. In this part we plan to formulate a new generalization of the classical Dyson series for the study of potential quantities of Dyson–Schwinger equations.

Classical Dyson series have provided a practical treatment to describe the behavior of a single quantum mechanical particle which moves in a given potential. It has been formulated in terms of a particular class of functionals on $C[0, t]$ of the form

$$F(y) := \exp\{\int_{(0,t)} \theta(s, y(s))ds\} \tag{4.9}$$

such that the complex valued function θ on $[0, t] \times \mathbb{R}^n$ is a fixed potential. This functional has been considered under a measure theoretic setting such that the standard Lebesgue–Stieltjes measure was replaced with other complex Borel measures and then it was shown that for each complex number with positive real part λ, the operators $K_\lambda(F_n)$ exist for each n such that $F_n(y) := (\int_{(0,t)} \theta(s, y(s))d\eta)^n$ and $K_\lambda(F) = \sum_{n\geq 0} a_n K_\lambda(F_n)$ [15]. In

[31] the compact topological abelian group structure on $\mathcal{H}^{\Phi}$ has been considered under a measure theoretic setting where the existence of different but equivalent measures on this space has been studied. The connection between the Haar integration theory on $\mathcal{H}^{\Phi}$ and the standard Riemann–Lebesgue integration theory on real valued functions has been concerned in [31] where as the result, the Johnson–Lapidus generalized Dyson series (which was made to improve Feynman's operational calculus) has been modified for the level of the Haar measure $\mu_{\rm Haar}$ on $\mathcal{H}^{\Phi}$. In the final part of this section we plan to improve our modification and give a new extension of the Johnson–Lapidus Theorem which works for the level of functionals on the space of continuous functions on $\mathcal{G}^{\Phi}$.

Proposition 4.4. *Let θ be a complex valued function on $\mathcal{G}^{\Phi} \times \mathbb{R}^2$ and $v(z) = \sum_{n\geq 0} a_n z^n$ with the radius of convergency strictly grater than $||\theta||_{\infty;\mu_{\rm Haar}}$. For the given functional F on $L^1(\mathcal{G}^{\Phi}, \mu_{\rm Haar})$ with the form*

$$F(f) := v(\int_{\mathcal{G}^{\Phi}} \theta(\Gamma, f(\Gamma)) d\mu_{\rm Haar}) \tag{4.10}$$

, there exists a family of operators $\{K_{\lambda}(F_n)\}_{n\in\mathbb{N}}$ such that λ's are complex numbers with positive real part and $F_n(f) := (\int_{\mathcal{G}^{\Phi}} \theta(\Gamma, f(\Gamma)) d\mu_{\rm Haar})^n$. In addition, we have

$$K_{\lambda}(F) = \sum_{n\geq 0} a_n K_{\lambda}(F_n). \tag{4.11}$$

Proof. The interrelationship between the Haar integration theory on $\mathcal{H}^{\Phi}$ with respect to the measure $\mu_{\rm Haar}$ and the Lebesgue integration theory with respect to the Borel measure has been considered in [Proposition 3.23 [31]]. It has led us to extend the Johnson-Lapidus generalized Dyson series for the level of the Haar measure $\mu_{\rm Haar}$ ([Proposition 3.26 [31]]). In addition, as we have discussed $\mathcal{G}^{\Phi}$ is a compact Hausdorff topological space. The density of the topological space $C_c(\mathcal{G}^{\Phi})$ of continuous functions on $\mathcal{G}^{\Phi}$ with compact support in $L^1(\mathcal{G}^{\Phi}, \mu_{\rm Haar})$ allows us to lift the Johnson-Lapidus generalized Dyson series onto $L^1(\mathcal{G}^{\Phi}, \mu_{\rm Haar})$. Thanks to [Theorem 1.1 [15]], we have the Proof. □

We can apply this class of functionals at the level of $\mathcal{G}^{\Phi}$ to explain the evolution of non-perturbative situations.

Corollary 4.5. *Let DSE be a non-perturbative type equation of the form (2.5) with the unique solution $X_{\rm DSE}$ (as the large graph) determined by the recursive relations (2.6). The functionals $K_{\lambda}(F)$ (determined by Proposition 4.4) interprets the evolution of the large graph $X_{\rm DSE}$ on the basis of its partial sums.*

Proof. For a given large graph $X_{\rm DSE}$ as the unique solution of a given equation DSE, let $X^{(n)} := X_1 + ... + X_n$ be the partial sum of the order n with the corresponding characteristic function $\chi^{(n)}$ which belongs to $L^1(\mathcal{G}^{\Phi}, \mu_{\rm Haar})$. Now it is enough to apply Proposition 4.4 for these characteristic functions. As the output, the integrand of the n-term encodes the following evolutions: a free evolution from $X^{(0)} = \mathbb{I}$ (empty graph) to $X^{(1)}$, interactions of particles in $X^{(1)}$ with the potential θ, free evolution from $X^{(1)}$ to $X^{(2)}$, and so on up to $n^{\rm th}$ integration with θ at the level $X^{(n)}$ followed by a free evolution from $X^{(n)}$ to $X_{\rm DSE}$. By integration we will concern all partial sums of the unique solution of the equation DSE. □

One immediate application of our generalized Dyson series is to provide a new treatment to compare the behavior of Dyson–Schwinger equations in different potentials. Proposition 4.4 and Corollary 4.5 provide a new mathematical process for the study of (optimal) potentials for interactions of particles which contribute to a single Feynman diagram or free evolutions of particles in solutions of Dyson–Schwinger equations.

5. Conclusion

The original effort in this research work is to build some new practical mathematical models which are useful for the explanation of dynamics and geometry of non-perturbative phenomena derived from Dyson–Schwinger equations in strong coupling constants. Here is a short list of the original achievements of our research effort.

(A) Thanks to the Connes–Kreimer renormalization Hopf algebra, we have studied the dynamics of non-perturbative aspects on the basis of new enriched versions of the renormalization group which are capable to encode the behavior of Dyson–Schwinger equations during the re-scaling of momentum parameter and bare strong coupling constant. Indeed, we have clarified the importance of a rigid abstract renormalization group method to control bare strong coupling constants before any regularization procedure. As we know, asymptotically free theories in Theoretical Physics enable us to have perturbatively calculable properties in the ultraviolet divergencies where the running coupling are small enough [6, 8]. Thanks to the structure of the multi-scale non-perturbative renormalization group (i.e. Corollary 2.5), we expect that our framework is useful to apply for models of non-perturbative asymptotic freedom.

(B) The geometry of quantum motions have been concerned where we have built a new class of spectral triples which are originated from solutions of Dyson–Schwinger equations. This new mathematical structure has potential to initiate the foundations of a theory of spectral geometry via Noncommutative Geometry in dealing with non-perturbative parameters.

(C) We have built a complex Banach algebraic structure on the space of (large) Feynman graphs of a given QFT-model physical theory where as the result we obtained a new interpretation of the concept of evolution at the level of Dyson–Schwinger equations on the basis of the Fourier transformation. Furthermore, we have obtained a new extension of the Johnson–Lapidus generalized Dyson series for the space $L^1(\mathcal{G}^\Phi, \mu_{\mathrm{Haar}})$. This result has addressed another method to evaluate infinite formal expansions originated from Dyson–Schwinger equations under generalized Dyson series.

We finish the conclusion part with addressing some open research topics in this direction which shows the potential of our construction for making new progresses. At first, an enrichment of the renormalization Hopf algebra on ribbon graphs has been formulated [20] which leads us to consider Dyson–Schwinger equations for ribbon graphs. Thanks to the built methodology in this research article, it will be reasonable to work on the construction of a new class of non-perturbative spectral triples which encodes non-perturbative aspects originated from expansions of ribbon graphs and develop our noncommutative geometric framework to the level of tensor and matrix models. At second, the Feynman path integral approach has shown its influence in the recent progresses around Quantum Gravity [13] and in addition, the appearance of a Hopf algebraic formalism in this modern QFT–type model has already been concerned [32, 33]. Thanks to these important progresses and the built methodology in this research work, it is useful to search on the generalizations of our framework to formulate an operator theoretic setting for Quantum Gravity.

Acknowledgment.

The author is grateful to Max Planck Institute for Mathematics for the support and hospitality.

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