A new continuum model for general relativistic viscous heat–conducting media*

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The lack of formulation of macroscopic equations for irreversible dynamics of viscous heat-conducting media compatible with the causality principle of Einstein's Special Relativity and the Euler-Lagrange structure of General Relativity is a long-lasting problem. In this paper, we propose a possible solution to this problem in the framework of SHTC equations. The approach does not rely on postulates of equilibrium irreversible thermodynamics but treats irreversible processes from the non-equilibrium point of view. Thus, each transfer process is characterized by a characteristic velocity of perturbation propagation in the non-equilibrium state, as well as by an intrinsic time/length scale of the dissipative dynamics. The resulting system of governing equations is formulated as a first-order system of hyperbolic equations with relaxation-type irreversible terms. Via a formal asymptotic analysis, we demonstrate that classical transport coefficients such as viscosity, heat conductivity, etc. are recovered in leading terms of our theory as effective transport coefficients. Some numerical examples are presented in order to demonstrate the viability of the approach.

1 INTRODUCTION

The lack of formulations of macroscopic equations for irreversible dynamics of viscous heat-conducting and resistive media compatible with the causality principle of Einstein's special relativity and the Euler-Lagrange structure of general relativity (GR) is a long lasting problem [1, 2, 3]. In this paper, we propose a possible solution to this problem in the framework of Symmetric Hyperbolic and Thermodynamically Compatible (SHTC) equations [4, 5, 6, 7, 8, 9]. Such an approach is not relying on postulates of equilibrium irreversible thermodynamics but treats irreversible processes from the non-equilibrium standpoint. Thus, each transfer process is characterized by a characteristic velocity of perturbation propagation c_{ch} in the non-equilibrium state as well as by an intrinsic time scale τ of the dissipative (irreversible) dynamics. The resulting system of governing equations is formulated as a first-order system of hyperbolic equations with relaxation-type irreversible terms and thus causal by construction. Via a formal asymptotic analysis, we demonstrate that classical transport coefficients such as viscosity and heat conductivity are recovered in leading terms of our theory as effective transport coefficients of the form $\sim \tau c_{ch}^2$. Some numerical examples will be presented in order to demonstrate the viability of the approach.

The overall time evolution described by SHTC equations is split into two parts, reversible and irreversible. The reversible part (all the differential terms) comprises most of the mathematical

structure of the governing equations. This part is associated with the most non-equilibrium state of the system, i.e. for the relaxation parameter approaching infinity, $\tau \to \infty$. On the other hand, when $\tau \to 0$, the system is driven towards the global thermodynamic equilibrium which is described by a system with a reduced structure (the Euler equations of an ideal inviscid fluid).

The reversible part conserves both the energy and the entropy and admits a variational formulation and thus, it is compatible with the Euler-Lagrange structure of the Einstein field equations of GR. An important feature of our approach is that the variational principle is formulated in the Lagrangian reference frame, while the final form of the governing equations is obtained after the passage from the Lagrangian to the Eulerian frame. This is discussed in Sec. 2. Finally, we recover a covariant form of the Eulerian equations, and the corresponding $3 + 1$ split ('*Valencia-type*') formulation [10, 11, 12], which is also well suited for carrying out numerical experiments for a first and direct validation of the proposed theory.

The irreversible part, which rises the entropy (second law of thermodynamics), is represented by algebraic relaxation-type source terms $\sim \tau^{-1}$, and can be viewed as gradients (with respect to the state variables) of a dissipation potential [4], see Sec. 3.

The relaxation character of the irreversible part, in particular, imposes a challenge for the numerical methods in the near equilibrium (diffusive) regime $\tau \ll 1$. Therefore, in the numerical simulations, we rely on the so-called ADER approach [13, 14, 15] which can be used to obtain the numerical solution in all regimes: near equilibrium $\tau \ll 1$, non-equilibrium $\tau \gg 1$, and intermediate $\tau \sim 1$, see [16, 17].

The SHTC approach was not developed as a non-equilibrium thermodynamics theory, but grew up from studying the admissible mathematical structure of macroscopic equations initiated by Godunov in [5]. Such a structure should simultaneously guarantee consistency with the principles of thermodynamics and the causality principle, and also should have good mathematical properties such as well-posedness of the initial value problem (Cauchy problem), which is in particular obligatory for a system of evolutionary equations to be solved numerically. Nevertheless, the SHTC approach shares many common features with other non-equilibrium thermodynamics theories. For example, it has been shown recently [4] that the SHTC equations admit a Hamiltonian formulation via Poisson brackets and thus, it can be seen as a particular realization of the GENERIC formulation of non-equilibrium thermodynamics [18, 19]. Moreover, as many theories, e.g. GENERIC, Extended Irreversible Thermodynamics (EIT) [20], Rational Extended Thermodynamics (RET) [21], the SHTC approach intends to identify new macroscopic fields (state parameters) and find evolution equations governing them in order to approach more and more non-equilibrium regime. The physical meaning of the new state parameters might be very different though from those used in RET and EIT, e.g. flux-type quantities in EIT and RET and density-type quantities in SHTC and GENERIC. Additionally, similar to RET, special care is given to the hyperbolicity of the governing equations.

We note that a distinguishing feature of the SHTC and Hamiltonian GENERIC approach is the exceptional role of the energy potential in formulating the governing equations. Indeed, the variational nature of the SHTC equations and the Hamiltonian nature of GENERIC imply a unique role of the Lagrangian and the Hamiltonian, respectively, which act as generating potentials (they generate the entire structure of the reversible part of the governing PDE system) and are intimately connected with the energy of the system. On the other hand, in theories such as EIT and RET, the role of the generating potential is given to the entropy. In SHTC and GENERIC, the entropy potential, which is intimately connected to the *dissipation potential* [4], is only responsible for generating the irreversible part of the time evolution.

Concerning the existing approaches to relativistic fluid mechanics, it is worthwhile to mention that there are also many theories for causal relativistic dissipation which can be roughly divided into two classes. The first class consists of models which are presented by mixed-order (first- and second-order) PDE systems such as the original covariant reformulation of the Navier-Stokes-Fourier equations put forward by Eckart [22] and Landau and Lifshitz [23] and which have been shown later to be *acausal* and unstable [2]. Recently, the causality and stability issues of the Eckart-Landau theory was addressed by several authors [24, 25, 26, 27, 28]. Thus, in

contrast to their predecessors [22, 23], Ván and Biró used the Lagrange multiplier approach to obtain a stable formulation in [24], Freistühler and Temple [25, 26] derived a causal and stable formulation which has a locally well-posed initial value problem and has a close connection with the second-order symmetric hyperbolic systems in the sense of Hughes-Kato-Marsden [29] and which is referred to as mixed-order symmetric hyperbolic system by the authors. Recently, Bemfica and co-authors [27, 28] proposed another causal, stable and locally well-posed extension of the Eckart-Landau theory, first in the case of conformal fluids [27] and later in the general case [28] which also can be derived from the relativistic Boltzmann kinetic equation via a perturbative expansion technique developed in [31]. See a more thorough review of the aforementioned approaches in [27]. The second class of models consists of those which are presented by first-order hyperbolic PDE systems such as Müller-Israel-Stewart theory [32, 33, 34, 35] which was originally established as a phenomenological extended thermodynamic theory but later it was shown [36] that it can benefit from the connection with the relativistic Boltzmann equation via the moment expansion method of Grad [37]. There are various state of the art extensions and modifications of the original Müller-Israel-Stewart theory, e.g. see [3, 38, 36, 39, 40, 41, 42]. Also, a quite general class of divergence-type theories [43, 3] can be related to the first-order PDE models. In particular, the divergence-type models are known to be symmetric hyperbolic by contraction due to the Godunov-Boillat theorem [4]. As it is shown on the example of viscous dissipation in [44], the theory proposed in this paper is equivalent to the Müller-Israel-Stewart theory up to the first-order terms in the out of equilibrium formal Chapman-Enskog expansion in the small relaxation parameter. However, the higher order terms are different as they differ within various versions of the Müller-Israel-Stewart theory [41] depending on the selected closure relations.

Let us finally name the two most important features that should help the reader to distinguish our approach within the aforementioned approaches to relativistic dissipative fluid mechanics. First of all, an attractive feature of our theory is that it admits a variational formulation so that the matter energy-momentum is equivalent to the GR canonical matter energy-momentum tensor. In other words, coupling of the matter and gravity occurs in a natural way, that is in contrast to the existing approaches, the energy-momentum tensor of our theory does not have to be added in an *ad hoc* manner to the Einstein field equations but emerges there as the Noether current via the variation of the Hilbert-Einstein action. The same can be said for coupling of matter and electromagnetic force as in the nonlinear electrodynamics of moving medium [17]. Additionally, due to the variational nature of the SHTC equations discussed here, the governing equations have a certain structure conditioned by the canonical structure of the Euler-Lagrange equations. This is convenient for designing of numerical methods and in particular, for developing of structure preserving methods. Let us also remark that despite the paramount role of the variational principle in relativistic physics, to the best of our knowledge, there were no much attempts to employ variational principle for deriving equations for relativistic dissipative continuum mechanics apart from the works [45, 46, 47]. However, at the end of the day, the viscosity law is postulated but not derived in these papers.

The second distinguishing feature of our approach is that it provides a *unified* description of fluids and solids [48, 16, 49, 50]. In contrast to the existing first-order hyperbolic approaches which rely on the kinetic theory of gases or parabolic theories which rely on the phenomenological Navier-Stokes-Fourier constitutive laws, our theory is not restricted by applications to only fluids but can be applied simultaneously to both relativistic liquids and solids (e.g. the star interior, the outer crust of the neutron stars [51]).

2 FORMULATION OF REVERSIBLE EQUATIONS

In this section, we give a variational formulation of the reversible part of the time evolution for relativistic viscous and elastoplastic heat-conducting media. As it will be clear from what

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In this work, we consider only first-order systems and the symemtric hyperbolicity is understood with respect to such systems, i.e. in the sense of Friedrichs [30, 5].

follows, our formulation involves two manifolds, the matter manifold and spacetime manifold. It is therefore not obvious on which manifold of the two one has to formulate the variational principle. In fact, the Euler-Lagrange equations are the same regardless whether the variations are performed on matter or spacetime manifold. However, the Euler-Lagrange equations are not the only equations we need in our theory. We also need the so-called *integrability conditions* $(7)_2$ and $(25)_2$ which form the evolution equations for the principal fields of the theory, which are the distortion field and the thermal impulse. It appears, that these equations cannot be rigorously derived if one works on the spacetime manifold (Eulerian frame) but they can only be obtained in an *ad hoc* manner. On the other hand, the formulation of the variational principle on the matter manifold (Lagrangian frame) allows us to obtain all equations in a rigorous mathematical way. Therefore, in what follows, we formulate the variational principle in the matter manifold (Lagrangian frame) and then transform the Euler-Lagrange equations and integrability conditions to the Eulerian frame associated with the spacetime manifold. In particular, we show that the matter energy-momentum tensor obtained in such a way is equivalent to the GR matter energy-momentum obtained in the standard way, i.e. by varying the action with respect to the spacetime metric $g_{\mu\nu}$.

Also, note that the governing equations are formulated in such a way that the Lagrangian density is left unspecified and has to be provided by the user in order to close the equations. Such a closure may depend on a particular application. This emphasizes an exceptional role of the energy potential in the formulation of the reversible equations, similar to the GENERIC formulation [18, 19].

2.1 Lagrangian equations of motion

In continuum mechanics, the motion of a continuous medium can be viewed as an embedding of the 4d matter-time manifold (4-continuum) $\mathcal{M}^4$ with a Lorentzian metric κ_{ab} in the 4d spacetime $\mathcal{V}^4$ with a general Lorentzian metric $g_{\mu\nu}$, see [44]. This embedding can be described in *Lagrangian coordinates* ξ^a associated with $\mathcal{M}^4$, i.e. the coordinates that are *comoving* and *codeforming* with the medium, or, alternatively, in generic non-comoving coordinates x^μ associated with $\mathcal{V}^4$, that are usually called *Eulerian* coordinates for convenience. Sometimes the Lagrangian coordinates are also named *material* coordinates, but in this work we prefer the first against the latter. The embedding implies that the following *one-to-one* relation

$$x^\mu = x^\mu(\xi), \qquad \xi^a = \xi^a(x), \quad \mu, a = 0, \dots, 3 \tag{1}$$

holds between Lagrangian and Eulerian coordinates, also called *motion*. In this work, by convention, we refer general Greek indexes to the non-comoving (Eulerian) system of coordinates, while lowercase Latin indexes $a, b, c = 0, 1, 2, 3$ to the comoving (Lagrangian) coordinates. Additionally, capital Latin letters $A, B, C = 1, 2, 3$ refer to the three purely spatial Lagrangian material coordinates, e.g. ξ^A.

We further specify the Lagrangian coordinates ξ^a by assuming that the three scalars ξ^A *label* the matter particles and hence label the particle worldlines, while $\xi^0 := \tau$ is defined to be the matter *proper time*, that is the time of the Lagrangian observer as measured from his comoving clock, i.e.

$$-d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2}$$

In the Lagrangian formalism, the motion (1), as seen by an Eulerian observer, represents deformation of the medium and hence, the gradients of the motion, also called *configuration gradients* in the relativistic elasticity literature, e.g. [52, 53, 54, 55, 56],

$$x^\mu_{\ a}(\xi) := \frac{\partial x^\mu}{\partial \xi^a}, \qquad \xi^a_{\ \mu}(x) := \frac{\partial \xi^a}{\partial x^\mu} \tag{3}$$

play a central role [44]. In particular, the 4-velocity of the material elements with respect to the Eulerian coordinate system x^μ is defined as the first column of the configuration gradient x^μ_a

$$u^\mu := x^\mu_{\ 0} = \frac{\partial x^\mu}{\partial \xi^0} = \frac{\partial x^\mu}{\partial \tau}, \qquad (4)$$

Thus, in the absence of other material fields (which will be introduced later), it is implied that the material Lagrangian density $\tilde{\Lambda}$ explicitly depends on the configuration gradient x^μ_a

$$\tilde{\Lambda}\left(\xi^a, x^\mu(\xi), x^\mu_{\ b}(\xi)\right) = \Lambda\left(x^\mu_{\ b}(\xi)\right) \qquad (5)$$

and does not explicitly depend on the unknowns ξ^a and the potentials $x^\mu(\xi)$. Hence, the first variation of the action $\mathcal{S} = \int \Lambda \, d\xi$ with respect to δx^μ gives the Euler-Lagrange equation

$$\partial_a \Lambda_{x^\mu_a} = 0, \qquad (6a)$$

where we introduced the notation $\Lambda_{x^\mu_a} = \frac{\partial \Lambda}{\partial x^\mu_a}$, and the Einstein convention of summation over repeated indexes is assumed. There are only 4 conservation laws in (6a) for 16 unknowns x^μ_a. The remaining 12 equations are indeed hidden within the *integrability conditions* of any configuration gradient, i.e.

$$\partial_b x^\mu_{\ a} - \partial_a x^\mu_{\ b} = 0, \quad a \neq b. \qquad (6b)$$

In fact, system (6b) consists of 24 equations, where 12 of them are effective *evolution* equations, i.e. those for $a, b = 0$, while the other 12 are the so-called *involution constraints*, e.g. see [44, 4], that are pure spatial constraints, conserved along the particle trajectories.

2.2 Eulerian equations of motion

Existence of the Lagrangian coordinates ξ^a is a mathematical idealization and their practical use for general fluid-like motion is usually very problematic. Therefore, our unified approach to fluids and solids [48, 16] relies on the reformulation of governing equations (6) in the Eulerian frame and later, on the replacement of the integrable (global) deformation field $\xi^a_{\ \mu}$ by the non-integrable (local) distortion field $A^a_{\ \mu}$ which can be seen as a local basis tetrad (or non-holonomic basis tetrad). Therefore, in this section, we formulate equations of motion in the Eulerian frame which are obtained from the Lagrangian governing equations (6) by changing the unknowns $\xi^a \to x^\mu$, see details in [44].

Thus, after a sequence of equation transformations, equations (6) read

$$-\nabla_\nu(|\xi| \, \xi^b_{\ \mu} \Lambda_{\xi^b_{\ \nu}}) = 0, \qquad u^\nu(\nabla_\nu \xi^a_{\ \mu} - \nabla_\mu \xi^a_{\ \nu}) = 0, \qquad (7)$$

where ∇_ν denotes a covariant derivative associated with the symmetric Levi-Civita connection of GR, and the first equation represents the conservation of the energy-momentum tensor-density $\Sigma^\nu_{\ \mu}$ of our theory

$$\Sigma^\nu_{\ \mu} := -|\xi| \, \xi^\mu_{\ a} \Lambda_{\xi^\nu_{\ a}} = |\xi| \, x^\nu_{\ a} \Lambda_{x^\mu_a}, \qquad |\xi| = \det(\xi^a_{\ \mu}). \qquad (8)$$

After introducing the Eulerian counterpart $\mathcal{L}$ of the material Lagrangian Λ:

$$\mathcal{L} := |\xi| \Lambda, \qquad (9)$$

the energy-momentum reads

$$-\Sigma^\nu_{\ \mu} = \xi^a_{\ \mu} \mathcal{L}_{\xi^a_{\ \nu}} - \mathcal{L}\delta^\nu_{\ \mu}, \qquad \mathcal{T}^\nu_{\ \mu} := \Sigma^\nu_{\ \mu}/\sqrt{-g}, \qquad g = \det(g_{\mu\nu}). \qquad (10)$$

In particular, it is shown in [44] that in the absence of other material and electromagnetic fields, $\mathcal{T}^\nu_{\ \mu}$ can be written in a conventional form

$$-\mathcal{T}^\nu_{\ \mu} = \mathcal{E}u^\nu u_\mu + ph^\nu_{\ \mu} + \sigma^\nu_{\ \mu}, \qquad h^\nu_{\ \mu} := \delta^\nu_{\ \mu} + u^\nu u_\mu, \qquad (11)$$

where $\mathcal{E} = \mathcal{L}/\sqrt{-g}$ is the total energy, $p := \rho\mathcal{E}_\rho - \mathcal{E}$ is the isotropic pressure, ρ is the rest mass density, $\sigma^\nu_\mu = \xi^a{}_\mu \mathcal{E}_{\xi^a{}_\nu}$ is the anisotropic part of the energy-momentum, h^ν_μ is the spatial projector.

The complete system of Eulerian governing equations for determining the 16 unknown fields $\xi^a{}_\mu$ reads

$$-\nabla_\nu(\xi^a{}_\mu \mathcal{E}_{\xi^a{}_\nu} - \mathcal{E}\delta^\nu_\mu) = 0, \qquad u^\nu(\nabla_\nu \xi^a{}_\mu - \nabla_\mu \xi^a{}_\nu) = 0. \tag{12}$$

Finally, we note that, as discussed in detail in [44, 57], the developed approach is geometrical in nature, in which the matter is considered as a non-Riemannian manifold in which the geometry is determined by the distortion field $A^a{}_\mu$ (non-holonomic frame field), which replaces the holonomic tetrad $\xi^a{}_\mu$ and plays the role of moving Cartan frames. In such a geometrical framework, the 4-velocity u^μ and the rest mass current $j^\mu := \sqrt{-g}\rho u^\mu$ are collinear by construction [44]. In other words, the proposed approach can be viewed as a generalization of Eckart's choice of the 4-velocity (Eckart's frame), see [22, 58, 3].

CONSISTENCY WITH GR. As it was shown in [44], our energy-momentum tensor (10) agrees well with the canonical matter energy-momentum tensor-density of GR (the source term in the Einstein field equations) which reads as

$$\sqrt{-g}\, T^\nu_\mu := -g^{\nu\lambda}\frac{\partial \mathcal{L}^{GR}}{\partial g^{\lambda\mu}} = g_{\mu\lambda}\frac{\partial \mathcal{L}^{GR}}{\partial g_{\lambda\nu}} = \sqrt{-g}\left(2g_{\mu\lambda}\mathcal{E}_{g_{\lambda\nu}} - \mathcal{E}\delta^\nu_\mu\right), \qquad \mathcal{L}^{GR} := \sqrt{-g}\mathcal{E}, \tag{13}$$

where $\mathcal{E}$ is the total energy. In particular, if we assume $\mathcal{L} = \mathcal{L}^{GR}$, it was shown in [44] that

$$-\mathcal{T}^\nu_\mu = \xi^a{}_\mu \mathcal{E}_{\xi^a{}_\nu} - \mathcal{E}\delta^\nu_\mu = -\left(2g_{\mu\lambda}\mathcal{E}_{g_{\lambda\nu}} - \mathcal{E}\delta^\nu_\mu\right) = -T^\nu_\mu. \tag{14}$$

This is an important result, saying that our way of formulating the variational principle, which consists in defining the action in the matter-time manifold $\mathcal{M}^4$ and varying the action with respect to the configuration gradient $\xi^a{}_\mu$, gives the energy-momentum tensor (14) equivalent to the conventional GR matter energy-momentum, which is obtained if the action is defined in the spacetime $\mathcal{V}^4$ and the variation is performed with respect to the spacetime metric $g_{\mu\nu}$.

Equality (14) also guarantees that the GR energy-momentum tensor T^ν_μ coincides identically with the SHTC energy-momentum $\mathcal{T}^\nu_\mu$ for arbitrary energy potential $\mathcal{E}$ even in the presence of extra fields, e.g. the thermal impulse needed for the heat conduction formulation.

In the end, we note that the proposed continuum theory does not rely on any specific assumptions of GR, e.g. *Einstein's equivalence principle*, and in fact, it can be viewed as a theory whose main feature is the causal description of dissipative phenomena, see Sec.2.4, i.e. it is compatible with the special relativity theory. Nevertheless, a side effect of using the configuration gradient $\xi^a{}_\mu$ (the frame field) as the principal field is that the SHTC energy-momentum tensor $\mathcal{T}^\nu_\mu$ is equivalent to the GR canonical matter energy-momentum tensor T^ν_μ for arbitrary background Lorentzian metric $g_{\mu\nu}$. Moreover, due to their specific structure, the governing equations are also generally covariant, e.g. see [44] and eqs.(23) and (24) below, for the torsion-less spacetimes.

2.3 Heat conduction

In this Section, we give a variational formulation for the reversible part of the relativistic version of the SHTC heat conduction equations [8, 4].

As it was shown in [4], non-relativistic SHTC equations can be viewed as a particular realization of the GENERIC formulation for non-equilibrium thermodynamics. It is therefore not surprising that the same heat conduction equations were also proposed by Öttinger [59, 60, 61] within the GENERIC approach in the relativistic context.

2.3.1 *Lagrangian equations.*

In the Lagrangian frame, let us consider a scalar potential $J(\xi^a)$ and the action integral

$$\mathcal{S} = \int \Lambda(J, \partial_a J) d\xi. \tag{15}$$

We assume that the Lagrangian Λ does not depend explicitly on the potential J itself, but only on its first derivatives $J_a := \partial_a J$. In analogy with non-relativistic equations [16, 17, 4], vector J_a is called relativistic *thermal impulse*. Furthermore, we shall single out the zeroth component $J_0 := -T$ of the gradient J_a, where, as will become clear later, T can be identified as the temperature in the medium.

The first variation of the action (15) with respect to J gives the Euler-Lagrange equation

$$\partial_a \Lambda_{J_a} = 0, \qquad \partial_b J_a - \partial_a J_b = 0, \tag{16}$$

which are accompanied by the *integrability condition* (second equation). The first equation in (16) will be associated with the entropy evolution equation and its more conventional form will be unveiled later. The second equation will be used to evolve the three spatial components of the thermal impulse.

2.3.2 *Eulerian equations.*

We now rewrite the Lagrangian equations (16) in the Eulerian frame x^μ. Thermal impulse J_a and Lagrangian Λ transform as

$$J_\mu = \xi^a{}_\mu J_a, \qquad \mathcal{L}(J_\mu) = |\xi| \Lambda(J_a). \tag{17}$$

In particular, keeping in mind that $u^\mu = x^\mu_0$, we have

$$u^\mu J_\mu = -T. \tag{18}$$

Furthermore, it follows from (17) that $|\xi| x^\mu_a \Lambda_{J_a} = \mathcal{L}_{J_\mu}$ and hence, using the identity $\partial_\mu(|\xi| x^\mu_a) = 0$ and that $\partial_a = x^\mu_a \partial_a$, one can rewrite (16)$_1$ as

$$0 = |\xi| x^\mu_a \partial_\mu \Lambda_{J_a} = \partial_\mu(|\xi| x^\mu_a \Lambda_{J_a}) = \partial_\mu \mathcal{L}_{J_\mu}, \tag{19}$$

that is the vector-density $\mathcal{L}_{J_\mu}$ conserves both ordinarily and covariantly:

$$\partial_\mu \mathcal{L}_{J_\mu} = 0, \qquad \text{and} \qquad \nabla_\mu \mathcal{L}_{J_\mu} = 0. \tag{20}$$

Let us now transform equation (16)$_2$. One may write (we consider only three genuine evolution equations, but not the spatial constraints $\partial_B J_A - \partial_A J_B = 0$)

$$\partial_\tau J_b - \partial_b J_0 = \partial_\tau(x^\mu_b J_\mu) - \partial_b(u^\mu J_\mu) = u^\lambda \partial_\lambda(x^\mu_b J_\mu) - x^\lambda_b \partial_\lambda(u^\mu J_\mu). \tag{21}$$

Then, using (18) and that $\partial_\lambda x^\mu_b = -x^\eta_b x^\mu_a \partial_\lambda \xi^a{}_\eta$, we can write (21) as

$$u^\lambda \partial_\lambda J_\nu - u^\lambda J_\mu x^\mu_a \partial_\lambda \xi^a{}_\nu + \partial_\nu T = 0, \tag{22}$$

which using the evolution equation for the configuration gradient $u^\lambda \partial_\lambda \xi^a{}_\nu + \xi^a{}_\lambda \partial_\nu u^\lambda = 0$ (see [44]), can be rewritten as

$$u^\lambda \partial_\lambda J_\nu + J_\mu \partial_\nu u^\mu = -\partial_\nu T. \tag{23}$$

Because the left hand-side is the Lie derivative of J_μ along the 4-velocity, while the right hand-side is the derivative of the proper scalar $T = -u^\mu J_\mu$, we may replace the ordinary derivatives with the covariant ones:

$$u^\lambda \nabla_\lambda J_\nu + J_\mu \nabla_\nu u^\mu = -\nabla_\nu T. \tag{24}$$

Using (18), this equation can be equivalently written as $u^\lambda(\nabla_\lambda J_\nu - \nabla_\nu J_\lambda) = 0$. Thus, the Eulerian form of the relativistic heat conduction equations (in the absence of the irreversible processes) reads

$$\nabla_\mu \mathcal{L}_{J_\mu} = 0, \qquad u^\lambda(\nabla_\lambda J_\nu - \nabla_\nu J_\lambda) = 0. \tag{25}$$

Note that within (16), there are only 4 evolution equations for 4 unknowns J_a and 3 pure spatial constraints $\partial_B J_A - \partial_A J_B = 0$.

2.3.3 *Energy-Momentum and Thermal Stress.*

We now demonstrate that the energy-momentum tensor has a contribution due to thermal impulse J_μ. Thus, during the Lagrange-to-Euler transformation the energy-momentum tensor density $\Sigma^\nu{}_\mu = \sqrt{-g}\mathcal{T}^\nu{}_\mu$ (10) transforms as

$$\Sigma^\nu{}_\mu = |\xi| x^\nu{}_a \left(\Lambda(x^\lambda{}_b, J_b)\right)_{x^\mu{}_a} = -|\xi|\, \xi^a{}_\mu \left(|\xi|^{-1}\mathcal{L}(\xi^b{}_\lambda, \xi^b{}_\lambda J_b)\right)_{\xi^a{}_\nu} = \mathcal{L}\delta^\nu{}_\mu - \xi^a{}_\mu \mathcal{L}_{\xi^a{}_\nu} - J_\mu \mathcal{L}_{J_\nu}.$$

Therefore, in the presence of heat conduction, the energy-momentum tensor $\mathcal{T}^\nu{}_\mu = \Sigma^\nu{}_\mu/\sqrt{-g}$ is

$$-\mathcal{T}^\nu{}_\mu = \xi^a{}_\mu \mathcal{E}_{\xi^a{}_\nu} + J_\mu \mathcal{E}_{J_\nu} - \mathcal{E}\delta^\nu{}_\mu, \tag{26}$$

where the term $J_\mu \mathcal{E}_{J_\nu}$ can be called the *thermal stress*.

2.3.4 *Entropy.*

In this section, we introduce an entropy of our theory and rewrite the conservation law $(25)_1$, which is in fact the entropy conservation law, in a conventional form. For brevity, we shall not consider the dependence of the energy potential on $\xi^a{}_\mu$ in this section.

First, we introduce the spatial projection w_μ of the thermal impulse J_μ

$$w_\mu := h^\lambda{}_\mu J_\lambda = J_\mu - Tu_\mu. \tag{27}$$

Let us also define total energy potentials (at the moment we use symbol E, while symbol $\mathcal{E}$ will appear later on, after a Legendre transform in (30))

$$E(J_\mu) = \bar{E}(T, J) = \check{E}(T, w), \tag{28}$$

where $E = \mathcal{L}/\sqrt{-g}$, $T = -u^\mu J_\mu$, $J := J_\mu J^\mu$, $w := w_\mu w^\mu = J + T^2$, and, in particular,

$$E_{J_\mu} = -\check{E}_T u^\nu + \check{E}_{w_\lambda}\frac{\partial w_\lambda}{\partial J_\nu} = -\check{E}_T u^\nu + \check{E}_{w_\lambda} h^\nu{}_\lambda = -\check{E}_T u^\nu + \check{E}_{w_\nu}. \tag{29}$$

Eventually, the entropy s can be introduced conventionally, i.e. as the Legendre conjugate to the temperature T. Thus, we introduce a new potential $\mathcal{E}(s, w_\mu) = T\check{E}_T - \check{E}$ and hence, one has

$$s := \check{E}_T, \qquad T = \mathcal{E}_s, \qquad \mathcal{E}_{w_\mu} = -\check{E}_{w_\mu}. \tag{30}$$

After that, conservation law $(25)_1$ reads

$$\nabla_\mu E_{J_\mu} = \nabla_\mu(-\check{E}_T u^\mu + \check{E}_{w_\mu}) = -\nabla_\mu(su^\mu + \mathcal{E}_{w_\mu}), \tag{31}$$

which represents the reversible part of the entropy evolution. Note that the non-advective part $\mathcal{E}_{w_\mu}$ of the entropy current is orthogonal to the 4-velocity

$$\mathcal{E}_{w_\mu} u_\mu = 0 \tag{32}$$

because $\mathcal{E}_{w_\mu} = -\check{E}_{w_\mu} = -2\check{E}_w w^\mu$ and $w^\mu u_\mu = 0$. In terms of the thermal impulse w_μ, eq. $(25)_2$ reads

$$u^\lambda(\nabla_\lambda w_\mu - \nabla_\mu w_\lambda) + h^\lambda{}_\mu Q_\lambda = 0, \tag{33}$$

where $Q_\mu := \nabla_\mu T + Tu^\lambda \nabla_\lambda u_\mu$ is the auxiliary temperature gradient vector, e.g. see [59, 3]. However, in the numerical simulation, it is more convenient to use equation $(25)_2$ for J_μ and then compute w_μ using (27) because, after the $3 + 1$ split, $(25)_2$ has the same structure as the non-relativistic SHTC heat conduction equation [4, 8].

2.3.5 *The symmetry of the thermal stress*

Let us now consider the question of symmetry of the energy-momentum. Thus, in new terms, the thermal stress can be written as

$$J_\mu E_{J_\nu} = -(w_\mu + Tu_\mu)(su^\nu + \mathcal{E}_{w_\nu}) = -(w_\mu \mathcal{E}_{w_\nu} + Tsu_\mu u^\nu + sw_\mu u^\nu + T\mathcal{E}_{w_\nu} u_\mu) \tag{34}$$

which is symmetric if and only if

$$sw^\nu = T\mathcal{E}_{w_\nu}. \tag{35}$$

This, in fact, implies that $\bar{E}_T = \check{E}_T + 2T\check{E}_w = s - 2T\mathcal{E}_w = 0$, that is $\bar{E}(T, J) = \bar{E}(J)$ does not depend explicitly on T and we can identify $E \equiv \bar{E}$ in (28).

2.3.6 *Heat flux*

Recall that, in the SHTC theory [8, 4, 16], the heat flux is introduced as

$$q^\mu := \mathcal{E}_s \mathcal{E}_{w_\mu} = T\mathcal{E}_{w_\mu} \tag{36}$$

hence, if condition (35) holds, the thermal stress (34) can be equivalently rewritten as

$$J^\mu E_{J_\nu} = -(s\mathcal{E}_s u^\mu u^\nu + w^\mu \mathcal{E}_{w_\nu} + q^\nu u^\mu + u^\nu q^\mu), \tag{37}$$

which is manifestly symmetric. Hence, the total energy-momentum for a relativistic heat-conducting fluid (anisotropic shear stress is omitted) reads

$$-\mathcal{T}^\nu{}_\mu = (s\mathcal{E}_s - \mathcal{E})u^\nu u_\mu + (-\rho\mathcal{E}_\rho - s\mathcal{E}_s + \mathcal{E})h^\nu{}_\mu - s\mathcal{E}_s u^\nu u_\mu - w_\mu \mathcal{E}_{w_\nu} - q^\nu u_\mu - u^\nu q_\mu, \tag{38}$$

so that we have

$$\mathcal{T}^\nu{}_\mu = \mathcal{E}u^\nu u_\mu + p\, h^\nu{}_\mu + w_\mu \mathcal{E}_{w_\nu} + q^\nu u_\mu + u^\nu q_\mu, \tag{39}$$

with $p := \rho\mathcal{E}_\rho + s\mathcal{E}_s - \mathcal{E}$ being the pressure. Here, the last two terms are conventional [58, 3], while the term $w_\mu \mathcal{E}_{w_\nu}$ is due to the non-equilibrium nature of the theory. The presence of such a term can be in particular justified based on macroscopic equations for non-equilibrium gas flows (moments equations) [62] derived from the Boltzmann kinetic equation. Such a term also appears in the GENERIC formulation of relativistic heat conduction [59, 61] which we found to be equivalent to the SHTC variational formulation. However, in the near-equilibrium settings (diffusive regime), this term can be ignored due to its smallness, see Sec. 3.4.

2.4 Hyperbolicity

Hyperbolic PDEs provide a natural framework for modeling time-dependent physical phenomena at a macroscale because they have a locally well-posed initial value problem and admit only finite speeds for perturbation propagation usually associated with the sound speeds in matter, which are subluminal (causality). Moreover, local well-posedness is also a critical property of a PDE system for its consistent numerical resolution. In particular, it is one of the goals of this paper to obtain a relativistic version of our hyperbolic equations for viscous momentum [48, 16] and heat transfer [8, 16, 4]. However, because hyperbolicity of a non-homogeneous first-order system is defined by only its leading order terms, we can consider this question even before the introduction of the irreversible source terms (low-order terms) in Sec. 3.

From the mathematical standpoint, hyperbolicity of a first-order non-linear PDE system in a domain Ω is equivalent to its *strong linear stability* at any point of Ω [63]. Recall that the *strong* linear stability implies not only that the characteristic velocities of the considered PDE system (eigenvalues) are *real* but also that a full set of eigenvectors exists, that is the quasilinear form of the PDE system is diagonalizable. In turn, this two conditions results in that the norm of Fourier-Laplace modes (the solution of the linearised PDE system) are *uniformly bounded* in time. Therefore, it is in principle equivalent to study hyperbolicity of a model in Eulerian and

Lagrangian frames (if only the two have the same number and types of unknowns) because their Fourier-Laplace modes are connected by the solution-dependent non-singular transformation (17) and thus, the growth type of the Fourier-Laplace mode cannot qualitatively change, i.e. from a uniformly bounded to unbounded behavior (blow-up solution) and *vice versa*.

Having said that, it has appeared that it is much easier to prove hyperbolicity, and even *symmetric hyperbolicity*, of the Lagrangian system (6) and (16) while hyperbolicity of its Eulerian counterpart that is summarized later in (41) and used in the numerical simulation in Sec.4 follows from the discussion above. In addition to the reasoning discussed at the beginning of Sec.2, this is another reason why we employ the Lagrangian frame in our theoretical considerations. Also, recall that for the first-order symmetric hyperbolic equations, a non-increasing energy functional can be easily obtained, see e.g. [64, 65, 1], which can be used to obtain estimates for the norm of the solution and its derivatives.

Lagrangian governing equations (6) and (16), after introducing new variables $m_\mu := \Lambda_{u^\mu}$, $S := \Lambda_T$, and a new potential $\mathcal{U} := u^\mu \Lambda_{u^\mu} + T\Lambda_T - \Lambda$ as Legendre conjugates, read as

$$\partial_\tau m_\mu - \partial_A \mathcal{U}_{x^\mu_A} = 0, \qquad \partial_\tau x^\mu_A - \partial_A \mathcal{U}_{m_\mu} = 0, \tag{40a}$$

$$\partial_\tau S + \partial_A \mathcal{U}_{J_A} = 0, \qquad \partial_\tau J_A + \partial_A \mathcal{U}_S = 0, \tag{40b}$$

where $\partial_\tau = \partial_0$ is the Lagrangian time derivative and τ is the proper time. This is exactly the system of conservation laws studied in [7, 4] and which is symmetrizable and compatible with the first law of thermodynamics (thermodynamical compatibility), i.e. it admits an extra conservation law for the potential $\mathcal{U}$:

$$\partial_\tau \mathcal{U} - \partial_A (\mathcal{U}_{m_\mu} \mathcal{U}_{\xi^A_\mu} - \mathcal{U}_S \mathcal{U}_{J_A}) = 0, \tag{40c}$$

which can be interpreted as the total energy conservation. Moreover, this system is *symmetric hyperbolic* [30] if the potential $\mathcal{U}$ is convex, see details in [4], that is its characteristic velocities (they give the perturbation propagation velocities) are guaranteed to be always real and the quasilinear form of (40) is diagonalizable. Moreover, the potential $\mathcal{U}$ is designed in such a way that the characteristic velocities coincide with the high-frequency limit of the sound speeds of the material under consideration, e.g. see (47), which can be taken from experimental data [66, 16, 49] or estimated from microscopic theories. Of course in reality, these sound speeds are always less than the speed of light and hence, the model is causal. Thus, similar to the Müller-Israel-Stewart theory [3], our theory has the longitudinal, transversal or shear, and thermal sound velocities, e.g. see [16].

3 RELATIVISTIC HEAT–CONDUCTING VISCOUS MEDIA

In this section, we finalize the formulation of the SHTC equations for relativistic heat-conducting viscous/elastoplastic media by specifying the irreversible part of the time evolution. Note that the differentiation between viscous and elastoplastic media is achieved via a proper choice of the dependence of the relaxation time τ_{sh}, see (49), on the state variables, e.g. see [67, 50], and does not depend on the reversible part of the time evolution. We then discuss thermodynamic consistency of the governing equations, close the system by providing an example of the energy potential, and recover *effective* shear viscosity and heat conductivity of our theory.

The system of SHTC governing equations for general relativistic heat-conducting viscous/elastoplastic media reads as follows

$$\nabla_\nu (\mathcal{E} u^\nu u_\mu + p\, h^\nu_\mu + A^a_\mu \mathcal{E}_{A^a_\nu} + w_\mu \mathcal{E}_{w_\nu} + q^\nu u_\mu + u^\nu q_\mu) = 0, \tag{41a}$$

$$u^\nu (\nabla_\nu A^a_\mu - \nabla_\mu A^a_\nu) = -\frac{1}{\theta_{sh}} G^{ab} g_{\mu\nu} \mathcal{E}_{A^b_\nu}, \tag{41b}$$

This is not always the case. For example, the Eulerian formulation of the Euler equations of ideal fluids requires 5 fields (density, momentum and energy), while its Lagrangian formulation requires extra nine fields for the deformation gradient, e.g. see [63].

$$\nabla_\mu(su^\mu + \mathcal{E}_{w_\mu}) = \frac{1}{T}\left(\frac{1}{\theta_{sh}}G^{ab}g_{\mu\nu}\mathcal{E}_{A^a_\mu}\mathcal{E}_{A^b_\nu} + \frac{1}{\theta_h}g_{\mu\nu}\mathcal{E}_{w_\nu}\mathcal{E}_{w_\mu}\right) \geqslant 0, \tag{41c}$$

$$u^\lambda(\nabla_\lambda w_\mu - \nabla_\mu w_\lambda) + h^\lambda_{\ \mu}Q_\lambda = -\frac{1}{\theta_h}g_{\mu\nu}\mathcal{E}_{w_\mu}, \tag{41d}$$

$$\nabla_\mu(\rho u^\mu) = 0, \tag{41e}$$

where

$$p = \rho\mathcal{E}_\rho + s\mathcal{E}_s - \mathcal{E}, \qquad T = \mathcal{E}_s, \qquad q^\mu = T\mathcal{E}_{w_\mu}, \qquad Q_\mu = \nabla_\mu T + Tu^\lambda\nabla_\lambda u_\mu, \tag{41f}$$

The left hand-side in (41) represents the reversible part of the time evolution derived in Sec.2.1 and 2.3, while the newly added relaxation source terms on the right hand-side of (41) represent the irreversible part and can be viewed as gradients of the quadratic dissipation potential [4]. The total energy $\mathcal{E}$ is left unspecified so far. However, one may clearly notice that the definition of the reversible and irreversible terms depends on the specification of the total energy $\mathcal{E}$ and will be provided in Sec. 3.3 as well as the specification of the relaxation parameters θ_{sh} and θ_h.

One of the non-trivial differences between the reversible equations (with zeros on the right hand-side) and irreversible ones is in the substitution of the configuration gradient $\xi^a_{\ \mu}$, which can be seen as a holonomic basis tetrad, by the distortion field $A^a_{\ \mu}$, which is a non-holonomic basis tetrad, i.e. it is not a gradient of the mapping (1). This is discussed in detail in [44] in the relativistic setting and in [57, 48, 16] in the non-relativistic setting. It is implied though that $|A| = |\xi|$.

One may clearly see that the GR version of SHTC equations (41) shares a lot of common structural features with the non-relativistic SHTC equations, e.g. [4]. We, therefore, expect that the relativistic equations (41) also admit a Hamiltonian formulation similar to [59, 18], that is the reversible part of the time evolution is generated by the anti-symmetric Poisson brackets, while the irreversible part can be generated by a symmetric bracket, see [18] and Sec.4 in [19]. However, to show this, one may need to obtain a genuinely 4-dimensional formulation of the GENERIC approach which currently relies on the explicit treatment of the time coordinate and therefore, is not truly 4-dimensional. Nevertheless, one may note that the irreversible part of the GR SHTC equations (41) is identical to its non-relativistic counterpart in [4] and thus, it can be seen as the gradient of the quadratic dissipation potential , see [4], and hence, it can also be viewed as it is generated by a symmetric dissipative bracket as in the branch of GENERIC developed in [59, 18].

3.1 Linear stability

One of the fundamental observations of the non-equilibrium thermodynamics is that an out-of-equilibrium system, if left free of external stimuli, tends towards the global equilibrium state [19]. From the mathematical standpoint, this is equivalent to that the equilibrium solution of (41) is linearly stable. Therefore, introduction of the dissipative relaxation terms in (41) should not contradict to this observation. To prove that the out-of-equilibrium solutions of (41) tend towards the equilibrium state, or that the perturbations of the equilibrium state decay in time, is again easy to do in the Lagrangian frame, that is for system (40) supplemented with the same type source terms as in (41) generated by the quadratic (convex) dissipation potential (see the discussion just before this subsection). Indeed, the resulting Lagrangian symmetric hyperbolic relaxation system, e.g. see [4], fulfills the conditions (symmetrizability and the non-negative definiteness of the dissipation matrix) of the class of first-order hyperbolic PDE systems studied by Yong and co-authors in [68, 69] for which it is proven that the equilibrium state is linearly stable.

3.2 Thermodynamic consistency

The choice of the entropy production source term in eq.(41c) is guided by both laws of thermodynamics, i.e. it has to be non-negative in order to guarantee the non-decreasing of the physical

entropy in irreversible process (second law of thermodynamics), but it also allows to conserve the total energy (first law of thermodynamics). Indeed, system (41) is an overdetermined PDE system, that is there are more equations than unknowns. Hence, if all equations are compatible with each other, one of the equations should be the consequence of the others. Thus, one can check that the 0-*th* equation in (41a), the total energy conservation law, is a linear combination of the remaining equations

$$\nabla_\nu(\mathcal{E}u^\nu u_0 + p\,h^\nu_0 + P^\nu_0) = -\frac{u^i}{u^0}\nabla_\nu(\mathcal{E}u^\nu u_i + p\,h^\nu_i + P^\nu_i) - \frac{\mathcal{E}_{A^a_\mu}}{u^0}u^\nu(\nabla_\nu A^a_\mu - \nabla_\mu A^a_\nu)$$

$$- \frac{\mathcal{E}_s}{u^0}\nabla_\mu(su^\mu + \mathcal{E}_{w_\mu}) - \frac{\mathcal{E}_{w_\mu}}{u^0}(u^\lambda(\nabla_\lambda w_\mu - \nabla_\mu w_\lambda) + h^\lambda_\mu Q_\lambda) - \frac{\mathcal{E}_\rho}{u^0}\nabla_\mu(\rho u^\mu), \quad (42)$$

where $P^\nu_\mu = A^a_\mu \mathcal{E}_{A^a_\nu} + w_\mu \mathcal{E}_{w_\nu} + q^\nu u_\mu + u^\nu q_\mu$. The source terms in (41) are designed in such a way that they canceled out during the summation (42), that is the total energy is indeed conserved.

Finally, we note that the rest mass conservation law (41e) is, in fact, not an independent equation, but can be derived from the configuration gradient $(12)_2$ or distortion evolution equation (41b), e.g. see [44]. Nevertheless, it is convenient to formally treat the rest mass density $\rho := \rho_0|A|/\sqrt{-g}$ as an independent variable [44], which should be taken into account in the derivatives $\mathcal{E}_{A^a_\mu}$ and $\mathcal{E}_\rho$. Here, ρ_0 is the reference mass density.

3.3 Closure: equation of state

As we have seen, the SHTC system of governing equations is formulated without specifying the Lagrangian density $\mathcal{L}$ or energy $\mathcal{E}$. We say that the energy potential $\mathcal{E}$ is the generating potential for system (41) which has to be specified in order to close the system.

Because $\mathcal{E}$ has to be a proper scalar, it has to depend on the the tensor fields via their invariants, including mixed invariants of two or more fields. The subsystems of (41) can be, therefore, non-linearly coupled via a specific closure, i.e. a specific choice of the potential $\mathcal{E}$. In this paper, however, we give an example of a rather simple closure (quadratic energy) which is discussed in the following subsection.

We assume the following decomposition of the total energy potential $\mathcal{E}$

$$\mathcal{E}(\rho, s, w_\mu, A^A_\mu) = \rho(1 + \varepsilon(\rho, S, w_\mu, A^A_\mu)), \tag{43}$$

where $S = s/\rho$ is the specific entropy (per unit of rest mass).

Furthermore, we shall need the material metric $G_{\mu\nu}$, see [44],

$$G_{\mu\nu} := G_{ab}A^a_\mu A^b_\nu = \kappa_{AB}A^A_\mu A^B_\nu, \qquad G_{ab} := \kappa_{ab} + U_a U_b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \kappa_{AB} & \\ 0 & & & \end{pmatrix}, \tag{44}$$

where we have defined the Lagrangian matter projector G_{ab} in the local relaxed frame, and $U^a = \frac{\partial \xi^a}{\partial \tau} = (1,0,0,0)$, $U_a = \kappa_{ab}U^b = (-1,0,0,0)$ is the Lagrangian 4-velocity. Following our papers on Newtonian continuum mechanics [16, 17], we shall decompose the material metric $G_{\mu\nu}$ into a traceless part $\mathring{G}_{\mu\nu}$ and a spherical part:

$$G_{\mu\nu} = \mathring{G}_{\mu\nu} + \frac{G^\lambda_\lambda}{3}h_{\mu\nu}, \quad \text{where} \quad \mathring{G}_{\mu\nu} := G_{\mu\nu} - \frac{G^\lambda_\lambda}{3}h_{\mu\nu}. \tag{45}$$

Note that, in this definition, $\mathring{G}_{\mu\nu}$ refers to the spatial projector $h_{\mu\nu}$ and not to the full spacetime metric $g_{\mu\nu}$. We then use the norm of the traceless part (here one has to use that $h^\mu_\mu = g^{\mu\lambda}h_{\lambda\mu} = 3$)

$$\mathring{G}^\lambda_\nu \mathring{G}^\nu_\lambda = I_2 - I_1^2/3, \qquad I_1 = G^\mu_\mu, \qquad I_2 = G^\mu_\nu G^\nu_\mu, \tag{46}$$

as an indication of the presence of non-volumetric (tangential) deformations, and define the specific energy $\varepsilon(\rho, S, w_\mu, A^A_\mu)$

$$\varepsilon(\rho, S, A^M_{\ \mu}, w^\mu) = \varepsilon^{eq}(\rho, S) + \frac{c^2_{sh}}{4}\mathring{G}^\lambda_{\ \nu}\mathring{G}^\nu_{\ \lambda} + \frac{\alpha^2_h}{2}w^\mu w_\mu, \tag{47}$$

where ε^{eq} is given by a hydrodynamic equation of state (EOS), which can be either the ideal or stiffened gas EOS (in the case of liquids or solids) or a general tabulated one. Here, c_{sh} denotes the sound speed of propagation of shear perturbation, that is the characteristic velocity of propagation of shear perturbation in the most non-equilibrium state, i.e. when the associated relaxation parameter τ_{sh} goes to infinity, $\tau_{sh} \to \infty$. The parameter α_h is related to the characteristic velocity of propagation of thermal perturbations c_h (so-called second sound) in the non-equilibrium state ($\tau_h \to \infty$) as $c^2_h = \alpha^2_h T/c_V$, where τ_h is the thermal relaxation time scale, and c_V is the specific heat capacity at constant volume [16]. For such a specification of $\mathcal{E}$, we have

$$\mathcal{E}_{A^A_{\ \mu}} = \rho\, c^2_{sh}\kappa_{AB}A^B_{\ \lambda}g^{\lambda\alpha}\mathring{G}_{\alpha\beta}h^{\beta\mu}, \qquad A^A_{\ \mu}\mathcal{E}_{A^A_{\ \nu}} = \rho\, c^2_{sh}\mathring{G}^\nu_{\ \lambda}G^\lambda_{\ \mu}, \tag{48a}$$

$$w_\mu\mathcal{E}_{w_\nu} = \rho\alpha^2_h w^\nu w_\mu, \qquad q^\mu = T\rho\alpha^2_h w^\mu, \qquad T = \mathcal{E}_s = \varepsilon^{eq}_S \tag{48b}$$

Based on the dimensional reasoning and asymptotic analysis performed in the following section, it is convenient to define the relaxation parameters θ_{sh} and θ_h as follows [44, 16, 4]

$$\theta_{sh} = \rho_0\,\tau_{sh}c^2_{sh}G^\lambda_{\ \lambda}/3, \qquad \theta_h = \rho\,\alpha^2_h\tau_h. \tag{49}$$

3.4 Asymptotic analysis

In this section, via a formal asymptotic analysis performed for the closure (47), we demonstrate that in the asymptotic relaxation limit $\tau_{sh} \to 0$ and $\tau_h \to 0$, the leading terms of our equations are identical to the relativistic Navier-Stokes-Fourier equations [3]. Although, the latter are parabolic and non-causal and known to be unstable [1, 2, 70], it is the high-order terms of our theory which should be responsible for the stability of the solution. Via a formal asymptotic analysis, we express the effective transport coefficients of the theory such as shear viscosity and heat conductivity in terms of the characteristic velocities c_{sh}, c_h and the relaxation times τ_{sh} and τ_h.

EFFECTIVE HEAT CONDUCTIVITY. Assuming that the thermal impulse can be expanded as

$$w_\mu = w^{[0]}_\mu + \tau_h w^{[1]}_\mu + \tau^2_h w^{[2]}_\mu + \dots, \tag{50}$$

we plug it in the equation (33), where the irreversible terms are now added:

$$u^\lambda\nabla_\lambda(w^{[0]}_\mu + \tau_h w^{[1]}_\mu) + (w^{[0]}_\lambda + \tau_h w^{[1]}_\lambda)\nabla_\mu u^\lambda + h^\lambda_{\ \mu}Q_\lambda + \dots = -\frac{\rho\alpha^2_h}{\theta_h}w_\mu, \tag{51}$$

where dots '...' mean higher order terms. It is convenient to define θ_h as in (53). Then, collecting terms with equal orders of power of τ_h, we obtain that at the zeroth order $w^{[0]}_\mu = 0$, while at the first order $w^{[1]}_\mu = -h^\lambda_{\ \mu}Q_\lambda$. Therefore, by truncating expansion (50) to first order, the solution can be approximated as $w_\mu = -\tau_h h^\lambda_{\ \mu}Q_\lambda$. Then, using that the heat flux 4-vector is defined as $q^\mu = T\mathcal{E}_{w_\mu}$ in (36), one has

$$q^\mu = T\mathcal{E}_{w_\mu} = T\rho\alpha^2_h w^\mu = -T\rho\alpha^2_h g^{\mu\nu}\tau_h h^\lambda_{\ \nu}Q_\lambda, \tag{52}$$

that is the effective heat conductivity κ^{eff} of our model in the diffusive regime ($\tau_h \ll 1$) can be defined as

$$\kappa^{eff} = \rho\, T\tau_h\alpha^2_h = \rho\, c_V\tau c^2_h. \tag{53}$$

EFFECTIVE SHEAR VISCOSITY. In [44], it was shown that, in the absence of heat conduction ($c_h = 0$), a formal asymptotic expansion reveals the structure of the leading terms of viscous stress $\sigma^V{}_\mu = A^A{}_\mu \mathcal{E}_{A^A{}_\nu}$ in the asymptotic relaxation limit when $\tau_{sh} \to 0$,

$$\sigma_{\mu\nu} = -\frac{1}{6}\rho_0 \tau_{sh} c_{sh}^2 \left(h_{\mu\lambda}\nabla_\nu u^\lambda + h_{\lambda\nu}\nabla_\mu u^\lambda - \frac{2}{3}(h^\alpha_\lambda \nabla_\alpha u^\lambda)h_{\mu\nu} + u^\lambda \nabla_\lambda h_{\mu\nu} \right), \tag{54}$$

which is equivalent to the Landau-Lifshitz version of the relativistic Navier-Stokes stress [71, 3] with the effective viscosity $\mu = \frac{1}{6}\rho_0 \tau_{sh} c_{sh}^2$.

4 NUMERICAL VALIDATION

4.1 3+1 formulation

In a few words, in the $3+1$ foliation of spacetime it is a usual procedure to project the 4d covariant governing equations into a set of 3d equations by using the so called *spatial* and *temporal* projection operators, $\gamma_{\mu\nu}$ and $N_{\mu\nu}$, respectively, such that $g_{\mu\nu} = \gamma_{\mu\nu} + N_{\mu\nu}$, e.g. see [3]. In this way, any symmetric rank-2 tensor can be decomposed in its spatial and temporal components, e.g.

$$T^{\mu\nu} = S^{\mu\nu} + S^\mu n^\nu + n^\mu S^\nu + \mathcal{U}n^\mu n^\nu, \tag{55}$$

where,

$$S^{\mu\nu} := \gamma^\mu_\alpha \gamma^\nu_\beta T^{\alpha\beta}, \qquad S^\mu := -\gamma^\mu_\alpha n_\beta T^{\alpha\beta}, \qquad \mathcal{U} := n_\alpha n_\beta T^{\alpha\beta} \tag{56}$$

In particular, such operators are defined after specifying on the spacetime $\mathcal{V}^4$ a proper field of local (Eulerian) observers travelling with non-constant 4-velocity n^μ. In particular, one can show that [3]

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_i \beta^i & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^i/\alpha^2 \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^i\beta^j/\alpha^2 \end{pmatrix},$$

$$n_\mu = -\alpha\nabla_\mu t = (-\alpha, 0_i), \quad n^\mu = (1/\alpha, -\beta^i/\alpha), \quad n_\mu n^\mu = -1,$$

$$\gamma_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu, \qquad N_{\mu\nu} := -n_\mu n_\nu,$$

where t is chosen to be the time coordinate, α is the *lapse function*, β_j is the *shift vector*, and γ_{ij} are the spatial components of the spatial metric $\gamma_{\mu\nu}$. The corresponding identities related to the medium 4-velocity u^μ are

$$u^\mu = \Gamma(n^\mu + v^\mu), \quad \Gamma := -n_\mu u^\mu = \alpha u^t = (1 - v_i v^i)^{-1/2} = (1 - v^2)^{-1/2},$$

$$\gamma \cdot u = (\delta^\mu_\nu + n^\mu n_\nu)u^\nu = \Gamma v^\mu, \quad v^i = u^i/\Gamma + \beta^i/\alpha,$$

where Γ is the Lorentz factor.

Then, after denoting with **V** the array of the so-called 33 *primitive variables*

$$\mathbf{V} := \left(\rho, v_j, p, A^i{}_j, J_j, \kappa_{AB}, \alpha, \beta^j, \gamma_{ij} \right), \tag{57}$$

where κ_{AB} are the material components of the matter-time metric κ_{ab}. The so-called 33 conserved variables $\mathbf{Q}(\mathbf{V})$ can be easily expressed in terms of the primitive variables via the relations

$$D := \rho\Gamma, \qquad \mathbf{S} := \rho h \Gamma^2 v, \qquad \mathcal{U} := \rho h \Gamma^2 - p. \tag{58}$$

Here, $\mathcal{U}$ is the conserved energy density, $h = 1 + \epsilon + p/\rho$ is the specific enthalpy and ϵ is the specific internal energy [3]. Then, the chosen state vector $\mathbf{Q}$ of conserved variables with respect to the PDE system (60) is defined as

$$\mathbf{Q} := \left(\sqrt{\gamma}\, D, \sqrt{\gamma}\, S_j, \sqrt{\gamma}\, \mathcal{U}, A^i{}_j, J_j, \kappa_{AB}, \alpha, \beta^j, \gamma_{ij} \right). \tag{59}$$

While the transformation from primitive to conserved variables is explicit and straightforward, in this work, the inversion of the primitive to conserved function is computed iteratively.

Eventually, after a standard $3 + 1$ foliation of spacetime [72, 3], system (41) projected into the Valencia-type formulation reads as follows:

$$\partial_t \left(\gamma^{\frac{1}{2}} D \right) + \partial_i \left[\gamma^{\frac{1}{2}} D \left(\alpha v^i - \beta^i \right) \right] = 0, \tag{60a}$$

$$\partial_t \left(\gamma^{\frac{1}{2}} S_j \right) + \partial_i \left[\gamma^{\frac{1}{2}} \left(\alpha S^i{}_j - \beta^i S_j \right) \right] - \gamma^{\frac{1}{2}} \left(\frac{1}{2} \alpha S^{ik} \partial_j \gamma_{ik} + S_i \partial_j \beta^i - \mathcal{U} \partial_j \alpha \right) = 0, \tag{60b}$$

$$\partial_t \left(\gamma^{\frac{1}{2}} \mathcal{U} \right) + \partial_i \left[\gamma^{\frac{1}{2}} \left(\alpha S^i - \beta^i \mathcal{U} \right) \right] - \gamma^{\frac{1}{2}} \left(\alpha S^{ij} K_{ij} - S^j \partial_j, \alpha \right) = 0, \tag{60c}$$

$$\partial_t A^i{}_j + \partial_j \left(\hat{v}^k A^i{}_k \right) + \hat{v}^k \left(\partial_k A^i{}_j - \partial_j A^i{}_k \right) = -\frac{1}{\theta_{sh}} A^i{}_\mu \mathring{G}^\mu{}_j, \tag{60d}$$

$$\partial_t J_i + \partial_i \left(\hat{v}^k J_k + \hat{T} \right) + \hat{v}^k \left(\partial_k J_i - \partial_i J_k \right) = -\frac{1}{\theta_h} J_i, \tag{60e}$$

$$\partial_t \kappa_{AB} + \hat{v}^k \partial_k \kappa_{AB} = 0, \qquad \partial_t \alpha = 0, \qquad \partial_t \beta^i = 0, \qquad \partial_t \gamma_{ij} = 0, \tag{60f}$$

where $\hat{T} = \alpha T / \Gamma$. In more detail, the components $S^i{}_j$, S_i and $\mathcal{U}$ are defined as the space-time decomposition (56) of the energy-stress tensor $\mathcal{T}^\gamma{}_\mu$ that appears in equation (41a). Also, we have introduced the definition of the so called *transport velocity* as $\hat{v}^i := (v^i + \beta^i)/\alpha$. Finally, we stress that due to the fact that system (41) is overdetermined, see (42), we are free to chose whether the total energy or entropy equation is discretized (we chose to discretize the energy equation). The remaining quantity is computed from the equation of state. Finally, for the static spacetimes, the Cowling approximation [72]

$$\alpha S^{ij} K_{ij} \equiv \frac{1}{2} S^{ik} \beta^j \partial_j \gamma_{ik} + S^j{}_i \partial_j \beta^i \tag{61}$$

can be used for the extrinsic curvature K_{ij} [3] in (60c).

4.2 Numerical examples

HEAT CONDUCTION. Here we solve a simple Riemann problem that involves heat conduction. The initial condition consists in a density jump from $\rho_L = 1$ to $\rho_R = 0.9$ located at $x = 0.5$. The other state variables are globally constant and are chosen as $v = J = 0$, $A = I$ and $p = 1$. We furthermore assume a flat Minkowski spacetime, hence the lapse function is $\alpha = 1$, the shift vector is $\beta^i = 0$ and the spatial metric tensor is $\gamma_{ij} = \delta_{ij}$. The remaining parameters of the model are the ideal gas EOS for ε^{eq} in (47) with $\gamma = \frac{5}{3}$ (the ratio of the specific heats), $c_{sh} = 0$, $\alpha_h = 0.8$, $c_V = 1$ and two different relaxation times $\tau_h \ll 1$ are considered, namely $\tau_h = 5 \cdot 10^{-3}$ and $\tau_h = 5 \cdot 10^{-4}$. The chosen initial data lead to a temperature jump of $T_L = \frac{1}{\gamma - 1} \frac{1}{\rho_L}$ to $T_R = \frac{1}{\gamma - 1} \frac{1}{\rho_R}$. Simulations are run on the domain $\Omega = [0, 1] \times [0, 1]$ until $t = 0.5$ using a fourth order ADER-DG scheme [15, 16, 73] with polynomial approximation degree $N = 3$ and 100×4 spatial elements. A fine grid reference solution is computed with a third order ADER-WENO finite volume scheme [14, 16, 73] on 1000 cells.

The obtained results are depicted in the left panel of Fig. 1, which show the typical behavior of heat conduction in the limit of the Fourier law. The agreement of the numerical simulation carried out with the high order discontinuous Galerkin finite element scheme and the finite volume reference solution is excellent.

RELATIVISTIC SOD SHOCK TUBE. In this last test problem, we solve a relativistic version of the Sod shock tube problem for the complete model (41), including viscosity and heat conduction. The initial data consist of a jump located in $x = 0.5$ with $\rho_L = 1$, $\rho_R = 0.125$, $v = J = 0$, $A = \sqrt[3]{\rho} I$, $p_L = 1$, $p_R = 0.1$, $\alpha = 1$, $\beta^i = 0$, $\gamma_{ij} = \delta_{ij}$. The remaining model parameters are chosen as the ideal gas EOS for ε^{eq} with $\gamma = \frac{5}{3}$, $c_V = 1$, $c_h = 0.01$ and $c_{sh} = 0.5$. Three different values of

relaxation times $\tau_{sh} = \tau_h \ll 1$ are used in the numerical simulations, namely $2 \cdot 10^{-1}$, $2 \cdot 10^{-2}$, and $2 \cdot 10^{-3}$. The computational domain is $\Omega = [0,1] \times [0,1]$ and is covered with 100×4 fourth order ADER-DG elements of polynomial approximation degree $N = 3$. Simulations are run until a final time of $t = 0.4$. Since the problem under consideration involves shock waves, we make use of the *a posteriori* subcell finite volume limiter presented in [74]. The computational results are compared against the exact solution of the Riemann problem of the ideal relativistic hydrodynamics (RHD) equations, which was kindly provided by Dr. Zanotti, see [75]. The obtained results are depicted in the right panel of Figure 1. For small relaxation times, an excellent agreement between the stiff relaxation limit of our model and the ideal relativistic Euler equations can be noted.
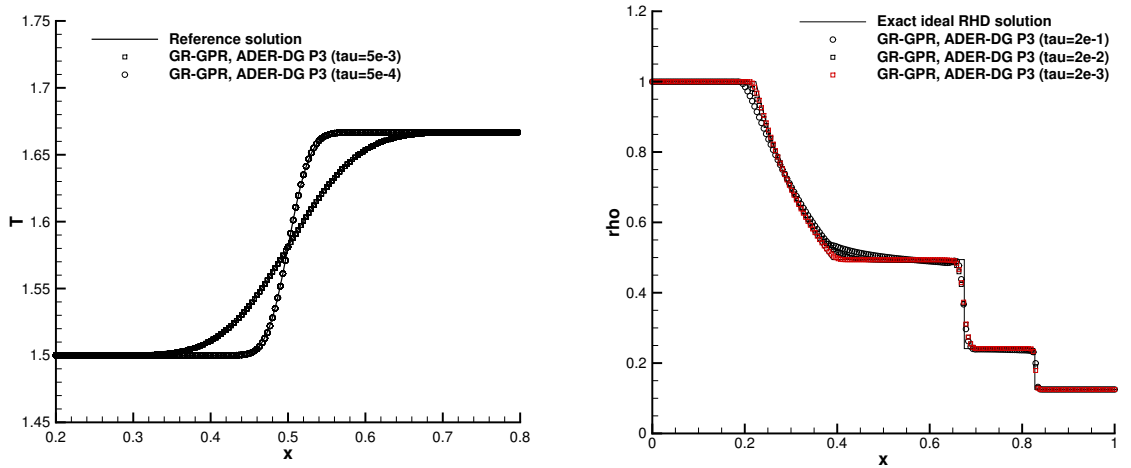
Figure 1: Left panel: Heat conduction based on the SHTC model. Temperature distribution at $t = 0.5$ for two different relaxation times $\tau_h = 5 \cdot 10^{-3}$ and $\tau_h = 5 \cdot 10^{-4}$, starting from a temperature jump initially located at $x = 0.5$. Right panel: Relativistic Sod shock tube problem solving the SHTC model (60) with viscosity and heat conduction with different relaxation times $\tau_h = \tau_{sh} = 2 \cdot 10^{-3}$, $2 \cdot 10^{-2}$, and $2 \cdot 10^{-1}$. As reference, also the exact solution of the Riemann problem of the ideal relativistic Euler equations (RHD) is shown.

5 CONCLUSION

We have presented a general relativistic formulation for viscous/elastoplastic heat-conducting continuous medium. Such a formulation is a generalization of the non-relativistic unified formulation for fluid and solid dynamics advanced recently in [48, 16, 49] and relies on the theory of first-order Symmetric Hyperbolic Thermodynamically Compatible (SHTC) equations [4, 5, 7, 8].

Both transport processes are considered from a non-equilibrium viewpoint, that is no local equilibrium assumptions such as Newton's law of viscosity or Fourier law of heat conduction are used. We provide a variational formulation for the reversible part of the time evolution which makes the model compatible with the Euler-Lagrange structure of the Einstein field equations. The irreversible part is represented by algebraic (no space and time derivatives) relaxation-type source terms and can be viewed as gradients of a dissipation potential intimately connected with the entropy, see [4]. We have observed that our variational formulation of the relativistic heat conduction provides equivalent equations to the Hamiltonian formulation by Öttinger [61, 59] within the GENERIC approach to non-equilibrium thermodynamics. The viscous part which is governed by the distortion evolution equation (41b) is different from that presented in [61, 59].

Nevertheless, at least in the Newtonian limit, it also admits a Hamiltonian formulation as proven in [4].

Via a formal asymptotic analysis, we have recovered the effective transport coefficients of our theory in the near equilibrium regime, see Sec. 3.4. Finally, we presented a 3+1 split of the governing equations in Sec. 4.1 which are then solved using the ADER-DG family of high-order numerical schemes [14, 15, 16, 73, 76] designed specifically for hyperbolic partial differential equations, see Sec. 4. We solved two one-dimensional Riemann problems in the special relativistic settings in order to demonstrate the physical consistency and mathematical regularity of the numerical solution.

Further research will concern the obtaining of general relativistic versions of the SHTC electrodynamics equations in moving medium [17, 4] with resistivity and of hyperbolic equations for mass transfer [77, 78].

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