HIGHER RANK ZETA FUNCTIONS OF CURVES II: EQUALITY WITH SL_n-ZETA FUNCTIONS

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1. Introduction

In [7, 10], a *non-abelian zeta function* $\zeta_{X,n}(s) = \zeta_{X/\mathbb{F}_q,n}(s)$ was defined for any smooth projective curve X over a finite field $\mathbb{F}_q$ and any integer $n \geq 1$ by

$$\zeta_{X,n}(s) \;=\sum_{[V]} \frac{|H^0(X,V)\smallsetminus\{0\}|}{|\mathrm{Aut}(V)|}\, q^{-\deg(V)s} \qquad (\Re(s)>1)\,, \tag{1}$$

where the sum is over the moduli stack of $\mathbb{F}_q$-rational semi-stable vector bundles V of rank n on X with degree divisible by n. Using the Riemann-Roch, duality and vanishing theorems for semi-stable bundles, it was shown that $\zeta_{X,n}(s)$ agrees with the usual Artin zeta function $\zeta_X(s)$ of $X/\mathbb{F}_q$ if $n=1$, that it has the form $P_{X,n}(T)/(1-T)(1-q^nT)$ for some polynomial $P_{X,n}(T)$ of degree $2g$ in T, where g is the genus of X and $T=q^{-ns}$, and that it satisfies the functional equation

$$\widehat{\zeta}_{X,n}(1-s) = \widehat{\zeta}_{X,n}(s)\,, \qquad \text{where} \quad \widehat{\zeta}_{X,n}(s) := q^{n(g-1)s}\cdot \zeta_{X,n}(s)\,.$$

It was also conjectured that $\zeta_{X,n}(s)$ satisfies the Riemann hypothesis (i.e., that all of its zeros have real part 1/2). In [12], Part I of this series, explicit formulas for $\zeta_{X,n}(s)$ and a proof of the Riemann hypothesis were given when $g=1$.

On the other hand, in [9, 10], a different approach to zeta functions for curves led to the so-called *group zeta function* $\widehat{\zeta}_X^{G,P}(s)$ of $X/\mathbb{F}_q$, associated to a connected split algebraic reductive group G and its maximal parabolic subgroup P. The precise definition, which is based on the theory of periods, will be recalled in §2. In this paper, we will be interested in the special case when $G=\mathrm{SL}_n$ and $P=P_{n-1,1}$, the subgroup of SL_n consisting of matrices whose final row vanishes except for its last entry, and will then write simply $\widehat{\zeta}_X^{SL_n}(s)$ for $\widehat{\zeta}_X^{G,P}(s)$. Our main result will be a proof of the following theorem, which was conjectured in [10] ("special uniformity conjecture").

Theorem 1. *The zeta functions $\widehat{\zeta}_{X,n}(s)$ and $\widehat{\zeta}_X^{SL_n}(s)$ coincide for all $n\geq 1$.*

This theorem should be seen (and cited) as a joint result of the present authors and of Sergey Mozgovoy and Markus Reineke, because it is proved by comparing a formula established here with a (harder) formula given in their paper [6]. Specifically, the proof of Theorem 1 consists of three steps:

(1) By analyzing the definition of $\widehat{\zeta}_X^{G,P}(s)$ for $G=\mathrm{SL}_n$, $P=P_{n-1,1}$, we will prove an explicit formula, giving $\widehat{\zeta}_X^{SL_n}(s)$ as a linear combination of the functions $\widehat{\zeta}_X(ns-k)$ for $0\leq k<n$ with rational functions of T as coefficients. The calculation is given in §§3–5.

(2) In [6], as recalled in §6, using the theory of Hall algebras and wall-crossing techniques, a formula for $\widehat{\zeta}_{X,n}(s)$ of the same general shape is proved.
(3) A short calculation, given in §7, shows that the two formulas agree.

The explicit formula is not very complicated, and we can state it here. Motivated by the Siegel-Weil formula for the total mass of vector bundles V of rank n and degree 0 on X (i.e., the number of such V's, weighted by the inverse of the number of their automorphisms), and in order to make a proper normalization, we define numbers $\widehat{v}_k$ $(k \geq 1)$ inductively by

$$\widehat{v}_k \;=\; \begin{cases} \lim_{s\to 1}(1-q^{1-s})\,\widehat{\zeta}_X(s) & \text{if } k \;=\; 1\,, \\ \widehat{\zeta}_X(k)\,\widehat{v}_{k-1} & \text{if } k \geq 2\,, \end{cases} \tag{2}$$

where $\widehat{\zeta}_X(s) = q^{s(g-1)}\zeta_X(s)$. Furthermore, as in [12]—where these functions were introduced for the purpose of writing down in a more structural way the non-abelian rank n zeta functions for elliptic curves over finite fields—we define rational functions $B_k(x)$ $(k \geq 0)$ either inductively by the formulas

$$B_k(x) \;=\; \begin{cases} 1 & \text{if } k = 0\,, \\ \sum\limits_{m=1}^{k} \widehat{v}_m \, \dfrac{B_{k-m}(q^m)}{1-q^m x} & \text{if } k \geq 1\,, \end{cases} \tag{3}$$

or in closed form (if $k \geq 1$) by

$$B_k(x) \;=\; \sum_{p=1}^{k} \sum_{\substack{k_1,\dots,k_p>0 \\ k_1+\cdots+k_p=k}} \frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{(1-q^{k_1+k_2})\dots(1-q^{k_{p-1}+k_p})} \cdot \frac{1}{1-q^{k_p}x}\,. \tag{4}$$

Then the formula that we will establish for $\widehat{\zeta}_X^{SL_n}(s)$ can be stated as follows:

Theorem 2. *With the above notations, we have*

$$\widehat{\zeta}_X^{SL_n}(s) \;=\; q^{\binom{n}{2}(g-1)} \sum_{k=0}^{n-1} B_k(q^{ns-k})\,B_{n-k-1}(q^{k+1-ns})\,\widehat{\zeta}_X(ns-k)\,. \tag{5}$$

Remarks. 1. In the definition (1) of the non-abelian zeta function $\zeta_{X,n}(s)$, vector bundles used are assumed to be of degrees divisible by the rank n. This definition is motivated by a work of Drinfeld [2] on counting super-cuspidal representations in rank two, and also because if we summed over all degrees as was originally done in [7], then the functional equation would still hold but the Riemann hypothesis would not.
2. The analogue of Theorem 1 for the case of number fields rather than function fields was proved by the first author several years ago by totally different techniques, using the theory of Eisenstein series and Arthur trace formulas (combine the "Global Bridge" on p. 295 and the discussion on p. 305 of [8] with the formulas on p. 284 of [11] and on p. 197 of [9]).
3. A proof of Theorem 1 for the cases $n = 2$ and $n = 3$ was given in [6], at a time when this paper was still in the preprint stage.

2. Zeta functions for (G, P)

Let G be a connected split reductive algebraic group of rank r with a fixed Borel subgroup B and associated maximal split torus T (over a base field). Denote by

$$\Big(V, \langle\cdot,\cdot\rangle, \Phi = \Phi^+ \cup \Phi^-, \Delta = \{\alpha_1, \ldots, \alpha_r\}, \varpi := \{\varpi_1, \ldots, \varpi_r\}, W\Big)$$

the associated root system. That is, V is the real vector space defined as the $\mathbb{R}$-span of rational characters of T, and as usual, is equipped with a natural inner product $\langle\cdot,\cdot\rangle$, with which we identify V with its dual V^*, $\Phi^+ \subset V$ is the set of positive roots, $\Phi^- := -\Phi^+$ the set of negative roots, $\Delta \subset V$ the set of simple roots, $\varpi \subset V$ the set of fundamental weights, and W the Weyl group. By definition, the fundamental weights are characterized by the formula $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$ for $i, j = 1, 2, \ldots, r$, where $\alpha^\vee := \frac{2}{\langle\alpha,\alpha\rangle}\,\alpha$ denotes the coroot of a root $\alpha \in \Phi$. We also define the Weyl vector ρ by $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$, and introduce a *coordinate system* on V (with respect to the base $\{\varpi_1, \ldots, \varpi_r\}$ of V and the vector ρ) by writing an element $\lambda \in V$ in the form

$$\lambda \;=\; \sum_{j=1}^{r}(1 - s_j)\varpi_j \;=\; \rho - \sum_{j=1}^{r} s_j \varpi_j\,,$$

thus fixing identifications of V and $V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$ with $\mathbb{R}^r$ and $\mathbb{C}^r$. In addition, for each Weyl element $w \in W$, we set $\Phi_w := \Phi^+ \cap w^{-1}\Phi^-$, i.e., the collection of positive roots whose w-images are negative.

As usual, by a *standard parabolic subgroup*, we mean a parabolic subgroup of G that contains the Borel subgroup B. From Lie theory (see e.g., [3]), there is an one-to-one correspondence between standard parabolic subgroups P of G and subsets Δ_P of Δ. In particular, if P is maximal, we may and will write $\Delta_P = \Delta \smallsetminus \{\alpha_p\}$ for a certain unique $p = p(P) \in \{1, \ldots, r\}$. For such a standard parabolic subgroup P, denote by V_P the $\mathbb{R}$-span of rational characters of the maximal split torus T_P contained in P, by V_P^* its dual space, and by $\Phi_P \subset V_P$ the set of non-trivial characters of T_P occurring in the space V. Then, by standard theory of reductive groups (see e.g., [1]), V_P admits a canonical embedding in V (and V_P^* admits a canonical embedding in V^*), which is known to be orthogonal to the fundamental weight ϖ_p, and hence Φ_P can be viewed as a subset of Φ. Set $\Phi_P^+ = \Phi^+ \cap \Phi_P$, $\rho_P = \frac{1}{2}\sum_{\alpha\in\Phi_P^+}\alpha$, and $c_P = 2\langle \varpi_p - \rho_P, \alpha_p^\vee \rangle$.

Now, let X be an integral regular projective curve of genus g over a finite field $\mathbb{F}_q$. In [10], motivated by the study of zeta functions for number fields,[1] for a connected split reductive algebraic group G, and its standard parabolic subgroup P as above (defined over the function field of X), the first author defined the *period of G for X* by

$$\omega_X^G(\lambda) := \sum_{w\in W} \frac{1}{\prod_{\alpha\in\Delta}(1 - q^{-\langle w\lambda-\rho,\alpha^\vee\rangle})} \prod_{\alpha\in\Phi_w} \frac{\widehat{\zeta}_X(\langle\lambda,\alpha^\vee\rangle)}{\widehat{\zeta}_X(\langle\lambda,\alpha^\vee\rangle + 1)}$$

[1]For number fields, the analogue of the two functions to be introduced below are special kinds of Eisenstein periods, defined as integrals of Eisenstein series over moduli spaces of semi-stable lattices. For details, see [9].

and the *period of* (G,P) *for* X by

$$\begin{aligned}\omega_X^{G,P}(s) \ &:= \ \mathrm{Res}_{\langle\lambda-\rho,\,\alpha^\vee\rangle=0,\ \alpha\in\Delta_P}\,\omega_X^G(\lambda)\big|_{s_p=s}\\ &= \ \mathrm{Res}_{s_r=0}\cdots\mathrm{Res}_{s_{p+1}=0}\mathrm{Res}_{s_{p-1}=0}\cdots\mathrm{Res}_{s_1=0}\,\omega_X^G(\lambda)\big|_{s_p=s}\,,\end{aligned}$$

where s is a complex variable[2] s and $1-s$ rather than s and $-n-s$. and where for the last equality we used the fact that $\langle\rho,\alpha^\vee\rangle=1$ for all $\alpha\in\Delta$ and the relation that $\langle\varpi_i,\alpha_j^\vee\rangle=\delta_{ij}$ for all $i,\,j\in\{1,\dots,r\}$. As proved in [4, 10], the ordering of taking residues along singular hyperplanes $\langle\lambda-\rho,\alpha^\vee\rangle=0$ for $\alpha\in\Delta_P$ does not affect the outcome, so that the definition is independent of the numbering of the simple roots.

To get the zeta function associated to (G,P) for X, certain normalizations should be made. For this purpose, write $\omega_X^G(\lambda) \ = \ \sum_{w\in W}T_w(\lambda)$, where, for each $w\in W$,

$$T_w(\lambda):=\frac{1}{\prod_{\alpha\in\Delta}(1-q^{-\langle w\lambda-\rho,\alpha^\vee\rangle})}\prod_{\alpha\in\Phi_w}\frac{\widehat{\zeta}_X(\langle\lambda,\alpha^\vee\rangle)}{\widehat{\zeta}_X(\langle\lambda,\alpha^\vee\rangle+1)}.$$

We must study the residue $\mathrm{Res}_{\langle\lambda-\rho,\,\alpha^\vee\rangle=0,\ \alpha\in\Delta_P}T_w(\lambda)$.

We care only about those elements $w\in W$ (we will call them *special*) that give non-trivial residues, namely, those satisfying the condition that $\mathrm{Res}_{\langle\lambda-\rho,\,\alpha^\vee\rangle=0,\ \alpha\in\Delta_P}T_w(\lambda)\not\equiv 0$. This can happen only if all singular hyperplanes are of one of the following two forms:

(1) $\langle w\lambda-\rho,\alpha^\vee\rangle=0$ for some $\alpha\in\Delta$, giving a simple pole of the rational factor $\frac{1}{\prod_{\alpha\in\Delta}(1-q^{-\langle w\lambda-\rho,\alpha^\vee\rangle})}$;
(2) $\langle\lambda,\alpha^\vee\rangle=1$ for some $\alpha\in\Phi_w$, giving a simple pole of the zeta factor $\widehat{\zeta}_X(\langle\lambda,\alpha^\vee\rangle)$.

For special $w\in W$, and $(k,h)\in\mathbb{Z}^2$, following [4] (see also [10]) we define

$$\begin{aligned}N_{P,w}(k,h)\ &:=\#\{\alpha\in w^{-1}\Phi^-\ :\ \langle\varpi_p,\alpha^\vee\rangle=k,\ \langle\rho,\alpha^\vee\rangle=h\}\\ M_P(k,h):=&\max_{w\text{ special}}\big(N_{P,w}(k,h-1)-N_{P,w}(k,h)\big)\,.\\ &=N_{P,w_0}(k,h-1)-N_{P,w_0}(k,h)\,,\end{aligned}\tag{6}$$

where w_0 is the longest element of the Weyl group and where the last equality is Corollary 8.7 of [5]. Note that $M_P(k,h)=0$ for almost all but finitely many pairs of integers (k,h), so it makes sense to introduce the product

$$D_X^{G,P}(s):=\prod_{k=0}^{\infty}\prod_{h=2}^{\infty}\widehat{\zeta}_X(kn(s-1)+h)^{M_P(k,h)}.\tag{7}$$

Following [9, 10], we define the *zeta function of* X *associated to* (G,P) by

$$\widehat{\zeta}_X^{G,P}(s):=q^{(g-1)\dim N_u(B)}\cdot D^{G,P}(s)\cdot\omega_X^{G,P}(s)\,.\tag{8}$$

Here $N_u(B)$ denote the nilpotent radical of the Borel subgroup B of G.

[2]We should warn the reader that in [8], [9] and [11] a different normalization is used, with the argument of $\omega_X^{G,P}$ (and later of $\zeta_X^{G,P}$) being given by $s=c_p(s_p-1)$ $(=n(s_p-1)$ in the special case $(G,P)=(\mathrm{SL}_n,P_{n-1,1}))$ rather than $s=s_p$ as chosen here. With the normalization used here the functional equation relates

Remark. For special $w \in W$, even after taking residues, there are some zeta factors $\widehat{\zeta}_X(ks+h)$ left in the denominator of $\mathrm{Res}_{\langle\lambda-\rho,\,\alpha^\vee\rangle=0,\,\alpha\in\Delta_P} T_w(\lambda)$. The reason for introducing the factor $D_X^{G,P}(s)$ in our normalization of the zeta functions, based on formulas in [4] and [10], is to clear up all of the zeta factors appearing in the denominators associated to special Weyl elements.

3. Specializing to SL_n

From now on, we will specialize to the case when G is the special linear group SL_n and P is the maximal parabolic subgroup $P_{n-1,1}$ consisting of matrices whose final row vanishes except for its last entry, corresponding to the ordered partition $(n-1)+1$ of n. Our purpose is to study the *zeta function of X associated to SL_n*

$$\widehat{\zeta}_X^{SL_n}(s) := \widehat{\zeta}_X^{SL_n,\,P_{n-1,1}}(s)\,. \tag{9}$$

As usual, we realize the root system A_{n-1} associated to SL_n as follows. Denote by $\{e_1,\dots,e_n\}$ the standard orthonormal basis of the Euclidean space $\mathbb{R}^n$. The positive roots are given by $\Phi^+ := \{e_i - e_j \mid 1 \le i < j \le n\}$, the simple roots by $\Delta = \{\alpha_1 := e_1 - e_2, \dots, \alpha_{n-1} := e_{n-1} - e_n\}$, and the Weyl vector by $\rho = \sum_{j=1}^n \frac{n+1-2j}{2}\, e_j$. We identify the Weyl group W with $\mathfrak{S}_n$, the symmetric group on n letters, by the assignment $w \mapsto \sigma_w$, where $w(e_i - e_j) = e_{\sigma_w(i)} - e_{\sigma_w(j)}$. For convenience, we will also write the corresponding Δ_P, Φ_P^+, ρ_P, ϖ_P and c_P simply as Δ', Φ'^+, ρ', ϖ' and c' respectively. We have

$$\Delta' = \{\alpha_1,\dots,\alpha_{n-2}\}, \qquad \Phi'^+ = \{e_i - e_j : 1 \le i < j \le n-1\}\,,$$

$$\rho' = \sum_{j=1}^{n-1} \frac{n-2j}{2}\, e_j\,, \qquad \varpi' = \varpi_{n-1} = \frac{1}{n}\sum_{j=1}^{n} e_j - e_n\,.$$

In addition, $\langle \rho, \alpha\rangle = 1$ for all $\alpha \in \Delta$, and $\alpha^\vee = \alpha$, $\langle\rho,\alpha\rangle = 1$ for all $\alpha\in\Phi^+$. Hence

$$\rho' = \rho - \frac{n}{2}\,\varpi'\,, \qquad c' = 2\langle \varpi' - \rho', \alpha_{n-1}\rangle = n\,.$$

Accordingly, for positive roots $\alpha_{ij} := e_i - e_j \in \Phi^+$, we have

$$\langle \rho, \alpha_{ij}\rangle = j - i, \qquad \langle \varpi', \alpha_{ij}\rangle = \delta_{jn} - \delta_{in}, \tag{10}$$

and, for $\lambda_s := (ns-n)\varpi' + \rho$,

$$\langle \lambda_s, \alpha_{ij}\rangle = \begin{cases} j-i & \text{if } i,\, j \ne n, \\ ns - i & \text{if } j = n, \\ -ns + j & \text{if } i = n. \end{cases} \tag{11}$$

To write down the zeta function $\widehat{\zeta}_X^{SL_n}(s)$ explicitly, we will express the multiple residues in the periods of $(SL_n, P_{n-1,1})$ as a single limit, after multiplying by suitable vanishing factors (to the period of SL_n). Indeed, since $\langle \lambda_s - \rho, \alpha_{n-1}\rangle = ns - n$, and

$$\lim_{\lambda\to\lambda_s} \left(1 - q^{-\langle \lambda-\rho,\alpha\rangle}\right) \equiv 0 \qquad (\forall\ \alpha \in \Delta')\,, \tag{12}$$

we have

$$\omega_X^{SL_n, P_{n-1,1}}(s) = \lim_{\lambda \to \lambda_s} \left(\prod_{\alpha \in \Delta'} (1 - q^{-\langle \lambda - \rho, \alpha \rangle}) \cdot \omega_X^{SL_n}(\lambda) \right). \tag{13}$$

Recall that $\omega_X^{SL_n}(\lambda) = \sum_{w \in W} T_w(\lambda)$. Accordingly, to pin down the non-zero contributions for the terms appearing in the limit, we should consider, for a fixed $w \in W$, the limit $\lim_{\lambda \to \lambda_s} \big(\prod_{\alpha \in \Delta'} (1 - q^{-\langle \lambda - \rho, \alpha \rangle}) \cdot T_w(\lambda) \big)$, or equivalently, for a fixed $\sigma \in \mathfrak{S}_n (\simeq W)$, the function

$$L_\sigma(s) = \lim_{\lambda \to \lambda_s} \left(\frac{\prod_{\alpha \in \Delta'} (1 - q^{-\langle \lambda - \rho, \alpha \rangle})}{\prod_{\beta \in \Delta} (1 - q^{-\langle \sigma\lambda - \rho, \beta \rangle})} \prod_{\alpha \in \Phi^+,\, \sigma(\alpha) < 0} \frac{\widehat{\zeta}_X(\langle \lambda, \alpha \rangle)}{\widehat{\zeta}_X(\langle \lambda, \alpha \rangle + 1)} \right). \tag{14}$$

In order for this limit $L_\sigma(s)$ to be non-zero, by (12), there should be a complete cancellation of all of the factors $(1 - q^{-\langle \lambda - \rho, \alpha \rangle})$ in the numerator of the first term in (14) that vanish at $\lambda = \lambda_s$ with either

(i) factors $\big(1 - q^{-\langle \sigma\lambda_s - \rho, \beta \rangle}\big)$ appearing in the denominator of the first term in (14), or else

(ii) the poles at $\lambda = \lambda_s$ of factors $\widehat{\zeta}_X\big(\langle \lambda, \alpha \rangle\big)$ appearing in the numerator of the second term in (14) for which $\langle \lambda_s, \alpha \rangle = 1$.

Since $\langle \cdot\,, \cdot \rangle$ is σ-invariant, for $\alpha \in \Delta'$, by (10), $\langle \sigma\lambda_s - \rho, \alpha \rangle = \langle \lambda_s, \sigma^{-1}\alpha \rangle - 1$. Hence, for $L_\sigma(s)$ to have a non-zero contribution to $\omega_X^{(SL_n,\, P_{n-1,1})}(s)$, the union of

$$A_\sigma := \big\{ \alpha \in \Delta' : \sigma\alpha \in \Delta \big\} \quad \text{and} \quad B_\sigma := \big\{ \alpha \in \Delta' : \sigma\alpha < 0 \big\} \tag{15}$$

must be of cardinality $n - 2$. Call such $\sigma \in \mathfrak{S}_n$ *special* and denote the collection of special permutations by $\mathfrak{S}_n^0$. Clearly, for $\sigma \in \mathfrak{S}_n$, we have $A_\sigma \cup B_\sigma \subset \Delta'$, and $A_\sigma \cup B_\sigma = \Delta'$ if and only if $\sigma \in \mathfrak{S}_n^0$. That is to say, the limit $L_\sigma(s)$ corresponding to the permutation $\sigma \in \mathfrak{S}_n$ can only be non-zero if σ is special, and in this case, we have $\Delta' = A_\sigma \sqcup B_\sigma$. This then completes the proof of the following

Lemma 3. *With the notations above,*

$$\omega_X^{SL_n, P_{n-1,1}}(s) = \sum_{\sigma \in \mathfrak{S}_n^0} L_\sigma(s)\,. \tag{16}$$

Here $\sigma \in \mathfrak{S}_n^0$ if and only if $A_\sigma \cup B_\sigma = \Delta'$.

The next lemma describes $L_\sigma(s)$ for special permutations σ.

Lemma 4. *For $\sigma \in \mathfrak{S}_n^0$, set*

$$R_\sigma(s) = \prod_{\substack{1 \le k \le n-1 \\ \sigma^{-1}\alpha_k \notin \Delta'}} \big(1 - q^{-\langle \sigma\lambda_s - \rho, \alpha_k \rangle}\big), \quad \widehat{\zeta}_\sigma^{[n]}(s) = \prod_{\substack{1 \le i \le n-1 \\ \sigma(i) > \sigma(n)}} \frac{\widehat{\zeta}_X(\langle \lambda_s, \alpha_{in} \rangle)}{\widehat{\zeta}_X(\langle \lambda_s, \alpha_{in} \rangle + 1)},$$

$$\widehat{\zeta}_\sigma^{[<n]}(s) := \left(\prod_{\substack{1 \le k \le n-2 \\ \sigma(k) > \sigma(k+1)}} (1 - q^{-\langle \lambda - \rho, \alpha_k \rangle}) \cdot \prod_{\substack{1 \le i < j \le n-1 \\ \sigma(i) > \sigma(j)}} \frac{\widehat{\zeta}_X(\langle \lambda, \alpha_{ij} \rangle)}{\widehat{\zeta}_X(\langle \lambda, \alpha_{ij} \rangle + 1)} \right) \Bigg|_{\lambda = \lambda_s}.$$

Then

$$L_\sigma(s) = \frac{1}{R_\sigma(s)} \cdot \widehat{\zeta}_\sigma^{[n]}(s) \cdot \widehat{\zeta}_\sigma^{[<n]}(s)\,. \tag{17}$$

Proof. This is obtained by regrouping the terms of (14) for special permutation $\sigma \in \mathfrak{S}_n^0$, following the discussions above. We first cancel the terms in the numerator of the first factor in (14) for $\alpha \in A_\sigma$ with the corresponding terms in the denominator for $\beta = \sigma\alpha$. The first factor $1/R_\sigma(s)$ in (17) is the value at $\lambda = \lambda_\sigma$ of the product of the remaining terms $\beta \in \Delta \smallsetminus \sigma A_\sigma$ in this denominator. The second factor $\widehat{\zeta}_\sigma^{[n]}(s)$ in (17) is the value at $\lambda = \lambda_\sigma$ of the product of the terms in the second factor in (14) for $\alpha \notin \Phi'^+$, i.e. $\alpha = e_i - e_n > 0$. The third factor $\widehat{\zeta}_\sigma^{[<n]}(s)$ in (17), which can also be written

$$\widehat{\zeta}_\sigma^{[<n]}(s) \;=\; \left(\prod_{\alpha\in B_\sigma}(1-q^{-\langle\lambda-\rho,\alpha\rangle})\cdot\prod_{\substack{\alpha\in\Phi'^+\\ \sigma(\alpha)<0}}\frac{\widehat{\zeta}_X(\langle\lambda,\alpha\rangle)}{\widehat{\zeta}_X(\langle\lambda,\alpha\rangle+1)}\right)\Bigg|_{\lambda=\lambda_s},$$

is obtained by collecting all the remaining zeta factors and rational factors appearing in the numerator. $\square$

The terms occurring in $\widehat{\zeta}_\sigma^{[<n]}(s)$ are of two types: for $\alpha \in B_\sigma$ we must combine the quantities $(1-q^{-\langle\lambda-\rho,\alpha_k\rangle})$ and $\frac{\widehat{\zeta}_X(\langle\lambda,\alpha_{ij}\rangle)}{\widehat{\zeta}_X(\langle\lambda,\alpha_{ij}\rangle+1)}$ before taking the limit as $\lambda \to \lambda_s$ because the first has a zero and the second has a pole, while in the remaining zeta-quotients from the second term in (17), corresponding to $\alpha \in \Phi'^+ \smallsetminus B_\sigma$, we could simply substitute $\lambda = \lambda_s$ instead of taking a limit. We can say this differently as follows. By abuse of notation we write simply $\widehat{\zeta}_X(1)$ for the limit as $s \to 1$ of $(1-q^{1-s})\widehat{\zeta}_X(s)$. (It should be written $\widehat{v}_1$, as defined in (2), but the "$\widehat{\zeta}_X(1)$" notation will let us write more uniform formulas.) Then the definition of $\widehat{\zeta}_\sigma^{[<n]}(s)$ can be rewritten using the first equation in (11) as

$$\widehat{\zeta}_\sigma^{[<n]}(s) \;=\; \prod_{k\ge1}\left(\frac{\widehat{\zeta}_X(k)}{\widehat{\zeta}_X(k+1)}\right)^{m_\sigma(k)} \;=\; \prod_{k\ge1}\widehat{\zeta}_X(k)^{n_\sigma(k)} \tag{18}$$

where

$$m_\sigma(k) \;=\; \sum_{\substack{1\le i<j\le n-1\\ \sigma(i)>\sigma(j),\; j-i=k}} 1 \;=\; \#\{\alpha\in\Phi'^+ : \sigma\alpha<0, \langle\rho,\alpha\rangle = k\} \tag{19}$$

and

$$n_\sigma(k) \;=\; m_\sigma(k) - m_\sigma(k-1)\,, \qquad n_\sigma(1) \;=\; m_\sigma(1) \;=\; \#B_\sigma\,. \tag{20}$$

Equation (18) gives an explicit formula for the third factor in (17), which, as one sees, does not depend on s at all. The other two factors in (17), which do depend on s, will be computed later, in Section 5.

Lemmas 3 and 4 calculate the third factor $\omega_X^{G,P}(s)$ in the definition (8) of $\widehat{\zeta}_X^{G,P}(s)$ in the special case $G = SL_n$, $P = P_{n-1,1}$, but since some of the numbers $n_\sigma(k)$ in (18) may be negative, the expression for this factor may still contain some zeta values in its denominator. These zeta values in the denominator will be cancelled when we include the second factor $D^{G,P}(s)$ in (8). Our next task is therefore to evaluate this expression explicitly in the case $(G,P) = (SL_n,\, P_{n-1,1})$. Then the formulas for $D^{G,P}(s)$ and $\widehat{\zeta}_X^{G,P}(s)$ can be written down explicitly as follows.

Lemma 5. *We have*

$$D^{SL_n,P_{n-1,1}}(s) \;=\; \prod_{k=2}^{n-1} \widehat{\zeta}_X(k)\,\cdot\,\widehat{\zeta}_X(ns). \tag{21}$$

and

$$\widehat{\zeta}_X^{SL_n}(s) \;=\; q^{\frac{n(n-1)}{2}(g-1)}\,\cdot\,D^{SL_n,P_{n-1,1}}(s)\,\cdot\,\omega_X^{(SL_n,P_{n-1,1})}(s)\,. \tag{22}$$

Proof. In view of the definitions (7) and (8), we must show that $M_P(k,h)$ equals 1 if $k=0$ and $2\le h<n$ or $k=1$ and $h=n$ and vanishes otherwise, which follows easily from (6) since here $w_0=\begin{pmatrix}1&2&\cdots&n\\ n&n-1&\cdots&1\end{pmatrix}$. $\square$

4. Special Permutations

In this section we describe special permutations explicitly. Recall from §3 that σ is special if and only if $A_\sigma\sqcup B_\sigma=\Delta'$, where A_σ and B_σ are defined as in (15). This implies that σ is special if and only if $\sigma(i+1)=\sigma(i)+1$ or $\sigma(i+1)<\sigma(i)$ for all $1\le i\le n-2$ (or equivalently, since σ is a permutation, if and only $\sigma(i+1)\le\sigma(i)+1$ for all $1\le i\le n-2$). Denote by $t_1>\cdots>t_m$ the distinct values of $\sigma(i)-i$ for $1\le i\le n-2$, and by I_ν $(1\le\nu\le m)$ the set of $i\in\{1,\dots,n-2\}$ with $\sigma(i)-i=t_\nu$. Then σ maps I_ν onto its image $I'_\nu=\sigma(I_\nu)$ by translation by t_ν, and we have $\bigcup I_\nu=\{1,\dots,n-1\}$ and $\bigcup I'_\nu=\{1,\dots,n\}\smallsetminus\{a\}$, where $a=\sigma(n)\in\{1,\dots,n\}$. It is easy to check[3] that $I_1<\cdots<I_m$ (in the sense that all elements of I_ν are less than all elements of $I_{\nu+1}$ if $1\le\nu\le m-1$) and $I'_1>\cdots>I'_m$ (in the same sense). These properties characterize special permutations and are illustrated in the figure at the end of the section, in which the lengths of the intervals I_ν with I'_ν above (respectively below) a are denoted by $k_1,\dots,k_p$ (resp. by $\ell_1,\dots,\ell_r$), so that $\sum_{i=1}^p k_i=n-a$, $\sum_{j=1}^r \ell_j=a-1$, and $p+r=m$. We will denote the corresponding special permutation by $\sigma(k_1,\dots,k_p;a;l_1,\dots,l_r)$ and also define two sequences of numbers $0=K_0<K_1<\cdots<K_p=n-a$ and $0=L_0<L_1<\cdots<L_r=a-1$ by

$$K_i=\;k_1+\cdots+k_i\quad(1\le i\le p),\qquad L_j=\;l_1+\cdots+l_j\quad(1\le j\le r). \tag{23}$$

Remark. Denote by $\mathfrak{S}_{n,a}$ $(a=1,\dots,n)$ the set of special permutations in $\mathfrak{S}_n$ with $\sigma(n)=a$. From the above description we find that $\mathfrak{S}_{n,a}\cong X_{n-a}\times X_{a-1}$ where X_K for $K\ge0$ is the set of ordered partitions of K (decompositions $K=k_1+\cdots+k_p$ with all $k_i\ge1$). Clearly the cardinality of X_K equals 1 if $K=0$ (in which case only $p=0$ can occur) and 2^{K-1} if $K\ge1$ (the ordered partitions of K are in 1:1 correspondence with the subsets of $\{1,\dots,K-1\}$, each such subset dividing the interval $[0,K]\subset\mathbb{R}$

[3]Indeed, let A denote the set of indices $i\in\{1,\dots,n-2\}$ with $\sigma(i+1)=\sigma(i)+1$. Then $\sigma(i)-i$ is constant when we pass from any $i\in A$ to $i+1$, so each set I_ν is a connected interval that is contained in A except for its right end-point i_0, which satisfies $\sigma(i_0+1)<\sigma(i_0)$, so that i_0+1 belongs to an I_μ satisfying $t_\mu<t_\nu$ and hence $\mu>\nu$. But then I_μ contains a point that is bigger than one of the points of I_ν and that has an image under σ that is smaller than the image of that point, and since all of these sets are connected intervals this means that all of I_μ lies to the right of all of I_ν and that all of I'_μ lies to the left of all of I'_ν, proving the assertion.

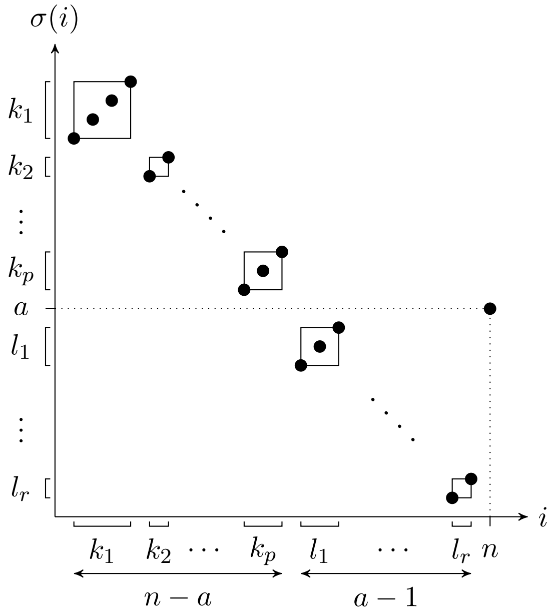

FIGURE 1. The special permutation $\sigma(k_1,\ldots,k_p;a;l_1,\ldots,l_r)$

into intervals of positive integral length), so $|\mathfrak{S}_{n,a}|$ equals 2^{n-2} for $a\in\{1,n\}$ and 2^{n-3} for $1<a<n$, and the whole set $\mathfrak{S}_n^0$ has cardinality $2^{n-3}(n+2)$.

5. PROOF OF THEOREM 2

In this section, we use the characterization of special permutations given in §4 to calculate the rational factor $R_\sigma(s)$ and the zeta factors $\widehat{\zeta}_\sigma^{[n]}(s)$ and $\widehat{\zeta}_\sigma^{[<n]}(s)$ appearing in Lemma 4 explicitly for special permutations σ. We begin with $R_\sigma(s)$.

Lemma 6. *For the special permutation $\sigma=\sigma(k_1,\ldots,k_p;a;l_1,\ldots,l_r)$, the quantity $R_\sigma(s)$ defined in Lemma 4 is given by*

$$\begin{aligned} R_\sigma(s) \;&=\; (1-q^{k_1+k_2})\cdots(1-q^{k_{p-1}+k_p})\cdot(1-q^{ns-n+a+k_p}) \\ &\qquad \times(1-q^{-ns+n-a+l_1+1})\,\cdot\,(1-q^{l_1+l_2})\cdots(1-q^{l_{r-1}+l_r})\,. \end{aligned}$$

Proof. By definition,

$$R_\sigma(s) \;=\; \prod_{\substack{1\le k\le n-1\\ \sigma^{-1}(\alpha_k)\notin\Delta'}} \left(1-q^{-\langle\sigma\lambda_s-\rho,\alpha_k\rangle}\right) \;=\; \prod_{\substack{1\le k\le n-1\\ \sigma^{-1}(\alpha_k)\notin\Delta'}} \left(1-q^{1-\langle\lambda_s,\sigma^{-1}\alpha_k\rangle}\right)$$

For each k occurring in this product, write $\sigma^{-1}(\alpha_k)=e_i-e_j=:\alpha_{ij}$. Then the condition $\alpha_{ij}\notin\Delta'$ says that the points $(i,\sigma(i)=k)$ and $(j,\sigma(j)=k+1)$ do *not* belong to the same square block in the picture of the graph of σ given in the last section. From that picture, we see that the k's occurring in the product, in decreasing order, together with the corresponding values of i and j, are given by the first three columns of the following table

k	$i=\sigma^{-1}(k)$	$j=\sigma^{-1}(k+1)$	$1-\langle\lambda_s,\alpha_{ij}\rangle$
$n-K_\mu\ (1\le\mu<p)$	$K_{\mu+1}$	$K_{\mu-1}+1$	$k_\mu+k_{\mu+1}$
a	n	$K_{p-1}+1$	$ns-n+a+k_p$
$a-1$	$n-a+l_1$	n	$-ns+n-a+l_1+1$
$n-L_\nu\ (1\le\nu<r)$	$L_{\nu+1}$	$L_{\nu-1}+1$	$l_\nu+l_{\nu+1}$

while the fourth column follows from equation (11). The lemma follows. $\square$

We next consider the zeta factor $\widehat{\zeta}_\sigma^{[n]}(s)$.

Lemma 7. *For the special permutation $\sigma=\sigma(k_1,\dots,k_p;a;l_1,\dots,l_r)$, the zeta factor $\widehat{\zeta}_\sigma^{[n]}(s)$ of $L_\sigma(s)$ is given by*

$$\widehat{\zeta}_\sigma^{[n]}(s) \;=\; \frac{\widehat{\zeta}_X(ns-n+a)}{\widehat{\zeta}_X(ns)}\;.$$

This lemma implies in particular that to normalize $\widehat{\zeta}_\sigma^{[n]}(s)$ we at least need to clear the denominator by multiplying by the zeta factor $\widehat{\zeta}_X(ns)$.

Proof. This is much easier. From $\lambda_s=(ns-n)\varpi+\rho$, we get $\langle\lambda_s,e_i-e_n\rangle = ns-i$. Moreover, by the graph in §4, for the special permutation $\sigma=\sigma(k_1,\dots,k_p;a;l_1,\dots,l_r)$, we have

$$\{e_i-e_n\,:\,1\le i<n,\ \sigma(i)>\sigma(n)\}\;=\;\{e_1-e_n,e_2-e_n,\dots,e_{n-a}-e_n\}\,.$$

Therefore, by the definition of $\widehat{\zeta}_\sigma^{[n]}(s)$ given in Corollary 4, we have

$$\begin{aligned}\widehat{\zeta}_\sigma^{[n]}(s) \;&=\; \prod_{\substack{\alpha=e_i-e_n,\,i\le n-1\\ \sigma(i)>\sigma(n)}} \frac{\widehat{\zeta}_X(\langle\lambda,\alpha\rangle)}{\widehat{\zeta}_X(\langle\lambda,\alpha\rangle+1)}\Big|_{\lambda=\lambda_s}\\ &=\;\prod_{i=1}^{n-a}\frac{\widehat{\zeta}_X(ns-i)}{\widehat{\zeta}_X(ns-i+1)} \;=\; \frac{\widehat{\zeta}_X(ns-n+a)}{\widehat{\zeta}_X(ns)}\end{aligned}$$

as asserted. $\square$

Finally, we treat the zeta factor $\widehat{\zeta}_\sigma^{[<n]}(s)$. However, with the normalization stated in Lemma 5, to obtain the group zeta function $\widehat{\zeta}_X^{SL_n}(s)$, it suffices to investigate the product $\widehat{\zeta}_\sigma^{[<n]}(s)\cdot\prod_{i\ge2}\widehat{\zeta}_X(i)^{-n(i)}$, or equivalently, by (18), the product $\widehat{\zeta}_X(1)^{\#B_\sigma}\prod_{i\ge2}\widehat{\zeta}_X(i)^{n_\sigma(i)-n(i)}$, which we write as $\prod_{i\ge1}\widehat{\zeta}_X(i)^{r_\sigma(i)}$ with

$$r_\sigma(k)\;=\;\begin{cases}\#\,B_\sigma & \text{if } k=1,\\ n_\sigma(k)-n(k) & \text{if } k\ge2,\end{cases}$$

where the numbers $n(k)$ are defined, in analogy with the numbers $n_\sigma(k)$ in Section 3 (equations (19) and (20)), by

$$m(k)\;=\;\#\{\alpha>0:\langle\rho,\alpha\rangle=k\}\,,\qquad n(k)\;=\;m(k)-m(k-1)\,.$$

Clearly $m(k)=n-k$ for $1\le k\le n$ and $n(k)=-1$ for $2\le k\le n$.

Lemma 8. *For the special permutation $\sigma = \sigma(k_1, \dots, k_p; a; l_1, \dots, l_r)$, we have*

$$\prod_{i\geq 1} \widehat{\zeta}_X(i)^{r_\sigma(i)} \;=\; \prod_{i=1}^{p} \widehat{v}_{k_i} \cdot \prod_{j=1}^{r} \widehat{v}_{l_j} \,. \tag{24}$$

In particular, $r_\sigma(k) \geq 0$.

Proof. This is based on a detailed analysis of $r_\sigma(k)$. Obviously,

$$r_\sigma(1) = \#\{\alpha \in \Delta' : \sigma\alpha < 0\} = \#\{(i, i+1) : 1 \leq i \leq n-2,\ \sigma(i) > \sigma(i+1)\}.$$

If $k \geq 2$, by definition,

$$\begin{aligned} m(k) - m_\sigma(k) &= \#\{\alpha > 0 : \langle\rho, \alpha\rangle = k\} - \#\{\alpha \in {\Phi'}^+ : \sigma\alpha < 0, \langle\rho, \alpha\rangle = k\} \\ &= \#\{e_i - e_n : \langle\rho, \alpha\rangle = k\} + \#\{\alpha \in {\Phi'}^+ : \sigma\alpha > 0, \langle\rho, \alpha\rangle = k\} \\ &= 1 + \#\{\alpha \in {\Phi'}^+ : \sigma\alpha > 0, \langle\rho, \alpha\rangle = k\}, \end{aligned}$$

since, by (10), $\{e_i - e_n : \langle\rho, \alpha\rangle = k\} = \{e_{n-k} - e_n\}$. Thus, by applying the characterization graph in §4 for special permutation $\sigma(k_1, \dots, k_p; a; l_1, \dots, l_r)$, we conclude that $\alpha = \alpha_{ij} \in {\Phi'}^+$ satisfying $\sigma\alpha > 0$ (or equivalently $\alpha = \alpha_{ij}$ satisfying $i < j \leq n-1$ and $\sigma(i) < \sigma(j)$) if and only if i and j belong to the same block, say I_μ for some μ, associated to $\sigma(k_1, \dots, k_p; a; l_1, \dots, l_r)$, and also $\sigma(j) \in I_\mu$ (or equivalently $j+1 \in I_\mu$), since otherwise $\sigma(\alpha_{ij}) < 0$.

Denote by $(m(k) - m_\sigma(k))_\mu$ (resp. $r_{\sigma,\mu}(k)$) the contribution to $m(k) - m_\sigma(k)$ (resp. to $r_\sigma(k)$) of the block I_μ. With the discussion above, we have

$$m(k) - m_\sigma(k) \;=\; \sum\nolimits_\mu (m(k) - m_\sigma(k))_\mu \qquad \text{and} \quad r_\sigma(k) = \sum\nolimits_\mu r_{\sigma,\mu}(k).$$

Fix some μ and let $I_\mu := \{a+1, a+2, \dots, a+b\}$ with $a,\, b \in \mathbb{Z}_{>0}$. Clearly, when $k = 1$, $r_{\sigma,\mu}(1) = \#\{(a+b-1, a+b)\} = 1$, since, for other $(i, i+1)$'s, $\sigma(i) < \sigma(i+1)$. Moreover, when $k \geq 2$, by (10) and the characterization of the graph again, we have

$$\begin{aligned} (m(k) - m_\sigma(k))_\mu &= \#\big\{(i, j) :\ i,\, j+1 \in I_\mu,\ i < j,\ j = i+k\big\} \\ &= \#\big\{(i, j) :\ a+1 \leq i < j < a+b,\ j = i+k\big\}. \end{aligned}$$

Note that, for each fixed i (with $a+1 \leq i < a+b$),

$$\#\Big\{(i, j) :\ a+1 \leq i < j < a+b,\ j = i+k\Big\} = \begin{cases} 1 & i+k < a+b \\ 0 & i+k \geq a+b \end{cases}.$$

Hence, $(m(k) - m_\sigma(k))_\mu \;=\; b - (k+1)$. This implies that, for all $k \geq 1$ $r_{\sigma,\mu}(k) = (m(k-1) - m_\sigma(k-1))_\mu - (m(k) - m_\sigma(k))_\mu = 1$. Consequently,

$$\prod_{i\geq 1} \widehat{\zeta}_X(k)^{r_{\sigma,\mu}(k)} \;=\; \widehat{\zeta}_X(1)\, \widehat{\zeta}_X(2) \cdots \widehat{\zeta}_X(b) \,.$$

Equation (24) follows. $\square$

Combining Lemmas 5, 6, 7, and 8, we get

$$\frac{\widehat{\zeta}_X^{SL_n}(s)}{q^{\frac{n(n-1)}{2}(g-1)}} = \prod_{i\geq 2}\widehat{\zeta}_X(i)^{-n(i)}\cdot\lim_{\lambda\to\lambda_s}\Big(\prod_{\alpha\in\Delta_P}(1-q^{-\langle\lambda-\rho,\alpha^\vee\rangle})\cdot\omega_X^{SL_n}(\lambda)\Big)$$

$$= \sum_{a=1}^{n}\sum_{\substack{k_1,\dots,k_p>0\\ k_1+\cdots+k_p=n-a}}\frac{\widehat{v}_{k_1}\dots\widehat{v}_{k_p}}{(1-q^{k_1+k_2})\dots(1-q^{k_{p-1}+k_p})}\cdot\frac{1}{1-q^{ns-n+a+k_p}}$$

$$\times\ \widehat{\zeta}(ns-n+a)$$

$$\times\sum_{\substack{l_1,\dots,l_r>0\\ l_1+\cdots+l_r=a-1}}\frac{1}{1-q^{-ns+n-a+1+l_1}}\cdot\frac{\widehat{v}_{l_1}\dots\widehat{v}_{l_r}}{(1-q^{l_1+l_2})\dots(1-q^{l_{r-1}+l_r})}\,.$$

This completes the proof of Theorem 2.

6. The theorem of Mozgovoy and Reineke

In the previous three sections we have given an explicit formula for the group zeta function associated to a curve over a finite field in the case $(G,P)=(SL_n,P_{n-1,1})$. As explained in the introduction, our main result (Theorem 1) will follow by comparing this formula with the explicit formula for the rank n non-abelian zeta function $\widehat{\zeta}_{X,n}(s)$ found by Mozgovoy and Reineke, namely:

Theorem (Theorem 7.2 of [6]**).** *The function $\widehat{\zeta}_{X,n}(s)$ is given by*

$$\widehat{\zeta}_{X,n}(s) = q^{\binom{n}{2}(g-1)}\sum_{h=1}^{n-1}\sum_{\substack{n_1,\dots,n_h>0\\ n_1+\cdots+n_h=n-1}}\frac{\widehat{v}_{n_1}\cdots\widehat{v}_{n_h}}{\prod_{j=1}^{h-1}(1-q^{n_j+n_{j+1}})}$$
$$\times\Big(\frac{\widehat{\zeta}_X(ns)}{1-q^{-ns+n_1+1}}$$
$$+\sum_{i=1}^{h-1}\frac{(1-q^{n_i+n_{i+1}})\cdot\widehat{\zeta}_X(ns-(n_1+\cdots+n_i))}{(1-q^{ns-(n_1+\cdots+n_{i-1})})(1-q^{-ns+n_1+\cdots+n_{i+1}+1})}$$
$$+\frac{\widehat{\zeta}_X(ns-n+1)}{1-q^{ns-(n_1+\cdots+n_{k-1})}}\Big)\,. \tag{25}$$

This already looks very similar to Theorem 2, and the precise equality of the two formulas will be verified in §7. But since the ideas leading to the expressions for the group zeta function and for the non-abelian zeta function are very different, and since the ideas of the proof in [6] are very interesting, we include a brief account of their calculation for the benefit of the interested reader. A reader who is interested only in the proof of the main result, or who is already familiar with the paper [6], can skip immediately to Section 7.

The first ingredient is that of semi-stable pairs and triples. Fix an integral regular projective curve X over a finite field $\mathbb{F}_q$. By a *pair* (E,s) over X we mean a vector bundle E on X together with a global section s of E on X. Such pairs form an $\mathbb{F}_q$-linear category, a morphism $(E,s)\to(E',s')$ being an element $(\lambda,f)\in\mathbb{F}_q\times\mathrm{Hom}_X(E,E')$ such that $f\circ s=\lambda\, s'$. A pair (E,s) is called *τ-semistable* ($\tau\in\mathbb{R}$) if $\mu(F)\leq\tau$ for any sub-bundle F of E and

$\mu(E/F) \geq \tau$ for any subbundle F of E with $s \in H^0(X, F)$. Here, as usual, $\mu(E)$ denotes the Mumford slope of E. For $(r, d) \in \mathbb{Z}_{>0} \times \mathbb{Z}$ we denote by $\mathcal{M}_X^\tau(r, d)$ the moduli stack of τ-semistable pairs (E, s) of rank r and degree d. If $\tau = d/r$, then this is the same as the usual slope semistability of E, so if we write $\mathcal{M}_X(r, d)$ for the moduli space of semistable bundles of rank r and degree d, then (cf. Corollary 3.7 of [6])

$$\sum_{(E,s)\in\mathcal{M}_X^{d/r}(r,d)} \frac{1}{\#\mathrm{Aut}(E, s)} \;=\; \frac{1}{q-1} \sum_{E\in\mathcal{M}_X(r,d)} \frac{q^{h^0(X,E)} - 1}{\#\mathrm{Aut}\, E}\,.$$

Next, we consider triples $\mathcal{E} = (E_0, E_1, s)$ consisting of two coherent sheaves E_0, E_1 on X and a morphism $s : E_1 \to E_0$. These triples form an abelian category which we denote by $\mathcal{A}$. The triple $\mathcal{E} = (E_0, E_1, s)$ is called μ_τ-*semistable* if $\mu_\tau(\mathcal{F}) \leq \mu_\tau(\mathcal{E})$ for any sub-object $\mathcal{F}$ of $\mathcal{E}$, where

$$\mu_\tau(\mathcal{E}) \;:=\; \frac{\deg E_0 + \deg E_1 + \tau \cdot \mathrm{rank} E_1}{\mathrm{rank} E_0 + \mathrm{rank} E_1}\,.$$

We also introduce $\chi(\mathcal{E}, \mathcal{F}) := \sum_{k=0}^2 (-1)^k \dim \mathrm{Ext}^k_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$. It is known that $\chi(\mathcal{E}, \mathcal{F}) \;=\; \chi(E_0, F_0) + \chi(E_1, F_1) - \chi(E_1, F_0)$, where as usual, $\chi(E, F) := \dim \mathrm{Hom}(E.F) - \dim \mathrm{Ext}^1(E, F)$. For $\alpha = (r, d)$, $\beta = (r', d') \in \mathbb{Z}_{>0} \times \mathbb{Z}$, set $\chi(\alpha) = d - (g-1)r$ and $\langle \alpha, \beta\rangle := 2(rd' - r'd)$. Similarly, for $\underline{\alpha} = (\alpha, v)$, $\underline{\beta} = (\beta, w)$ with v, $w \in Z_{\geq 0}$ we set $\langle \underline{\alpha}, \underline{\beta}\rangle := \langle \alpha, \beta\rangle - v\,\chi(\beta) + w\,\chi(\alpha)$.

The next ingredients are Hall algebras and integration maps. Let $K_0(\mathrm{St}_{\mathbb{F}_q})$ be the Grothendieck ring of finite type stacks over $\mathbb{F}_q$ with affine stabilizers and $\mathbb{L}$ be the Lefschetz motive. We introduce the coefficient ring $R = K_0(\mathrm{St}_{\mathbb{F}_q})[\mathbb{L}^{\pm 1/2}]$ and define the *quantum affine plane* $\mathbb{A}_0$ to be the completion of the algebra $R[x_1, x_2^{\pm 1}]$ with the multiplication

$$x^\alpha \circ x^\beta \;:=\; (-\mathbb{L}^{1/2})^{\langle \alpha, \beta\rangle} x^{\alpha+\beta}\,.$$

(Here the completion is defined by requiring that for $f = \sum_{\alpha\in\mathbb{N}\times\mathbb{Z}} f_\alpha x^\alpha \in \mathbb{A}_0$ and any $t \in \mathbb{R}$ there are only finitely many (r, d) with $f_{r,d} \neq 0$ and $\frac{d}{r+1} < t$.) If we further denote by $\mathcal{A}_0$ the category of coherent sheaves on X and by $H(\mathcal{A}_0)$ its associated *Hall algebra*, whose multiplication $[E] \circ [F]$ counts extensions from $\mathrm{Ext}^1(F, E)$, then we have a morphism of algebras

$$\begin{array}{rccc} I: & H(\mathcal{A}_0) & \longrightarrow & \mathbb{A}_0 \\ & E & \mapsto & (-\mathbb{L}^{1/2})^{\chi(E,E)} \cdot \frac{x^{\mathrm{ch}(E)}}{[\mathrm{Aut} E]}\,, \end{array}$$

which we call the *integration map*. Here $\mathrm{ch}(E) := (\mathrm{rank} E, \deg E)$. Similarly, if we introduce a second quantum affine plane $\mathbb{A}$ as the completion of the algebra $R[x_1, x_2^{\pm 1}, x_3]$ with the multiplication

$$x^{\underline{\alpha}} \circ x^{\underline{\beta}} \;:=\; (-\mathbb{L}^{1/2})^{\langle \underline{\alpha}, \underline{\beta}\rangle} x^{\underline{\alpha}+\underline{\beta}}\,,$$

then we have an integration map on the Hall algebra $H(\mathcal{A})$

$$\begin{array}{rccc} \underline{I}: & H(\mathcal{A}) & \longrightarrow & \mathbb{A} \\ & \mathcal{E} & \mapsto & (-\mathbb{L}^{1/2})^{\chi(\mathcal{E},\mathcal{E})} \cdot \frac{x^{\mathrm{cl}(\mathcal{E})}}{[\mathrm{Aut}\mathcal{E}]}\,, \end{array}$$

where $\mathrm{cl}(E) := (\mathrm{rank} E_0, \deg E_0, \mathrm{rank} E_1)$. We have $\underline{I}|_{H(\mathcal{A}_0)} = I$. The map $\underline{I}$ is not an algebra morphism in general, but if $\mathrm{Ext}^2(\mathcal{F}, \mathcal{E}) = 0$, then $\underline{I}(\mathcal{E} \circ \mathcal{F}) = \underline{I}(\mathcal{E})\underline{I}(\mathcal{F})$.

The last and most important ingredient of the proof in [6] is a *wall-crossing formula*. For $\alpha=(r,d)\in\mathbb{Z}_{>0}\times\mathbb{Z}$ and $\tau\in\mathbb{R}$, let

$$\mathfrak{u}(\alpha) \;:=\; (-\mathbb{L}^{-1/2})^{\chi(\alpha,\alpha)+d}\,[\mathcal{M}_X(\alpha)]$$

be the motivic class of $\mathcal{M}_X(\alpha)$ counting semi-stable bundles E on X with $\operatorname{ch}E=\alpha$, and similarly set

$$\mathfrak{f}_\tau(\alpha) \;=\; (\mathbb{L}-1)(-\mathbb{L}^{-1/2})^{\chi(\alpha,\alpha)+d}[\mathcal{M}_X^\tau(\alpha)]\,.$$

We introduce the two generating series

$$\mathfrak{u}_\tau \;=\; 1+\sum_{\mu(\alpha)=\tau}\mathfrak{u}(\alpha)\,x^\alpha\in\mathbb{A}_0\,,\qquad \mathfrak{f}_\tau \;=\; \sum_\alpha \mathfrak{f}_\tau(\alpha)\,x^{(\alpha,1)}\in\mathbb{A}\,.$$

Then the rank n non-abelian zeta function for X can be expressed as

$$\zeta_{X,n}(s) \;=\; (q-1)\sum_{k\ge0}[\mathcal{M}_X(n,kn)]q^{-sk} \;=\; q^{\frac{n(n-1)}{2}(g-1)}\sum_{k\ge0}\mathfrak{f}_k(n,kn)q^{-ks}\,.$$

We can also identify the moduli stack $\mathcal{M}_X^\infty(1,d)$ with the Hilbert scheme Hilb^dX or with Sym^dX, the d-th symmetric product of X. Consequently,

$$\mathfrak{f}_\infty \;:=\; x_1x_3\sum_{d\ge0}[\mathrm{Sym}^dX]\,x_2^d \;=\; x_1x_3\,Z_X(x_2)$$

where $Z_X(t)$ is the Artin zeta function with $\zeta_X(s)=Z_X(q^{-s})$. (This can be interpreted as the limiting special case of $\mathfrak{f}_\tau$ as $\tau\to\infty$, since the condition of semistability with respect to τ of a pair (E,s) in the limit $\tau\to\infty$ is equivalent to the requirement that $\operatorname{coker}(s)$ is finite.) Finally, set

$$\mathfrak{u}_{\ge\tau} \;:=\; \overrightarrow{\prod_{\tau'\ge\tau}}\mathfrak{u}_{\tau'}\,,$$

where the product is taken in the decreasing slope order, and, for an element $\mathfrak{g}=\sum_\alpha g_\alpha x^{(\alpha,1)}\in\mathbb{A}$, set

$$\mathfrak{g}\big|_{\mu\le\tau} \;:=\; \sum_{\mu(\alpha)<\tau} g_\alpha x^{(\alpha,1)}\,.$$

Then, using the theory of Hall algebras and wall-crossing techniques, the main result (Theorem 5.4) of [6] is the identity

$$\mathfrak{f}_\tau=\Big(\mathfrak{u}_{>\tau}^{-1}\circ f_\infty\circ\mathfrak{u}_{\ge\tau}\Big)\Big|_{\mu\le\tau}\qquad(\tau\in\mathbb{R})\,.$$

Equation (25) is obtained from this basic formula by a somewhat involved combinatorial discussion, using a "Zagier-type formula" (i.e., one based on the combinatorics in [13]) for the motivic classes of moduli spaces of semi-stable bundles.

7. Proof of Theorem 1 and structure of the function $\zeta_{X,n}(s)$

To complete the proof of Theorem 1, we verify the term-by-term equality of the sums appearing in (5) and (25). Clearly, the factor $q^{\binom{n}{2}(g-1)}$ is the same in both cases. Both sums have the form of a linear combination of $\widehat{\zeta}_X(ns-k)$ with $0\le k\le n-1$, so we only have to check the equality of the coefficients. The case $k=0$ is immediate: since $B_0(x)$ is identically 1, the coefficient of $\widehat{\zeta}_X(ns)$ in the sum in (5) is $B_{n-1}(q^{1-ns})$, which by formula (4) is identical with the coefficient of $\widehat{\zeta}_X(ns)$ in the sum in (25). (Set $p=h$, $k_i = n_{h+1-i}$.) The case $k=n-1$ is exactly similar, or can be deduced from the case $k=0$ by noticing that (5) is invariant under $k\to n-1-k$, $s\to 1-s$ and (25) under $n_j \to n_{h+1-j}$, $i\to h-i$, and $s\to 1-s$. If $0<k<n-1$ then the coefficient of $\widehat{\zeta}_X(ns-k)$ in the sum in (25) can be rewritten

$$\sum_{0<i<h<n} \sum_{\substack{n_1+\cdots n_i=k\\ n_{i+1}+\cdots+n_h=n-1-k}} \left(\frac{\widehat{v}_{n_1}\cdots\widehat{v}_{n_i}}{\prod_{j=1}^{i-1}(1-q^{n_j+n_{j+1}})}\cdot\frac{1}{1-q^{ns-k+n_i}}\right.$$
$$\left.\times\ \frac{\widehat{v}_{n_{i+1}}\cdots\widehat{v}_{n_h}}{\prod_{j=i+1}^{h-1}(1-q^{n_j+n_{j+1}})}\cdot\frac{1}{1-q^{-ns+k+n_{i+1}+1}}\right),$$

and since the tuples $(n_1,\ldots,n_i)$ with sum k and the tuples $(n_{i+1},\ldots,n_h)$ with sum $n-k-1$ are independent, this equals $B_k(q^{ns-k})B_{n-k-1}(q^{k+1-ns})$ as required. This completes the comparison of formulas (5) and (25) and hence the proof of Theorem 1.

We end the paper by looking briefly at the structure of the explicit formula for the higher rank zeta function $\zeta_{X,n}(s)$, and in particular check that it implies the known properties of this zeta function as listed in the opening paragraph. One of these properties was the functional equation $\widehat{\zeta}_{X,n}(1-s) = \widehat{\zeta}_{X,n}(s)$, which, as we have already said, follows immediately from (5) by interchanging k and $n-k-1$ and using the known functional equation $\widehat{\zeta}_X(1-s) = \widehat{\zeta}_X(s)$. The other concerned the form of $\zeta_{X,n}(s)$. Here it is more convenient to work with the variables $t=q^{-s}$ and $T=q^{-ns}=t^n$, writing $\zeta_X(s)$ and $\zeta_{X,n}(s)$ as $Z_X(t)$ and $Z_{X,n}(T)$, respectively, and similarly $\widehat{\zeta}_X(s)=\widehat{Z}_X(t)$ and $\widehat{\zeta}_{X,n}(s)=\widehat{Z}_{X,n}(T)$ with $\widehat{Z}_X(t)=t^{1-g}Z_X(t)$, $\widehat{Z}_{X,n}(T) = T^{1-g}Z_{X,n}(T)$. It is well known that $Z_X(t)$ has the form $P(t)/(1-t)(1-qt)$ where $P(t)=P_X(t)$ is a polynomial of degree $2g$, and the assertion is that $Z_{X,n}(T)$, which from the definition (1) is just a power series in T, has the corresponding form $P_n(T)/(1-T)(1-q^nT)$ where $P_n(T)=P_{X,n}(T)$ is again a polynomial of degree $2g$. In these terms, the formula for the rank n zeta function becomes

$$q^{-\binom{n}{2}(g-1)}\,\widehat{Z}_{X,n}(T) \;=\; \sum_{k=0}^{n-1} B_k(q^{-k}T^{-1})\,\widehat{Z}_X(q^kT)\,B_{n-k-1}(q^{k+1}T)\,. \tag{26}$$

From this it is clear that $\widehat{Z}_{X,n}(T)$ is a rational function of T and grows at most like $\mathrm{O}(T^{g-1})$ as $T\to\infty$ and like $\mathrm{O}(T^{1-g})$ as $T\to 0$, since the definition of the function $B_k(x)$ shows that it is bounded at both 0 and ∞, so the only non-trivial assertion is that $\widehat{Z}_{X,n}(T)$ has at most simple poles at $T=1$ and $T=q^{-n}$ and no other poles. From the definition of $B_k(x)$ and the properties of $\widehat{Z}_X(t)$ we see that every term in (26) has simple poles at $T=1,q^{-1},\ldots,q^{-n}$ (the first factor has simple poles at q^{-i} with $0\le i<k$, the second at $i=k$ and $i=k+1$, and the third at $k+1<i\le n$), so the only thing that needs to be checked is that the residues at q^{-i} for $0<i<n$ sum to 0. Denote by R_i $(0\le i\le n)$ the limiting value as $T\to q^{-i}$ of the right-hand side of (26) multiplied by $1-q^iT$, and by $R_{i,k}$ the corresponding

contribution from the kth term, so that $R_i = \sum_{k=0}^{n-1} R_{i,k}$. Suppose that $0 < i < n$. Then for $0 \le k \le i-2$ we find

$$R_{i,k} \;=\; B_k(q^{i-k})\,\widehat{Z}_X(q^{k-i})\,\widehat{v}_{i-k-1}\,B_{n-i}(q^{i-k-1})$$

and for $k = i-1$ we find

$$R_{i,i-1} \;=\; B_{i-1}(q)\,\widehat{v}_1\,B_{n-i}(1)\,.$$

Since $\widehat{Z}_X(q^{k-i})\widehat{v}_{i-k-1} = \widehat{v}_{i-k}$, these formulas can be written uniformly as

$$R_{i,k} \;=\; B_k(q^{i-k})\,\widehat{v}_{i-k}\,B_{n-i}(q^{i-k-1}) \qquad (0 \le k \le i-1).$$

The formulas in the other two cases can be computed similarly, but this is not necessary since the above-mentioned symmetry of the terms in (26) under $(k,T) \mapsto (n-1-k, q^{-n}T^{-1})$ implies that $R_{i,k} = -R_{n-i,n-k-1}$ and hence $R_i = S_i - S_{n-i}$ with $S_i = \sum_{k=0}^{i-1} R_{i,k}$. But the formula just proved for $R_{i,k}$ for $0 \le k \le i-1$ can be rewritten as

$$R_{i,k} \;=\; \sum_{1\le s<r\le n} \;\; \sum_{\substack{n_1,\dots,n_r\ge 1\\ n_1+\cdots+n_r=n\\ n_1+\cdots+n_{s-1}=k,\; n_s=i-k}} \frac{\widehat{v}_{n_1}\cdots\widehat{v}_{n_r}}{(1-q^{n_1+n_2})\cdots(1-q^{n_{r-1}+n_r})},$$

so

$$S_i \;=\; \sum_{1\le s<r\le n} \;\; \sum_{\substack{n_1,\dots,n_r\ge 1\\ n_1+\cdots+n_r=n\\ n_1+\cdots+n_s=i}} \frac{\widehat{v}_{n_1}\cdots\widehat{v}_{n_r}}{(1-q^{n_1+n_2})\cdots(1-q^{n_{r-1}+n_r})}\,,$$

which is visibly symmetric under $i \mapsto n-i$ by replacing n_j by n_{r+1-j} and s by $r+1-s$. This completes the proof of vanishing of R_i for $0 < i < n$, and by essentially the same calculation we also get the corresponding formulas

$$R_n \;=\; -\,R_0 \;=\; \sum_{r=1}^{n} \;\; \sum_{\substack{n_1,\dots,n_r\ge 1\\ n_1+\cdots+n_r=n}} \frac{\widehat{v}_{n_1}\cdots\widehat{v}_{n_r}}{(1-q^{n_1+n_2})\cdots(1-q^{n_{r-1}+n_r})}$$

for the two remaining coefficients R_i describing the poles of $\zeta_{X,n}(s)$.

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